ETH Zürich SS 2015
D-MATH
Prof. J. Teichmann **August 2015**

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

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Question	Maximum	Points	Check
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam ⏲: 180 min.

Closed book examination: no notes, no books, no calculator, etc. allowed.

Important ℹ:

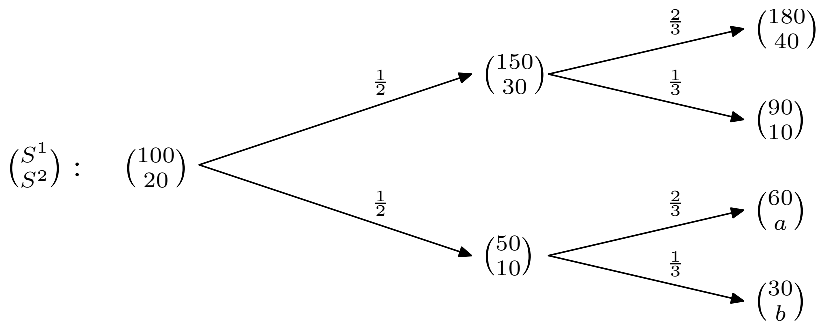
- ☞ Please put your student card on the table.
- ☞ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ☞ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ☞ Take a new sheet for each question and write your name on every sheet.
- ☞ Unless otherwise stated, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas from the lecture or from the exercise classes without derivation.
- ☞ Simplify your results as far as possible.
- ☞ Most of the subquestions can be solved independently of each other.

★★★ Good luck! ★★★

Mathematical Foundations For Finance

Exam

Question 1. Consider a market (S^0, S^1, S^2) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \{0,1,2\}}$ generated by the assets. Let $r = 0$ ($S^0 \equiv 1$) and S^1 and S^2 evolve as in the figure.



Assume for subquestions (a), (b) and (c), that $a = 13$ and $b = 6$.

(a) Show that the submarkets (S^0, S^1) and (S^0, S^2) are free of arbitrage.

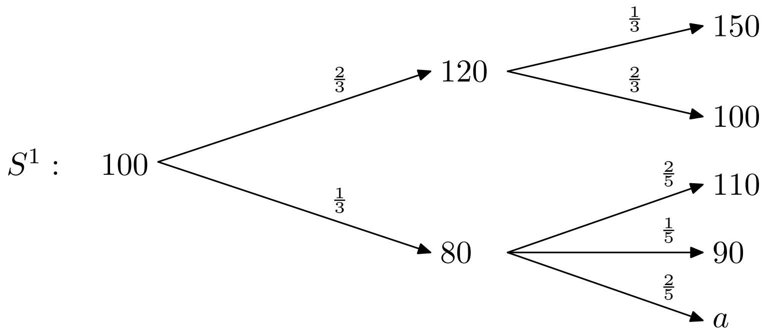
(b) Calculate the price at time $t = 0$ of the claim with payoff $(13 - S_2^2)^+$ and maturity $T = 2$ in the market (S^0, S^2).

(c) Show that the market (S^0, S^1, S^2) is not arbitrage-free by explicitly constructing an arbitrage strategy.

(d) Let a and b be arbitrary in $\mathbb{R}$. Let us fix a. Find all values of b such that the market (S^0, S^1, S^2) is free of arbitrage.

For these values of b, calculate the arbitrage free price of the claim with payoff $(13 - S_2^2)^+$ and maturity $T = 2$ for general $a \in \mathbb{R}$ in the market (S^0, S^1, S^2).

Question 2. We consider a multinomial model with two time periods ($T = 2$). The market consists of a riskless asset $\widetilde{S^0}$ that grows at a rate $r = 0$, and one risky asset whose evolution is given by the tree below. The numbers on the branches give the probability of following a particular branch. The tree shows the evolution of both the discounted and undiscounted price of the asset since the value of one unit of the bank account stays constant ($\widetilde{S^0} \equiv 1$).



Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with $\Omega = \{u\} \times \{u, d\} \cup \{d\} \times \{u, m, d\}$, $\mathcal{F} = 2^{\Omega}$. The filtration is given by $\mathbb{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F})$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \{u\} \times \{u, d\}, \{d\} \times \{u, m, d\}, \Omega\}$. The probability $\mathbb{P}$ defined as follows :

$$\mathbb{P}\left[\{u\} \times \{u, d\}\right] = p_u = \frac{2}{3}, \qquad \mathbb{P}\left[\{d\} \times \{u, m, d\}\right] = p_d = \frac{1}{3}$$

$$\mathbb{P}[\{(i, j)\}] = p_i \times p_{i,j} \quad \text{for} \quad i \in \{u, d\}, j \in \{u, m, d\}$$

