ETH Zürich HS 2014
D-MATH
Prof. J. Teichmann **January 2015**

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** □ **MScQF** □ **SAV** □ **Other** □
Matriculation number	

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Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam ⏲: 180 min.

Closed book examination: no notes, no books, no calculator, etc. allowed.

Important ℹ:

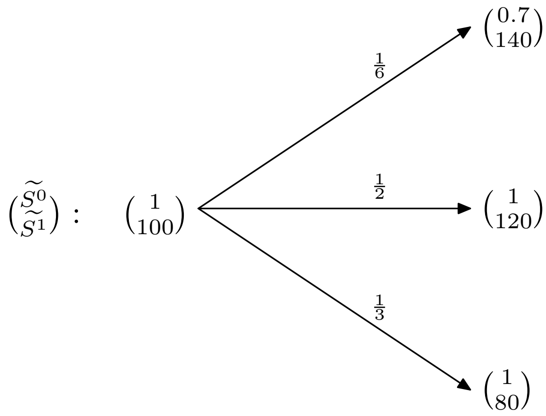
- ☞ Please put your student card on the table.
- ☞ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ☞ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ☞ Take a new sheet for each question and write your name on every sheet.
- ☞ Unless otherwise stated, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas from the lecture or from the exercise classes without derivation.
- ☞ Simplify your results as far as possible.
- ☞ Most of the subquestions can be solved independently of each other.

★★★ Good luck! ★★★

Mathematical Foundations For Finance

Exam

Question 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. We consider the following one period market model with stochastic interest rate for the bank account. The price processes for the bank account $\widetilde{S}^0$ and the risky asset $\widetilde{S}^1$ are given by the following tree:



(a) Find all equivalent martingale measures for the discounted price process $S^1 = \frac{\widetilde{S}^1}{\widetilde{S}^0}$. Is the market arbitrage-free ? Complete ?

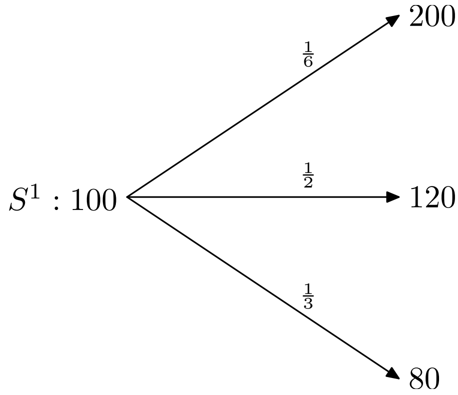
(b) (i) Consider a put option on the stock, with maturity $T = 1$ and strike $\widetilde{K} = 100$. Its payoff is $\widetilde{P} = \left(\widetilde{K} - \widetilde{S}^1_1\right)^+$. Is it attainable ?

(ii) Compute a super-replicating strategy for the put defined in (i), with price $\theta^0_0 = \sup_{\mathbb{Q} \in \mathbb{P}_e(S^1)} \mathbb{E}_{\mathbb{Q}} \left[\left(\widetilde{K} - \widetilde{S}^1_1\right)^+ \right]$.

(c) (i) We assume that the put is sold for 12 at time t=0. Show that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{P})$ is complete, where $\widetilde{P}$ is as in b).

(ii) We now consider a call on the stock with maturity $T = 1$ and strike $\widetilde{K} = 100$. Write down the put-call parity and find a super-replicating strategy for the call in the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{P})$.

Solution 1. (a) First let us consider the discounted price tree:



The process is adapted for the filtration it generates, and the probability space is finite. The martingale measures for S^1 that are equivalent to $\mathbb{P}$ are given by triple $(q_u, q_m, q_d) \in (0,1)^3$ such that :

$$\mathbb{E}_{\mathbb{Q}}\left[S_1^1 \mid \mathcal{F}_0\right] = S_0^1$$

That is to say, $\mathbb{Q}$ is an EMM for S^1if and only if

$$\begin{cases} 200q_u + 120q_m + 80q_d & = 100 \\ q_u + q_m + q_d & = 1 \\ (q_u, q_m, q_d) & \in (0,1)^3 \end{cases} \Leftrightarrow \begin{cases} 8q_u - 4q_d & = -2 \\ q_u + q_m + q_d & = 1 \\ (q_u, q_m, q_d) & \in (0,1)^3 \end{cases}$$

And we finally get:

$$\mathbb{Q} \in \mathbb{P}_e\left(S^1\right) \Leftrightarrow \begin{cases} q_u & \in \left(0, \frac{1}{6}\right) \\ q_m & = \frac{1}{2} - 3q_u \\ q_d & = \frac{1}{2} + 2q_u \end{cases}$$

and

$$\begin{aligned} \mathbb{P}_e\left(S^1\right) &= \left\{\left(\lambda, \frac{1}{2} - 3\lambda, \frac{1}{2} + 2\lambda\right) \mid \lambda \in \left(0, \frac{1}{6}\right)\right\} \\ &= \left\{\left(\frac{1}{6} - \frac{1}{3}\mu, \mu, \frac{5}{6} - \frac{2}{3}\mu\right) \mid \mu \in \left(0, \frac{1}{2}\right)\right\} \\ &= \left\{\left(\frac{1}{2}\nu - \frac{1}{4}, \frac{5}{4} - \frac{3}{2}\nu, \nu\right) \mid \nu \in \left(\frac{1}{2}, \frac{5}{6}\right)\right\} \end{aligned}$$

It is a non-empty, open set. The market is therefore arbitrage-free and incomplete.

(b) (i) Let us compute the set of arbitrage-free prices for this claim:

$$\begin{aligned} \left\{\mathbb{E}_{\mathbb{Q}}\left[\frac{\left(\widetilde{K} - \widetilde{S_1^1}\right)^+}{\widetilde{S_1^0}}\right] \mid \mathbb{Q} \in \mathbb{P}_e\left(S^1\right)\right\} &= \left\{\mathbb{E}_{\mathbb{Q}}\left[\left(\widetilde{K} - \widetilde{S_1^1}\right)^+\right] \mid \mathbb{Q} \in \mathbb{P}_e\left(S^1\right)\right\} \\ &= \left\{20\left(\frac{1}{2} + 2\lambda\right) \mid \lambda \in \left(0, \frac{1}{6}\right)\right\} \\ &= \left\{10 + 40\lambda \mid \lambda \in \left(0, \frac{1}{6}\right)\right\}. \end{aligned}$$

The function that maps an equivalent martingale measure to the expectation under this measure of the claim payoff is not constant. By the criteria of attainability (LN p 49) the put is not attainable.

