ETH Zürich, HS 2017
D-MATH
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January 2018

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

Leave blank

Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- Please put your student card on the table.
- Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- Take a new sheet for each question and write your name on every sheet.
- Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- Simplify your results as much as possible.
- Most of the subquestions can be solved independently of each other.

⋆⋆⋆ Good luck! ⋆⋆⋆

Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no cross or more than one cross in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Do not fill in

correct	wrong	no answer

Do not fill in

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points)**

For each of the following eight subquestions, there is exactly one correct answer. For each correct answer you get 1 point, for each wrong answer you get -0.5 point, and for no answer you get 0 points. You get at least 0 points for the whole exercise. **Please use the printed form for your answers.** It is enough to indicate your answer by a cross; you do not need to explain your choice.

Throughout subquestions (a) to (d), let $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a finite time horizon $T \in \mathbb{N}$ and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}^1$. Let $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}^1_0 := s^1_0 > 0$. The discounted market is denoted by (S^0, S^1).

(a) Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with $u > r > d > -1$. Which of the following is **not** always true?

(1) $(\widetilde{S}^0, \widetilde{S}^1)$ is free of arbitrage.

(2) $(\frac{\widetilde{S}^0}{\widetilde{S}^1}, 1)$ is free of arbitrage.

(3) $(\widetilde{S}^0, S^1)$ is free of arbitrage.

(b) Suppose that there exists a self-financing trading strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with the property that $P\left[V_T(\varphi) < 0\right] = 0$ and $P\left[V_T(\varphi) > 0\right] > 0$. Then

(1) (S^0, S^1) is not free of arbitrage.

(2) φ is an arbitrage opportunity.

(3) φ is an arbitrage opportunity provided that its cost process $C(\varphi)$ is constant.

(c) Suppose now that $(\widetilde{S}^0, \widetilde{S}^1)$ is a multinomial model on the canonical path space. Which of the following statements is **not** true?

(1) Every self-financing trading strategy $\varphi \mathrel{\widehat{=}} (V_0, \vartheta)$ is admissible.

(2) $(\widetilde{S}^0, \widetilde{S}^1)$ does not admit any attainable payoffs.

(3) If S^1 is a local martingale under some $Q \approx P$, then (S^0, S^1) is free of arbitrage.

(d) Let $Z = (Z_k)_{k=0,1,\dots,T}$ be the density process of some $Q \in \mathbb{P}_e(S^1)$ with respect to P. Then

(1) Z is a Q-martingale.

(2) $E_Q\left[S^1_k \,\middle|\, \mathcal{F}_{k-1}\right] = E_P\left[Z_k S^1_k \,\middle|\, \mathcal{F}_{k-1}\right]$.

(3) $E_P\left[Z_k\right] = 1$ for all $k \in \{0, 1, \dots, T\}$.

Throughout subquestions (e) to (h), W denotes a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions of P-completeness and right-continuity.

(e) Let $M = (M_t)_{t \geq 0}$ be an arbitrary local $(P, \mathbb{F})$-martingale and $f : \mathbb{R} \to \mathbb{R}$ a C^4 function. Which of the following statements is true?

(1) $\big(f(M_t)\big)_{t \geq 0}$ is a possibly discontinuous $(P, \mathbb{F})$-semimartingale.

(2) $\big(f(M_t)\big)_{t \geq 0}$ is a $(P, \mathbb{F})$-semimartingale as well as a local $(P, \mathbb{F})$-martingale.

(3) $\big(f(M_t)\big)_{t \geq 0}$ is a local $(P, \mathbb{F})$-martingale.

(f) Let $H = (H_t)_{t \geq 0}$ be a predictable process and fix a $T > 0$. Which of the following assertions is true?

(1) If H is continuous, then $\int_0^T H_s dW_s$ is normally distributed.

(2) $\int_0^T H_s dW_s$ is not normally distributed for any discontinuous H.

(3) If H is deterministic and bounded, then $\int_0^T H_s dW_s$ is normally distributed.

(g) Which of the following statements about W is **not** true?

(1) P-almost all paths of W are nowhere differentiable.

(2) $\lim_{t \to \infty} \frac{W_t}{\sqrt{t}} = 1$ P-a.s.

(3) W is a local $(P, \mathbb{F})$-martingale with the Markov property.

(h) Let $X = (X_t)_{t \geq 0}$ be a general $(P, \mathbb{F})$-semimartingale null at 0 and let $\mathcal{E}(X)$ denote the stochastic exponential of X. Which of the following is **not** true?

(1) $\mathcal{E}(X)$ is the unique solution of $dZ_t = Z_{t-} dX_t$ with $Z_0 = 1$.

(2) $\mathcal{E}(X)$ is necessarily a $(P, \mathbb{F})$-semimartingale.

(3) We have the explicit formula $\mathcal{E}(X) = \exp\big(X - \frac{1}{2}[X]\big)$.

