ETH Zürich SS 2015
D-MATH
Prof. J. Teichmann **August 2015**

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** ☐ **MScQF** ☐ **SAV** ☐ **Other** ☐
Matriculation number	

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Question	Maximum	Points	Check
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam ⏲: 180 min.

Closed book examination: no notes, no books, no calculator, etc. allowed.

Important ℹ:

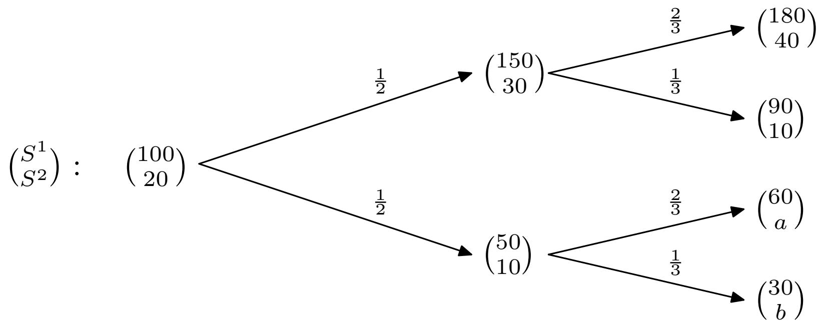
- ☞ Please put your student card on the table.
- ☞ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ☞ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ☞ Take a new sheet for each question and write your name on every sheet.
- ☞ Unless otherwise stated, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas from the lecture or from the exercise classes without derivation.
- ☞ Simplify your results as far as possible.
- ☞ Most of the subquestions can be solved independently of each other.

★★★ Good luck! ★★★

Mathematical Foundations For Finance

Exam

Question 1. Consider a market (S^0, S^1, S^2) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\{0,1,2\}}$ generated by the assets. Let $r = 0$ ($S^0 \equiv 1$) and S^1 and S^2 evolve as in the figure.



Assume for subquestions (a), (b) and (c), that $a = 13$ and $b = 6$.

(a) Show that the submarkets (S^0, S^1) and (S^0, S^2) are free of arbitrage.

(b) Calculate the price at time $t = 0$ of the claim with payoff $(13 - S_2^2)^+$ and maturity $T = 2$ in the market (S^0, S^2).

(c) Show that the market (S^0, S^1, S^2) is not arbitrage-free by explicitly constructing an arbitrage strategy.

(d) Let a and b be arbitrary in $\mathbb{R}$. Let us fix a. Find all values of b such that the market (S^0, S^1, S^2) is free of arbitrage.

For these values of b, calculate the arbitrage free price of the claim with payoff $(13 - S_2^2)^+$ and maturity $T = 2$ for general $a \in \mathbb{R}$ in the market (S^0, S^1, S^2).

Solution 1. (a) By Theorem 2.2.1 the market (S^0, S^i) is arbitrage free if and only if there exists an EMM $\mathbb{Q}^i$. Let $\Omega = \{u, d\} \times \{u, d\}$ where u and d denote upward and downward price movements, $S_t^i = S_0^i \Pi_{k=1}^t Y_k^i$ and write an equivalent probability measure $\mathbb{Q}^i$ in terms of transition probabilities $\mathbb{Q}^i((x_1, x_2)) = q_{x_1}^i q_{x_1,x_2}^i$.

Now write the martingale condition

$$\mathbb{E}_{\mathbb{Q}^i}[Y_k^i | \mathcal{F}_{k-1}] = 1, \quad \forall i \in \{1, 2\}, \ \forall k \in \{1, 2\}.$$

Inserting the values of Y_k^i and solving for the probabilities yields

$$q_{x_1}^1 = \frac{1}{2}, \quad q_{x_1,u}^1 = \frac{2}{3}, \quad \forall x_1 \in \{u, d\},$$
$$q_u^2 = \frac{1}{2}, \quad q_{u,u}^2 = \frac{2}{3}, \quad q_{d,u}^2 = \frac{4}{7}.$$

We have, $q_{x_1}^1, q_{x_1,u}^1, q_u^2, q_{u,u}^2, q_{d,u}^2 \in (0, 1)$, so $\mathbb{Q}$ is indeed equivalent to $\mathbb{P}$.

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(b) We find the price at time 0 by calculating the expected value of the payoff under the EMM $\mathbb{Q}^2$:

$$\mathbb{E}_{\mathbb{Q}^2}[(13 - S_2^2)^+] = 3q_u^2 q_{u,d}^2 + 7q_d^2 q_{d,d}^2 = \frac{1}{2}\left(3\frac{1}{3} + 7\frac{3}{7}\right) = 2.$$

(c) The market (S^0, S^1, S^2) is arbitrage-free if and only if there exists an equivalent martingale measure for S^1 and S^2. There exists only one EMM for S^1, and we have $q_{d,u}^1 \neq q_{d,u}^2$. Computing the expected value of S_2^2 conditioned on $S_1^2 = 10$ under $\mathbb{Q}^1$, we see that in the second period, in case of a down move during the first period, S^2 is underpriced in comparison to S^1. We have indeed,

$$\mathbb{E}_{\mathbb{Q}^1}[S_2^2 | S_1^2 = 10] = 13\frac{2}{3} + 6\frac{1}{3} = \frac{32}{3} > 10.$$

We therefore want to buy S^2 and sell S^1 in this scenario. Therefore consider the self-financing portfolio with 0 starting wealth given by

$$\vartheta_1 = 0, \text{ and } \vartheta_2 = (-1, 5)1_{\{S_1^1 = 50\}}.$$

The gains process at time 2 of this strategy is

$$G_2(\vartheta) = 5 \cdot 1_{\{S_2^1 = 60\}},$$

which is certainly non-negative and is strictly positive with positive probability.

