Question 1

The correct answers are:

(a) (3)

(b) (2)

(c) (2)

(d) (1)

(e) (1)

(f) (3)

(g) (2)

(h) (2)

Question 2

(a) By definition, $\mathbb{P}_e(S^1)$ is given by all probability measures Q on $(\Omega, \mathcal{F})$ that are equivalent to P and satisfy $E_Q\left[S_1^1\right] = S_0^1$. Since Ω is a finite set, all such probability measures can be characterized by the probability vectors $(q_1, q_2, q_3) \in (0,1)^3$ with $q_1 + q_2 + q_3 = 1$ and

$$\begin{aligned} E_Q\left[S_1^1\right] = S_0^1 \quad &\iff \quad E_Q\left[\widetilde{S}_0^1 \frac{Y_1}{(1+r)}\right] = \widetilde{S}_0^1 \quad \iff \quad E_Q\left[Y_1\right] = 1 + r \\ &\iff \quad q_1(1+d) + q_2(1+m) + q_3(1+u) = 1 + r \\ &\iff \quad q_1 d + q_2 m + q_3 u = r \\ &\iff \quad -0.2q_1 + 0.1q_2 + 0.3q_3 = 0.1. \end{aligned} \tag{1}$$

Setting $q_1 = \alpha$, we obtain that $q_3 = 1 - \alpha - q_2$ from the condition $q_1 + q_2 + q_3 = 1$ and thus from (1) that

$$0.1q_2 = 0.1 + 0.2\alpha - 0.3(1 - \alpha - q_2) \quad \iff \quad -0.2q_2 = -0.2 + 0.5\alpha \quad \iff \quad q_2 = 1 - \frac{5}{2}\alpha.$$

It thus follows that

$$q_3 = 1 - \alpha - 1 + \frac{5}{2}\alpha = \frac{3}{2}\alpha.$$

Since we must have $(q_1, q_2, q_3) \in (0,1)^3$, we can only take $\alpha \in (0, 2/5)$. So

$$\mathbb{P}_e(S^1) = \left\{ Q_\alpha \mathrel{\widehat{=}} \left(\alpha, 1 - \frac{5}{2}\alpha, \frac{3}{2}\alpha\right) \ : \ \alpha \in \left(0, \frac{2}{5}\right) \right\}.$$

Parametrising instead $q_2 := \alpha$, analogous computations lead to

$$\mathbb{P}_e(S^1) = \left\{ Q_\alpha \mathrel{\widehat{=}} \left(\frac{2}{5} - \frac{2}{5}\alpha, \alpha, \frac{3}{5} - \frac{3}{5}\alpha\right) \ : \ \alpha \in (0,1) \right\},$$

and parametrising instead $q_3 := \alpha$ to

$$\mathbb{P}_e(S^1) = \left\{ Q_\alpha \mathrel{\widehat{=}} \left(\frac{2}{3}\alpha, 1 - \frac{5}{3}\alpha, \alpha\right) \ : \ \alpha \in \left(0, \frac{3}{5}\right) \right\}.$$

(b) Since every martingale is a local martingale (with respect to the same probability measure and filtration), we clearly have that $\mathbb{P}_e(S^1) \subseteq \mathbb{P}_{e,loc}(S^1)$. To show the opposite inclusion, we note that S^1 is bounded P-a.s. by a fixed constant C because Ω is finite, and so is then $(S^1)^\tau$ for any $\mathbb{F}$-stopping time τ. Fix a $Q \in \mathbb{P}_{e,loc}(S^1)$ and let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for S^1. Then each $(S^1)^{\tau_n}$ is a $(Q,\mathbb{F})$-martingale and Q-a.s. bounded by C because $Q \approx P$. So the dominated convergence theorem then gives

$$E_Q\left[S_1^1\right] = E_Q\left[\lim_{n\to\infty} S_{1\wedge\tau_n}^1\right] = \lim_{n\to\infty} E_Q\left[S_{1\wedge\tau_n}^1\right] = \lim_{n\to\infty} S_{0\wedge\tau_n}^1 = S_0^1.$$

So S^1 is in fact a $(Q,\mathbb{F})$-martingale, which means that $Q \in \mathbb{P}_e(S^1)$. This shows that $\mathbb{P}_{e,loc}(S^1) \subseteq \mathbb{P}_e(S^1)$ and concludes the proof.

