ETH Zürich, HS 2020
D-MATH
Prof. Dr. M. Schweizer **January 2021**

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Important!

Enter the first two letters of your surname and your first name, as well as the last six digits of your matriculation number in the boxes below. Write the same information and **only this information** clearly at the top of each additional sheet of paper that you submit.

Surname	**First name**	**Matriculation (legi) number**	**Exam number**
first *two letters of each*		***last*** *six digits only*	*leave blank*

Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration: 180 min.

Closed book examination: no notes, no books, no calculator, no mobile phones, etc. allowed.

Important:

- ◇ Please put your student card on the table.
- ◇ Only pen and paper are allowed on the table. Please **do not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ◇ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question, but try to solve as many questions as possible.
- ◇ Take a new sheet for each question.
- ◇ Except for Question 1, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas and results from the lecture or from the exercise classes without derivation.
- ◇ Simplify your results as much as possible.
- ◇ Most of the subquestions can be solved independently of each other.

$\star\star\star$ Good luck! $\star\star\star$

Answer Sheet for Question 1

Please use this sheet to answer Question 1. Indicate the correct answer by ✗. If there is no ✗ or more than one ✗ in a line, this will be interpreted as "no answer".

	answer (1)	answer (2)	answer (3)
(a)			
(b)			
(c)			
(d)			
(e)			
(f)			
(g)			
(h)			

Leave blank

correct	wrong	no answer

Leave blank

	1st corr.	2nd corr.
correct		
wrong		
no answer		
Points		

Question 1 (8 **Points**)

For each of the following eight subquestions, there is **exactly one** correct answer. For each correct answer, you get 1 point; for each wrong answer, you get -0.5 point; and for no answer, you get 0 points. You get at least 0 points overall for the question. **Please use the printed form on the previous page for your answers.** It is enough to indicate your answer by ✗; you need not explain your choice.

Throughout subquestions (a) to (d), let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ or shortly $(\widetilde{S}^0, \widetilde{S}^1)$ be an undiscounted financial market in discrete time with a finite time horizon $T \in \mathbb{N}$ and with the filtration $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,T}$ generated by $\widetilde{S}^1$. Furthermore, $\widetilde{S}^0_k := (1+r)^k$ for $k = 0, 1, \dots, T$ and constants $r > -1$ and $\widetilde{S}^1_0 := s^1_0 > 0$. The prices discounted by $\widetilde{S}^0$ are denoted by S^0 and S^1, respectively.

(a) Which of the following statements is correct?

(1) If the market (S^0, S^1) is complete, then it is arbitrage-free.

(2) If the market (S^0, S^1) is arbitrage-free and complete, then every contingent claim $H \in L^0_+(\mathcal{F}_T)$ admits a unique replicating strategy.

(3) If (S^0, S^1) is arbitrage-free but not complete, then there exists a contingent claim $H \in L^0_+(\mathcal{F}_T)$ that admits infinitely many arbitrage-free price processes.

(b) Suppose that S^1 satisfies $S^1_0 = 1$ and $\log S^1_1 = Z$, where Z is an exponentially distributed random variable, i.e. the density f_Z of Z is given by $f_Z(z) = \exp(-z)$ for $z \geq 0$. Then:

(1) The market (S^0, S^1) is arbitrage-free but not complete.

(2) The market (S^0, S^1) admits an arbitrage.

(3) The market (S^0, S^1) is arbitrage-free and complete.

(c) The market (S^0, S^1) is **not** arbitrage-free if:

(1) The set of all equivalent local martingale measures for S^1 is non-empty.

(2) There exists a self-financing, admissible strategy $\varphi \mathrel{\widehat{=}} (0, \vartheta)$ with $V_T(\varphi) \geq 0$ P-a.s. and $P[V_T(\varphi) > 0] > 0$.

(3) S^1 is a $(P, \mathbb{F})$-martingale.

(d) Let τ be an $\mathbb{F}$-stopping time and $X = (X_k)_{k=0,1,\dots,T}$ a $(P, \mathbb{F})$-martingale. Then:

(1) $X^\tau = (X^\tau_k)_{k=0,1,\dots,T}$ is a $(P, \mathbb{F})$-martingale.

(2) $X^\tau = (X^\tau_k)_{k=0,1,\dots,T}$ is a $(P, \mathbb{F})$-supermartingale, but not a $(P, \mathbb{F})$-martingale.

(3) $X^\tau = (X^\tau_k)_{k=0,1,\dots,T}$ is a local $(P, \mathbb{F})$-martingale, but not a $(P, \mathbb{F})$-martingale.

Throughout subquestions (e) to (h), W denotes a Brownian motion and N a Poisson process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions of P-completeness and right-continuity.

(e) Define a process $M = (M_t)_{t\geq 0}$ by $M_t = \sum_{k=0}^{N_t}(W_{k+1} - W_k)$. Then we have P-a.s. for all $t \geq 0$ that:

(1) $[M]_t = \sum_{k=0}^{N_t} (W_{k+1} - W_k)^2$.

(2) $[M]_t = N_t$.

(3) $[M]_t = N_t W_{N_t+1}^2$.