Question 2 (8 Points)

Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1)$ consisting of a bank account and one stock. The movements of the bank account $\widetilde{S}^0$ and the stock price $\widetilde{S}^1$ are described by the trees below, where the numbers beside the branches denote transition probabilities.

$$\begin{tikzcd}[row sep=small]
\widetilde{S}^0: & 1 \arrow[r, "1"] & 1+r \arrow[r, "1"] & (1+r)(1+2r) \\
 & & & (1+u)(1+2u) \\
 & & 1+u \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\
 & & & (1+u)(1+2d) \\
\widetilde{S}^1: & 1 \arrow[uur, "\frac{1}{2}"] \arrow[ddr, "\frac{1}{2}"'] & & \\
 & & & (1+d)(1+u) \\
 & & 1+d \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\
 & & & (1+d)(1+d)
\end{tikzcd}$$

Note that the interest rate is $2r$ in the second period.

More precisely, let $(\Omega, \mathcal{F}, P)$ be the probability space given by $\Omega := \{-1, 1\}^2$, $\mathcal{F} := 2^\Omega$ and the probability measure P defined by $P[\{(x_1, x_2)\}] := p_{x_1} p_{x_1, x_2}$, where

$$p_1 = p_{-1} := \frac{1}{2} \quad \text{and} \quad p_{1,1} = p_{1,-1} = p_{-1,1} = p_{-1,-1} := \frac{1}{2}.$$

Next, let $u > d$ and $d, r > -0.5$ and consider the random variables Y_1 and Y_2 given by

$$\begin{aligned}
Y_1\big((1,1)\big) &:= 1+u, & Y_1\big((-1,1)\big) &:= 1+d, \\
Y_1\big((1,-1)\big) &:= 1+u, & Y_1\big((-1,-1)\big) &:= 1+d, \\
Y_2\big((1,1)\big) &:= 1+2u, & Y_2\big((-1,1)\big) &:= 1+u, \\
Y_2\big((1,-1)\big) &:= 1+2d, & Y_2\big((-1,-1)\big) &:= 1+d.
\end{aligned}$$

The bank account process $\widetilde{S}^0$ and the stock price process $\widetilde{S}^1$ are then given by $\widetilde{S}^0_k = \prod_{j=1}^k (1+jr)$ and $\widetilde{S}^1_k = \prod_{j=1}^k Y_j$ for $k \in \{0, 1, 2\}$, respectively. Finally, the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,2}$ is defined by $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 := \sigma(Y_1)$ and $\mathcal{F}_2 := \sigma(Y_1, Y_2) = 2^\Omega = \mathcal{F}$.

(a) Prove in detail that the market $(\widetilde{S}^0, \widetilde{S}^1)$ is free of arbitrage if and only if both $d < r < u$ and $d < 2r < u$ are satisfied. (2 pt)

(b) Now let $u = \frac{1}{3}$, $r = \frac{1}{9}$ and $d = -\frac{1}{3}$. Compute the arbitrage-free price at time $k = 0$ of a claim whose undiscounted payoff is given by $\widetilde{H} = \max\{0, \widetilde{S}^1_2 - \widetilde{K}\}$ with $\widetilde{K} = \frac{8}{9}$. (2 pt)

(c) Suppose that $u = \frac{1}{3}$, $r = \frac{1}{6}$ and $d = -\frac{1}{3}$. Give an example of a self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ satisfying $P[V_2(\varphi) \geq 2] = 0.25$ and $V_2(\varphi) \geq 0$ P-a.s. (2 pt)

(d) Suppose again that $u = \frac{1}{3}$, $r = \frac{1}{6}$ and $d = -\frac{1}{3}$. Does there exist a self-financing strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with $V_2(\varphi) \geq 2$ P-a.s.? Justify your answer by either providing an example of such a strategy or by formally arguing that such a strategy does not exist. (2 pt)

Question 3 (8 **Points**)

In a game between a gambler and a croupier, suppose that the total capital in play is 1. The capital held by the gambler after the n-th round is denoted by $X_n \in [0,1]$. The capital held by the croupier after the n-th round is thus equal to $1 - X_n$. We assume $X_0 = p \in (0,1)$.

The rules of the game are such that after n rounds, the probability for the gambler to win the $(n+1)$-th round is equal to X_n; if he does, he gains half of the capital the croupier held after the n-th round, while if he loses he gives half of his capital to the croupier. Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ with $\mathcal{F}_n = \sigma(X_i, 1 \le i \le n)$.