(d) For the whole market to be free of arbitrage we need $\mathbb{Q}^1$ and $\mathbb{Q}^2$ to coincide. It is therefore sufficient that (by Corollary 2.1.4 in the lecture notes)

$$\frac{2}{3} = q_{d,u}^2 = \frac{r-d}{u-d} = \frac{-\left(\frac{b}{10} - 1\right)}{\frac{a}{10} - 1\left(\frac{b}{10} - 1\right)} = \frac{10-b}{a-b}.$$

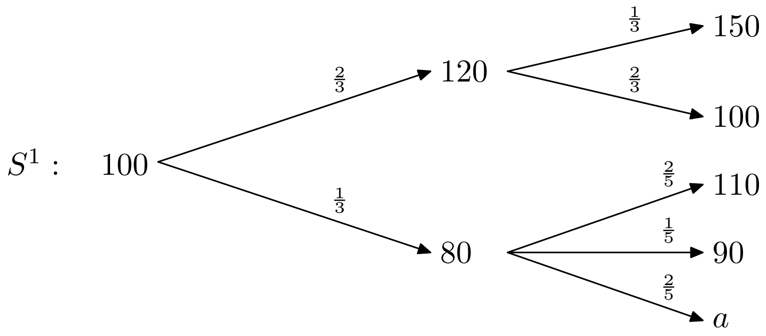
Solving for b yields

$$b = 30 - 2a.$$

For pricing we again have to calculate the expected value:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^1}[(13 - S_2^2)^+] &= \frac{1}{2}\left(\frac{1}{3}(13 - (30 - 2a))^+ + \frac{2}{3}(13 - a)^+ + \frac{1}{3}3\right) \\
&= \frac{1}{6}\left(2\left(a - \frac{17}{2}\right)^+ + 2(13 - a)^+ + 3\right) = \begin{cases} \frac{29-2a}{6}, & a < \frac{17}{2}, \\ 2, & \frac{17}{2} \leq a \leq 13, \\ \frac{2a-14}{6}, & 13 < a. \end{cases}
\end{aligned}$$

Question 2. We consider a multinomial model with two time periods ($T = 2$). The market consists of a riskless asset $\widetilde{S^0}$ that grows at a rate $r = 0$, and one risky asset whose evolution is given by the tree below. The numbers on the branches give the probability of following a particular branch. The tree shows the evolution of both the discounted and undiscounted price of the asset since the value of one unit of the bank account stays constant ($\widetilde{S^0} \equiv 1$).



Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with $\Omega = \{u\} \times \{u, d\} \cup \{d\} \times \{u, m, d\}$, $\mathcal{F} = 2^\Omega$. The filtration is given by $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F})$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \{u\} \times \{u, d\}, \{d\} \times \{u, m, d\}, \Omega\}$. The probability $\mathbb{P}$ defined as follows :

$$\mathbb{P}\left[\{u\} \times \{u, d\}\right] = p_u = \frac{2}{3}, \qquad \mathbb{P}\left[\{d\} \times \{u, m, d\}\right] = p_d = \frac{1}{3}$$

$$\mathbb{P}[\{(i, j)\}] = p_i \times p_{i,j} \quad \text{for} \quad i \in \{u, d\}, j \in \{u, m, d\}$$

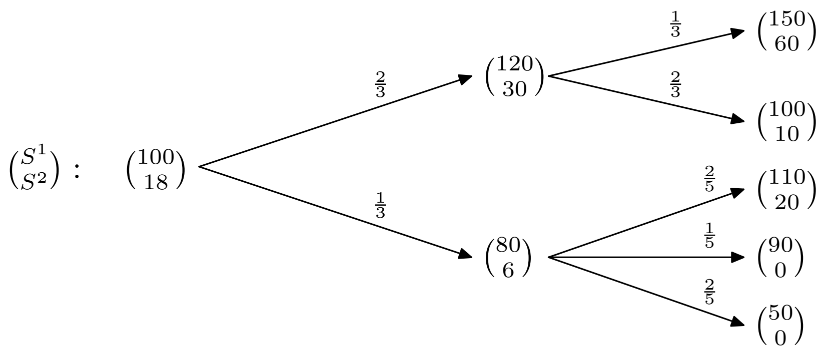
with $p_{u,u} = \frac{1}{3}$, $p_{u,d} = \frac{2}{3}$, $p_{d,u} = \frac{2}{5}$, $p_{d,m} = \frac{1}{5}$, $p_{d,d} = \frac{2}{5}$.

a) Assume $a = 80$, is the market arbitrage-free ? Find the equivalent martingale measures for S^1 or construct an arbitrage opportunity and prove in details that it is one. If the market is arbitrage-free, is it complete ?

b) Assume $a = 50$, is the market arbitrage free ? Find the equivalent martingale measures for S^1 or construct an arbitrage opportunity and prove in details that it is one. If the market is arbitrage-free, is it complete ?

c) Assume $a = 50$.

  (i) A second asset S^2 can be traded on the market. The dynamics of (S^1, S^2) are given by the tree below. Show that the market $\left(S^1, S^2\right)$ is complete.



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(ii) We consider an asian option on the stock S^1. On this two periods model for the market (S^1, S^2), the payoff of the asian option with strike K is given by

$$C_K^A = \left(\frac{1}{2} \sum_{j=1}^{2} S_j^1 - K \right)^+$$

What is the non-arbitrage price at time $t=0$ of C_{90}^A in the market (S^0, S^1, S^2)?

Solution 2. a) The interest rate is equal to 0, so the value of the riskless asset is constant, equal to 1. We see that the risky asset in case of a down movement in the price during the first period does not lose value with probability 1. An arbitrage strategy would be to wait for the first period, and if the price went down, to invest in the stock and borrow from the bank. The self-financing strategy that we use is $\phi = (\phi_0^0, \theta_1, \theta_2)$, with $\phi_0^0 = 0$, $\theta_1 = 0$, $\theta_2((u,x)) = 0$, $\theta_2((d,y)) = 1$ for $x \in \{u, d\}$, $y \in \{u, m, d\}$. The terminal value of the portfolio is :

$$V_2(\phi) = \phi_0^0 + \theta_1 \Delta S_1^1 + \theta_2 \Delta S_2^1 = \begin{cases} 0 & \text{if } \omega = (u,u) \\ 0 & \text{if } \omega = (u,d) \\ 30 & \text{if } \omega = (d,u) \\ 10 & \text{if } \omega = (d,m) \\ 0 & \text{if } \omega = (d,d) \end{cases} .$$

