Question 1

The correct answers are:

(a) (3)

(b) (1)

(c) (2)

(d) (3)

(e) (1)

(f) (3)

(g) (2)

(h) (3)

Question 2

(a) By the fundamental theorem of asset pricing (Theorem II.2.1 in the lecture notes), the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free if and only if there exists an EMM Q for the discounted stock price process S^1.

Any probability measure Q equivalent to P on $\mathcal{F}_2$ can be described by

$$Q[\{(x_1, x_2)\}] := q_{x_1} q_{x_1,x_2},$$

where $q_{x_1}, q_{x_1,x_2} \in (0,1)$ satisfy that $\sum_{x_1 \in \{-1,1\}} q_{x_1} = 1$ and $\sum_{x_2 \in \{-1,1\}} q_{x_1,x_2} = 1$ for all $x_1 \in \{-1,1\}$. Next, since $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(Y_1)$ and Y_1 only takes two values, S^1 is a $(Q, \mathbb{F})$-martingale if and only if $q_1, q_{1,1}, q_{-1,1} \in (0,1)$ and

$$E_Q\left[\frac{Y_1}{1+r}\right] = 1, \quad E_Q\left[\frac{Y_2}{1+2r}\,\middle|\, Y_1 = 1+u\right] = 1 \quad \text{and} \quad E_Q\left[\frac{Y_2}{1+2r}\,\middle|\, Y_1 = 1+d\right] = 1.$$

This is equivalent to $q_1, q_{1,1}, q_{-1,1} \in (0,1)$ and

$$\begin{aligned}
q_1(1+u) + (1-q_1)(1+d) &= 1+r & &\iff & q_1 &= \frac{r-d}{u-d}, \\
q_{1,1}(1+2u) + (1-q_{1,1})(1+2d) &= 1+2r & &\iff & q_{1,1} &= \frac{r-d}{u-d}, \\
q_{-1,1}(1+u) + (1-q_{-1,1})(1+d) &= 1+2r & &\iff & q_{-1,1} &= \frac{2r-d}{u-d}.
\end{aligned}$$

In conclusion, the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free if and only if

$$\frac{r-d}{u-d} \in (0,1) \quad \text{and} \quad \frac{2r-d}{u-d} \in (0,1) \qquad \iff \qquad d < r < u \quad \text{and} \quad d < 2r < u.$$

(b) Since we indeed have that $u > r > d$ and $u > 2r > d$, the market $(\widetilde{S}^0, \widetilde{S}^1)$ is arbitrage-free. We know from (a) that the EMM for S^1 is in fact unique, so all claims are attainable in this model. The arbitrage-free price $V_0^{\widetilde{H}}$ of an arbitrary payoff $\widetilde{H} \in L^0_+(\mathcal{F}_2)$ in the above model is therefore given by

$$V_0^{\widetilde{H}} = \frac{1}{(1+r)(1+2r)} E_Q\left[\widetilde{H}\,\middle|\,\mathcal{F}_0\right] = \frac{1}{(1+r)(1+2r)} E_Q\left[\widetilde{H}\right], \tag{1}$$

where the second equality follows from $\mathcal{F}_0$ being P-trivial.

Since one can notice that we have in fact that $\widetilde{K} = (1+u)(1+d)$, we can conclude that the payoff $\widetilde{H}$ is non-zero if and only if $\omega = (1,1)$. Using this as well as the form of the EMM from (a), we can reduce (1) to

$$\begin{aligned}
V_0^{\widetilde{H}} &= \frac{1}{(1+r)(1+2r)} q_1 q_{1,1} \left((1+u)(1+2u) - \frac{8}{9}\right) \\
&= \frac{9}{10}\frac{9}{11}\left(\frac{2}{3}\right)^2 \left(\frac{20}{9} - \frac{8}{9}\right) = \frac{24}{55}.
\end{aligned}$$