(c) The set of all arbitrage-free prices for $\widetilde{C}(\widetilde{S}_1^1)$ is given by

$$M = \left\{ E_Q\left[\frac{\widetilde{C}(\widetilde{S}_1^1)}{1+r}\right] \ : \ Q \in \mathbb{P}_e(S^1) \right\}.$$

Using the parametrisation of $P_e(S^1)$ from (a), we compute

$$E_{Q_\alpha}\left[\frac{\widetilde{C}(\widetilde{S}_1^1)}{1+r}\right] = \frac{1}{1.1}\left(\alpha \times 0 + \left(1 - \frac{5}{2}\alpha\right) \times 0 + \frac{3}{2}\alpha \times 2\right) = \frac{30}{11}\alpha.$$

So we conclude that $M = \left(0, \frac{12}{11}\right)$ because $\alpha \in \left(0, \frac{2}{5}\right)$.

(d) We have from (c) that

$$\sup_{Q \in \mathbb{P}_e(S^1)} E_Q\left[\frac{1}{1+r}\widetilde{C}(\widetilde{S}_1^1)\right] = \sup_{\alpha \in (0,2/5)} E_{Q_\alpha}\left[\frac{1}{1+r}\widetilde{C}(\widetilde{S}_1^1)\right] = \sup_{\alpha \in (0,2/5)} \frac{30}{11}\alpha = \frac{12}{11}.$$

The value $\frac{12}{11}$ is clearly attained for $\alpha = \frac{2}{5}$, which means that it is attained under the probability measure Q^* characterized by the probability vector $(\frac{2}{5}, 0, \frac{3}{5})$. Q^* is clearly not equivalent to P, but since $P[\{\omega\}] = 0$ implies $Q^*[\{\omega\}] = 0$, Q^* is absolutely continuous with respect to P. (In fact, $P[\{\omega\}] = 0$ is never true so that by the logical fact that an empty premise implies every conclusion, any probability measure on $(\Omega, \mathcal{F})$ is absolutely continuous with respect to P.)

S^1 is also a $(Q^*, \mathbb{F})$-martingale since

$$E_{Q^*}\left[S_1^1\right] = \frac{1}{1.1} \times \left(\frac{2}{5} \times 8 + \frac{3}{5} \times 13\right) = \frac{1}{1.1} \times \frac{16+39}{5} = \frac{1}{1.1} \times 11 = 10 = S_0^1.$$

The Q^*-integrability of S^1 is trivial since S^1 is bounded, and adaptedness does not depend on the probability measure.

Question 3

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of simple random variables of the form

$$X_n = \sum_{i=1}^{n} x_{i,n}\mathbb{1}_{A_{i,n}}$$

for some constants $x_{1,n},\ldots,x_{n,n} \geq 0$ and some sets $A_{1,n},\ldots,A_{n,n} \in \mathcal{F}$ with $X_n \uparrow X$ pointwise as $n \to \infty$. We have seen that such a sequence exists, for instance in the solution to Exercise 2.3 of the exercise sheets. We compute

$$\begin{aligned} E_Q[X_n] &= \sum_{i=1}^{n} x_{i,n}E_Q\left[\mathbb{1}_{A_{i,n}}\right] = \sum_{i=1}^{n} x_{i,n}Q[A_{i,n}] = \sum_{i=1}^{n} x_{i,n}E_P\left[\mathcal{D}\mathbb{1}_{A_{i,n}}\right] \\ &= E_P\left[\mathcal{D}\sum_{i=1}^{n} x_{i,n}\mathbb{1}_{A_{i,n}}\right] = E_P[\mathcal{D}X_n]. \end{aligned} \tag{2}$$

By the monotone convergence theorem, we immediately obtain that $E_Q[X_n] \uparrow E_Q[X]$ as $n \to \infty$. But since $\mathcal{D} > 0$, we also clearly have that $\mathcal{D}X_n \uparrow \mathcal{D}X$ and another application of the monotone convergence theorem thus gives that $E_P[\mathcal{D}X_n] \uparrow E_P[\mathcal{D}X]$. Therefore, taking the limit on both sides of (2) gives $E_Q[X] = E_P[\mathcal{D}X]$ as desired.

(b) We compute

$$E_Q[Y] = E_P[\mathcal{D}Y] = E_P[E_P[\mathcal{D}Y \,|\, \mathcal{F}_k]] = E_P[Y E_P[\mathcal{D} \,|\, \mathcal{F}_k]] = E_P[Z_k Y].$$

The first equality uses (a), the second one uses the tower property of conditional expectation, the third one the $\mathcal{F}_k$-measurability and nonnegativity of Y, and the last one the definition of Z_k.