(ii) We want to find $\left(\theta_1^0, \theta_1^1\right)$ such that $\theta_1^0 \widetilde{S_1^0} + \theta_1^1 \widetilde{S_1^1} \geqslant \left(\widetilde{K} - \widetilde{S_1^1}\right)^+$ $\mathbb{P}$-a.s., that is to say:

$$\begin{cases} 0.7\theta_1^0 + 140\theta_1^1 & \geqslant 0 \\ \theta_1^0 + 120\theta_1^1 & \geqslant 0 \\ \theta_1^0 + 80\theta_1^1 & \geqslant 20, \end{cases}$$

and such that $\left(\theta_1^0 - \theta_0^0\right) \widetilde{S_0^0} + \theta_1^1 \widetilde{S_0^1} = 0$, with $\theta_0^0 = \sup_{\mathbb{Q} \in \mathbb{P}_e(S^1)} \mathbb{E}_{\mathbb{Q}} \left[\left(\widetilde{K} - \widetilde{S_1^1}\right)^+ \right] = \frac{50}{3}$, the optimal super-replication price (notice that for claim having non-zero values in the two lower states only, discounted and undiscounted prices are the same). We have then $\theta_1^0 = \frac{50}{3} - 100\theta_1^1$. Substituting in the system gives:

$$\begin{cases} 0.7\left(\frac{50}{3} - 100\theta_1^1\right) + 140\theta_1^1 & \geqslant 0 \\ \frac{50}{3} - 100\theta_1^1 + 120\theta_1^1 & \geqslant 0 \\ \frac{50}{3} - 100\theta_1^1 + 80\theta_1^1 & \geqslant 20, \end{cases}$$

$$\begin{cases} \frac{35}{3} + 70\theta_1^1 & \geqslant 0 \\ \frac{50}{3} + 20\theta_1^1 & \geqslant 0 \\ \frac{50}{3} - 20\theta_1^1 & \geqslant 20, \end{cases}$$

We get finally $\theta_1^1 = -\frac{1}{6}$. The super-replicating strategy with price $\theta_0^0 = \frac{50}{3}$, holds $\theta_1^0 = \frac{100}{3}$ units of the bank account and $\theta_1^1 = -\frac{1}{6}$ units of stocks at time 1.

(c) (i) Let us compute the set of martingale measures for the market $\left(S^1, \frac{\widetilde{P}}{\widetilde{S^0}}\right)$. The equivalent martingale for this market are to be taken from $\mathbb{P}_e\left(S^1\right)$. We compute for $\mathbb{Q} \in \mathbb{P}_e\left(S^1\right)$:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\frac{\widetilde{P}}{\widetilde{S^0}} \mid \mathcal{F}_0\right] = 12 &\Leftrightarrow \left(\frac{1}{2} + 2\lambda\right) 20 = 12 \\ &\Leftrightarrow \lambda = \frac{1}{20}. \end{aligned}$$

Therefore there exists a unique equivalent martingale measure for the new market. It is then complete.

(ii) **Method 1, Super-replication in** $\left(\widetilde{S^0}, \widetilde{S^1}\right)$ **:** We have $\left(\widetilde{S_1^1} - \widetilde{K}\right)^+ - \left(\widetilde{K} - \widetilde{S_1^1}\right)^+ = \widetilde{S_1^1} - \widetilde{K}$, so to super-replicate the call we just have to super-replicate a put (we just computed how to do that), hold a stock, and super-replicate $-\widetilde{K}$, selling $\frac{\widetilde{K}}{\max\{\widetilde{S_1^0}(\omega) \mid \omega \in \Omega\}} = \widetilde{K}$ units of the bank account. A possible superreplicating strategy starts with a wealth $\theta_0^0 = \frac{50}{3}$, and holds $\theta_1^0 = -\frac{200}{3}$ units of the bank account and $\theta_1^1 = \frac{5}{6}$ units of stocks in the first period.

Method 2, Super-replication in $\left(\widetilde{S^0}, \widetilde{S^1}, \widetilde{P}\right)$ **:** We have $\left(\widetilde{S_1^1} - \widetilde{K}\right)^+ - \left(\widetilde{K} - \widetilde{S_1^1}\right)^+ = \widetilde{S_1^1} - \widetilde{K}$, so to super-replicate the call we just have to buy a put and a stock at time 0 and short sell $\widetilde{K}$ units of the bank account (which optimally super-replicate the payoff $-\widetilde{K}$ at time 1). Initial cost is $\theta_0^0 = 12$.

Method 3, exact replication in the complete market $\left(\widetilde{S^0}, \widetilde{S^1}, \widetilde{P}\right)$ **:** We have $\theta_0^0 = \mathbb{E}_{\mathbb{Q}_{\frac{1}{20}}}\left[\frac{\left(\widetilde{S_1^1} - \widetilde{K}\right)^+}{\widetilde{S_1^0}}\right] = \frac{69}{7}$. The replicating strategy $\phi \hat{=} \left(\theta_0^0, (\theta_1^1, \theta_1^2)\right)$ satisfies :

$$\begin{cases} \frac{69}{7} + 100\theta_1^1 - 12\theta_1^2 & = \frac{40}{0.7} \\ \frac{69}{7} + 20\theta_1^1 - 12\theta_1^2 & = 20 \\ \frac{69}{7} - 20\theta_1^1 + 8\theta_1^2 & = 0, \end{cases}$$

which yields : $\theta_1^1 = \frac{13}{28}$ and $\theta_1^2 = -\frac{2}{28}$.