(c) Note that we have $u = 2r$, so the market is not free of arbitrage by (a). The idea is to sell the stock in the case of an "down" movement in the first period, since that is the sub-market which is not arbitrage-free. To this end, consider the strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$, where

$$\vartheta_1^1 := 0, \quad \vartheta_2^1((1,1)) := \vartheta_2^1((1,-1)) := 0, \quad \vartheta_2^1((-1,1)) := \vartheta_2^1((-1,-1)) := -c,$$

where $c > 0$ is to be determined. Then ϑ is predictable and we have, using $u = 2r$,

$$\begin{aligned}
V_2(\varphi)((1,1)) &= 0 + 0 \times \Delta S_1^1((1,1)) + 0 \times \Delta S_2^1((1,1)) = 0, \\
V_2(\varphi)((1,-1)) &= 0 + 0 \times \Delta S_1^1((1,-1)) + 0 \times \Delta S_2^1((1,-1)) = 0, \\
V_2(\varphi)((-1,1)) &= 0 + 0 \times \Delta S_1^1((-1,1)) - c \times \Delta S_2^1((-1,1)) \\
&= -c \times \left(\frac{(1+d)(1+2r)}{(1+r)(1+2r)} - \frac{1+d}{1+r} \right) = -c \times 0 = 0, \\
V_2(\varphi)((-1,-1)) &= 0 + 0 \times \Delta S_1^1((-1,-1)) - c \times \Delta S_2^1((-1,-1)) \\
&= -c \times \left(\frac{(1+d)(1+d)}{(1+r)(1+2r)} - \frac{1+d}{1+r} \right) \\
&= -c \times \left(\frac{1+d}{1+r} \times \frac{d-2r}{1+2r} \right) = c \times \frac{12}{21} \times \frac{6}{12} = c \times \frac{2}{7}.
\end{aligned}$$

Choosing c large enough, i.e. $c \times \frac{2}{7} \geq 2$ or $c \geq 7$ gives the desired strategy because

$$P[\{(-1,1)\}] = 1/2 \times 1/2 = 0.25.$$

(d) Intuitively, such a strategy does not exist because the sub-market corresponding to the first period as well as the sub-market corresponding to the second period after an “up” movement in the first period are free of arbitrage.

Formally, one can do this by contradiction. Suppose that there exists a trading strategy $\varphi \widehat{=} (0, \vartheta)$ with $V_2(\varphi) \geq 2$ P-a.s. Since the sub-market corresponding to the second period after an “up” movement in the first period is free of arbitrage, $V_2(\varphi) \geq 2$ P-a.s. implies that $V_1(\varphi)((1,1)) = V_1(\varphi)(1,-1) \geq 2$. Since $\Delta S_2^1((-1,1)) = 0$ (see the solution of (c)), it also follows that $V_1(\varphi)((-1,1)) = V_1(\varphi)((-1,-1)) \geq 2$. Together, this gives $V_1(\varphi) \geq 2$ P-a.s. But since the first period market admits a unique EMM Q^*, we have that

$$V_0(\varphi) = E_{Q^*}\left[V_1(\varphi)\right] \geq 2,$$

which is a contradiction to the requirement that $V_0(\varphi) = 0$.

Question 3

(a) Let A_n be the event that the gambler wins the n-th round. Then by definition

$$X_{n+1} = X_n + \frac{1}{2}(1 - X_n)\mathbb{1}_{A_{n+1}} - \frac{1}{2}X_n\mathbb{1}_{A_{n+1}^c} = \frac{1}{2}(1 + X_n)\mathbb{1}_{A_{n+1}} + \frac{1}{2}X_n\mathbb{1}_{A_{n+1}^c}.$$

By definition X is adapted. Also, $0 \le X_n \le 1$ for all $n \ge 0$, which directly implies the integrability of X_n. Finally we compute

$$\begin{aligned} E\left[X_{n+1} \,|\, F_n\right] &= \frac{1}{2}(1 + X_n)E\left[\mathbb{1}_{A_{n+1}} \,|\, \mathcal{F}_n\right] + \frac{1}{2}X_n E\left[\mathbb{1}_{A_{n+1}^c} \,\middle|\, \mathcal{F}_n\right] \\ &= \frac{1}{2}(1 + X_n)X_n + \frac{1}{2}X_n(1 - X_n) = X_n. \end{aligned}$$

