Question 1

The correct answers are:

(a) (2)

(b) (3)

(c) (3)

(d) (2)

(e) (1)

(f) (3)

(g) (2)

(h) (1)

Question 2

(a) Any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_2$ can be described by

$$\mathbb{Q}[\{(x_1, x_2)\}] := q_{x_1} q_{x_1,x_2},$$

where $q_{x_1}, q_{x_1,x_2} \in (0,1)$ with $\sum_{x_1 \in \{u,d\}} q_{x_1} = 1$ and $\sum_{x_2 \in \{u,d\}} q_{x_1,x_2} = 1$ for all $x_1 \in \{u,d\}$. Since $r = 0$, S^1 is a $\mathbb{Q}$-martingale if and only if $\widetilde{S}^1$ is, and that is equivalent to

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[Y_1] = 1 \quad &\text{and} \quad \mathbb{E}_{\mathbb{Q}}[Y_2 | \, Y_1 = 1.02] = 1 \\ &\text{and} \quad \mathbb{E}_{\mathbb{Q}}[Y_2 \,|\, Y_1 = 0.98] = 1, \end{aligned} \tag{1}$$

since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and Y_1 only takes two values. This is equivalent to

$$\begin{aligned} q_u \times 1.02 + (1 - q_u) \times 0.98 = 1 \quad &\iff \quad q_u = \frac{1}{2}, \\ q_{u,u} \times \frac{103}{102} + (1 - q_{u,u}) \times \frac{101}{102} = 1 \quad &\iff \quad q_{u,u} = \frac{1}{2}, \\ q_{d,u} \times \frac{100}{98} + (1 - q_{d,u}) \times \frac{97}{98} = 1 \quad &\iff \quad q_{d,u} = \frac{1}{3}. \end{aligned} \tag{2}$$

(b) The density process $Z = (Z_k)_{k=0,1,2}$ is defined by

$$Z_k := \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \middle| \mathcal{F}_k \right], \quad k = 0, 1, 2.$$

Since $\mathcal{F}_0$ is trivial, we have

$$Z_0 = \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right] = \mathbb{E}_{\mathbb{Q}}[1] = 1.$$

Since $\mathbb{Q} \approx \mathbb{P}$, we have $Z_k > 0$ $\mathbb{P}$-a.s. for $k = 0, 1, 2$. Therefore, we can define

$$D_k := \frac{Z_k}{Z_{k-1}} \quad \text{for } k = 1, 2.$$

Moreover, we know from the lecture (p. 42 in the lecture notes) that the D_k, $k = 1, 2$, play the role of "one step conditional densities" of $\mathbb{Q}$ with respect to $\mathbb{P}$. Therefore, by plugging in the values for p_{x_1} and q_{x_1}, where $x_1 \in \{u, d\}$, we get

$$\begin{aligned} D_1((u,u)) = D_1((u,d)) &= \frac{q_u}{p_u} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}, \\ D_1((d,u)) = D_1((d,d)) &= \frac{q_d}{p_d} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}, \end{aligned} \tag{3}$$

and for p_{x_1,x_2} and q_{x_1,x_2}, where $(x_1, x_2) \in \{u, d\}^2$, we get

$$\begin{aligned} D_2((u,u)) &= \frac{q_{u,u}}{p_u} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}, \qquad & D_2((u,d)) &= \frac{q_{u,d}}{p_d} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}, \\ D_2((d,u)) &= \frac{q_{d,u}}{p_u} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}, \qquad & D_2((d,d)) &= \frac{q_{d,d}}{p_d} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2. \end{aligned} \tag{4}$$

The random variables D_1 and D_2 are **not** independent under $\mathbb{P}$, since for example we have

$$P\left[D_1 = \frac{3}{2}\right] \times P\left[D_2 = \frac{3}{2}\right] = \frac{1}{3} \times \frac{2}{9} = \frac{2}{27} > 0$$

while

$$P\left[D_1 = \frac{3}{2},\ D_2 = \frac{3}{2}\right] = P[\emptyset] = 0.$$

(c) In the current setup, an arbitrage opportunity is an admissible, self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with $\vartheta = (\vartheta_k)_{k=0,1,2}$ such that $\vartheta_0 = 0$ and

$$V_2(\varphi) \geq 0 \ \mathbb{P}\text{-a.s.}$$
$$\mathbb{P}\big[V_2(\varphi) > 0\big] > 0.$$

If such a strategy does not exist, we say that the market is arbitrage-free.