So $V_2(\phi) \geqslant 0$ $\mathbb{P}$-a.s and since $\mathbb{P}[V_2(\phi) > 0] = \mathbb{P}[\{(d,u),(d,m)\}] = \frac{1}{5} > 0$, the strategy is indeed an arbitrage.

b) Now $a = 50$, so there is an actual possibility for losses in the second period. An equivalent martingale measure is such that the discounted price process is a martingale. We need to find $q_u, q_d, q_{u,u}, q_{u,d}, q_{d,u}, q_{d,m}, q_{d,d} \in (0,1)$, defined similarly as the parameters of $\mathbb{P}$. We have the following equations :

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[S_1^1 \middle| \mathcal{F}_0\right] &= S_0^1 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\left[S_2^1 \middle| \mathcal{F}_1\right] = S_1^1 \\ \Leftrightarrow \mathbb{E}_{\mathbb{Q}}\left[S_1^1\right] &= S_0^1 \quad \text{and} \quad \begin{cases} \mathbb{E}_{\mathbb{Q}}\left[S_2^1 \middle| S_1^1 = 120\right] = 120 \\ \mathbb{E}_{\mathbb{Q}}\left[S_2^1 \middle| S_1^1 = 80\right] = 80 \end{cases} \end{aligned}$$

We obtain the second line with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_1 = \sigma(S_1^1)$. This gives the following equations.

$$\begin{cases} 120 q_u + 80 q_d = 100 \\ q_u + q_d = 1 \end{cases} \qquad \begin{cases} 150 q_{u,u} + 100 q_{u,d} = 120 \\ q_{u,u} + q_{u,d} = 1 \end{cases} \qquad \begin{cases} 110 q_{d,u} + 90 q_{d,m} + 50 q_{d,d} = 80 \\ q_{d,u} + q_{d,m} + q_{d,d} = 1 \end{cases}$$

This gives the solutions : $q_u = q_d = \frac{1}{2}$, $q_{u,u} = \frac{2}{5}$, $q_{u,d} = \frac{3}{5}$, and $(q_{d,u}, q_{d,m}, q_{d,d}) \in \left\{ \left(\lambda, \frac{3}{4} - \frac{3}{2}\lambda, \frac{1}{4} + \frac{1}{2}\lambda\right) \;\middle|\; \lambda \in \left(0, \frac{1}{2}\right) \right\}$. The set of EMM is non-empty, but contains more than one element, so the market is arbitrage free but incomplete.

c) (i) We are looking for an equivalent martingale measure for $\left(S^1, S^2\right)$. We have seen in question b) how an equivalent martingale measure for $\left(S^1\right)$ needs to be. So we need to find one among these such that the price process of S^2 is a martingale as well.

$$\mathbb{E}_{\mathbb{Q}^\lambda}\left[S_1^2 \mid \mathcal{F}_0\right] = \frac{1}{2} \cdot 30 + \frac{1}{2} \cdot 6 = 18 = S_0^2$$

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$$\mathbb{E}_{\mathbb{Q}^\lambda}\left[S_2^2 \mid S_1^1 = 120\right] = \frac{2}{5}\cdot 60 + \frac{3}{5}\cdot 10 = 30 = S_1^2 \cdot \mathbb{1}_{\{S_1^1=120\}}$$
$$\mathbb{E}_{\mathbb{Q}^\lambda}\left[S_2^2 \mid S_1^1 = 80\right] = 20\lambda$$

For $\lambda = \frac{3}{10}$, we have $\mathbb{E}_{\mathbb{Q}^\lambda}\left[S_2^2 \mid S_1^1 = 80\right] = S_1^2 \cdot \mathbb{1}_{\{S_1^1=80\}}$. There exists a unique martingale measure for $\left(S^1, S^2\right)$, $\mathbb{Q}^* = \mathbb{Q}^\lambda$ for $\lambda = \frac{3}{10}$. This market is complete.

(ii) The payoff of the asian option with payoff C_{90}^A is the following :

$$C_{90}^A\left((u,u)\right) = 45,\ C_{90}^A\left((u,d)\right) = 20,\ C_{90}^A\left((d,u)\right) = 5,\ C_{90}^A\left((d,m)\right) = C_{90}^A\left((d,d)\right) = 0.$$

The price of this option in the complete market, is the expectation of its payoff under the equivalent martingale measure $\mathbb{Q}^*$.

$$\mathbb{E}_{\mathbb{Q}^*}\left[C_{90}^A\right] = \frac{1}{2}\cdot\frac{2}{5}\cdot 45 + \frac{1}{2}\cdot\frac{3}{5}\cdot 20 + \frac{1}{2}\cdot\frac{3}{10}\cdot 5 = 15 + \frac{3}{4} = \frac{63}{4}.$$

Question 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space with an augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by a Brownian motion $(W_t)_{t\geqslant 0}$. Consider a discounted Bachelier market (S^0, S^1) on this space, i.e.,

$$S_t^0 = 1, \quad \text{and} \quad S_t^1 = S_0^1 + \sigma W_t, \quad t \geq 0$$

for some $S_0^1, \sigma > 0$.

Let $T > 0$ and $K \in \mathbb{R}$. Compute the replicating (self-financing) strategy for $h(S_T^1)$ by computing the value process and using Itô's formula, where h is defined as

(a) $h(y) = y^3 - 6y^2 + 11y - 6$.

(b) $h(y) = 1_{\{S\geq K\}}$.

(c) $h(y) = \max\{K, y\}$.