(c) We compute

$$E_P[Y] = E_P\left[Z_k\frac{1}{Z_k}Y\right] = E_Q\left[\frac{1}{Z_k}Y\right].$$

The first equality is obvious because $Z_k > 0$ P-a.s., and the second one follows from (b) since Y/Z_k is nonnegative by nonnegativity of Z_k and Y and also $\mathcal{F}_k$-measurable as a ratio of two $\mathcal{F}_k$-measurable random variables.

(d) By the definition of conditional expectation, we need to show that

$$E_Q\big[E_Q[U_k \,|\, \mathcal{F}_j]\,\mathbb{1}_A\big] = E_Q\left[\frac{1}{Z_j}E_P[Z_kU_k \,|\, \mathcal{F}_j]\,\mathbb{1}_A\right]$$

for all $A \in \mathcal{F}_j$. We fix $A \in \mathcal{F}_j$ and compute

$$\begin{aligned} E_Q\big[E_Q[U_k \,|\, \mathcal{F}_j]\,\mathbb{1}_A\big] &= E_Q[U_k\mathbb{1}_A] = E_P[Z_kU_k\mathbb{1}_A] = E_P\big[E_P[Z_kU_k \,|\, \mathcal{F}_j]\,\mathbb{1}_A\big] \\ &= E_Q\left[\frac{1}{Z_j}E_P[Z_kU_k \,|\, \mathcal{F}_j]\,\mathbb{1}_A\right]. \end{aligned}$$

The first and the third equality follow from the definition of conditional expectation, the second one from (b), and the last one uses (c) with the fact that $E_P[Z_kU_k \,|\, \mathcal{F}_j]\,\mathbb{1}_A$ is nonnegative by the nonnegativity of U_k and Z_k and also $\mathcal{F}_j$-measurable. Indeed, a conditional expectation with respect to $\mathcal{F}_j$ is $\mathcal{F}_j$-measurable and $\mathbb{1}_A$ is also $\mathcal{F}_j$-measurable since $A \in \mathcal{F}_j$ by assumption.

(e) If N is $\mathbb{F}$-adapted, then ZN is $\mathbb{F}$-adapted since the product of measurable functions is a measurable function. Conversely, if ZN is $\mathbb{F}$-adapted, then N is $\mathbb{F}$-adapted for the same reason since $N = \frac{1}{Z}ZN$ and $Z > 0$. The same argument shows that Z is nonnegative if and only if ZN is nonnegative.

Now, N is Q-integrable if and only if ZN is P-integrable because for any $k \in \{0, 1, \dots, T\}$, we have by (b) that

$$E_P[|Z_k N_k|] = E_P[Z_k |N_k|] = E_Q[|N_k|].$$

Finally, $N \geq 0$ satisfies the martingale property under Q if and only if $ZN \geq 0$ satisfies the martingale property under P. Indeed, note that by (d), we have for any $k \in \{1, \dots, T\}$ that

$$E_Q[N_k \,|\, \mathcal{F}_{k-1}] = \frac{1}{Z_{k-1}} E_P[Z_k N_k \,|\, \mathcal{F}_{k-1}],$$

which gives that

$$\begin{aligned} E_Q[N_k \,|\, \mathcal{F}_{k-1}] = N_{k-1} \qquad &\Longleftrightarrow \qquad \frac{1}{Z_{k-1}} E_P[Z_k N_k \,|\, \mathcal{F}_{k-1}] = N_{k-1} \\ &\Longleftrightarrow \qquad E_P[Z_k N_k] = Z_{k-1} N_{k-1} \end{aligned}$$

since $Z_{k-1} > 0$.

Question 4

(a) Z^σ is clearly positive by definition for all $\sigma > -1$. Furthermore, using the fact that $N_t - N_s \sim \text{Poi}(\lambda(t-s))$ and the knowledge of the moment generating function of the Poisson distribution, we compute

$$E\left[\frac{Z_t^\sigma}{Z_s^\sigma}\,\middle|\,\mathcal{F}_s\right] = E\left[e^{(N_t-N_s)\log(1+\sigma)-\lambda\sigma(t-s)}\,\middle|\,\mathcal{F}_s\right] = e^{-\lambda\sigma(t-s)}E\left[e^{(N_t-N_s)\log(1+\sigma)}\right]$$
$$= e^{-\lambda\sigma(t-s)}e^{\lambda(t-s)(1+\sigma-1)} = 1.$$

Here we do not have to worry about the integrability of Z^σ when verifying the martingale condition since $Z^\sigma > 0$ for $\sigma > -1$. Using the martingale property of Z^σ, it also follows for any $\sigma > -1$ and $t \in [0,T]$ that

$$E_P\left[Z_t^\sigma\right] = E_P\left[Z_0^\sigma\right] = 1$$

since $N_0 = 0$ P-a.s.

