ETH Zürich HS 2014
D-MATH
Prof. J. Teichmann **January 2015**

Examination

Mathematical Foundations for Finance

MATH, MScQF, SAV

Please fill in the following table

Last name	
First name	
Programme of study	**MATH** ☐ **MScQF** ☐ **SAV** ☐ **Other** ☐
Matriculation number	

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Question	**Maximum**	**Points**	**Check**
1	8		
2	8		
3	8		
4	8		
5	8		
Total	**40**		

Instructions

Duration of exam ⏲: 180 min.

Closed book examination: no notes, no books, no calculator, etc. allowed.

Important ℹ:

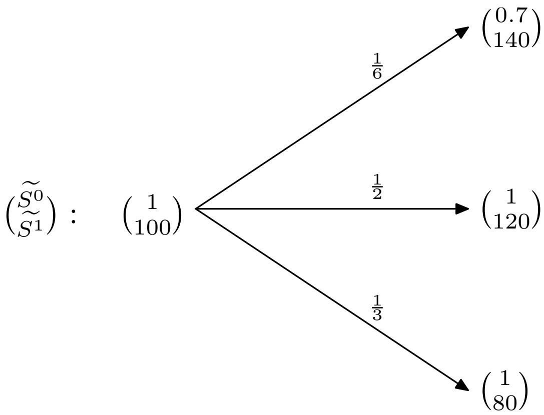
- ☞ Please put your student card on the table.
- ☞ Only pen and paper are allowed on the table. Please do **not** write with a **pencil** or a **red** or **green** pen. Moreover, please do not use **whiteout**.
- ☞ Start by reading all questions and answer the ones which you think are easier first, before proceeding to the ones you expect to be more difficult. Don't spend too much time on one question but try to solve as many questions as possible.
- ☞ Take a new sheet for each question and write your name on every sheet.
- ☞ Unless otherwise stated, all results have to be **explained/argued** by indicating intermediate steps in the respective calculations. You can use known formulas from the lecture or from the exercise classes without derivation.
- ☞ Simplify your results as far as possible.
- ☞ Most of the subquestions can be solved independently of each other.

★★★ Good luck! ★★★

Mathematical Foundations For Finance

Exam

Question 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. We consider the following one period market model with stochastic interest rate for the bank account. The price processes for the bank account $\widetilde{S^0}$ and the risky asset $\widetilde{S^1}$ are given by the following tree:



(a) Find all equivalent martingale measures for the discounted price process $S^1 = \frac{\widetilde{S^1}}{\widetilde{S^0}}$. Is the market arbitrage-free ? Complete ?

(b) (i) Consider a put option on the stock, with maturity $T = 1$ and strike $\widetilde{K} = 100$. Its payoff is $\widetilde{P} = \left(\widetilde{K} - \widetilde{S_1^1}\right)^+$. Is it attainable ?

(ii) Compute a super-replicating strategy for the put defined in (i), with price $\theta_0^0 = \sup_{\mathbb{Q} \in \mathbb{P}_e(S^1)} \mathbb{E}_{\mathbb{Q}} \left[\left(\widetilde{K} - \widetilde{S_1^1}\right)^+ \right]$.

(c) (i) We assume that the put is sold for 12 at time t=0. Show that the market $(\widetilde{S^0}, \widetilde{S^1}, \widetilde{P})$ is complete, where $\widetilde{P}$ is as in b).

(ii) We now consider a call on the stock with maturity $T = 1$ and strike $\widetilde{K} = 100$. Write down the put-call parity and find a super-replicating strategy for the call in the market $(\widetilde{S^0}, \widetilde{S^1}, \widetilde{P})$.

Question 2. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{u, m, d\} \times \{u, d\}$, $\mathcal{F} = 2^\Omega$ and $\mathbb{P}$ defined by $\mathbb{P}(\{(x_1, x_2)\}) = p_{x_1} p_{x_1,x_2}$ for

$$p_u = 0.2, \quad p_m = 0.3, \quad p_d = 0.5,$$
$$p_{u,u} = p_{u,d} = 0.5, \quad p_{m,u} = 0.4, \quad p_{m,d} = 0.6,$$
$$p_{d,u} = 0.75, \quad \text{and} \quad p_{d,d} = 0.25.$$

Now define the random variables Y_1 and Y_2 by

$$Y_1((u, x_2)) = 1 + y_u^1, \quad Y_1((m, x_2)) = 1 + y_m^1, \quad Y_1((d, x_2)) = 1 + y_d^1,$$
$$Y_2((x_1, u)) = 1 + y_u^2 \quad \text{and} \quad Y_2((x_1, d)) = 1 + y_d^2.$$

for $x_1 \in \{u, m, d\}$, $x_2 \in \{u, d\}$, $(y_u^1, y_m^1, y_d^1) = (0.4, 0.2, -0.2)$ and $(y_u^2, y_d^2) = (0.1, -0.2)$.

Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a financial market consisting of a bank account and one stock defined as

$$\widetilde{S}_0^0 = \widetilde{S}_1^0 = \widetilde{S}_2^0 = 1,$$
$$\widetilde{S}_0^1 = 1, \quad \widetilde{S}_1^1 = Y_1 \quad \text{and} \quad \widetilde{S}_2^1 = Y_1 Y_2.$$

Define the discounted price processes: $S^0 \equiv 1$, and $S^1 = \frac{\widetilde{S}^1}{\widetilde{S}^0}$.

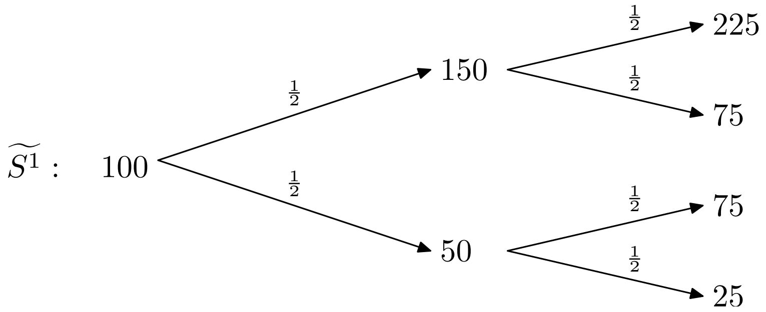
(a) Draw a tree showing the evolution of the asset S^1.

(b) Find the set of equivalent martingale measures for (S^0, S^1).

(c) Define the claim with payoff $H_K = 1_{\{S_2^1 \geq K\}}$ for $K \in \mathbb{R}$ at maturity $T = 2$. For which values of $K \geq 1$ is H_K attainable?

(d) Calculate the super-replication price $\pi_S(H_K)$ of the claim defined above in c) as a function of K for $K \geq 1$.

Question 3. Consider the market $(\widetilde{S}^0, \widetilde{S}^1)$ on some probability space with filtration generated by the assets. Let $\widetilde{S^0_t} = (1+r)^t$ and



for $r = 0.25$. Let $\widetilde{H}_t = (\widetilde{K} - \widetilde{S}^1_t)^+$ with $\widetilde{K} = 150$. Do not forget to explain how you found your solution in the following problems!

(a) Draw a tree of the discounted value process corresponding to the European option V^{Eu} with (undiscounted) payoff $\widetilde{H}_2$ at time $t = 2$.

(b) Draw a tree of the discounted value process corresponding to the American option V^{Am} with maturity 2 and (undiscounted) payoff process $\widetilde{H}$.

Hint : Recall that the discounted value process of an American option with maturity T and with discounted payoff process $(H_k)_{k\in\{0,\ldots T\}}$ is given by the recursive scheme $V_t^{Am} = \max\{H_t, \mathbb{E}_{\mathbb{Q}}\left[V_{t+1}^{Am} \mid \mathcal{F}_t\right]\}$ for $t \in \{0, \ldots T-1\}$, $\mathbb{Q}$ an equivalent martingale measure for the market, and $V_T^{Am} = H_T$.

(c) Determine whether there exist self-financing portfolios φ^{Am} and φ^{Eu} such that $V_t(\varphi^{\mathrm{Am}}) = V_t^{\mathrm{Am}}$ and $V_t(\varphi^{\mathrm{Eu}}) = V_t^{\mathrm{Eu}}$ for all $t \in \{0, 1, 2\}$. Find them if they exist.

Question 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which exists a Brownian motion $(W_t)_{t \geqslant 0}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ be the $\mathbb{P}$-augmented filtration generated by W.

We consider a market model with two assets whose price processes are the following:

$$\begin{cases} \widetilde{S^0_t} &= e^{rt}, \\ \widetilde{S^1_t} &= S^1_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right), \end{cases}$$

for $t \geqslant 0$, with $\mu \in \mathbb{R}$, $\sigma > 0$ and $S^1_0 > 0$, $r > 0$.