(b) Note that

$$Y_{n+1} = (1 + X_n)\mathbb{1}_{A_{n+1}} + X_n\mathbb{1}_{A_{n+1}^c} - X_n = \mathbb{1}_{A_{n+1}}.$$

So by definition

$$P\left[Y_{n+1} = 0 \,|\, \mathcal{F}_n\right] = E\left[\mathbb{1}_{A_{n+1}^c} \,\middle|\, \mathcal{F}_n\right] = P\left[A_{n+1}^c \,|\, \mathcal{F}_n\right] = 1 - X_n$$

and

$$P\left[Y_{n+1} = 1 \,|\, \mathcal{F}_n\right] = E\left[\mathbb{1}_{A_{n+1}} \,|\, \mathcal{F}_n\right] = P\left[A_{n+1} \,|\, \mathcal{F}_n\right] = X_n.$$

Clearly Y_n are $\{0,1\}$-valued random variables with $P\left[Y_n = 1\right] = E\left[X_n\right] = p$ for all $n > 0$.

(c) We directly compute

$$\begin{aligned} E\left[X_{n+1}^2 \,|\, \mathcal{F}_n\right] &= E\left[\frac{1}{4}\left(1 + X_n\right)^2 \mathbb{1}_{A_{n+1}} \,\middle|\, \mathcal{F}_n\right] + E\left[\frac{1}{4}X_n^2\mathbb{1}_{A_{n+1}^c} \,\middle|\, \mathcal{F}_n\right] \\ &= \frac{1}{4}\left(1 + X_n\right)^2 E\left[\mathbb{1}_{A_{n+1}} \,|\, \mathcal{F}_n\right] + \frac{1}{4}X_n^2 E\left[\mathbb{1}_{A_{n+1}^c} \,\middle|\, \mathcal{F}_n\right] \\ &= \frac{(1 + X_n)^2 X_n + X_n^2(1 - X_n)}{4} = \frac{3X_n^2 + X_n}{4}. \end{aligned}$$

Using the hint, we obtain that $E\left[X_n^2\right] \to E\left[Z^2\right]$ and $E\left[X_n\right] \to E\left[Z\right]$. Because X is a martingale, we have that $E\left[X_n\right] = E\left[X_0\right] = p$, which implies that $E\left[Z\right] = p$. From the above expression for $E\left[X_{n+1}^2 \,\middle|\, \mathcal{F}_n\right]$, we obtain after taking expectations, letting $n \to \infty$ and using $E\left[X_n^2\right] \to E\left[Z^2\right]$ that

$$E\left[Z^2\right] = E\left[\frac{3Z^2 + Z}{4}\right] \quad \Longleftrightarrow \quad E\left[\frac{Z^2}{4}\right] = E\left[\frac{Z}{4}\right] \quad \Longleftrightarrow \quad E\left[Z^2\right] = E\left[Z\right].$$

(d) The fact that $Y_n \to Z$ P-a.s. follows immediately from the fact that $X_n \to Z$ P-a.s. Now since Y_n is $\{0,1\}$-valued for all n, Z is also $\{0,1\}$-valued. Observe that

$$\begin{aligned} \{G_n \text{ occurs only finitely many times}\} &= \{Z = 0\}, \\ \{L_n \text{ occurs only finitely many times}\} &= \{Z = 1\}. \end{aligned}$$

Moreover, $P\left[Z = 1\right] = E\left[Z\right] = p$ and $P\left[Z = 0\right] = 1 - p$. The result thus follows.

Question 4

(a) We know from the lecture that

$$W_t^2 = 2\int_0^t W_s dW_s + t.$$

This can also be calculated by applying Itô's formula to the function $f : x \mapsto x^2$ and the (semi)martingale W.