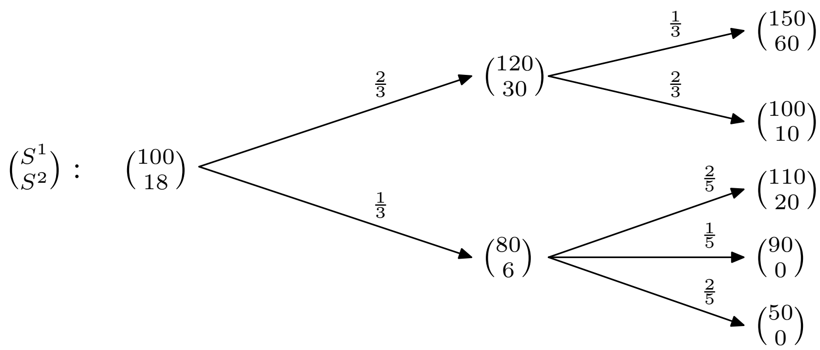
with $p_{u,u} = \frac{1}{3}$, $p_{u,d} = \frac{2}{3}$, $p_{d,u} = \frac{2}{5}$, $p_{d,m} = \frac{1}{5}$, $p_{d,d} = \frac{2}{5}$.

a) Assume $a = 80$, is the market arbitrage-free ? Find the equivalent martingale measures for S^1 or construct an arbitrage opportunity and prove in details that it is one. If the market is arbitrage-free, is it complete ?

b) Assume $a = 50$, is the market arbitrage free ? Find the equivalent martingale measures for S^1 or construct an arbitrage opportunity and prove in details that it is one. If the market is arbitrage-free, is it complete ?

c) Assume $a = 50$.

   (i) A second asset S^2 can be traded on the market. The dynamics of (S^1, S^2) are given by the tree below. Show that the market $\left(S^1, S^2\right)$ is complete.



Please see next sheet!

(ii) We consider an asian option on the stock S^1. On this two periods model for the market (S^1, S^2), the payoff of the asian option with strike K is given by

$$C_K^A = \left(\frac{1}{2} \sum_{j=1}^{2} S_j^1 - K \right)^+$$

What is the non-arbitrage price at time $t = 0$ of C_{90}^A in the market (S^0, S^1, S^2)?

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Question 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space with an augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by a Brownian motion $(W_t)_{t\geqslant 0}$. Consider a discounted Bachelier market (S^0, S^1) on this space, i.e.,

$$S_t^0 = 1, \quad \text{and} \quad S_t^1 = S_0^1 + \sigma W_t, \quad t \geq 0$$

for some $S_0^1, \sigma > 0$.

Let $T > 0$ and $K \in \mathbb{R}$. Compute the replicating (self-financing) strategy for $h(S_T^1)$ by computing the value process and using Itô's formula, where h is defined as

(a) $h(y) = y^3 - 6y^2 + 11y - 6$.

(b) $h(y) = 1_{\{S\geq K\}}$.

(c) $h(y) = \max\{K, y\}$.

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Question 4. Let $T > 0$ be a fixed time horizon and $W = (W_t)_{t\in[0,T]}$ a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by W. Consider the Black-Scholes model with bank account price process $\widetilde{S}^0$ and stock price process $\widetilde{S}^1$ satisfying the SDEs

$$\frac{d\widetilde{S}^1_t}{\widetilde{S}^1_t} = \mu dt + \sigma dW_t,$$
$$\frac{d\widetilde{S}^0_t}{\widetilde{S}^0_t} = r dt,$$

for some fixed $r, \mu, \sigma > 0$ and with $\widetilde{S}^0_0 = \widetilde{S}^1_0 = 1$. Denote by $\widetilde{V}_0^{\mathrm{Call},\widetilde{K}}$ and $\widetilde{V}_0^{\mathrm{Put},\widetilde{K}}$ respectively the undiscounted prices at time zero of European call and put options with undiscounted strike $\widetilde{K} > 0$.

For any two constants $0 < K_1 < K_2$, set $\widetilde{K}_i = e^{rT} K_i$ and consider the following two strategies: First the (long) butterfly option giving the payoff

$$\widetilde{H}^{\mathrm{BF}} = \left(\widetilde{S}^1_T - \widetilde{K}_1\right)^+ - 2\left(\widetilde{S}^1_T - \frac{\widetilde{K}_1 + \widetilde{K}_2}{2}\right)^+ + \left(\widetilde{S}^1_T - \widetilde{K}_2\right)^+$$

at time T. Consider also the (short) straddle position yielding the payoff

$$\widetilde{H}^{\mathrm{Straddle}} = -\left(\widetilde{S}^1_T - \frac{\widetilde{K}_1 + \widetilde{K}_2}{2}\right)^+ - \left(\frac{\widetilde{K}_1 + \widetilde{K}_2}{2} - \widetilde{S}^1_T\right)^+$$

at time T.

Finally, define $\widetilde{H} = \widetilde{H}^{\mathrm{Straddle}} - \widetilde{H}^{\mathrm{BF}}$.

(a) Find the value of $\widetilde{H}$ at time 0 in terms of the difference $\Delta K = (K_2 - K_1)$ as well as $\widetilde{V}_0^{\mathrm{Call},\widetilde{K}_i}$ and $\widetilde{V}_0^{\mathrm{Put},\widetilde{K}_j}$ for $i, j \in \{1, 2\}$.