(f) Which of the following processes is **not** a $(P, \mathbb{F})$-martingale?

(1) $X = (X_t)_{t\geq 0}$ defined by $X_t := N_t - [N]_t$.

(2) $X = (X_t)_{t\geq 0}$ defined by $X_t := W_t^2 - t$.

(3) $X = (X_t)_{t\geq 0}$ defined by $X_t := t^2 W_t^2 - t^3$.

(g) Let $Q \overset{loc}{\approx} P$ on $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where $\mathbb{G}$ is a filtration such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \geq 0$. Which of the following statements is true?

(1) If X is a $(P, \mathbb{F})$-semimartingale, then X is a $(Q, \mathbb{G})$-semimartingale.

(2) If X is a $(P, \mathbb{F})$-semimartingale, then X is a $(Q, \mathbb{F})$-semimartingale.

(3) If X is a $(P, \mathbb{G})$-semimartingale, then X is a $(Q, \mathbb{F})$-semimartingale.

(h) Let $T > 0$. There exists a measure $Q \approx P$ on $\mathcal{F}_T$ such that:

(1) $W - \int W_s dN_s$ is a $(Q, \mathbb{F})$-Brownian motion on $[0, T]$.

(2) $W - \int N_{s-} ds$ is a $(Q, \mathbb{F})$-Brownian motion on $[0, T]$.

(3) $W - \int N_{s-} dW_s$ is a $(Q, \mathbb{F})$-Brownian motion on $[0, T]$.

Question 2 (8 **Points**)

Let $(\Omega, \mathcal{F}, \mathbb{F}, P, \widetilde{S}^0, \widetilde{S}^1)$ be a single-period trinomial model on the canonical path space. More specifically, $\Omega = \{1, 2, 3\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$, $\widetilde{S}^0_k = (1+r)^k$, $k = 0, 1$, and $\widetilde{S}^1_0 = 10$, $\widetilde{S}^1_1 = 10Y$, where Y is a random variable on $(\Omega, \mathcal{F})$ with $P[Y = 1+d] = p_1$, $P[Y = 1+m] = p_2$ and $P[Y = 1+u] = p_3$ with $p_1, p_2, p_3 > 0$ and $p_1+p_2+p_3 = 1$. Let $d = -0.2$, $m = 0.1$, $u = 0.3$ and $r = 0.1$. As in the lecture, we drop the tildes for quantities discounted with $\widetilde{S}^0$.

(a) Find the set $\mathbb{P}_e(S^1)$ of equivalent martingale measures for S^1.

(b) Prove that $\mathbb{P}_e(S^1) = \mathbb{P}_{e,loc}(S^1)$, where $\mathbb{P}_{e,loc}(S^1)$ is the set of all equivalent *local* martingale measures for S^1.

(c) Find the set of all arbitrage-free prices for a European call option with (undiscounted) payoff $\widetilde{C}(x) = (x - 11)^+$.

(d) Show that

$$\sup_{Q \in \mathbb{P}_e(S^1)} E_Q\left[\frac{1}{1+r}\widetilde{C}(\widetilde{S}^1_1)\right] = E_{Q^*}\left[\frac{1}{1+r}\widetilde{C}(\widetilde{S}^1_1)\right]$$

for some probability measure Q^* which is absolutely continuous with respect to P but *not* equivalent to P. Prove or disprove whether S^1 is a $(Q^*, \mathbb{F})$-martingale.

Question 3 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$, $T \in \mathbb{N}$, where we set $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $\mathcal{D} > 0$ be a random variable on $(\Omega, \mathcal{F})$ with $E_P[\mathcal{D}] = 1$. Define a probability measure Q on $(\Omega, \mathcal{F})$ by $Q[A] = E_P[\mathcal{D}\mathbb{1}_A]$ for $A \in \mathcal{F}$. Finally, define the process $Z = (Z_k)_{k=0,1,\dots,T}$ by $Z_k := E_P[\mathcal{D} \,|\, \mathcal{F}_k]$.

(a) Show that we have for any random variable $X \geq 0$ on $(\Omega, \mathcal{F})$ that $E_Q[X] = E_P[\mathcal{D}X]$.

(b) Using the result from (a), show that we have for any $\mathcal{F}_k$-measurable random variable $Y \geq 0$ that $E_Q[Y] = E_P[Z_k Y]$.

(c) Let $k \in \{1, \dots, T\}$. Using the result from (b), show that we have for any $\mathcal{F}_k$-measurable random variable $Y \geq 0$ that

$$E_P[Y] = E_Q\left[\frac{1}{Z_k} Y\right].$$

(d) Let $k \in \{1, \dots, T\}$ and $j \leq k$. Using the results from (b) and (c), show that we have for any $\mathcal{F}_k$-measurable random variable $U_k \geq 0$ that

$$E_Q[U_k \,|\, \mathcal{F}_j] = \frac{1}{Z_j} E_P[Z_k U_k \,|\, \mathcal{F}_j] \quad Q\text{-a.s.}$$

(e) Show that a stochastic process $N = (N_k)_{k=0,1,\dots,T}$ is a nonnegative $(Q, \mathbb{F})$-martingale if and only if the product ZN is a nonnegative $(P, \mathbb{F})$-martingale.