Solution 3. Since S^1 is a $\mathbb{P}$-martingale, the value process is calculated as

$$V_t = \mathbb{E}[h(S_T^1)|\mathcal{F}_t].$$

Using independence of Brownian motion increments, this simplifies to

$$V_t = \mathbb{E}[h(y + \sigma(W_T - W_t))]|_{y=S_t^1} = \mathbb{E}[h(y + \sigma W_{T-t})]|_{y=S_t^1} = \mathbb{E}[h(y + \sigma\sqrt{T-t}Z)]\Big|_{y=S_t^1} = f(t,y)|_{y=S_t^1},$$

where $Z \sim \mathcal{N}(0,1)$. If $f \in \mathcal{C}^{1,2}$, by Itô's formula and $[S^1]_t = \sigma^2 t$ we see that f solves the heat equation

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2} = 0,$$

then the replicating strategy is given by (V_0, ϑ) for

$$\vartheta_t = \frac{\partial f}{\partial y}(t, S_t^1).$$

(a) Plug in to the above:

$$\begin{aligned} V_t &= \mathbb{E}[(y + \sigma W_{T-t})^3 - 6(y + \sigma W_{T-t})^2 + 11(y + \sigma W_{T-t}) - 6]\big|_{y=S_t^1} \\ &= (S_t^1)^3 + 3S_t^1\sigma^2(T-t) - 6(S_t^1)^2 - 6\sigma^2(T-t) + 11S_t^1 - 6 = f(t, S_t^1) \end{aligned}$$

f is $\mathcal{C}^{1,2}$ and its derivatives are given by

$$\frac{\partial f}{\partial t}(t,y) = -3y\sigma^2 + 6\sigma^2 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}(t,y) = +6y - 12,$$

which means that f solves the above equation. Hence (V_0, ϑ) is given by

$$V_0 = (S_0^1)^3 + 3S_0^1\sigma^2 T - 6(S_0^1)^2 - 6\sigma^2 T + 11S_0^1 - 6$$

and

$$\vartheta_t = 3(S_t^1)^2 + 3\sigma^2(T-t) - 12S_t^1 + 11.$$

(b) We proceed similarly.

$$V_t = \mathbb{E}[1_{\{y+\sigma W_{T-t}\geq K\}}]\big|_{y=S_t^1} = \mathbb{E}[1_{\{W_{T-t}\leq \frac{y-K}{\sigma}\}}]\big|_{y=S_t^1} = \mathbb{E}[1_{\{Z\leq \frac{y-K}{\sigma\sqrt{T-t}}\}}]\big|_{y=S_t^1} = \Phi\left(\frac{S_t^1 - K}{\sigma\sqrt{T-t}}\right) = f(t, S_t^1),$$

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where Φ is the cumulative distribution function for a standard normal variable and Z is a standard normal random variable. Let ϕ be the density function of a standard normal distribution. Again $f \in \mathcal{C}^{1,2}$ and we can calculate the derivatives of f to obtain

$$\frac{\partial f}{\partial t}(t,y) = \frac{y-K}{2\sigma(T-t)^{3/2}}\phi\left(\frac{y-K}{\sigma\sqrt{T-t}}\right)$$

and

$$\frac{\partial^2 f}{\partial y^2}(t,y) = -\frac{y-K}{\sigma^3(T-t)^{3/2}}\phi\left(\frac{y-K}{\sigma\sqrt{T-t}}\right).$$

We conclude that the replicating portfolio is (V_0, ϑ) with

$$V_0 = \Phi\left(\frac{S_0^1 - K}{\sigma\sqrt{T}}\right) \quad \text{and} \quad \vartheta_t = \frac{1}{\sigma\sqrt{T-t}}\phi\left(\frac{S_t^1 - K}{\sigma\sqrt{T-t}}\right).$$

(c) Observe first that $\max\{K, y\} = (y-K)^+ + K$. If we can price a European call option with strike K, we are done. First calculate the value process:

$$\begin{aligned}
V_t &= \\
&= \mathbb{E}[(y + \sigma W_{T-t} - K)^+]|_{y=S_t^1} + K \\
&= \mathbb{E}[(y + \sigma\sqrt{T-t}Z - K)^+]|_{y=S_t^1} + K \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (S_t^1 - K + \sigma\sqrt{T-t}z)^+ e^{-\frac{z^2}{2}}\,dz + K \\
&= \frac{1}{\sqrt{2\pi}}\int_{\frac{K-S_t^1}{\sigma\sqrt{T-t}}}^{\infty} (S_t^1 - K + \sigma\sqrt{T-t}z) e^{-\frac{z^2}{2}}\,dz + K \\
&= (S_t^1 - K)\Phi\left(\frac{S_t^1 - K}{\sigma\sqrt{T-t}}\right) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\exp\left(-\frac{(S_t^1-K)^2}{2\sigma^2(T-t)}\right) + K \\
&= (S_t^1 - K)\Phi\left(\frac{S_t^1 - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{S_t^1 - K}{\sigma\sqrt{T-t}}\right) + K \\
&= f(t, S_t),
\end{aligned}$$

where Z is a standard normal random variable.

Calculate the derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,y) &= -\frac{\sigma}{2\sqrt{T-t}}\phi\left(\frac{y-K}{\sigma\sqrt{T-t}}\right) \\
\frac{\partial f}{\partial y}(t,y) &= \Phi\left(\frac{y-K}{\sigma\sqrt{T-t}}\right) \quad \text{and} \\
\frac{\partial^2 f}{\partial y^2}(t,y) &= \frac{1}{\sigma\sqrt{T-t}}\phi\left(\frac{y-K}{\sigma\sqrt{T-t}}\right),
\end{aligned}$$

which means f solves the heat equation and the replicating portfolio is given by (V_0, ϑ) with

$$\begin{aligned}
V_0 &= K + (S_0^1 - K)\Phi\left(\frac{S_0^1 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0^1 - K}{\sigma\sqrt{T}}\right), \\
\vartheta_t &= \Phi\left(\frac{S_t^1 - K}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

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Question 4. Let $T > 0$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by W. Consider the Black-Scholes model with bank account price process $\widetilde{S}^0$ and stock price process $\widetilde{S}^1$ satisfying the SDEs

$$\begin{aligned}\frac{d\widetilde{S}^1_t}{\widetilde{S}^1_t} &= \mu dt + \sigma dW_t,\\ \frac{d\widetilde{S}^0_t}{\widetilde{S}^0_t} &= r dt,\end{aligned}$$

for some fixed $r, \mu, \sigma > 0$ and with $\widetilde{S}^0_0 = \widetilde{S}^1_0 = 1$. Denote by $\widetilde{V}_0^{\mathrm{Call},\widetilde{K}}$ and $\widetilde{V}_0^{\mathrm{Put},\widetilde{K}}$ respectively the undiscounted prices at time zero of European call and put options with undiscounted strike $\widetilde{K} > 0$.