(b) First, since $Q^\sigma \approx P$ for any $\sigma > -1$, we immediately obtain that $\{N_0 \neq 0\}$ is a Q^σ-nullset because it is a P-nullset and that

$$\{\omega \in \Omega : [0,T] \ni t \mapsto N_t(\omega) \text{ is not RCLL with jumps of size } 1\}$$

is a Q^σ-nullset because it is P-nullset. So $N_0 = 0$ Q^σ-a.s. and Q^σ-almost all trajectories of N are RCLL with jumps of size 1.

Now we compute the conditional moment generating function of the increment $N_t - N_s$, $0 \le s \le t \le T$, under Q^σ. It is given by

$$E_{Q^\sigma}\left[e^{u(N_t-N_s)}\,\middle|\,\mathcal{F}_s\right] = \frac{1}{Z_s^\sigma}E_P\left[Z_t^\sigma e^{u(N_t-N_s)}\,\middle|\,\mathcal{F}_s\right]$$
$$= e^{-N_s\log(1+\sigma)+\lambda\sigma s}E_P\left[e^{N_t\log(1+\sigma)-\lambda\sigma t}e^{u(N_t-N_s)}\,\middle|\,\mathcal{F}_s\right]$$
$$= e^{-\lambda\sigma(t-s)}E_P\left[e^{(N_t-N_s)\log(1+\sigma)}e^{u(N_t-N_s)}\,\middle|\,\mathcal{F}_s\right]$$
$$= e^{-\lambda\sigma(t-s)}E_P\left[e^{(N_t-N_s)(\log(1+\sigma)+u)}\right] = e^{-\lambda\sigma(t-s)}e^{\lambda(t-s)(e^{\log(1+\sigma)+u}-1)}$$
$$= e^{-\lambda\sigma(t-s)}e^{\lambda(t-s)((1+\sigma)e^u-1)} = e^{-\lambda(1+\sigma)(t-s)(e^u-1)}.$$

The first equality follows from the Bayes formula, the third from the $\mathcal{F}_s$-measurability of $e^{N_s\log(1+\sigma)}$, the fourth from the independence of $N_t - N_s$ of $\mathcal{F}_s$ under P, and the fifth from the fact that $E_P\left[e^{(\log(1+\sigma)+u)(N_t-N_s)}\right]$ is the moment generating function of $\text{Poi}(\lambda(t-s))$ evaluated at $\log(1+\sigma)+u$.

The last expression above is in fact the moment generating function of $\text{Poi}(\lambda(1+\sigma)(t-s))$ and thus shows that $N_t - N_s \sim \text{Poi}(\lambda(1+\sigma)(t-s))$. Furthermore, since the expression does not depend on $\omega \in \Omega$, we can also conclude that $e^{u(N_t-N_s)}$ is independent of $\mathcal{F}_s$ under Q^σ, by which we can conclude the same about $N_t - N_s$ since is can be written as a continuous (therefore measurable) transformation of $e^{u(N_t-N_s)}$. We can thus conclude that N is $(Q^\sigma, \mathbb{F})$-Poisson process with parameter $\lambda(1+\sigma) > 0$.

(c) Since X and Y are predictable and satisfy

$$E_P\left[\int_0^T X_t^2 d[\widetilde{N}]_t\right] < \infty \quad \text{and} \quad E_P\left[\int_0^T Y_t^2 d[\widetilde{N}]_t\right],$$

the stochastic integrals $\int_0^T X_t d\widetilde{N}_t$ and $\int_0^T Y_t d\widetilde{N}_t$ are well defined. By squaring out, we obtain

$$\left(\int_0^T X_t d\widetilde{N}_t\right)\left(\int_0^T Y_t d\widetilde{N}_t\right)$$
$$= \frac{1}{2}\left(\left(\int_0^T X_t d\widetilde{N}_t + \int_0^T Y_t d\widetilde{N}_t\right)^2 - \left(\int_0^T X_t d\widetilde{N}_t\right)^2 - \left(\int_0^T Y_t d\widetilde{N}_t\right)^2\right)$$
$$= \frac{1}{2}\left(\left(\int_0^T (X_t+Y_t) d\widetilde{N}_t\right)^2 - \left(\int_0^T X_t d\widetilde{N}_t\right)^2 - \left(\int_0^T Y_t d\widetilde{N}_t\right)^2\right).$$