Question 2. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{u, m, d\} \times \{u, d\}$, $\mathcal{F} = 2^\Omega$ and $\mathbb{P}$ defined by $\mathbb{P}(\{(x_1, x_2)\}) = p_{x_1} p_{x_1,x_2}$ for

$$p_u = 0.2, \quad p_m = 0.3, \quad p_d = 0.5,$$
$$p_{u,u} = p_{u,d} = 0.5, \quad p_{m,u} = 0.4, \quad p_{m,d} = 0.6,$$
$$p_{d,u} = 0.75, \quad \text{and} \quad p_{d,d} = 0.25.$$

Now define the random variables Y_1 and Y_2 by

$$Y_1((u, x_2)) = 1 + y_u^1, \quad Y_1((m, x_2)) = 1 + y_m^1, \quad Y_1((d, x_2)) = 1 + y_d^1,$$
$$Y_2((x_1, u)) = 1 + y_u^2 \quad \text{and} \quad Y_2((x_1, d)) = 1 + y_d^2.$$

for $x_1 \in \{u, m, d\}$, $x_2 \in \{u, d\}$, $(y_u^1, y_m^1, y_d^1) = (0.4, 0.2, -0.2)$ and $(y_u^2, y_d^2) = (0.1, -0.2)$.

Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a financial market consisting of a bank account and one stock defined as

$$\widetilde{S}_0^0 = \widetilde{S}_1^0 = \widetilde{S}_2^0 = 1,$$
$$\widetilde{S}_0^1 = 1, \quad \widetilde{S}_1^1 = Y_1 \quad \text{and} \quad \widetilde{S}_2^1 = Y_1 Y_2.$$

Define the discounted price processes: $S^0 \equiv 1$, and $S^1 = \frac{\widetilde{S}^1}{\widetilde{S}^0}$.

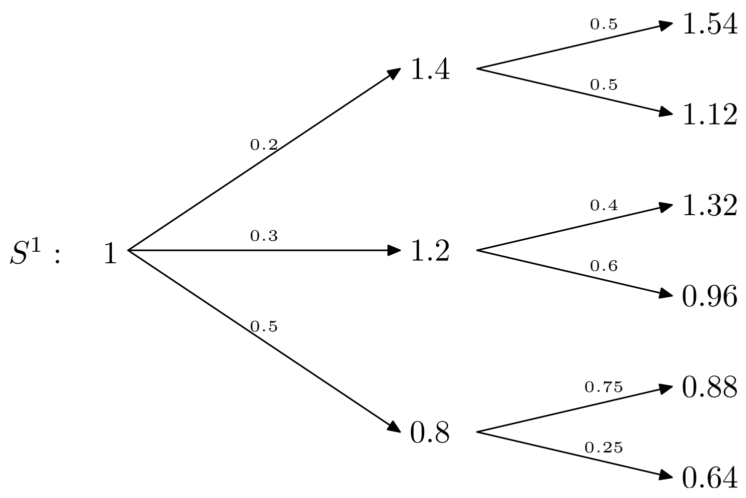
(a) Draw a tree showing the evolution of the asset S^1.

(b) Find the set of equivalent martingale measures for (S^0, S^1).

(c) Define the claim with payoff $H_K = 1_{\{S_2^1 \geq K\}}$ for $K \in \mathbb{R}$ at maturity $T = 2$. For which values of $K \geq 1$ is H_K attainable?

(d) Calculate the super-replication price $\pi_S(H_K)$ of the claim defined above in c) as a function of K for $K \geq 1$.

Solution 2. (a) The evolution tree of the risky asset is:



(b) Write $\mathbb{Q}(\{(x_1, x_2)\}) = q_{x_1} q_{x_1,x_2}$. The size of the increments in the second period do not depend on the outcome of the first, and, since they are binomial with parameters y_u^2, y_d^2 for the movements up and down respectively, Corollary 2.1.4 then states that the probabilities q_{x_1,x_2} are uniquely determined by

$$q_{x_1,u} = \frac{r - y_d^2}{y_u^2 - y_d^2} = \frac{-(-0.2)}{0.1 - (-0.2)} = \frac{2}{3} \quad \forall x_1 \in \{u, m, d\}.$$

For the first period, write down the martingale property:

$$1 = E[S_1^1] = (1 + 0.4)q_u + (1 + 0.2)q_m + (1 - 0.2)q_d.$$

Using the fact that the probabilities sum to 1 and parametrizing $q_m = \lambda$ yields

$$q_u = \frac{1}{3}(1 - 2\lambda) \quad \text{and} \quad q_d = \frac{1}{3}(2 - \lambda).$$

Equivalence to P means $q_u, q_m, q_d \in (0, 1)$, which is equivalent to $\lambda \in (0, 1/2)$.

The equivalent martingale measures also have two other parametrizations. The first one is obtained by setting $q_u = \lambda_1$. This yields

$$q_m = \frac{1 - 3\lambda_1}{2}, \quad \text{and} \quad q_d = \frac{1 + \lambda_1}{2}, \text{ for } \lambda_1 \in \left(0, \frac{1}{3}\right).$$

Finally, $q_d = \lambda_2$ gives

$$q_m = 2 - 3\lambda_2 \quad \text{and} \quad q_u = 2\lambda_2 - 1, \text{ for } \lambda_2 \in \left(\frac{1}{2}, \frac{2}{3}\right).$$

(c) Parametrize the set of EMMs by λ and calculate the expected value of H_K:

$$E_{\mathbb{Q}^\lambda}[H_K] = \begin{cases} 0, & 1.54 < K, \\ q_u q_{u,u} & 1.32 < K \leq 1.54, \\ q_u q_{u,u} + q_m q_{m,u} & 1.12 < K \leq 1.32, \\ q_u + q_m q_{m,u} & 1 \leq K \leq 1.12 \end{cases}$$

$$= \begin{cases} 0, & 1.54 < K, \\ \frac{2}{9}(1 - 2\lambda), & 1.32 < K \leq 1.54, \\ \frac{2}{9}(1 + \lambda), & 1.12 < K \leq 1.32, \\ \frac{1}{3}, & 1 \leq K \leq 1.12. \end{cases}$$

By the characterization of attainable payoffs we know that H_K is attainable if and only if $\lambda \mapsto E_{\mathbb{Q}^\lambda}[H_K]$ is constant, which is true only for $K \in [1, 1.12]$ and $K > 1.54$..