(a) What is the probability under the historical probability measure $\mathbb{P}$ that a call on the risky asset $\widetilde{S^1}$, with strike $\widetilde{K}$ and maturity T is exercised ?

(b) There exists a unique probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ such that the process $S^1 = \frac{\widetilde{S^1}}{\widetilde{S^0}}$ is a $\mathbb{Q}$-martingale on $[0, T]$. Give the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}_{|\mathcal{F}_T}$.

(c) We now consider two calls with same maturity T and strike $\widetilde{K_1}$ and $\widetilde{K_2}$, with $\widetilde{K_1} < \widetilde{K_2}$. Let $\widetilde{C}(\widetilde{K_1})$ and $\widetilde{C}(\widetilde{K_2})$ be the undiscounted price processes of these claims.

(i) Show the following relations:

$$\widetilde{C}_t(\widetilde{K_2}) \leqslant \widetilde{C}_t(\widetilde{K_1}), \text{ for } t \in [0, T]$$

$$\widetilde{C}_t(\widetilde{K_1}) - \widetilde{C}_t(\widetilde{K_2}) \leqslant \frac{\widetilde{K_2} - \widetilde{K_1}}{e^{r(T-t)}}, \text{ for } t \in [0, T]$$

(ii) Let $\widetilde{K_3} = \lambda \widetilde{K_1} + (1 - \lambda)\widetilde{K_2}$ for $\lambda \in [0, 1]$. Show that $\widetilde{C}_t(\widetilde{K_3}) \leqslant \lambda \widetilde{C}_t(\widetilde{K_1}) + (1 - \lambda)\widetilde{C}_t(\widetilde{K_2})$.

(d) Let us a consider a power option that pays $\left(\widetilde{S^1_T}\right)^p$ at maturity T, for $p = 3$. Compute the discounted price process and the replicating strategy for this option.

Question 5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, on which we have a Brownian motion $(W_t)_{t \geqslant 0}$. Let $\mathbb{F}$ be the $\mathbb{P}$-augmented filtration generated by W. We consider a Bachelier market model with two assets. The riskless asset is such that $\widetilde{S^0} \equiv 1$ (no interest rate), and the risky asset has the following price process:

$$\widetilde{S_t^1} = S_t^1 = S_0^1 + \sigma W_t, \text{ for } t \geqslant 0, \text{ with } \sigma > 0, \text{ and } S_0^1 > 0,$$

so the discounted price process is already a martingale under the historical measure $\mathbb{P}$ which is here the only equivalent martingale measure. The market is therefore complete and arbitrage-free.

We have seen in the lecture what European call and put options are. Gap call and put options are small modifications of these options.

- For $z > K \geqslant 0$ a gap call option with maturity T has terminal payoff: $C_{K,z}^G(S_T^1) = \left(S_T^1 - K\right) \mathbb{1}_{\{S_T^1 \geqslant z\}}$.
- For $K > z \geqslant 0$ a gap put option with maturity T has terminal payoff: $P_{K,z}^G(S_T^1) = \left(K - S_T^1\right) \mathbb{1}_{\{S_T^1 < z\}}$.

a) Let $a > b > c > 0$ with $a - b = b - c$, we consider the claim $h(S_T^1)$ with payoff at time T given by the function h:

$$h(x) = \begin{cases} a - x & \text{for } x \in (-\infty, b] \\ x - c & \text{for } x \in [b, \infty) \ . \end{cases}$$

Express this payoff as a linear combination of a gap call and a gap put.

b) Compute the price process V^{GC} of a gap call with strike K_1 and threshold $z_1 > K_1 \geqslant 0$ and V^P of a gap put with strike K_2 and threshold $0 \leqslant z_2 < K_2$. What is the price process V^S of the contingent claim with payoff $h(S_T^1)$?

c) The price process V^S can be written as:

$$V_t^S = u(S_t^1, T - t), \text{for } t \in [0, T],$$

for some continuous function u. We assume that u satisfies the following partial differential equation:

$$\frac{\partial u}{\partial s}(y, s) = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2}(y, s).$$

Find a replicating strategy for $h(S_T^1)$.

Hint : You can use that the density function ϕ of a standard normal distribution (i.e.$\sim \mathcal{N}(0, 1)$) is symmetric : $\phi(x) = \phi(-x)$ for $x \in \mathbb{R}$, and that the cumulative distribution function of a standard normal random variable Φ satisfies : $\Phi(-x) = 1 - \Phi(x)$ for $x \in \mathbb{R}$. Furthermore $\phi'(x) = -x\phi(x)$.