By applying Itô's formula to the function $f(x) = \exp(\alpha x)$ and the (semi)martingale W, we also get that

$$e^{\alpha W_t} = 1 + \alpha \int_0^t e^{\alpha W_s} dW_s + \frac{1}{2}\alpha^2 \int_0^t e^{\alpha W_s} ds.$$

By the bilinearity of quadratic covariation and the fact that the processes $Y = (Y_t)_{t\geq 0}$ and $Z = (Z_t)_{t\geq 0}$ given respectively by

$$Y_t := t \quad \text{and} \quad Z_t := \frac{1}{2}\alpha^2 \int_0^t e^{\alpha W_s} ds$$

are continuous and of finite variation, we obtain that

$$\begin{aligned}
\left[W^2, e^{\alpha W}\right]_t &= \left[2\int W dW, \alpha \int e^{\alpha W} dW\right]_t = 2\alpha \left[\int W dW, \int e^{\alpha W} dW\right]_t \\
&= 2\alpha \int_0^t W_s e^{\alpha W_s} d[W,W]_s = 2\alpha \int_0^t W_s e^{\alpha W_s} ds.
\end{aligned}$$

Since both W^2 and $\exp(\alpha W)$ are clearly continuous, we also have for all $t \geq 0$ that

$$\left\langle W^2, \exp(\alpha W)\right\rangle_t = \left[W^2, \exp(\alpha W)\right]_t \quad P\text{-a.s.}$$

(b) The process $(W_t, t)_{t\geq 0}$ is a continuous semimartingale and since $f \in C^{2,1}$ and $(t)_{t\geq 0}$ is additionally a finite variation process, Itô's lemma gives us that

$$\begin{aligned}
Y_t &= f(0,0) + \int_0^t \left(\frac{\partial f}{\partial t}(W_s, s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(W_s, s)\right) ds + \int_0^t \frac{\partial f}{\partial x}(W_s, s) dW_s \\
&= \int_0^t \frac{\partial f}{\partial x}(W_s, s) dW_s,
\end{aligned}$$

where the second equality holds by our assumptions on f. We know from the exercise sheets that since $\frac{\partial f}{\partial x}(W_s, s)$ is continuous and adapted, it is also predictable and locally bounded, and thus in $L^2_{\text{loc}}(W)$. So Y is at least a local $(P, \mathbb{F})$-martingale.

But f is bounded by assumption, so Y is clearly square-integrable. Moreover, we know that every bounded local martingale is a true martingale. More specifically, if $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for Y, then, since $|Y_t| \leq a$ for all $t \geq 0$ and some $a \geq 0$, dominated convergence theorem gives for all $0 \leq s \leq t$ that

$$E[Y_t \,|\, \mathcal{F}_s] = E\left[\lim_{n\to\infty} Y_{\tau_n \wedge t} \,\middle|\, \mathcal{F}_s\right] = \lim_{n\to\infty} E[Y_{\tau_n \wedge t} \,|\, \mathcal{F}_s] = \lim_{n\to\infty} Y_{\tau_n \wedge s} = Y_s.$$

Since Y is a (square-integrable) $(P, \mathbb{F})$-martingale and null at 0, we have that

$$E[Y_t] = E[E[Y_t \,|\, \mathcal{F}_0]] = Y_0 = 0,$$

which means that $\text{Var}[Y_t] = E\left[Y_t^2\right]$. We compute

$$\begin{aligned}
\text{Var}[Y_t] &= E\left[\left(\int_0^t \frac{\partial f}{\partial x}(W_s, s) dW_s\right)^2\right] = E\left[\int_0^t \left(\frac{\partial f}{\partial x}(W_s, s)\right)^2 ds\right] \\
&= \int_0^t E\left[\left(\frac{\partial f}{\partial x}(W_s, s)\right)^2\right] ds,
\end{aligned}$$

where the second equality uses the isometry property of stochastic integrals and the third uses Fubini's theorem.

(c) Showing that M is a $(P, \mathbb{F})$-martingale is straightforward. Define $X := \mathbb{1}_{\{a \leq W_T \leq b\}}$ and note that $M_t = E[X \mid \mathcal{F}_t]$. Then M_t is $\mathcal{F}_t$-measurable by the definition of conditional expectation, and so M is adapted on $[0, T]$. Because X is bounded by 1, so is M, and so M is integrable on $[0, T]$. To show the martingale property, we compute

$$E[M_t \mid \mathcal{F}_s] = E[E[X \mid \mathcal{F}_t] \mid \mathcal{F}_s] = E[X \mid \mathcal{F}_s] = M_s,$$

and hence conclude that M is a $(P, \mathbb{F})$-martingale.