For any two constants $0 < K_1 < K_2$, set $\widetilde{K}_i = e^{rT}K_i$ and consider the following two strategies: First the (long) butterfly option giving the payoff

$$\widetilde{H}^{\mathrm{BF}} = \left(\widetilde{S}^1_T - \widetilde{K}_1\right)^+ - 2\left(\widetilde{S}^1_T - \frac{\widetilde{K}_1 + \widetilde{K}_2}{2}\right)^+ + \left(\widetilde{S}^1_T - \widetilde{K}_2\right)^+$$

at time T. Consider also the (short) straddle position yielding the payoff

$$\widetilde{H}^{\mathrm{Straddle}} = -\left(\widetilde{S}^1_T - \frac{\widetilde{K}_1 + \widetilde{K}_2}{2}\right)^+ - \left(\frac{\widetilde{K}_1 + \widetilde{K}_2}{2} - \widetilde{S}^1_T\right)^+$$

at time T.

Finally, define $\widetilde{H} = \widetilde{H}^{\mathrm{Straddle}} - \widetilde{H}^{\mathrm{BF}}$.

(a) Find the value of $\widetilde{H}$ at time 0 in terms of the difference $\Delta K = (K_2 - K_1)$ as well as $\widetilde{V}_0^{\mathrm{Call},\widetilde{K}_i}$ and $\widetilde{V}_0^{\mathrm{Put},\widetilde{K}_j}$ for $i, j \in \{1, 2\}$.

(b) Sketch how the payoff $\widetilde{H}$ depends on the final value of the stock price $\widetilde{S}^1_T$ (draw the curve $\widetilde{S}^1_T \mapsto \widetilde{H}(\widetilde{S}^1_T)$). Qualitative and quantitative details should be clear from the figure or explained in text (slopes, intercepts,...).

Hint: Sketching $\widetilde{H}^{\mathrm{Straddle}}$ and $\widetilde{H}^{\mathrm{BF}}$ separately could help visualizing, but is not necessary.

(c) Let $\widetilde{K} > 0$. Compute a replicating strategy for a European put option with strike $\widetilde{K}$ and maturity T. Derive and prove your result. (Don't forget to compute $\widetilde{V}_0^{\mathrm{Put},\widetilde{K}}$ as well.)

Solution 4. (a) The simplest way to solve this is to use the following simplification

$$\begin{aligned}\widetilde{H}^{\mathrm{Straddle}} - \widetilde{H}^{\mathrm{BF}} &= -\left(\frac{\widetilde{K}_1 + \widetilde{K}_2}{2} - \widetilde{S}^1_T\right)^+ - (\widetilde{S}^1_T - \widetilde{K}_1)^+\\ &\quad + \left(\widetilde{S}^1_T - \frac{\widetilde{K}_1 + \widetilde{K}_2}{2}\right)^+ - (\widetilde{S}^1_T - \widetilde{K}_2)^+\\ &= \widetilde{S}^1_T - \frac{\widetilde{K}_1 + \widetilde{K}_2}{2} - (\widetilde{S}^1_T - \widetilde{K}_1)^+ - (\widetilde{S}^1_T - \widetilde{K}_2)^+\\ &= -(\widetilde{K}_1 - \widetilde{S}^1_T)^+ - \frac{\widetilde{K}_2 - \widetilde{K}_1}{2} - (\widetilde{S}^1_T - \widetilde{K}_2)^+.\end{aligned}$$

Then we can just price two of the terms directly with $V_0^{\text{Put},\widetilde{K}_2}$ and $V_0^{\text{Call},\widetilde{K}_1}$ and the constant term is simply discounted. We get

$$V_0(\widetilde{H}) = -V_0^{\text{Put},\widetilde{K}_1} - \frac{K_2 - K_1}{2} - V_0^{\text{Call},\widetilde{K}_2} = -V_0^{\text{Put},\widetilde{K}_1} - \frac{\Delta K}{2} - V_0^{\text{Call},\widetilde{K}_2}.$$

Another way is to start out with the expression for $\widetilde{H}$ given before simplification, value everything in terms of put and call option prices and finally use the put-call parity

$$V_t^{\text{Call},\widetilde{K}} - V_t^{\text{Put},\widetilde{K}} = S_t^1 - \widetilde{K}e^{-r(T-t)}$$

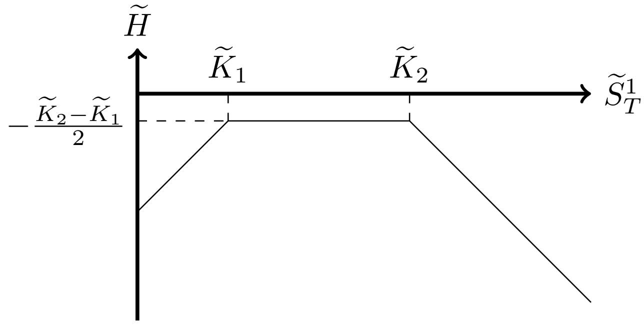
to obtain the value in terms of the sought put and call options. This is mathematically the same procedure as above and is accomplished as follows: Write the value of the terms of $\widetilde{H}$ before simplification to obtain

$$V_0(\widetilde{H}) = -V_0^{\text{Put},(\widetilde{K}_1+\widetilde{K}_2)/2} - V_0^{\text{Call},\widetilde{K}_1} + V_0^{\text{Call},(\widetilde{K}_1+\widetilde{K}_2)/2} - V_0^{\text{Call},\widetilde{K}_2}.$$

Now simplify using the put-call parity twice to obtain

$$\begin{aligned} V_0(\widetilde{H}) &= S_0^1 - \frac{K_1 + K_2}{2} - V_0^{\text{Call},\widetilde{K}_1} - V_0^{\text{Call},\widetilde{K}_2} \\ &= S_0^1 - \frac{K_1 + K_2}{2} + K_1 - S_0^1 - V_0^{\text{Put},\widetilde{K}_1} - V_0^{\text{Call},\widetilde{K}_2} \\ &= -V_0^{\text{Put},\widetilde{K}_1} - \frac{\Delta K}{2} - V_0^{\text{Call},\widetilde{K}_2} \\ &= -V_0^{\text{Put},\widetilde{K}_2} + \frac{\Delta K}{2} - V_0^{\text{Call},\widetilde{K}_1}. \end{aligned}$$

(b) Using the simplification from the previous question, the payoff can be drawn using three lines as below. The function $\tilde{H}(\widetilde{S}_T^1)$ is constant on $[\widetilde{K}_1, \widetilde{K}_2]$ and has value $-\frac{\widetilde{K}_2-\widetilde{K}_1}{2}$ on this interval.