For the other parametrizations, similar calculations yield the expressions

$$E_{\mathbb{Q}^{\lambda_1}}[H_K] = \begin{cases} 0, & 1.54 < K, \\ \frac{2}{3}\lambda_1, & 1.32 < K \leq 1.54, \\ \frac{1}{3}(1 - \lambda_1), & 1.12 < K \leq 1.32, \\ \frac{1}{3}, & 1 \leq K \leq 1.12, \end{cases}$$

and

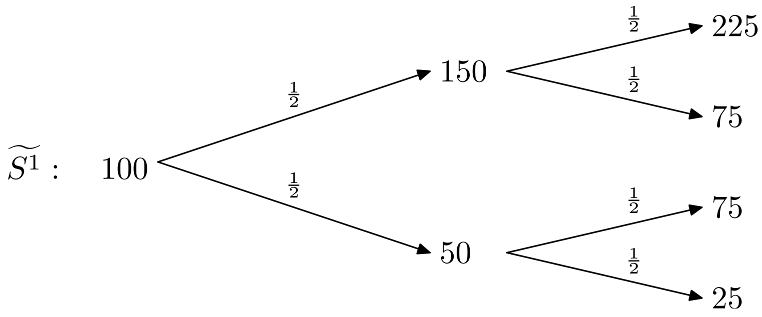
$$E_{\mathbb{Q}^{\lambda_2}}[H_K] = \begin{cases} 0, & 1.54 < K, \\ \frac{2}{3}(2\lambda_2 - 1), & 1.32 < K \leq 1.54, \\ \frac{2}{3}(1 - \lambda_2), & 1.12 < K \leq 1.32, \\ \frac{1}{3}, & 1 \leq K \leq 1.12, \end{cases}$$

but the conclusion is the same.

(d) We find the super-replication price by using the identity

$$\pi_S(H_K) = \sup_{\lambda \in (0,1/2)} E_{\mathbb{Q}^\lambda}[H_K] = \begin{cases} 0, & 1.54 < K, \\ \frac{2}{9}, & 1.32 < K \leq 1.54, \\ \frac{1}{3}, & 1.12 < K \leq 1.32, \\ \frac{1}{3}, & 1 \leq K \leq 1.12. \end{cases}$$

Question 3. Consider the market $(\widetilde{S}^0, \widetilde{S}^1)$ on some probability space with filtration generated by the assets. Let $\widetilde{S_t^0} = (1+r)^t$ and



for $r = 0.25$. Let $\widetilde{H}_t = (\widetilde{K} - \widetilde{S}_t^1)^+$ with $\widetilde{K} = 150$. Do not forget to explain how you found your solution in the following problems!

(a) Draw a tree of the discounted value process corresponding to the European option V^{Eu} with (undiscounted) payoff $\widetilde{H}_2$ at time $t = 2$.

(b) Draw a tree of the discounted value process corresponding to the American option V^{Am} with maturity 2 and (undiscounted) payoff process $\widetilde{H}$.

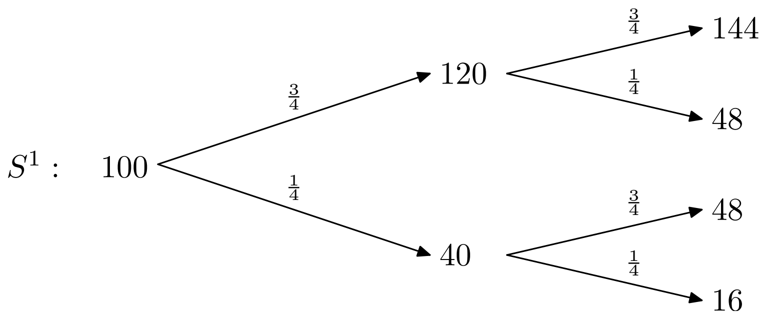
Hint : Recall that the discounted value process of an American option with maturity T and with discounted payoff process $(H_k)_{k\in\{0,\ldots T\}}$ is given by the recursive scheme $V_t^{Am} = \max\{H_t, \mathbb{E}_{\mathbb{Q}}\left[V_{t+1}^{Am} \mid \mathcal{F}_t\right]\}$ for $t \in \{0, \ldots T-1\}$, $\mathbb{Q}$ an equivalent martingale measure for the market, and $V_T^{Am} = H_T$.

(c) Determine whether there exist self-financing portfolios φ^{Am} and φ^{Eu} such that $V_t(\varphi^{\mathrm{Am}}) = V_t^{\mathrm{Am}}$ and $V_t(\varphi^{\mathrm{Eu}}) = V_t^{\mathrm{Eu}}$ for all $t \in \{0, 1, 2\}$. Find them if they exist.

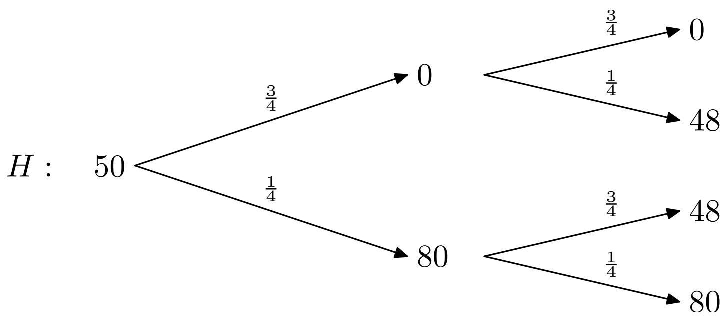
Solution 3. The market can be written as a binomial model with $u = 0.5$, $d = -0.5$ and $r = 0.25$. According to Corollary 2.1.4, the unique EMM Q will therefore assign the corresponding probabilities

$$q_u = \frac{r-d}{u-d} = \frac{0.25-(-0.5)}{0.5-(-0.5)} = \frac{3}{4} \quad \text{and} \quad q_d = 1 - q_u = \frac{1}{4}.$$

