D–MATH

Exam Quantitative Risk Management

401-3629-00S

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Problems

1. Risk & risk measures [11 Points]

(a) **[3 Points]**
Let X be a random variable with a cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 - x^{-\theta} & \text{if} \quad x \geq 1 \end{cases}$$

for a parameter $\theta > 1$. Calculate $\mathrm{VaR}_\alpha(X)$ and $\mathrm{ES}_\alpha(X)$ for $\alpha \in (0,1)$.

(b) **[3 Points]**
Let X_1, X_2 be independent and identically distributed random variables taking the values 100 with probability $p \in (0,1)$ and 0 with probability $1-p$. For which $\alpha \in (0,1)$ does one have $\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$? Explain your answer.

(c) **[3 Points]**
Consider two stocks, A and B, with current values 500 and 200, respectively. The monthly log-returns of the two stocks in % over the last 5 months are given in the following table:

lag k	5	4	3	2	1
log-return of A at lag k in %	10	-5	-3	15	4
log-return of B at lag k in %	12	-10	1	10	2

Use historical simulation to estimate $\mathrm{VaR}_{0.8}$ of the linearized loss L^Δ of a portfolio consisting of one share of A and two shares of B over the next month.

(d) **[2 Points]**
Let X and Y be two random variables on the same probability space such that

$$X \sim \text{Log-Norm}(\mu_1, \sigma_1^2) \quad \text{and} \quad Y \sim \text{Log-Norm}(\mu_2, \sigma_2^2).$$

Is it true that $XY \sim \text{Log-Norm}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$? Explain your answer.

2. Multivariate models [9 Points]

Suppose the losses $L_1, \ldots, L_d$ of d financial assets are described by the factor model

$$L_i = \rho_i Z_0 + \sqrt{1 - \rho_i^2} Z_i, \quad i = 1, \ldots, d,$$

where $Z_0, Z_1, \ldots, Z_d$ are i.i.d. $N(0, \sigma^2)$-distributed random variables and $\rho_i \in (0,1)$.

(a) **[2 Points]**
Determine the distribution of the random vector $L = (L_1, \ldots, L_d)$.

(b) **[2 Points]**
Calculate $\mathrm{VaR}_\alpha(L_i)$, $i = 1, \ldots, d$.

(c) **[2 Points]**
Calculate $\mathrm{VaR}_\alpha\left(\frac{1}{d}\sum_{i=1}^d L_i\right)$.

(d) **[3 Points]**
Does investing capital equally in the d financial assets decrease the portfolio's VaR compared to investing the whole capital in one asset? Explain your answer using the results from b) and c).

3. Extreme value theory **[10 Points]**

Let X be a random variable with a cumulative distribution function

$$F(x) = \begin{cases} \dfrac{3x+1}{3x+2} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

(a) **[1 Point]**
Does X have a density? If yes, can you derive it?

(b) **[1 Point]**
Find all $k \in \mathbb{N} = \{1, 2, \dots\}$ such that $\mathbb{E}[|X|^k] < \infty$.

(c) **[3 Points]**
Does F belong to $\mathrm{MDA}(H_\xi)$ for a standard generalized extreme value distribution H_ξ? If yes, what is ξ and what are the normalizing sequences?

(d) **[2 Points]**
Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x \mid X > u]$, $x \geq 0$.

(e) **[3 Points]**
Does there exist a parameter $\xi \in \mathbb{R}$ and a function β such that

$$\lim_{u\to\infty} \sup_{x>0} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0,$$

where $G_{\xi,\beta}$ denotes the cumulative distribution function of a generalized Pareto distribution? If yes, for which ξ and β does this hold?

4. Copulas & dependence **[10 Points]**

(a) **[2 Points]**
Compute the upper tail dependence coefficient λ_u of the two-dimensional copula

$$C(u,v) = 1 - \left((1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta\right)^{1/\theta}, \quad u, v \in (0,1),$$

for $\theta \in [1, \infty)$.

(b) **[3 Points]**
Let (X, Y) be a two-dimensional random vector with Exp(1)-marginals and copula $C(u, v)$ given in a). Compute the cumulative distribution function of (X, Y).

(c) **[5 Points]**
Let (X, Y) be a two-dimensional random vector with a cumulative distribution function

$$F(x, y) = \frac{x^2}{\sqrt{x^4(1 + e^{-y})^2 + 1 + 2x^2}}$$

defined on $\mathbb{R}^+ \times \mathbb{R}$. Compute the marginal distributions and the copula of (X, Y).

5. Practical questions **[10 Points]**

(a) **[3 Points]**
Describe how one can test univariate distributions with graphical tests.

(b) **[3 Points]**
Describe what a p-factor model is.

(c) **[4 Points]**
Name advantages and disadvantages of the multivariate normal distribution as a model for financial log-returns.