Furthermore, by the definition of M, we have that

$$\begin{aligned} M_t &= P[a \leq W_T \leq b \mid \mathcal{F}_t] = P[a - W_t \leq W_T - W_t \leq b - W_t \mid \mathcal{F}_t] \\ &= P\left[\frac{a - W_t}{\sqrt{T-t}} \leq \frac{W_T - W_t}{\sqrt{T-t}} \leq \frac{b - W_t}{\sqrt{T-t}} \,\middle|\, \mathcal{F}_t\right] = P\left[\frac{a - x}{\sqrt{T-t}} \leq \frac{W_T - W_t}{\sqrt{T-t}} \leq \frac{b - x}{\sqrt{T-t}}\right]\Bigg|_{x = W_t} \\ &= g(W_t, t), \end{aligned}$$

where the second to last equality uses that $W_T - W_t$ is independent of $\mathcal{F}_t$ and that W_t is $\mathcal{F}_t$-measurable. Since we have that $\frac{W_T - W_t}{\sqrt{T-t}} \sim \mathcal{N}(0,1)$, we can compute, for $t < T$,

$$g(W_t, t) = \Phi\left(\frac{b - W_t}{\sqrt{T-t}}\right) - \Phi\left(\frac{a - W_t}{\sqrt{T-t}}\right).$$

(d) Following the approach from the example on page 103 in the lecture notes, applying Itô's formula to the $C^{2,1}$ function

$$g(x, t) = \Phi\left(\frac{b - x}{\sqrt{T-t}}\right) - \Phi\left(\frac{a - x}{\sqrt{T-t}}\right)$$

and the continuous semimartingale $(W_t, t)_{t \geq 0}$ gives

$$M_t = g(0,0) + \int_0^t g_x(W_s, s) dW_s + \int_0^t \left(g_t + \frac{1}{2} g_{xx}\right)(W_s, s) ds \quad \text{for } 0 \leq t \leq T.$$

Since M is a $(P, \mathbb{F})$-martingale, the "ds" integral must vanish as we know that every continuous local martingale of finite variation is constant. We therefore do not even have to calculate g_t and g_{xx}, and we obtain that

$$M_t = g(0,0) + \int_0^t g_x(W_s, s) dW_s.$$

Denoting by ϕ the density of the standard normal distribution $\mathcal{N}(0,1)$, we obtain for M_0 and $\psi = (\psi_t)_{t \in [0,T]}$ that

$$M_0 = \Phi\left(\frac{b}{\sqrt{T}}\right) - \Phi\left(\frac{a}{\sqrt{T}}\right), \quad \psi_t = \frac{1}{\sqrt{T-t}}\left(\phi\left(\frac{a - W_t}{\sqrt{T-t}}\right) - \phi\left(\frac{b - W_t}{\sqrt{T-t}}\right)\right).$$

Question 5

(a) Define $f(x,y) = \log(x+y)$. We have

$$f_x = \frac{1}{x+y}, \quad f_y = \frac{1}{x+y}, \quad f_{yy} = -\frac{1}{(x+y)^2}.$$

Applying Itô's formula to f and the continuous semimartingale $(\widetilde{S}^0, \widetilde{S}^1)$, we obtain

$$dZ_t = \frac{1}{\widetilde{S}_t^0 + \widetilde{S}_t^1} d\widetilde{S}_t^0 + \frac{1}{\widetilde{S}_t^0 + \widetilde{S}_t^1} d\widetilde{S}_t^1 - \frac{1}{2} \frac{1}{(\widetilde{S}_t^0 + \widetilde{S}_t^1)^2} d\langle \widetilde{S}_t^1 \rangle.$$