(a) Show that $X = (X_n)_{n\in\mathbb{N}}$ is an $\mathbb{F}$-martingale. (2 pt)

(b) For any $n \ge 0$, let $Y_{n+1} = 2X_{n+1} - X_n$. Find the distribution of Y_{n+1} conditional on $\mathcal{F}_n$, i.e. show that

$$P\left[Y_{n+1} = 0 \,|\, \mathcal{F}_n\right] = 1 - X_n \quad \text{and} \quad P\left[Y_{n+1} = 1 \,|\, \mathcal{F}_n\right] = X_n.$$

Then express the (unconditional) distribution of Y_n. (2 pt)

(c) Show that

$$E\left[X_{n+1}^2\right] = E\left[\frac{3X_n^2 + X_n}{4}\right].$$

Because X is a bounded martingale, one can prove that X_n converges P-a.s. *as well as* in L^2 towards a limit Z. Deduce that

$$E\left[Z^2\right] = E\left[Z\right] = p.$$

Hint: $X_n \to Z$ *in* L^2 *implies that* $E\left[X_n^2\right] \to E\left[Z^2\right]$. (2 pt)

(d) Let $G_n = \{Y_n = 1\}$, $L_n = \{Y_n = 0\}$. Show that $Y_n \to Z$ P-a.s. and derive the distribution of Z. Deduce that

$$P\left[G_n \text{ occurs only finitely many times}\right] = 1 - p,$$
$$P\left[L_n \text{ occurs only finitely many times}\right] = p.$$

(2 pt)

Question 4 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $W = (W_t)_{t\geq 0}$ a Brownian motion with respect to P and $\mathbb{F}$.

(a) Calculate $\langle W^2, \exp(\alpha W)\rangle$ and $[W^2, \exp(\alpha W)]$ for $\alpha \in \mathbb{R}$. (2 pt)

(b) Let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, $(x,t) \mapsto f(x,t)$ be a bounded $C^{2,1}$ function with $f(0,0) = 0$ and satisfying

$$\frac{\partial f}{\partial t}(x,t) = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t) \quad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Define the process $Y = (Y_t)_{t\geq 0}$ by $Y_t := f(W_t, t)$. Is this process a $(P, \mathbb{F})$-martingale? Why is it square-integrable? Compute $\operatorname{Var}[Y_t]$. (2 pt)

(c) Fix a $T > 0$. Show that the process $M = (M_t)_{t\in[0,T]}$ defined by $M_t := P[a \leq W_T \leq b \,|\, \mathcal{F}_t]$ is a $(P, \mathbb{F})$-martingale on $[0,T]$ and that it admits a unique representation as

$$M_t = g(W_t, t) \quad \text{for } t \in [0,T]$$

and for some measurable function $g : \mathbb{R} \times [0,T] \to \mathbb{R}$. Express g for $t < T$ using Φ, the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. (3 pt)

(d) For the martingale M in (c), explicitly construct $M_0 \in \mathbb{R}$ and a process $\psi \in L^2_{\text{loc}}(W)$ such that

$$M = M_0 + \int \psi \, dW.$$

If you did not find the explicit form of the function g in (c), express the above representation of M using a generic $C^{2,1}$ function $g : \mathbb{R} \times [0,T) \to \mathbb{R}$. (1 pt)

Question 5 (8 **Points**)

Let $T \in (0, \infty)$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s; 0 \leq s \leq T)$. Consider the Black–Scholes model, where the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy

$$
\begin{aligned}
d\widetilde{S}^0_t &= \widetilde{S}^0_t r dt, & \widetilde{S}^0_0 &= 1,\\
d\widetilde{S}^1_t &= \widetilde{S}^1_t \left(\mu dt + \sigma dW_t\right), & \widetilde{S}^1_0 &= S^1_0,
\end{aligned}
$$

with constants $\mu, r \in \mathbb{R}$, $\sigma > 0$, and $\widetilde{S}^1_0 = S^1_0 > 0$.

(a) Let $Z_t := \log(\widetilde{S}^0_t + \widetilde{S}^1_t)$. Derive the SDE satisfied by Z. The final result should only contain functions of Z_t and t, but neither $\widetilde{S}^0_t$ nor $\widetilde{S}^1_t$. (2 pt)

(b) Consider the *n-th root of the stock option*, whose undiscounted payoff at time T is given by

$$\widetilde{H}_n = \left(\widetilde{S}^1_T\right)^{1/n},$$

for $n \in \mathbb{N}$.

(i) Compute the undiscounted arbitrage-free price $\widetilde{V}^{\widetilde{H}_n}_t$ at time t.

(ii) Find the replicating strategy for $\widetilde{H}_n$.

(3 pt)

(c) Consider the stochastic process

$$X_t = \int_0^t W_s dW_s \quad \text{for } 0 \leq t \leq T.$$

It is known that $\mathcal{E}(X)$ is a $(P, \mathbb{F})$-martingale (you do not need to prove this). Consider the equivalent measure $\widehat{Q} \approx P$ on $\mathcal{F}_T$ given by

$$E_P\left[\frac{d\widehat{Q}}{dP}\,\middle|\,\mathcal{F}_t\right] = \mathcal{E}(X)_t \quad \text{for } 0 \leq t \leq T.$$

Find the SDE satisfied by $\widetilde{S}^1$ under $\widehat{Q}$; it **must** involve a $\widehat{Q}$-Brownian motion. Additionally, show that

$$E_{\widehat{Q}}\left[W_t\right] = 0 \quad \text{for all } t \in [0, T].$$

(3 pt)