Question 4 (8 **Points**)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ a filtration on $(\Omega, \mathcal{F})$. Let $N = (N_t)_{t\in[0,T]}$ be a $(P, \mathbb{F})$-Poisson process with parameter $\lambda > 0$. Recall that this means $N_0 = 0$ P-a.s. and

(PP1) For $0 \leq s < t \leq T$, the *increment* $N_t - N_s$ is independent of $\mathcal{F}_s$ under P and *Poisson-distributed* with parameter $\lambda(t-s)$ under P, i.e.

$$P[N_t - N_s = k] = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for P-almost all $\omega \in \Omega$, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant, $\mathbb{N}_0$-valued and increases by jumps of size 1.

Recall also that the moment generating function $\phi \colon \mathbb{R} \to \mathbb{R}_+$ of $M \sim \mathrm{Poi}(\eta)$, $\eta > 0$, is given by

$$\phi(u) = E\left[e^{uM}\right] = e^{\eta(e^u - 1)}.$$

Finally, let us fix $\sigma > -1$ and define $Z^\sigma = (Z_t^\sigma)_{t\in[0,T]}$ by $Z_t^\sigma := e^{N_t \log(1+\sigma) - \lambda\sigma t}$.

(a) Show that Z^σ is a positive $(P, \mathbb{F})$-martingale with $E_P[Z_t^\sigma] = 1$ for all $t \in [0, T]$, meaning that it is a density process with respect to P of some probability measure $Q^\sigma \approx P$ on $\mathcal{F}_T$.

(b) Let Q^σ be the probability measure whose density process with respect to P is given by Z^σ. Show that N is a $(Q^\sigma, \mathbb{F})$-Poisson process. What is the parameter of N under Q^σ?

(c) Let $\widetilde{N} = (\widetilde{N})_{t\in[0,T]}$ be the compensated $(P, \mathbb{F})$-Poisson process defined by $\widetilde{N}_t := N_t - \lambda t$. Using the isometry property of the stochastic integral, show that

$$E_P\left[\left(\int_0^T X_t d\widetilde{N}_t\right)\left(\int_0^T Y_t d\widetilde{N}_t\right)\right] = E_P\left[\int_0^T X_t Y_t dN_t\right]$$

for any predictable $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$ with

$$E_P\left[\int_0^T X_t^2 d[\widetilde{N}]_t\right] < \infty \quad \text{and} \quad E_P\left[\int_0^T Y_t^2 d[\widetilde{N}]_t\right] < \infty.$$

Question 5 (8 **Points)**

Let $T > 0$ denote a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by W and augmented by the P-nullsets in $\sigma(W_s, 0 \leq s \leq T)$. Consider the Black–Scholes model with time-dependent interest rate r, i.e., the undiscounted bank account price process $\widetilde{S}^0 = (\widetilde{S}^0_t)_{t\in[0,T]}$ and the undiscounted stock price process $\widetilde{S}^1 = (\widetilde{S}^1_t)_{t\in[0,T]}$ satisfy the SDEs

$$\frac{d\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu_1 dt + \sigma_1 dW_t, \quad \widetilde{S}^1_0 = 1,$$
$$\frac{d\widetilde{S}^0_t}{\widetilde{S}^0_t} = r(t)dt, \quad \widetilde{S}^0_0 = 1,$$

where $r\colon [0,T] \to \mathbb{R}$ is of the form $r(t) = \sum_{i=0}^{n-1} d_i \mathbb{1}_{(t_i,t_{i+1}]}(t)$, for some constants $d_i \in \mathbb{R}$ and a partition $0 = t_0 < \cdots < t_n = T$, $\mu_1 \in \mathbb{R}$ and $\sigma_1 > 0$.

(a) Show that every $(P,\mathbb{F})$-supermartingale $M = (M_t)_{t\in[0,T]}$ with $E[M_t] = C$ for all $t \in [0,T]$ and some $C \in \mathbb{R}$ is a true $(P,\mathbb{F})$-martingale.

(b) Let $\lambda\colon [0,T] \to \mathbb{R}$ be in $L^2_{loc}(W)$. Show that the process $Z = (Z_t)_{t\in[0,T]}$ defined by

$$Z_t := \mathcal{E}\left(\int \lambda(s) dW_s\right)_t$$

is a true $(P,\mathbb{F})$-martingale.

Hint: You can use that P-almost all sample paths of $\left(\int_0^t \lambda(s)dW_s\right)_{t\in[0,T]}$ are continuous.

(c) Find the density process Z of a probability measure $Q \approx P$ on $\mathcal{F}_T$ such that the discounted price process $S^1 := \widetilde{S}^1/\widetilde{S}^0$ is a $(Q,\mathbb{F})$-martingale, and compute the dynamics of S^1 under the measure Q.

(d) Hedge the discounted power option for $p > 0$, i.e., find (V_0, ϑ) such that

$$H := \frac{\left(\widetilde{S}^1_T\right)^p}{\widetilde{S}^0_T} = V_0 + \int_0^T \vartheta_t dS^1_t.$$