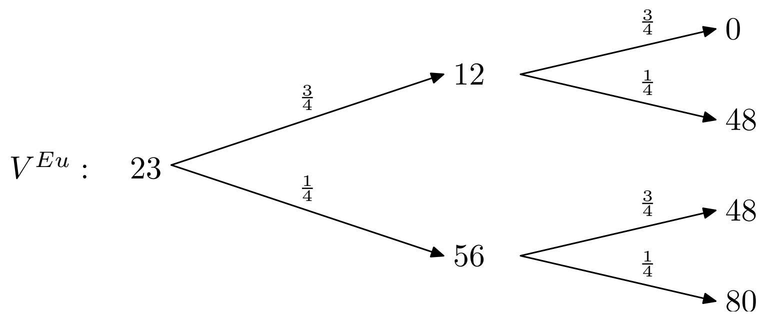
We will also need the discounted values for all problems. The discounted process is then



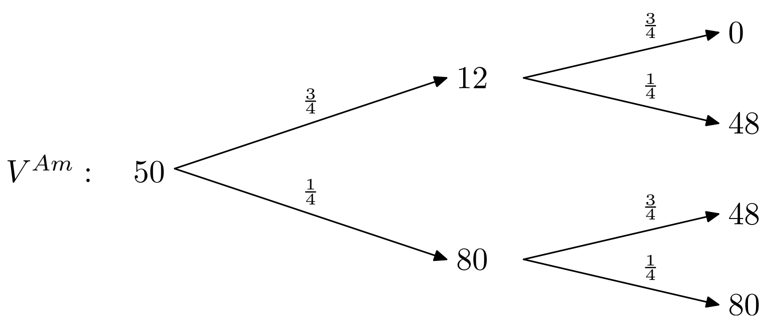
with probabilities given under Q. Discounting $\widetilde{H}$ gives $H_t = (150 \cdot 0.8^t - S_t)^+$, i.e.,



(a) We calculate the values backwards from H_2 in every node by computing the (conditional) expectation of the following two nodes, e.g., the value in the first node below is calculated as $\frac{3}{4}12 + \frac{1}{4}56 = 23$.



(b) To calculate the American option we proceed similarly, but in each step we compare the expectation of the following two nodes with the current value of H and choose the largest. This is the case since we can at any point stop and get H, but only do so if it gives more profit than is expected from continuing. For example, the value in the first node below is calculated as $\max\{\frac{3}{4}12 + \frac{1}{4}80, 50\} = 50$.



(c) Start with the European option and let ϑ_2^u and ϑ_2^d be the strategies in the second period given that the stock price increased respectively decreased in the first period. Furthermore, let ϑ_1 be the strategy for the first period. Using the relation $V_t = V_0 + G_t(\vartheta)$ we have 6 equations in the variables V_0, ϑ_1, ϑ_2^u and ϑ_2^d (2 for the first time step and 4 for the second).

More precisely, we have

$$\begin{bmatrix} 1 & 20 & & \\ 1 & -60 & & \\ 1 & 20 & 24 & \\ 1 & 20 & -72 & \\ 1 & -60 & & 8 \\ 1 & -60 & & -24 \end{bmatrix} \begin{bmatrix} V_0 \\ \vartheta_1 \\ \vartheta_2^u \\ \vartheta_2^d \end{bmatrix} = \begin{bmatrix} 12 \\ 56 \\ 0 \\ 48 \\ 48 \\ 80 \end{bmatrix},$$

which is solved by $(V_0, \vartheta_1, \vartheta_2^u, \vartheta_2^d) = (23, -\frac{11}{20}, -\frac{1}{2}, -1)$.

Note that the four last equations are enough to find the solution, but the first two have to be satisfied since the equality is required for all time points, not only the last.

For the American option the idea is similar, but even at the first timestep the equations cannot be fulfilled since it would require

$$\begin{aligned} 50 + 20\vartheta_1 &= 12, \\ 50 - 60\vartheta_1 &= 80, \end{aligned}$$

which has no solution. We conclude that there cannot exist such a self-financing portfolio.

Question 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which exists a Brownian motion $(W_t)_{t\geqslant 0}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ be the $\mathbb{P}$-augmented filtration generated by W.

We consider a market model with two assets whose price processes are the following:

$$\begin{cases} \widetilde{S_t^0} &= e^{rt}, \\ \widetilde{S_t^1} &= S_0^1 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right), \end{cases}$$

for $t \geqslant 0$, with $\mu \in \mathbb{R}$, $\sigma > 0$ and $S_0^1 > 0$, $r > 0$.

(a) What is the probability under the historical probability measure $\mathbb{P}$ that a call on the risky asset $\widetilde{S^1}$, with strike $\widetilde{K}$ and maturity T is exercised ?

(b) There exists a unique probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ such that the process $S^1 = \frac{\widetilde{S^1}}{\widetilde{S^0}}$ is a $\mathbb{Q}$-martingale on $[0,T]$. Give the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}{}_{|\mathcal{F}_T}$.

(c) We now consider two calls with same maturity T and strike $\widetilde{K_1}$ and $\widetilde{K_2}$, with $\widetilde{K_1} < \widetilde{K_2}$. Let $\widetilde{C}(\widetilde{K_1})$ and $\widetilde{C}(\widetilde{K_2})$ be the undiscounted price processes of these claims.

(i) Show the following relations:

$$\widetilde{C}_t(\widetilde{K_2}) \leqslant \widetilde{C}_t(\widetilde{K_1}), \text{ for } t \in [0,T]$$

$$\widetilde{C}_t(\widetilde{K_1}) - \widetilde{C}_t(\widetilde{K_2}) \leqslant \frac{\widetilde{K_2} - \widetilde{K_1}}{e^{r(T-t)}}, \text{ for } t \in [0,T]$$

(ii) Let $\widetilde{K_3} = \lambda\widetilde{K_1} + (1-\lambda)\widetilde{K_2}$ for $\lambda \in [0,1]$. Show that $\widetilde{C}_t(\widetilde{K_3}) \leqslant \lambda\widetilde{C}_t(\widetilde{K_1}) + (1-\lambda)\widetilde{C}_t(\widetilde{K_2})$.