Taking P-expectations on both sides, using linearity and applying the isometry property of the stochastic integral to all three terms on the right-hand side ($\widetilde{N}$ is a $(P, \mathbb{F})$-martingale as shown in Exercise 9.2 (a)), we obtain

$$\begin{aligned}
&E_P\left[\left(\int_0^T X_t d\widetilde{N}_t\right)\left(\int_0^T Y_t d\widetilde{N}_t\right)\right] \\
&= \frac{1}{2}\left(E_P\left[\int_0^T (X_t+Y_t)^2 d[\widetilde{N}]_t\right] - E_P\left[\int_0^T X_t^2 d[\widetilde{N}]_t\right] - E_P\left[\int_0^T Y_t^2 d[\widetilde{N}]_t\right]\right) \\
&= E_P\left[\int_0^T X_t Y_t d[\widetilde{N}]_t\right] = E_P\left[\int_0^T X_t Y_t dN_t\right],
\end{aligned}$$

as desired. The last equality follows from the fact that $[\widetilde{N}] = N$, as shown in Exercise 9.2 (b) in the exercise sheet.

Question 5

(a) Since M is a $(P, \mathbb{F})$-supermartingale, we have for all $0 \leq s \leq t \leq T$ that $M_s - E_P[M_t \,|\, \mathcal{F}_s] \geq 0$ P-a.s. Taking the expectation of the left-hand side gives

$$E_P\big[M_s - E_P[M_t \,|\, \mathcal{F}_s]\big] = E_P[M_s] - E_P[M_t] = C - C = 0.$$

But every nonnegative random variable with zero P-expectation is P-a.s. equal to zero (as shown in Exercise 1.2 in the exercise sheets). So we have $E_P[M_t \,|\, \mathcal{F}_s] = M_s$ P-a.s., which is the martingale property of M. Integrability and adaptedness follow from the fact that M is a $(P, \mathbb{F})$-supermartingale.

(b) Let us define the process $Y = (Y_t)_{t\in[0,T]}$ by

$$Y_t = \int_0^t \lambda(s) dW_s.$$

Using the hint and the assumption that $\lambda \in L^2_{loc}(W)$, we see that Y is in fact a continuous local $(P, \mathbb{F})$-martingale. The process Z is thus explicitly given by

$$Z_t = \exp\left(Y_t - \frac{1}{2}\langle Y\rangle_t\right) = \exp\left(\int_0^t \lambda(s) dW_s - \frac{1}{2}\int_0^t \lambda^2(s) ds\right) \tag{3}$$

and it is in particular also a local $(P, \mathbb{F})$-martingale. Taking expectations on both sides of (3) and using the fact that

$$\int_0^t \lambda(s) dW_s \sim \mathcal{N}\left(0, \int_0^t \lambda^2(s) ds\right)$$

because λ is a deterministic function (see Exercise 12.3 (b) in the exercise sheets), we obtain that $E[Z_t] = 1$ for all $t \in [0, T]$. In particular, Z is integrable. But since Z is also positive, we can apply Fatou's lemma to show that it is in fact a $(P, \mathbb{F})$-supermartingale. Indeed, let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence for Z. Then we have for all $0 \leq s \leq t \leq T$ that

$$E_P[Z_t \,|\, \mathcal{F}_s] = E_P\left[\liminf_{n\to\infty} Z_t^{\tau_n} \,\middle|\, \mathcal{F}_s\right] \leq \liminf_{n\to\infty} E_P[Z_t^{\tau_n} \,|\, \mathcal{F}_s] = \liminf_{n\to\infty} Z_s^{\tau_n} = Z_s.$$

But according to (a), every $(P, \mathbb{F})$-supermartingale with a constant expectation is a true $(P, \mathbb{F})$-martingale and we are done.

Alternatively, and more simply, one could recall the *Novikov's condition* that is briefly mentioned in the lecture notes and which says that if $L = (L_t)_{t\in[0,T]}$ is a continuous local $(P, \mathbb{F})$-martingale with $L_0 = 0$ and $E_P\big[\frac{1}{2}\langle L\rangle_T\big] < \infty$, then $\mathcal{E}(L)$ is a true $(P, \mathbb{F})$-martingale on $[0, T]$. In our case, we have that

$$E_P\left[e^{\frac{1}{2}\langle Y\rangle_T}\right] = E_P\left[e^{\frac{1}{2}\int_0^T \lambda^2(s)ds}\right] = e^{\frac{1}{2}\int_0^T \lambda^2(s)ds} < \infty,$$

so the result follows.