The slope is ± 1 where the line is not constant. The payoff for $\widetilde{S}_T^1 = 0$ is $-\frac{\widetilde{K}_1+\widetilde{K}_2}{2}$.

(c) Denote by $\left(V_t^{\text{Put},\widetilde{K}}\right)_{t\in[0,T]}$ the discounted price process of the put option. We have seen in the lecture that the unique equivalent martingale measure $\mathbb{Q}$ for the market is given by the following probability change

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\left(-\frac{\mu - r}{\sigma}W_T - \frac{(\mu - r)^2}{2\sigma^2}T\right).$$

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Under $\mathbb{Q}$, the process W^* defined by $W_t^* = W_t + \frac{\mu-r}{\sigma}t$ is a Brownian motion. To get $V_t^{\mathrm{Put},\widetilde{K}}$ we compute the conditional expectation given $\mathcal{F}_t$ of the discounted payoff $\left(K - S_T^1\right)^+$ under $\mathbb{Q}$. Using that the process S^1 satisfies $S_t^1 = S_0^1 \exp\left(\sigma W_t + (\mu - r - \frac{1}{2}\sigma^2)t\right)$ we obtain

$$
\begin{aligned}
V_t^{\mathrm{Put},\widetilde{K}} &= \mathbb{E}_{\mathbb{Q}}\left[\left(K - S_T^1\right)^+ \mid \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\left(K - S_t^1 \exp\left(\sigma(W_T - W_t) + (\mu - r - \frac{1}{2}\sigma^2)(T-t)\right)\right)^+ \Bigm| \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\left(K - S_t^1 \exp\left(\sigma(W_T + \frac{\mu-r}{\sigma}T - W_t - \frac{\mu-r}{\sigma}t) - \frac{1}{2}\sigma^2(T-t)\right)\right)^+ \Bigm| \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\left(K - S_t^1 \exp\left(\sigma(W_T^* - W_t^*) - \frac{1}{2}\sigma^2(T-t)\right)\right)^+ \Bigm| \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\left(K - y \exp\left(\sigma\sqrt{T-t}X - \frac{1}{2}\sigma^2(T-t)\right)\right) \mathbb{1}_{\left\{X < \frac{\log(K/y) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right\}}\right]\Bigg|_{y=S_t^1} \\
&= \int_{-\infty}^{\frac{\log(K/S_t^1)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \left(K - S_t^1 \exp\left(\sigma\sqrt{T-t}x - \frac{1}{2}\sigma^2(T-t)\right)\right) \phi(x)\,\mathrm{d}x \\
&= K\Phi\left(\frac{\log(K/S_t^1) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - S_t^1 \int_{-\infty}^{\frac{\log(K/S_t^1)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(x^2 - 2x\sigma\sqrt{T-t} + \sigma^2(T-t)\right)\right) \mathrm{d}x \\
&= K\Phi\left(\frac{\log(K/S_t^1) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - S_t^1 \int_{-\infty}^{\frac{\log(K/S_t^1)-\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) \mathrm{d}u \\
&= K\Phi\left(\frac{\log(K/S_t^1) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - S_t^1\Phi\left(\frac{\log(K/S_t^1) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&:= f(t, S_t^1),
\end{aligned}
$$

where $X \sim \mathcal{N}(0,1)$ and we have set $u = x - \sigma\sqrt{T-t}$. We have therefore

$$
V_0^{\mathrm{Put},\widetilde{K}} = K\Phi\left(\frac{\log(K/S_0^1) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - S_0^1\Phi\left(\frac{\log(K/S_0^1) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right).
$$

The process $V_t^{\mathrm{Put},\widetilde{K}}$ is a martingale, f is twice differentiable in space and differentiable in time. Computations give

$$
\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0,
$$

and so the position in the risky asset of the replicating portfolio is given by a direct differentiation

$$
\theta_t := \frac{\partial f}{\partial x}(t, S_t^1) = -\Phi\left(\frac{\log(K/S_t^1) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right).
$$

Question 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. We consider a trinomial market model, with two assets and $T \geqslant 2$ periods. The riskless asset (bank account) has the following price process

$$\widetilde{S_k^0} = (1+r)^k, \text{ for } k \in \{0, 1, ..., T\},$$

and the risky asset price process is defined as

$$\widetilde{S_k^1} = \widetilde{S_0^1} \prod_{i=1}^{k} Y_i, \text{ for } k \in \{0, 1, ..., T\},$$

where the Y_i are i.i.d. and taking values in $\{1+u,\ 1+m,\ 1+d\}$, each one with strictly positive probability under $\mathbb{P}$, and where $m = r \geqslant 0$, $u > m > d > -1$ and $\widetilde{S_0^1} > 0$.

We denote the discounted price process by $S^1 = \frac{\widetilde{S^1}}{\widetilde{S^0}}$. Let $\mathbb{F} = (\mathcal{F}_k)_{k \in \{0,1,\ldots,T\}}$ be the filtration generated by $(Y_k)_{k \in \{0,1,\ldots,T\}}$.

a) Assume for this subquestion that $T = 2$. Is the market arbitrage-free ? Is it complete ? Compute the set $\mathbb{P}_e\left(S^1\right)$ of equivalent martingale measures for the discounted price process.

b) Assume for this subquestion that $T = 1$, $\widetilde{S_0^0} = 1$, $\widetilde{S_0^1} = 100$, $u = 0.1$, $m = r = 0$, $d = -0.1$. Find a super-replication strategy with the lowest possible initial wealth V_0 for the claim with maturity $T = 1$ and payoff $\widetilde{C} = \left|\widetilde{S_1^1} - 95\right|$.