(d) Let us a consider a power option that pays $\left(\widetilde{S_T^1}\right)^p$ at maturity T, for $p = 3$. Compute the discounted price process and the replicating strategy for this option.

Solution 4. (a) A call with strike $\widetilde{K}$ and maturity T is exercised if and only if $\widetilde{S_T^1} \geqslant \widetilde{K}$. The probability of this event is:

$$\begin{aligned} \mathbb{P}\left[\widetilde{S_T^1} \geqslant \widetilde{K}\right] &= \mathbb{P}\left[S_0^1 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right) \geqslant \widetilde{K}\right] \\ &= \mathbb{P}\left[W_T \geqslant \frac{1}{\sigma}\left(\log\left(\frac{\widetilde{K}}{S_0^1}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T\right)\right] \\ &= 1 - \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{\widetilde{K}}{S_0^1}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T\right)\right) \\ &= \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\left(\mu - \frac{1}{2}\sigma^2\right)T - \log\left(\frac{\widetilde{K}}{S_0^1}\right)\right)\right). \end{aligned}$$

(b) Let us define the process $Z = (Z_t)_{t\in[0,T]}$ by:

$$Z_t = \exp\left(-\frac{\mu - r}{\sigma}W_t - \frac{(\mu - r)^2}{2\sigma^2}t\right)$$

which is a true $\mathbb{P}$-martingale. By the theorem of Girsanov, the process $\left(\widetilde{W_t}\right)_{t\in[0,T]} = \left(W_t + \frac{\mu - r}{\sigma}t\right)_{t\in[0,T]}$ is a $\mathbb{Q}$-Brownian motion. We have:

$$S_t^1 = S_0^1 \exp\left(\sigma\widetilde{W_t} - \frac{1}{2}\sigma^2 t\right),$$

which is a martingale under $\mathbb{Q}$ on [0,T].

(c) (i) For $x \in [0, \widetilde{K_2}]$, $\left(x - \widetilde{K_2}\right)^+ = 0$, so $\left(x - \widetilde{K_1}\right)^+ \geqslant \left(x - \widetilde{K_2}\right)^+$, for $x \in [\widetilde{K_2}, \infty)$, $\left(x - \widetilde{K_2}\right)^+ = x - \widetilde{K_2}$, and $\left(x - \widetilde{K_1}\right)^+ = x - \widetilde{K_1}$ and since $\widetilde{K_2} > \widetilde{K_1}$, it holds that $\left(x - \widetilde{K_1}\right)^+ \geqslant \left(x - \widetilde{K_2}\right)^+$.

Conditional expectations are monotonous, so we have for all $t \in [0, T]$:

$$\widetilde{C}_t(\widetilde{K_2}) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(\widetilde{S^1_T} - \widetilde{K_2}\right)^+ \mid \mathcal{F}_t\right] \leqslant \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}\left(\widetilde{S^1_T} - \widetilde{K_1}\right)^+ \mid \mathcal{F}_t\right] = \widetilde{C}_t(\widetilde{K_1})$$

Furthermore if we notice that $\left(x - \widetilde{K_1}\right)^+ - \left(x - \widetilde{K_2}\right)^+ \leqslant K_2 - K_1$, taking conditional expectation given $\mathcal{F}_t$:

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\left(\widetilde{S^1_T} - \widetilde{K_1}\right)^+ - \left(\widetilde{S^1_T} - \widetilde{K_2}\right)^+ \mid \mathcal{F}_t\right] &\leqslant e^{-r(T-t)}\left(\widetilde{K_2} - \widetilde{K_1}\right) \\ \widetilde{C}_t(\widetilde{K_1}) - \widetilde{C}_t(\widetilde{K_2}) &\leqslant e^{-r(T-t)}\left(\widetilde{K_2} - \widetilde{K_1}\right) \end{aligned}$$

(ii) **Method 1 :** Let $\lambda \in (0,1)$ and $\widetilde{K_3} = \lambda\widetilde{K_1} + (1-\lambda)\widetilde{K_2}$. Studying the different cases we get:

$$\begin{cases} \left(x - \widetilde{K_3}\right)^+ = 0 < \lambda\left(x - \widetilde{K_1}\right)^+ + (1-\lambda)\left(x - \widetilde{K_2}\right)^+ & \text{for } x \in [0, \widetilde{K_3}] \\ \left(x - \widetilde{K_3}\right)^+ = \lambda\left(x - \widetilde{K_1}\right) + (1-\lambda)\left(x - \widetilde{K_2}\right) < \lambda\left(x - \widetilde{K_1}\right)^+ + (1-\lambda)\left(x - \widetilde{K_2}\right)^+ & \text{for } x \in (\widetilde{K_3}, \widetilde{K_2}) \\ \left(x - \widetilde{K_3}\right)^+ = \left(x - \widetilde{K_3}\right) = \lambda\left(x - \widetilde{K_1}\right)^+ + (1-\lambda)\left(x - \widetilde{K_2}\right)^+ & \text{for } x \in [\widetilde{K_2}, \infty) \end{cases}$$

and then $\left(x - \widetilde{K_3}\right)^+ \leqslant \lambda\left(x - \widetilde{K_1}\right)^+ + (1-\lambda)\left(x - \widetilde{K_2}\right)^+$ for all $x \in \mathbb{R}$. Taking conditional expectation given $\mathcal{F}_t$ with respect to $\mathbb{Q}$ gives:

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\left(\widetilde{S^1_T} - \widetilde{K_3}\right)^+ \mid \mathcal{F}_t\right] &\leqslant e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\lambda\left(\widetilde{S^1_T} - \widetilde{K_1}\right)^+ + (1-\lambda)\left(\widetilde{S^1_T} - \widetilde{K_2}\right)^+ \mid \mathcal{F}_t\right] \\ \widetilde{C}_t(\widetilde{K_3}) &\leqslant \lambda\widetilde{C}_t(\widetilde{K_1}) + (1-\lambda)\widetilde{C}_t(\widetilde{K_2}). \end{aligned}$$

The price of the call at time t is a convex function of the strike.