(c1) Let us write $u = 0.02$ and $d = -0.02$. If $r = -0.02$, then for the first time step, from $k = 0$ to $k = 1$, the stock grows in each state of the world at least as fast as the bank account, but with positive probability faster since $u > d = r$. Therefore, the obvious arbitrage opportunity consists in borrowing money at time $k = 0$ from the bank account to buy, say, one stock and selling the stock at time $k = 1$.

In mathematical terms, this means that we consider the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$, where ϑ is given by $\vartheta_0 = 0$, $\vartheta_1 = 1$ and $\vartheta_2 = 0$, which is deterministic and therefore a fortiori predictable. Moreover, we have

$$V_1(\varphi) = G_1(\vartheta) = S_1^1 - S_0^1 \geq 0 \ \mathbb{P}\text{-a.s.},$$

because $S_1^1 = \widetilde{S}_1^1/\widetilde{S}_1^0$ takes the values $\frac{102}{1-0.02}$ and $\frac{98}{1-0.02}$ which are both at least $100 = S_0^1$, and

$$V_2(\varphi) = G_2(\vartheta) = G_1(\vartheta) = S_1^1 - S_0^1 \geq 0 \ \mathbb{P}\text{-a.s.},$$

so, the strategy φ is admissible, and

$$\mathbb{P}[V_2(\varphi) > 0] = \mathbb{P}[Y_1 = 1.02] > 0$$

as required. So, φ is an arbitrage opportunity.

(c2) Let us write $u = 0.02$ and $d = -0.02$, $u \mid u = \frac{1}{102}$ and $d \mid u = -\frac{1}{102}$, and $u \mid d = \frac{2}{98}$ and $d \mid u = -\frac{1}{98}$. We have that $d < r < u$ and $(d \mid d) < r < (u \mid d)$, so by Corollary 2.3 in the lecture notes, we know that the first time step from $k = 0$ to $k = 1$ does not admit arbitrage and neither does the second time step from $k = 1$ to $k = 2$ under the condition that the stock price went down on the first step. However, we have $(d \mid u) < (u \mid u) < r$. This means that the stock grows strictly less than the bank account after it goes up in the first time step. Therefore, the arbitrage opportunity consists in short-selling, say, one share of stock at time $k = 1$ if the stock price went up at the initial time step, and investing the money into the bank account until the time horizon $k = 2$, and doing nothing if the stock price goes down in the first time step.

In mathematical terms, this means that we consider the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$, where ϑ is given by $\vartheta_0 = 0$, $\vartheta_1 = 0$ and $\vartheta_2 = -1_A$, where $A = \{Y_1 = 1.02\} \in \mathcal{F}_1$, so ϑ is predictable. We have

$$V_1(\varphi) = G_1(\vartheta) = 0 \ \mathbb{P}\text{-a.s.}$$

and

$$V_2(\varphi) = G_2(\vartheta) = \Delta G_2(\vartheta) = \vartheta_2 \left(S_2^1 - S_1^1\right) = -1_A \left(S_2^1 - S_1^1\right) \geq 0 \ \mathbb{P}\text{-a.s.},$$

because on A,

$$S_2^1 - S_1^1 = \frac{103}{1.01^2} - \frac{102}{1.01} = \frac{103 - 102 \times 1.01}{(1.01)^2} < 0,$$

so, the strategy φ is admissible, and since

$$\mathbb{P}[V_2(\varphi) > 0] = \mathbb{P}[A] = \frac{2}{3} > 0,$$

the strategy φ is an arbitrage opportunity.

Question 3

(a) From $Y_2 = X_2$ and the equalities

$$Y_k = \max\{X_k, E_Q[Y_{k+1} \mid \mathcal{F}_k]\}, \; k = 0, 1,$$

it follows that Y is a Q-supermartingale dominating X. Indeed, $Y_2 = X_2$ and

$$Y_k = \max\{X_k, E_Q[Y_{k+1} \mid \mathcal{F}_k]\} \geq X_k, \; k = 0, 1,$$

so Y dominates X. Moreover, we see that Y_2 is $\mathcal{F}_2$-measurable and Q-integrable, because X_2 is, and

$$Y_1 = \max\{X_1, E_Q[Y_2 \mid \mathcal{F}_1]\}$$

is $\mathcal{F}_1$-measurable and Q-integrable, because X_1 and $E_Q[Y_2 \mid \mathcal{F}_1]$ are, which further implies that Y_0 is $\mathcal{F}_0$-measurable and Q-integrable, so the process Y is Q-integrable and $\mathbb{F}$-adapted, and we check from its definition that

$$Y_k = \max\{X_k, E_Q[Y_{k+1} \mid \mathcal{F}_k]\} \geq E_Q[Y_{k+1} \mid \mathcal{F}_k], \; k = 0, 1.$$