(c) We first compute the P-dynamics of the discounted price process $S^1 = \widetilde{S}^1/\widetilde{S}^0$. Direct application of Itô's formula (or the product rule) to the semimartingale $(\widetilde{S}^0, \widetilde{S}^1)$ and the C^2 function $\mathbb{R}^2_{++} \ni (x, y) \mapsto x/y$ yields

$$\frac{dS_t^1}{S_t^1} = \big(\mu_1 - r(t)\big)dt + \sigma_1 dW_t. \tag{4}$$

Define $\lambda\colon [0, T] \mapsto \mathbb{R}$ by $\lambda(s) := (\mu_1 - r(s))/\sigma_1$ and the process $Z = (Z_t)_{t\in[0,T]}$ by

$$Z_t := \mathcal{E}\left(-\int \lambda(s) dW_s\right)_t.$$

Since λ is left-continuous and bounded, (b) gives that Z is a positive $(P, \mathbb{F})$-martingale with $E_P[Z_t] = 1$ for all $t \in [0, T]$, and thus the density process of some measure $Q \approx P$. Girsanov's theorem gives that

$$W_t^Q := W_t - \left\langle W, -\int \lambda(s) dW_s \right\rangle_t = W_t + \int_0^t \lambda(s) ds = W_t + \int_0^t \frac{\mu_1 - r(s)}{\sigma_1} ds$$

is a $(Q, \mathbb{F})$-Brownian motion. By (4), the Q-dynamics of S^1 are given by

$$\frac{dS_t^1}{S_t^1} = (\mu_1 - r(t))dt - (\mu_1 - r(t))dt + \sigma_1 d\left(W_t + \int_0^t \frac{\mu_1 - r(s)}{\sigma_1} ds\right) = \sigma_1 dW_t^Q.$$

In other words, $S^1 = \mathcal{E}(\sigma_1 W^Q)$, which is a $(Q, \mathbb{F})$-martingale.

(d) The arbitrage-free price at time t of the discounted payoff H is given by

$$\begin{aligned}
V_t &= E_Q\left[\frac{(\widetilde{S}_T^1)^p}{\widetilde{S}_T^0} \,\middle|\, \mathcal{F}_t\right] = \left(\widetilde{S}_T^0\right)^{p-1} E_Q\left[(S_T^1)^p \,\middle|\, \mathcal{F}_t\right] = \left(\widetilde{S}_T^0\right)^{p-1} (S_t^1)^p E_Q\left[\left(\frac{S_T^1}{S_t^1}\right)^p \,\middle|\, \mathcal{F}_t\right] \\
&= e^{(p-1)\int_0^T r(s)ds} \left(S_t^1\right)^p E_Q\left[e^{p\sigma_1(W_T^Q - W_t^Q) - p\frac{\sigma_1^2}{2}(T-t)} \,\middle|\, \mathcal{F}_t\right] \\
&= e^{(p-1)\int_0^T r(s)ds - p\frac{\sigma_1^2}{2}(T-t)} \left(S_t^1\right)^p E_Q\left[e^{p\sigma_1(W_T^Q - W_t^Q)}\right] \\
&= e^{(p-1)\int_0^T r(s)ds - p\frac{\sigma_1^2}{2}(T-t)} \left(S_t^1\right)^p e^{\frac{p^2\sigma_1^2(T-t)}{2}} \\
&=: v(t, S_t^1),
\end{aligned}$$

where we have used that $W_T^Q - W_t^Q$ is independent of $\mathcal{F}_t$ under Q and normally distributed with mean 0 and variance $T - t$. Consequently, the delta of H is given by

$$\vartheta_t = \left.\frac{\partial v}{\partial x}\right|_{(t,x)=(t,S_t^1)} = p e^{(p-1)\int_0^T r(s)ds - p\frac{\sigma_1^2}{2}(T-t)} (S_t^1)^{p-1} e^{\frac{p^2\sigma_1^2(T-t)}{2}}, \tag{5}$$

and the price at time 0 is

$$V_0 = v(0, S_0^1) = e^{(p-1)\int_0^T r(s)ds - p\frac{\sigma_1^2}{2}T} e^{\frac{p^2\sigma_1^2 T}{2}}.$$