We are back in the general case, for any $T \geqslant 2$, any $\widetilde{S_0^1} > 0$, and any u, m, r, d such that $m = r \geqslant 0$ and $u > m > d > -1$. Let f be a convex function. Define recursively the function $\tilde{v}$

$$\begin{cases} \tilde{v}(T, x) &= f(x), \text{ for } x \in \mathbb{R}^+, \\ \tilde{v}(k, x) &= \frac{p^* \tilde{v}(k+1, x(1+u)) + (1-p^*)\tilde{v}(k+1, x(1+d))}{1+r}, \quad \text{for } x \in \mathbb{R}^+ \text{ and } 0 \leqslant k \leqslant T-1, \end{cases}$$

where $p^* = \frac{r-d}{u-d}$. Define now the function ξ as follows

$$\xi(k+1, x) = \frac{\tilde{v}(k+1, x(1+u)) - \tilde{v}(k+1, x(1+d))}{x(u-d)}, \quad \text{for } x \in \mathbb{R}^+ \text{ and } 0 \leqslant k \leqslant T-1.$$

c) (i) Show that for all $k \in \{0, 1, ..., T\}$, the function $\tilde{v}(k, .)$ is convex.

(ii) Show that for $x \in \mathbb{R}^+$,

$$\frac{\tilde{v}(k+1, x(1+y))}{1+r} \leqslant \tilde{v}(k, x) + \xi(k+1, x)\left(\frac{x(1+y)}{1+r} - x\right),$$

for $y \in \{u, m, d\}$ and $0 \leqslant k \leqslant T-1$.

(iii) Let $\phi \hat{=} (\tilde{v}(0, S_0), \theta)$ be the self-financing strategy that starts with $\tilde{v}(0, S_0)$ initial capital and use the strategy $\theta_{k+1} = \xi(k+1, \widetilde{S_k^1})$ for $0 \leqslant k \leqslant T-1$ on the stock. Show that ϕ is a super-replicating strategy for the contingent claim with payoff $f(\widetilde{S_T^1})$ at time T.

Hint: Show by induction that $\widetilde{V}_k \geqslant \tilde{v}(k, \widetilde{S_k^1})$ *for all* $k \in \{0, 1, ...T\}$*, where* $\widetilde{V}$ *is the undiscounted value process of the portfolio with strategy* ϕ*.*

Solution 5. (a) The set Ω is finite, so $\widetilde{S_k^1}$ is integrable under any probability measure. Furthermore since the filtration is generated by the Y_i's, the process $\widetilde{S^1}$ is adapted to $\mathbb{F}$. For $\mathbb{Q}$

to be a martingale measure for S^1, equivalent to $\mathbb{P}$, $\mathbb{Q}$ needs to satisfy:

$$\begin{cases} \mathbb{Q} & \approx \mathbb{P} \\ \mathbb{E}_{\mathbb{Q}}\left[\frac{\widetilde{S_2^1}}{\widetilde{S_2^0}} \,\middle|\, \mathcal{F}_1\right] & = \frac{\widetilde{S_1^1}}{\widetilde{S_1^0}} \\ \mathbb{E}_{\mathbb{Q}}\left[\frac{\widetilde{S_1^1}}{\widetilde{S_1^0}} \,\middle|\, \mathcal{F}_0\right] & = \frac{\widetilde{S_0^1}}{\widetilde{S_0^0}} \end{cases} \Leftrightarrow \begin{cases} \mathbb{Q} & \approx \mathbb{P} \\ \mathbb{E}_{\mathbb{Q}}\left[Y_2\right] & = 1+r \\ \mathbb{E}_{\mathbb{Q}}\left[Y_1\right] & = 1+r \end{cases}.$$

Writing

$\mathbb{Q}[Y_1 = 1+u] = q_u$, $\quad \mathbb{Q}[Y_1 = 1+m] = q_m$, $\quad \mathbb{Q}[Y_1 = 1+d] = q_d$
$\mathbb{Q}[Y_2 = 1+u \mid Y_1 = 1+y] = q_{y,u}$, $\quad \mathbb{Q}[Y_2 = 1+m \mid Y_1 = 1+y] = q_{y,m}$, $\quad \mathbb{Q}[Y_2 = 1+d \mid Y_1 = 1+y] = q_{y,d}$,

for $y \in \{u, m, d\}$, we get that the previous system is equivalent to :

$$\begin{cases} (q_u, q_m, q_d) & \in (0,1)^3 \\ q_u u + q_m m + q_d d & = r \\ q_u + q_m + q_d & = 1 \end{cases} \quad \text{and} \quad \begin{cases} (q_{y,u}, q_{y,m}, q_{y,d}) & \in (0,1)^3 \\ q_{y,u} u + q_{y,m} m + q_{y,d} d & = r \\ q_{y,u} + q_{y,m} + q_{y,d} & = 1 \end{cases} \quad \text{for } y \in \{u, m, d\}$$

Solving the system gives :

$$\begin{cases} q_u & \in \left(0, \frac{r-d}{u-d}\right) \\ q_m & = 1 - \frac{u-d}{r-d} q_u \\ q_d & = \frac{u-r}{r-d} q_u \end{cases} \quad \text{and} \quad \begin{cases} q_{y,u} & \in \left(0, \frac{r-d}{u-d}\right) \\ q_{y,m} & = 1 - \frac{u-d}{r-d} q_{y,u} \\ q_{y,d} & = \frac{u-r}{r-d} q_{y,u} \end{cases}$$

The set of EMM for S^1 is

$$\begin{aligned} \mathbb{P}_e\left(S^1\right) = \Bigg\{ & \left(q_u, 1 - \frac{u-d}{r-d} q_u, \frac{u-r}{r-d} q_u\right), \left(q_{u,u}, 1 - \frac{u-d}{r-d} q_{u,u}, \frac{u-r}{r-d} q_{u,u}\right), \\ & \left(q_{m,u}, 1 - \frac{u-d}{r-d} q_{m,u}, \frac{u-r}{r-d} q_{m,u}\right), \left(q_{d,u}, 1 - \frac{u-d}{r-d} q_{d,u}, \frac{u-r}{r-d} q_{d,u}\right) \Bigg| \\ & (q_u, q_{u,u}, q_{m,u}, q_{d,u}) \in \left(0, \frac{r-d}{u-d}\right) \Bigg\}. \end{aligned}$$

