D–MATH

Exam Quantitative Risk Management

401-3629-00L

Please do not turn the page yet!

Question 1 (10 Pts)

a) Let X be a random variable with cdf

$$F(x) = \begin{cases} 1 - e^{-\lambda\sqrt{x}} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

for a parameter $\lambda > 0$. Calculate $\mathrm{VaR}_\alpha(X)$ and $\mathrm{AVaR}_\alpha(X)$ for $\alpha \in (0,1)$. (4 Pts)

b) Let X be a random variable with a continuous cdf. Show that

$$\mathrm{ES}_\alpha(X) = \frac{\mathbb{E}[X] + \alpha \mathrm{ES}_{(1-\alpha)}(-X)}{1-\alpha}$$

for all $\alpha \in (0,1)$. (2 Pts)

c) Denote by $\mathcal{L}$ the set of all random variables X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}\left[e^{\lambda X}\right] < \infty$ for every $\lambda \in \mathbb{R}_+$. Define the risk measure $\rho\colon \mathcal{L} \to \mathbb{R}$ by

$$\rho(X) = \log \mathbb{E}\left[e^X\right].$$

Which properties of a coherent risk measure does ρ have? Explain your answers.

(4 Pts)

Question 2 (10 Pts)

a) Consider a d-dimensional random vector $X = (X_1, \dots, X_d) \sim S_d(\psi)$ for some $d \geq 1$. Show that the components $X_1, \dots, X_d$ of X are symmetrically distributed. (2 Pts)

b) Consider a d-dimensional random vector $X = (X_1, \dots, X_d) \sim M_d(\mu, \Sigma, \hat{F}_W)$ for some $d \geq 1$ and a positive definite $d \times d$-matrix Σ. Let $a \in \mathbb{R}^d$ and suppose that $\mathbb{E}|X_i| < \infty$, for all $i = 1, \dots, d$. Can you derive a formula for $\mathrm{ES}_\alpha \left(a^T X\right)$ in terms of $\mathrm{ES}_\alpha \left(X_1\right)$?

(5 Pts)

c) Consider a d-dimensional random vector $X = (X_1, \dots, X_d) \sim E_d(\mu, \Sigma, \psi)$ for some $d \geq 1$ and a positive semidefinite $d \times d$-matrix Σ. Show that for every bounded measurable function $f \colon \mathbb{R} \to \mathbb{R}$ there exists a function $F \colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\mathbb{E}\left[f(m + \lambda^T X)\right] = F(m + \lambda^T \mu, \lambda^T \Sigma \lambda)$$

for all $m \in \mathbb{R}$ and $\lambda \in \mathbb{R}^d$. (3 Pts)

Question 3 (9 Pts)
Let X be a random variable with cdf

$$F(x) = 1 - \frac{1}{(x+2)^5}, \quad x \geq -1.$$

a) Does X have a density? If yes, can you derive it? (1 Pts)

b) Find all $k \in \mathbb{N} = \{1, 2, \dots\}$ such that $\mathbb{E}[|X|^k] < \infty$. (1 Pts)

c) Does F belong to MDA(H_ξ) for a standard generalized extreme value distribution H_ξ? If yes, what is ξ and what are the normalizing sequences? (3 Pts)

d) Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x \mid X > u]$, $x \geq 0$. (2 Pts)

e) Does there exist a parameter $\xi \in \mathbb{R}$ and a function β such that

$$\lim_{u \to \infty} \sup_{x > 0} |F_u(x) - G_{\xi, \beta(u)}(x)| = 0,$$

where $G_{\xi,\beta}$ denotes the cdf of a generalized Pareto distribution? If yes, for which ξ and β does this hold? (2 Pts)

Question 4 (11 Pts)

a) Compute the lower tail dependence coefficient λ_l of the two-dimensional copula

$$C(u,v) = \frac{1}{\theta} \log\left(1 + \frac{(e^{\theta u} - 1)(e^{\theta v} - 1)}{e^{\theta} - 1}\right), \quad u, v \in (0,1),$$

for $\theta \in \mathbb{R}\backslash\{0\}$. (2 Pts)

b) Is $C(u,v)$ given in a) still a copula for $\theta \to -\infty$ and $\theta \to \infty$? Justify your answers. (4 Pts)

c) Let (X, Y) be a two dimensional random vector with cdf

$$F(x,y) = \frac{1}{1 + e^{-x} + e^{-y}}$$

defined on $\mathbb{R}^2$. Compute the marginal distributions and the copula of (X, Y). (5 Pts)

Question 5 (10 Pts)

a) How can one test multivariate normality? (2 Pts)

b) Describe how one can estimate risk measures with a GARCH(1,1) model. (2 Pts)

c) Name advantages and disadvantages of Archimedean copulas. (3 Pts)

d) Name different operational risk categories. (3 Pts)