Thus, the process Y is a Q-supermartingale. Assume that Z is another Q-supermartingale dominating X; then $Z_2 \geq X_2 = Y_2$. So, we have

$$Z_1 \geq E_Q[Z_2 \mid \mathcal{F}_1] \geq E_Q[Y_2 \mid \mathcal{F}_1]$$

whence

$$Z_1 \geq \max\{X_1, E_Q[Y_2 \mid \mathcal{F}_1]\} = Y_1.$$

Hence, $Z_1 \geq Y_1$. Repeating the same reasoning, we get

$$Z_0 \geq E_Q[Z_1 \mid \mathcal{F}_0] \geq E_Q[Y_1 \mid \mathcal{F}_0]$$

whence

$$Z_0 \geq \max\{X_0, E_Q[Y_1 \mid \mathcal{F}_0]\} = Y_0.$$

(b) We have

$$\tau = \inf\{k \in \{0, 1, 2\} : Y_k = X_k\} = \inf\{k \in \{0, 1, 2\} : Y_k - X_k = 0\}.$$

Since $Y_2 = X_2$, τ is indeed $\{0, 1, 2\}$-valued and since $Y - X$ is $\mathbb{F}$-adapted, we conclude that τ is a stopping time with respect to $\mathbb{F}$ as the hitting time of $Y - X$ to $\{0\}$.

Alternatively, we could argue that

$$\{\tau = 0\} = \{Y_0 = X_0\} \in \mathcal{F}_0$$

and

$$\{\tau = 1\} = \{Y_0 > X_0\} \cap \{Y_1 = X_1\} \in \mathcal{F}_1$$

and

$$\{\tau = 2\} = \{Y_0 > X_0\} \cap \{Y_1 > X_1\} \cap \{Y_2 = X_2\} \in \mathcal{F}_2.$$

(c) By definition, $Y_k = \max\{X_k, E_Q[Y_{k+1} \mid \mathcal{F}_k]\}$ and $Y_k > X_k$ on the set $\{k+1 \leq \tau\}$. Consequently, $Y_k = E_Q[Y_{k+1} \mid \mathcal{F}_k]$ on the set $\{k+1 \leq \tau\}$ and taking $\mathcal{F}_k$-conditional expectations on both sides of (1), we get

$$E_Q[Y^\tau_{k+1} - Y^\tau_k \mid \mathcal{F}_k] = 1_{\{k+1 \leq \tau\}}(E_Q[Y_{k+1} \mid \mathcal{F}_k] - Y_k)$$

because $\{k+1 \leq \tau\} \in \mathcal{F}_k$ ($\{k+1 \leq \tau\}$ is the complement of $\{\tau \leq k\}$). Hence,

$$E[Y^\tau_{k+1} - Y^\tau_k \mid \mathcal{F}_k] = 0,$$

which proves that Y^τ has the martingale property. From

$$Y_k^\tau = Y_{\tau\wedge k} = Y_0 + \sum_{j=1}^{k} 1_{\{j\le\tau\}}(Y_j - Y_{j-1}),\ k = 1, 2,$$

it follows that the process Y^τ is $\mathbb{F}$-adapted and Q-integrable, because $(1_{\{k\le\tau\}})_{k=0,1,2}$ is a bounded $\mathbb{F}$-predictable process and the process Y is $\mathbb{F}$-adapted and Q-integrable by (a). Thus, Y^τ is a Q-martingale.

(d) Since Y^τ is a martingale by (c), τ is a $\{0, 1, 2\}$-valued $\mathbb{F}$-stopping time by (b), and $\mathcal{F}_0$ is trivial by our assumption, we have

$$Y_0 = Y_0^\tau = E_Q[Y_2^\tau \mid \mathcal{F}_0] = E_Q[Y_2^\tau] = E_Q[Y_\tau] = E_Q[X_\tau]$$

as claimed.

