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11 August 2018

Quantitative Risk Management

Important:

- Put your student card on the table
- Begin each exercise on a new sheet of paper, and write your name on each sheet
- Only pen and paper are allowed

Please fill in the following table.

Last name	
First name	
Student number (if available)	

Please do <u>not</u> fill in the following table.

Question	**Points**	**Control**	**Maximum**
#1			10
#2			8
#3			10
#4			12
#5			10
Total			50

Question 1 (10 Pts)

a) Define the notion of a coherent risk measure, and give a financial interpretation of each axiom of coherence. (4 Pts)

b) Let $X \sim \mathrm{Par}(\theta, \kappa)$ with cdf

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^{\theta}, \quad x \geq 0,$$

for parameters $\kappa > 0$ and $\theta > 1$. Calculate $\mathrm{VaR}_\alpha(X)$ and $\mathrm{AVaR}_\alpha(X)$. (3 Pts)

c) Let L be a d-dimensional random vector whose components $L_1, \ldots, L_d$ are normally distributed with means $\mu_1, \ldots, \mu_d \in \mathbb{R}$ and variances $\sigma_1^2, \ldots, \sigma_d^2 > 0$. Fix a level $\alpha \in (1/2, 1)$. Is $\mathrm{VaR}_\alpha(L_1 + \cdots + L_d)$ larger if the copula of the random vector L is the independence copula or the comonotonicity copula? (3 Pts)

Question 2 (8 Pts)

a) Is every normal variance mixture distribution elliptical? Explain your answer. (4 Pts)

b) Let d financial returns be modeled by the components $X_1, \ldots, X_d$ of a d-dimensional random vector X. Assume X has an elliptical distribution such that $\mathbb{E}[X_i^2] < \infty$ for all $i = 1, \ldots, d$, and $\mathbb{E}[X_1] = \cdots = \mathbb{E}[X_d]$. We want to show that the minimum variance portfolio also minimizes Value-at-Risk. More precisely, denote

$$\Delta := \left\{ w \in \mathbb{R}^d : \sum_{i=1}^{d} w_i = 1 \right\}$$

and fix a level $\alpha \in (1/2, 1)$. Then show that the two minimization problems

$$\min_{w \in \Delta} \mathrm{Var}\left(\sum_{i=1}^{d} w_i X_i\right) \quad \text{and} \quad \min_{w \in \Delta} \mathrm{VaR}_\alpha\left(-\sum_{i=1}^{d} w_i X_i\right)$$

have the same minimizer $w^* \in \Delta$. (4 Pts)

Question 3 (10 Pts)

a) Let (X, Y) be a two-dimensional random vector with $\mathrm{Exp}(1)$-marginals and copula

$$C(u, v) = uv + (1 - u)(1 - v)uv.$$

Does (X, Y) have a density? If yes, can you compute it? (3 Pts)

b) Calculate Spearman's rank correlation between X and Y given in a). (2 Pts)

c) Calculate the coefficient of upper tail dependence λ_u between X and Y given in a). (2 Pts)

d) Let (X, Y) be a two dimensional random vector with cdf

$$\frac{1 - e^{-x} - e^{-y} + e^{-x-y}}{1 - e^{-x-y}}$$

on $\mathbb{R}_+^2$. What are the marginal distributions and copula of (X, Y)? (3 Pts)

Question 4 (12 Pts)

Let X be a non-negative random random variable with cdf

$$F_X(x) = \frac{x}{x+1}, \quad x \geq 0.$$

a) Does X have a density? If yes, can you derive it? (2 Pts)

b) Find all $k \in \mathbb{N}$ such that $\mathbb{E}[|X|^k] < \infty$. (2 Pts)

c) Does F_X belong to $\mathrm{MDA}(H_\xi)$ for a generalized extreme value distribution H_ξ? If yes, what is H_ξ and what are the normalizing sequences? (3 Pts)

d) Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x \mid X > u]$, $x \geq 0$. (2 Pts)

e) Does there exist a parameter $\xi \in \mathbb{R}$ and a function β such that

$$\lim_{u \to \infty} \sup_{x > 0} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0$$

for a generalized Pareto distribution $G_{\xi,\beta}$? If yes, for which ξ and β does this hold? (3 Pts)

Question 5 (10 Pts)

a) Write down the specification of a GARCH(1,1) model. (2 Pts)

b) Which stylized facts of daily log-returns can a GARCH(1,1) model capture? (2 Pts)

c) Let the distribution of a d-dimensional random vector X be given by univariate marginal cdf's $F_1, \ldots, F_d$ and a Gaussian copula C_P^{Ga}. Describe an algorithm for simulating X. Justify your approach. (3 Pts)

d) Describe the Peaks-Over-Threshold method. (3 Pts)