(b) Sketch how the payoff $\widetilde{H}$ depends on the final value of the stock price $\widetilde{S}^1_T$ (draw the curve $\widetilde{S}^1_T \mapsto \widetilde{H}(\widetilde{S}^1_T)$). Qualitative and quantitative details should be clear from the figure or explained in text (slopes, intercepts,...).

Hint: Sketching $\widetilde{H}^{\mathrm{Straddle}}$ and $\widetilde{H}^{\mathrm{BF}}$ separately could help visualizing, but is not necessary.

(c) Let $\widetilde{K} > 0$. Compute a replicating strategy for a European put option with strike $\widetilde{K}$ and maturity T. Derive and prove your result. (Don't forget to compute $\widetilde{V}_0^{\mathrm{Put},\widetilde{K}}$ as well.)

Question 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. We consider a trinomial market model, with two assets and $T \geqslant 2$ periods. The riskless asset (bank account) has the following price process

$$\widetilde{S_k^0} = (1+r)^k\,, \text{ for } k \in \{0, 1, ..., T\},$$

and the risky asset price process is defined as

$$\widetilde{S_k^1} = \widetilde{S_0^1} \prod_{i=1}^{k} Y_i, \text{ for } k \in \{0, 1, ..., T\},$$

where the Y_i are i.i.d. and taking values in $\{1+u,\ 1+m,\ 1+d\}$, each one with strictly positive probability under $\mathbb{P}$, and where $m = r \geqslant 0$, $u > m > d > -1$ and $\widetilde{S_0^1} > 0$.

We denote the discounted price process by $S^1 = \frac{\widetilde{S^1}}{\widetilde{S^0}}$. Let $\mathbb{F} = (\mathcal{F}_k)_{k \in \{0,1,\ldots,T\}}$ be the filtration generated by $(Y_k)_{k \in \{0,1,\ldots,T\}}$.

a) Assume for this subquestion that $T = 2$. Is the market arbitrage-free ? Is it complete ? Compute the set $\mathbb{P}_e\left(S^1\right)$ of equivalent martingale measures for the discounted price process.

b) Assume for this subquestion that $T = 1$, $\widetilde{S_0^0} = 1$, $\widetilde{S_0^1} = 100$, $u = 0.1$, $m = r = 0$, $d = -0.1$. Find a super-replication strategy with the lowest possible initial wealth V_0 for the claim with maturity $T = 1$ and payoff $\widetilde{C} = \left|\widetilde{S_1^1} - 95\right|$.

We are back in the general case, for any $T \geqslant 2$, any $\widetilde{S_0^1} > 0$, and any u, m, r, d such that $m = r \geqslant 0$ and $u > m > d > -1$. Let f be a convex function. Define recursively the function $\tilde{v}$

$$\begin{cases} \tilde{v}(T, x) &= f(x), \text{ for } x \in \mathbb{R}^+, \\ \tilde{v}(k, x) &= \frac{p^* \tilde{v}(k+1, x(1+u)) + (1-p^*)\tilde{v}(k+1, x(1+d))}{1+r}, \quad \text{for } x \in \mathbb{R}^+ \text{ and } 0 \leqslant k \leqslant T-1, \end{cases}$$

where $p^* = \frac{r-d}{u-d}$. Define now the function ξ as follows

$$\xi(k+1, x) = \frac{\tilde{v}(k+1, x(1+u)) - \tilde{v}(k+1, x(1+d))}{x(u-d)}, \quad \text{for } x \in \mathbb{R}^+ \text{ and } 0 \leqslant k \leqslant T-1.$$

c) (i) Show that for all $k \in \{0, 1, ..., T\}$, the function $\tilde{v}(k, .)$ is convex.

(ii) Show that for $x \in \mathbb{R}^+$,

$$\frac{\tilde{v}(k+1, x(1+y))}{1+r} \leqslant \tilde{v}(k, x) + \xi(k+1, x)\left(\frac{x(1+y)}{1+r} - x\right),$$

for $y \in \{u, m, d\}$ and $0 \leqslant k \leqslant T-1$.

(iii) Let $\phi \hat{=} (\tilde{v}(0, S_0), \theta)$ be the self-financing strategy that starts with $\tilde{v}(0, S_0)$ initial capital and use the strategy $\theta_{k+1} = \xi(k+1, \widetilde{S_k^1})$ for $0 \leqslant k \leqslant T-1$ on the stock. Show that ϕ is a super-replicating strategy for the contingent claim with payoff $f(\widetilde{S_T^1})$ at time T.

Hint: Show by induction that $\widetilde{V}_k \geqslant \tilde{v}(k, \widetilde{S_k^1})$ for all $k \in \{0, 1, ...T\}$, where $\widetilde{V}$ is the undiscounted value process of the portfolio with strategy ϕ.