This set is non empty, and it is not a singleton, so by the FTAP (LN p34, p52) the market is arbitrage-free and incomplete.

b) Notice first that $r = 0$. **Method 1, direct computation:** We are looking for a θ_1 such that the following system is satisfied:

$$\begin{cases} V_0 + 10\theta_1 & \geqslant 15 \\ V_0 & \geqslant 5 \\ V_0 - 10\theta_1 & \geqslant 5 \end{cases}$$
$$\Leftrightarrow \begin{cases} V_0 & \geqslant 10 \\ V_0 + 10\theta_1 & \geqslant 15 \\ V_0 - 10\theta_1 & \geqslant 5 \end{cases}$$

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Taking $V_0 = 10$ and $\theta_1 = \frac{1}{2}$, gives a super replicating strategy $\phi \hat{=} (V_0, \theta)$. Adding the first and third equation of the first system shows that the claim cannot be replicated with a lower initial wealth.

Method 2, using the question c)'s result: We can compute the super-replication price as

$$V_0 = \sup \left\{ \mathbb{E}_{\mathbb{Q}} \left[\left| \widetilde{S}_1^1 - 95 \right| \right] \;\middle|\; \mathbb{Q} \in \mathbb{P}_e \left(S^1 \right) \right\}.$$

By question 1a) and using the values given:

$$\begin{aligned} V_0 &= \sup \left\{ 10\lambda + 5 \;\middle|\; \lambda \in \left(0, \frac{1}{2} \right) \right\} \\ &= 10 \end{aligned}$$

We have $v(1, x) = |x - 95|$, so

$$\xi(1, x) = \frac{v(1, 1.1x) - v(1, 0.9x)}{0.2x},$$

and $\theta_1 = \xi(1, S_0^1) = \frac{15-5}{20} = \frac{1}{2}$. A super-replication strategy with lowest possible starting wealth is then $\phi \hat{=} (V_0, \theta_1)$.

c) (i) We prove this by backward induction: $\tilde{v}(T, .) = f(.)$ is convex. Let us assume that $\tilde{v}(k+1, .)$ is convex for $k \in \{0, 1, ..., T-1\}$. We have $0 < 1 + d < 1 + r = 1 + m < 1 + u$ so the functions $x \mapsto \frac{\tilde{v}(k+1,(1+u)x)}{1+r}$ and $x \mapsto \frac{\tilde{v}(k+1,(1+d)x)}{1+r}$ are convex, and $p^* \in (0, 1)$ so $\tilde{v}(k, .)$ as a convex combination of convex functions is convex. We conclude that $\tilde{v}(k, .)$ is convex for $k \in \{0, 1, ..., T\}$.

(ii) Let $x \in \mathbb{R}^+$, and $k \in \{0, 1, ..., T-1\}$.
We have :

$$\begin{aligned} \tilde{v}(k, x) + \xi(k+1, x) \left(\frac{x(1+u)}{1+r} - x \right) &= \frac{p^*}{1+r} \tilde{v}(k+1, x(1+u)) + \frac{(1-p^*)}{1+r} \tilde{v}(k+1, x(1+d)) \\ &+ \frac{\tilde{v}(k+1, x(1+u)) - \tilde{v}(k+1, x(1+d))}{(u-d)(1+r)} (u-r) \\ &= \tilde{v}(k+1, x(1+u)) \left(\frac{r-d}{(u-d)(1+r)} + \frac{u-r}{(u-d)(1+r)} \right) \\ &+ \tilde{v}(k+1, x(1+d)) \left(\frac{u-r}{(u-d)(1+r)} - \frac{u-r}{(u-d)(1+r)} \right) \\ &= \frac{1}{1+r} \tilde{v}(k+1, x(1+u)), \end{aligned}$$

Same computations give: $\tilde{v}(k, x) + \xi(k+1, x) \left(\frac{x(1+d)}{1+r} - x \right) = \tilde{v}(k+1, x(1+d))$.

To prove the last inequality, remark that m can be written as the following convex combination $m = \frac{u-m}{u-d} d + \frac{m-d}{u-d} u = (1-p^*)d + p^* u$, since $m = r$. By convexity of $\tilde{v}(k, .)$ for all $k \in \{0, 1, ..., T\}$ we have:

$$\begin{aligned} \tilde{v}(k, x) + \xi(k+1, x) \left(\frac{x(1+r)}{1+r} - x \right) &= \tilde{v}(k, x) \\ &= \frac{p^*}{1+r} \tilde{v}(k+1, x(1+u)) + \frac{(1-p^*)}{1+r} \tilde{v}(k+1, x(1+d)) \end{aligned}$$

$$\geqslant \frac{1}{1+r}\tilde{v}(k+1, x(1+m)),$$

which yields the desired result.

(iii) If we denote by $\widetilde{V_k}$ the undiscounted price at time k of our portfolio with strategy ϕ, we have $\widetilde{V_0} = \tilde{v}(0, S_0)$ by construction. Assume $\widetilde{V_k} \geqslant \tilde{v}(k, \widetilde{S^1_k})$ for some $k \in \{0, 1, ..., T-1\}$. By definition of the price process, and by b) (ii),

$$\begin{aligned}\frac{\widetilde{V}_{k+1}}{1+r} &= \widetilde{V}_k + \xi\left(k+1, \widetilde{S}^1_k\right)\left(\frac{\widetilde{S}^1_{k+1}}{1+r} - \widetilde{S}^1_k\right) \\ &\geqslant \tilde{v}(k+1, \widetilde{S}^1_k).\end{aligned}$$

We conclude that $\widetilde{V}_T \geqslant \tilde{v}(T, \widetilde{S}^1_T) = f(\widetilde{S}^1_T)$ and ϕ is a super-replicating strategy.