(e) For the process $f(S)$, the Q-integrability is assumed and the $\mathbb{F}$-adaptedness follows from the fact that the function f is measurable combined with the fact that the process S is $\mathbb{F}$-adapted. By Jensen's inequality,

$$E_Q[f(S_{k+1}) \mid \mathcal{F}_k] \ge f(E_Q[S_{k+1} \mid \mathcal{F}_k]) = f(S_k),\ k = 0, 1,$$

as claimed.

(f) Because $\sigma \equiv 2$ is in $\mathcal{T}$, we get

$$V^{C,Eu} = E_Q[X_2] \le V^{C,Am}.$$

On the other hand, the function $x \mapsto (x-K)^+$ is convex; so by (e), X is a Q-submartingale. So for all $\sigma \in \mathcal{T}$, the optional stopping theorem gives

$$E_Q[X_2] \ge E_Q[X_\sigma]$$

and therefore

$$V^{C,Eu} = E_Q[X_2] \ge \sup_{\sigma\in\mathcal{T}} E_Q[X_\sigma] = V^{C,Am}.$$

Hence, $V^{C,Am} = V^{C,Eu}$.

Question 4

(a) By Itô's formula,

$$d(W_t^4) = 4W_t^3 dW_t + \frac{1}{2}12W_t^2 d\langle W, W\rangle_t = 4W_t^3 dW_t + 6W_t^2 dt. \tag{5}$$

Since $4W^3 \in L^2(W^T)$ for $T < \infty$, we have $\left(\int 4W^3 dW\right)^T \in \mathcal{M}_0^2$, so that in particular,

$$E\left[\int_0^T 4W_t^3 dW_t\right] = 0.$$

Hence,

$$E[W_T^4] = E\left[\int_0^T 6W_t^2 dt\right] = 6\int_0^T E[W_t^2]dt = 6\int_0^T t dt = 3T^2,$$

where we used Fubini's theorem to change the order of integration.

(b) By Itô's formula,

$$d(W_t^2) = 2W_t dW_t + dt.$$

Since the finite variation term dt plays no role, we get

$$\langle W^2, W^2\rangle_T = \left\langle \int 2W dW, \int 2W dW \right\rangle_T = 4\int_0^T W_t^2 dt.$$

(c) By Itô's formula, it follows that

$$X_t = X_0 + \int_0^t \frac{\partial f}{\partial x}(s, W_s)\,\mathrm{d}W_s + \int_0^t \left(\frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s)\right)\mathrm{d}s.$$

Since W is a continuous $(P, \mathbb{F})$-martingale and since the integrand $\left(\frac{\partial f}{\partial x}(t, W_t)\right)_{t\geq 0}$ is continuous and adapted, and thus an element of $L^2_{\text{loc}}(W)$, we have that

$$\int_0^\cdot \left(\frac{\partial f}{\partial x}(s, W_s)\right)\mathrm{d}W_s$$

is a local $(P, \mathbb{F})$-martingale. Moreover, the process X is a (continuous) local $(P, \mathbb{F})$-martingale because the assumption on f gives

$$\int_0^t \left(\frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, W_s)\right)\mathrm{d}s = 0, \qquad \forall t \geq 0.$$

We know that $[W^T] = [W]^T$ so that

$$[W^T]_t = t \wedge T$$

for all $t \geq 0$. Hence

$$E\left[\int_0^\infty \left(\frac{\partial f}{\partial x}(s, W_s)\right)^2 d\left[W^T\right]_s\right] = E\left[\int_0^\infty \left(\frac{\partial f}{\partial x}(s, W_s)\right)^2 d(s \wedge T)\right]$$
$$= E\left[\int_0^T \left(\frac{\partial f}{\partial x}(s, W_s)\right)^2 ds\right] < \infty,$$

and thus $\frac{\partial f}{\partial x}(s, W_s) \in L^2\left(W^T\right)$. Since $W^T \in \mathcal{M}_0^2$, this directly implies that $\int \frac{\partial f}{\partial x}(s, W_s) dW_s^T$ is in $\mathcal{M}_0^2$. Moreover, since

$$\int \frac{\partial f}{\partial x}(s, W_s) dW_s^T = \left(\int \frac{\partial f}{\partial x}(s, W_s) dW_s\right)^T,$$

we get that $\left(\int \frac{\partial f}{\partial x}(s, W_s) dW_s\right)^T \in \mathcal{M}_0^2$. So, $(X_t)_{0\leq t\leq T} = \left(\int_0^t \frac{\partial f}{\partial x}(s, W_s) dW_s\right)_{0\leq t\leq T}$ is a (square-integrable) martingale on $[0, T]$.