Method 2 : The function $\widetilde{K} \mapsto (x - \widetilde{K})^+$ is convex, conditional expectations are linear, so the functions $\widetilde{K} \mapsto \mathbb{E}_{\mathbb{Q}}\left[(x - \widetilde{K})^+ \mid \mathcal{F}_t\right]$ are convex for all $t \in [0, T]$, and we have the result.

(d) Let $(V_t)_{t\in[0,T]}$ be the discounted price process of the claim. We compute:

$$\begin{aligned} V_t &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S^1_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right)\right)^3 \mid \mathcal{F}_t\right] \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S^1_0 \exp\left(\sigma\left(W_T + \frac{\mu - r}{\sigma}T\right) - \frac{1}{2}\sigma^2 T + rT\right)\right)^3 \mid \mathcal{F}_t\right] \\ &= e^{2rT}\left(S^1_t\right)^3 \mathbb{E}_{\mathbb{Q}}\left[e^{3\sigma\left(\widetilde{W_T} - \widetilde{W_t}\right) - \frac{3}{2}\sigma^2(T-t)}\right] \\ &= \left(S^1_t\right)^3 e^{3\sigma^2(T-t)+2rT} \end{aligned}$$

We define $u(y,s) = y^3 e^{3s\sigma^2+2rT}$. Then $V_t = u(S_t^1, T-t)$. u is infinitely differentiable on $\mathbb{R}^2$, so we can apply Itô to V:

$$\begin{aligned}
V_t &= V_0 + \int_0^t 3\left(S_t^1\right)^2 e^{3\sigma^2(T-t)+2rT}\,\mathrm{d}S_t^1 \\
&+ \int_0^t \left(-3\sigma^2\left(S_t^1\right)^3 e^{3\sigma^2(T-t)+2rT} + \frac{1}{2}\sigma^2\left(S_t^1\right)^2 6\left(S_t^1\right) e^{3\sigma^2(T-t)+2rT}\right)\mathrm{d}t \\
&= V_0 + \int_0^t 3\left(S_t^1\right)^2 e^{3\sigma^2(T-t)+2rT}\,\mathrm{d}S_t^1.
\end{aligned}$$

We get the initial wealth necessary to replicate the portfolio $V_0 = \left(S_0^1\right)^3 e^{3\sigma^2 T+2rT}$, and the amounts of stock to hold over time: $\theta_t = 3\left(S_t^1\right)^2 e^{3\sigma^2(T-t)+2rT}$.

Question 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, on which we have a Brownian motion $(W_t)_{t \geqslant 0}$. Let $\mathbb{F}$ be the $\mathbb{P}$-augmented filtration generated by W. We consider a Bachelier market model with two assets. The riskless asset is such that $\widetilde{S^0} \equiv 1$ (no interest rate), and the risky asset has the following price process:

$$\widetilde{S^1_t} = S^1_t = S^1_0 + \sigma W_t, \text{ for } t \geqslant 0, \text{ with } \sigma > 0, \text{ and } S^1_0 > 0,$$

so the discounted price process is already a martingale under the historical measure $\mathbb{P}$ which is here the only equivalent martingale measure. The market is therefore complete and arbitrage-free.

We have seen in the lecture what European call and put options are. Gap call and put options are small modifications of these options.

- For $z > K \geqslant 0$ a gap call option with maturity T has terminal payoff: $C^G_{K,z}(S^1_T) = \left(S^1_T - K\right) \mathbb{1}_{\{S^1_T \geqslant z\}}$.
- For $K > z \geqslant 0$ a gap put option with maturity T has terminal payoff: $P^G_{K,z}(S^1_T) = \left(K - S^1_T\right) \mathbb{1}_{\{S^1_T < z\}}$.

a) Let $a > b > c > 0$ with $a - b = b - c$, we consider the claim $h(S^1_T)$ with payoff at time T given by the function h:

$$h(x) = \begin{cases} a - x & \text{for } x \in (-\infty, b] \\ x - c & \text{for } x \in [b, \infty) \, . \end{cases}$$

Express this payoff as a linear combination of a gap call and a gap put.

b) Compute the price process V^{GC} of a gap call with strike K_1 and threshold $z_1 > K_1 \geqslant 0$ and V^P of a gap put with strike K_2 and threshold $0 \leqslant z_2 < K_2$. What is the price process V^S of the contingent claim with payoff $h(S^1_T)$?

c) The price process V^S can be written as:

$$V^S_t = u(S^1_t, T - t), \text{for } t \in [0, T],$$

for some continuous function u. We assume that u satisfies the following partial differential equation:

$$\frac{\partial u}{\partial s}(y, s) = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2}(y, s).$$

Find a replicating strategy for $h(S^1_T)$.

Hint : You can use that the density function ϕ of a standard normal distribution (i.e.$\sim \mathcal{N}(0,1)$) is symmetric : $\phi(x) = \phi(-x)$ for $x \in \mathbb{R}$, and that the cumulative distribution function of a standard normal random variable Φ satisfies : $\Phi(-x) = 1 - \Phi(x)$ for $x \in \mathbb{R}$. Furthermore $\phi'(x) = -x\phi(x)$.

Solution 5. (a) One can rewrite the payoff h as follows:

$$\begin{aligned} h(x) &= (a - x)\mathbb{1}_{\{x < b\}} + (x - c)\mathbb{1}_{\{x \geqslant b\}} \\ &= P^G_{a,b}(x) + C^G_{c,b}(x). \end{aligned}$$

The payoff of the option at time T is then : $P^G_{a,b}(S^1_T) + C^G_{c,b}(S^1_T)$.