Since

$$d\widetilde{S}_t^1 = \mu \widetilde{S}_t^1 dt + \sigma \widetilde{S}_t^1 dW_t$$

it follows that

$$d\langle \widetilde{S}_t^1 \rangle = \sigma^2 (\widetilde{S}_t^1)^2 dt.$$

Subsequently, we get

$$dZ_t = \left(\frac{r\widetilde{S}_t^0}{\widetilde{S}_t^0 + \widetilde{S}_t^1} + \frac{\mu \widetilde{S}_t^1}{\widetilde{S}_t^0 + \widetilde{S}_t^1} - \frac{1}{2} \frac{\sigma^2 (\widetilde{S}_t^1)^2}{(\widetilde{S}_t^0 + \widetilde{S}_t^1)^2} \right) dt + \frac{\sigma \widetilde{S}_t^1}{\widetilde{S}_t^0 + \widetilde{S}_t^1} dW_t.$$

Moreover, we know that $\widetilde{S}_t^0 = e^{rt}$ and $\widetilde{S}_t^1 = e^{Z_t} - e^{rt}$, so after plugging these into the above equation, we obtain

$$dZ_t = \left(\frac{re^{rt}}{e^{Z_t}} + \frac{\mu(e^{Z_t} - e^{rt})}{e^{Z_t}} - \frac{1}{2} \frac{\sigma^2 (e^{Z_t} - e^{rt})^2}{e^{2Z_t}} \right) dt + \frac{\sigma(e^{Z_t} - e^{rt})}{e^{Z_t}} dW_t.$$

(b) (i) First, recall that

$$\widetilde{S}_t^1 = S_0^1 e^{\sigma W_t^* + \left(r - \frac{1}{2}\sigma^2\right)t},$$

and that

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t$$

is a Brownian motion under the unique EMM Q. Then we compute

$$\begin{aligned}
\widetilde{V}_t^{\widetilde{H}_n} &= \widetilde{S}_t^0 E_Q \left[\frac{(\widetilde{S}_T^1)^{1/n}}{\widetilde{S}_T^0} \,\middle|\, \mathcal{F}_t \right] = \frac{\widetilde{S}_t^0}{\widetilde{S}_T^0} (\widetilde{S}_t^1)^{1/n} E_Q \left[\left(\frac{\widetilde{S}_T^1}{\widetilde{S}_t^1} \right)^{1/n} \,\middle|\, \mathcal{F}_t \right] \\
&= \frac{\widetilde{S}_t^0}{\widetilde{S}_T^0} (\widetilde{S}_t^1)^{1/n} E_Q \left[\exp\left(\frac{\sigma}{n}(W_T^* - W_t^*) + \frac{1}{n}\left(r - \frac{\sigma^2}{2}\right)(T-t) \right) \,\middle|\, \mathcal{F}_t \right] \\
&= \frac{\widetilde{S}_t^0}{\widetilde{S}_T^0} (\widetilde{S}_t^1)^{1/n} \exp\left(\frac{\sigma^2}{2n^2}(T-t) + \frac{1}{n}\left(r - \frac{\sigma^2}{2}\right)(T-t) \right) \\
&= \frac{\widetilde{S}_t^0}{\widetilde{S}_T^0} (\widetilde{S}_t^1)^{1/n} \left(\frac{\widetilde{S}_T^0}{\widetilde{S}_t^0} \right)^{1/n} \exp\left(\frac{\sigma^2}{2n^2}(T-t)\left(\frac{1}{n} - 1\right) \right) \\
&= (\widetilde{S}_t^1)^{1/n} \exp\left(\left(r + \frac{\sigma^2}{2n^2}\right)(T-t)\left(\frac{1}{n} - 1\right) \right) \\
&= \widetilde{S}_t^0 (\widetilde{S}_t^1)^{1/n} \exp\left(\left(rT + \frac{\sigma^2}{2n^2}(T-t)\right)\left(\frac{1}{n} - 1\right) \right)
\end{aligned}$$

where in the fourth equality we compute the expectation by evaluating the moment generating function of $\mathcal{N}(0, T-t)$ at the point $\frac{\sigma}{n}$ as seen during the exercise sessions, i.e.,

$$E_Q \left[\exp\left(\frac{\sigma}{n}(W_T^* - W_t^*) \right) \right] = \exp\left(\frac{1}{2} \frac{\sigma^2}{n^2}(T-t) \right).$$