Question 5

(a) By Girsanov's theorem, for

$$W_t^{Q'} := W_t + \frac{\mu + r}{\sigma} t = W_t + \int_0^t \frac{\mu + r}{\sigma} ds,$$

a measure Q' under which $W^{Q'}$ is a Brownian motion is given by the Radon–Nikodým derivative

$$\frac{dQ'}{dP} = \mathcal{E}\left(- \int \frac{\mu + r}{\sigma} dW \right)_T = \exp\left(- \frac{\mu + r}{\sigma} W_T - \frac{1}{2} \left(\frac{\mu + r}{\sigma} \right)^2 T \right).$$

(b) The unique equivalent martingale measure Q for the *discounted* stock price S (and a Q-Brownian motion W^Q) is obtained by replacing $\frac{\mu+r}{\sigma}$ with $\frac{\mu-r}{\sigma}$ in (a) (Lecture notes p. 117). Under the measure Q, the *undiscounted* stock price process $\widetilde{S}$ is given by

$$\widetilde{S}_t = e^{rt} S_t := e^{rt} S_0 \exp(\sigma W_t^Q - \frac{1}{2} \sigma^2 t), \ t \in [0, T],$$

where $S_t := S_0 \exp(\sigma W_t^Q - \frac{1}{2} \sigma^2 t)$ represents the *discounted* stock price at time $t \in [0, T]$ under the measure Q. We have

$$\widetilde{S}_T = e^{r(T-t)} \widetilde{S}_t \exp\left(\sigma(W_T^Q - W_t^Q) - \frac{1}{2} \sigma^2 (T - t)\right), \ t \in [0, T].$$

The *discounted* value V_t of a power option at time t with undiscounted payoff $h(\widetilde{S}_T) = \widetilde{S}_T^p$ is the payoff's discounted $\mathcal{F}_t$-conditional Q-expected value, i.e.,

$$V_t = E_Q[e^{-rT} h(\widetilde{S}_T) | \mathcal{F}_t] = E_Q[e^{-rT} \widetilde{S}_T^p | \mathcal{F}_t].$$

We have

$$e^{-rT} \widetilde{S}_T^p = e^{prT - rT} \left(e^{-rT} \widetilde{S}_T \right)^p = e^{r(p-1)T} S_T^p, \tag{6}$$

where

$$\begin{aligned} S_T^p &= S_t^p \exp\left(\sigma p (W_T^Q - W_t^Q) - \frac{1}{2} \sigma^2 p (T - t) \right) \\ &= S_t^p \exp\left(\sigma p (W_T^Q - W_t^Q) - \frac{1}{2} \sigma^2 p^2 (T - t) \right) \exp\left(\frac{1}{2} \sigma^2 p (p - 1)(T - t) \right). \end{aligned}$$

The middle factor has $\mathcal{F}_t$-conditional Q-expectation 1; so we get

$$\begin{aligned} V_t &= S_t^p \exp\left(\frac{1}{2} \sigma^2 p (p - 1)(T - t) + r(p - 1)T \right) \\ &= e^{-rt} \widetilde{S}_t^p \exp\left(\left(\frac{1}{2} \sigma^2 p + r \right) (p - 1)(T - t) \right), \end{aligned}$$

where we used that $S_t^p = e^{-rt} \widetilde{S}_t^p e^{-r(p-1)t}$; c.f. (6). The *undiscounted* value at time t is

$$\widetilde{V}_t = e^{rt} V_t = \widetilde{S}_t^p \exp\left(\left(\frac{1}{2} \sigma^2 p + r \right) (p - 1)(T - t) \right).$$

(c) We know from the lecture (notes page 123) that for the value process

$$V_t = v(t, S_t),$$

the hedging strategy is

$$\vartheta_t = \frac{\partial v}{\partial x}(t, S_t), \ \ \eta_t = V_t - \vartheta_t S_t.$$

Since

$$V_t = S_t^p \exp\left(\frac{1}{2}\sigma^2 p(p-1)(T-t) + r(p-1)T\right),$$

we can compute

$$\vartheta_t = pS_t^{p-1} \exp\left(\frac{1}{2}\sigma^2 p(p-1)(T-t) + r(p-1)T\right)$$

and then obtain

$$\eta_t = (1-p)S_t^p \exp\left(\frac{1}{2}\sigma^2 p(p-1)(T-t) + r(p-1)T\right).$$