(b) The price process of the contingent claim is obtained by taking conditional expectation of the terminal value of the contingent claim under the pricing measure. We get for the gap call:

$$\begin{aligned}
V_t^{GC} &= \mathbb{E}_{\mathbb{P}}\left[\left(S_T^1 - K_1\right)\mathbb{1}_{\{S_T^1 \geqslant z_1\}} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\left(\sigma\left(W_T - W_t\right) + S_t^1 - K_1\right)\mathbb{1}_{\{W_T - W_t \geqslant \frac{1}{\sigma}(z_1 - S_t^1)\}} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\left(\sigma\sqrt{T-t}X + y - K_1\right)\mathbb{1}_{\{X \geqslant \frac{1}{\sigma\sqrt{T-t}}(z_1 - y)\}}\right]\Big|_{y = S_t^1} \\
&= \int_{\frac{1}{\sigma\sqrt{T-t}}(z_1 - S_t^1)}^{\infty}\left(\sigma\sqrt{T-t}x + S_t^1 - K_1\right)\phi(x)\,\mathrm{d}x \\
&= \sigma\sqrt{T-t}\int_{\frac{1}{\sigma\sqrt{T-t}}(z_1 - S_t^1)}^{\infty} x\phi(x)\,\mathrm{d}x + \left(S_t^1 - K_1\right)\left(1 - \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(z_1 - S_t^1\right)\right)\right) \\
&= \sigma\sqrt{T-t}\int_{\frac{1}{\sigma\sqrt{T-t}}(z_1 - S_t^1)}^{\infty} x\phi(x)\,\mathrm{d}x + \left(S_t^1 - K_1\right)\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(S_t^1 - z_1\right)\right) \\
&= \sigma\sqrt{T-t}\phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(z_1 - S_t^1\right)\right) + \left(S_t^1 - K_1\right)\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(S_t^1 - z_1\right)\right),
\end{aligned}$$

where $X \sim \mathcal{N}(0,1)$ is a standard normal random variable, ϕ is the density function of a standard normal distribution, and Φ is its cumulative distribution function. We get similarly for the gap put:

$$\begin{aligned}
V_t^{GP} &= \mathbb{E}_{\mathbb{P}}\left[\left(K_2 - S_T^1\right)\mathbb{1}_{\{S_T^1 < z_2\}} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\left(K_2 - S_t^1 - \sigma\left(W_T - W_t\right)\right)\mathbb{1}_{\{W_T - W_t < \frac{1}{\sigma}(z_2 - S_t^1)\}} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\left(K_2 - y - \sigma\sqrt{T-t}X\right)\mathbb{1}_{\{X < \frac{1}{\sigma\sqrt{T-t}}(z_2 - y)\}}\right]\Big|_{y = S_t^1} \\
&= \int_{-\infty}^{\frac{1}{\sigma\sqrt{T-t}}(z_2 - S_t^1)}\left(K_2 - S_t^1 - \sigma\sqrt{T-t}x\right)\phi(x)\,\mathrm{d}x \\
&= \left(K_2 - S_t^1\right)\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(z_2 - S_t^1\right)\right) - \sigma\sqrt{T-t}\int_{-\infty}^{\frac{1}{\sigma\sqrt{T-t}}(z_2 - S_t^1)} x\phi(x)\,\mathrm{d}x \\
&= \left(K_2 - S_t^1\right)\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(z_2 - S_t^1\right)\right) + \sigma\sqrt{T-t}\phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(z_2 - S_t^1\right)\right),
\end{aligned}$$

where $X \sim \mathcal{N}(0,1)$.

The price process of our contingent claim is the sum of these two prices processes for $K_1 = c$, $K_2 = a$, $z_1 = z_2 = b$. We get:

$$\begin{aligned}
V_t^S &= \left(a - S_t^1\right)\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(b - S_t^1\right)\right) + \sigma\sqrt{T-t}\phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(b - S_t^1\right)\right) \\
&+ \sigma\sqrt{T-t}\phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(S_t^1 - b\right)\right) + \left(S_t^1 - c\right)\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(S_t^1 - b\right)\right) \\
&= \left(S_t^1 - c\right) + \left(a + c - 2S_t^1\right)\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(b - S_t^1\right)\right) + 2\sigma\sqrt{T-t}\phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(S_t^1 - b\right)\right) \\
&= u(S_t^1, T-t).
\end{aligned}$$

(c) The function u defined above is infinitely differentiable on $\mathbb{R} \times [0, T)$, we apply Itô's formula, and with our assumption:

$$
\begin{aligned}
V_T^S &= V_0^S + \int_0^T \frac{\partial u}{\partial y}\left(S_t^1, T-t\right) \mathrm{d}S_t^1 + \int_0^T \left(-\frac{\partial u}{\partial s}\left(S_t^1, T-t\right) + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2}\left(S_t^1, T-t\right)\right) \mathrm{d}t \\
&= V_0^S + \int_0^T \frac{\partial u}{\partial y}\left(S_t^1, T-t\right) \mathrm{d}S_t^1 \\
&= V_0^S + \int_0^T \left(1 - \frac{a+c-2S_t^1}{\sigma\sqrt{T-t}}\phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(b-S_t^1\right)\right) - 2\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(b-S_t^1\right)\right)\right. \\
&\left. -2\frac{1}{\sigma\sqrt{T-t}}\left(S_t^1-b\right)\phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(S_t^1-b\right)\right)\right) \mathrm{d}S_t^1 \\
&= V_0^S + \int_0^T \left(1 - 2\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(b-S_t^1\right)\right)\right) \mathrm{d}S_t^1.
\end{aligned}
$$

The trading strategy is given by

$$
\begin{aligned}
V_0^S &= \left(S_0^1 - c\right) + \left(a+c-2S_0^1\right)\Phi\left(\frac{1}{\sigma\sqrt{T}}\left(b-S_0^1\right)\right) + 2\sigma\sqrt{T}\phi\left(\frac{1}{\sigma\sqrt{T}}\left(S_0^1-b\right)\right), \\
\theta_t &= 1 - 2\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(b-S_t^1\right)\right).
\end{aligned}
$$