(ii) By writing $\widetilde{V}_t^{\widetilde{H}_n} = \widetilde{v}(t, \widetilde{S}_t^1)$ and using $\vartheta_t^{\widetilde{H}_n} = \frac{\partial \widetilde{v}}{\partial \widetilde{x}}(t, \widetilde{S}_t^1)$, we get

$$\begin{aligned}
\vartheta_t^{\widetilde{H}_n} &= \frac{1}{n}\left(\widetilde{S}_t^1 \exp\left(\left(r + \frac{\sigma^2}{2n}\right)(T-t)\right)\right)^{\frac{1}{n}-1} = \frac{1}{n}\left(S_t^1 \exp\left(rT + \frac{\sigma^2}{2n}(T-t)\right)\right)^{\frac{1}{n}-1}, \\
\eta_t^{\widetilde{H}_n} &= \frac{\widetilde{V}_t^{\widetilde{H}_n} - \vartheta_t^{\widetilde{H}_n}\widetilde{S}_t^1}{\widetilde{S}_t^0} = \frac{1}{\widetilde{S}_t^0}(\widetilde{S}_t^1)^{1/n} \exp\left(\left(r + \frac{\sigma^2}{2n}\right)(T-t)\left(\frac{1}{n} - 1\right)\right)\left(1 - \frac{1}{n}\right) \\
&= (S_t^1)^{1/n} \exp\left(\left(rT + \frac{\sigma^2}{2n}(T-t)\right)\left(\frac{1}{n} - 1\right)\right)\left(1 - \frac{1}{n}\right).
\end{aligned}$$

(c) By Girsanov's theorem, the process $\widehat{W} = (\widehat{W}_t)_{t\geq 0}$ given by

$$\widehat{W}_t = W_t - \langle W, X\rangle_t = W_t - \left\langle \int dW, \int W dW \right\rangle_t = W_t - \int_0^t W_s ds$$

is a Brownian motion with respect to $\widehat{Q}$ and $\mathbb{F}$. In the differential form, the above can be expressed as

$$d\widehat{W}_t = dW_t - W_t dt.$$

We can therefore write

$$\begin{aligned}
d\widetilde{S}_t^1 &= \widetilde{S}_t^1(\mu dt + \sigma dW_t) = \widetilde{S}_t^1(\mu dt + \sigma dW_t - \sigma W_t dt + \sigma W_t dt) \\
&= \widetilde{S}_t^1\big((\mu + \sigma W_t)dt + \sigma(dW_t - W_t dt)\big) = \widetilde{S}_t^1\big((\mu + \sigma W_t)dt + \sigma d\widehat{W}_t\big) \\
&= \widetilde{S}_t^1(\mu + \sigma W_t)dt + \widetilde{S}_t^1 \sigma d\widehat{W}_t.
\end{aligned}$$

In order to show that $E_{\widehat{Q}}[W_t] = 0$, we write $W_t = \widehat{W}_t + \int_0^t W_s ds$ and compute

$$E_{\widehat{Q}}[W_t] = E_{\widehat{Q}}\left[\widehat{W}_t\right] + \int_0^t E_{\widehat{Q}}[W_s]\, ds = \int_0^t E_{\widehat{Q}}[W_s]\, ds.$$

Now, it is clear that $E_{\widehat{Q}}[W_t] = f(t)$ for some deterministic function $f : \mathbb{R}_+ \to \mathbb{R}$. Also, since $\widehat{Q} \approx P$ on $\mathcal{F}_T$ and $W_0 = 0$ P-a.s., we must also have that $W_0 = 0$ $\widehat{Q}$-a.s., which means that for $t \in [0, T]$,

$$f(t) = \int_0^t f(s)ds, \quad f(0) = 0 \quad \iff \quad f'(t) = f(t), \quad f(0) = 0.$$

But this is just a simple ODE, whoso unique solution can easily be seen to given by $f(t) = 0$, and we can therefore conclude that $E_{\widehat{Q}}[W_t] = 0$.