Stochastik - Lösungsskizze (BSc D-MAVT / BSc D-MATH / BSc D-MATL)

1. a) 2. **b)** 1. **c)** 3. **d)** 1. **e)** 2. **f)** 2. **g)** 3. **h)** 3. **i)** 3. **j)** 1.

2. Wir bezeichnen mit C bzw. V das Ereignis, dass ein Kunde Schokoladen- bzw. Vanilleeis in seinen Becher hinzufügt. Sei W das Ereignis, dass ein Kunde den Eisverkäufer weiterempfielt. Dann gilt $P(C) = P(V) = 0.6$ und $P(W|C) = 0.7$ bzw. $P(W^c|C^c) = 0.1$ aus der Aufgabenstellung.

a) Mit dem Satz der totalen W'keit und der Rechenregel für Komplenentärereignisse erhalten wir

$$\begin{aligned} P(W) &= P(W|C)P(C) + P(W|C^c)P(C^c) \\ &= 0.7 \cdot 0.6 + (1-0.1) \cdot (1-0.6) = 0.78. \end{aligned}$$

b) Mit dem Satz von Bayes' und dem Ergebnis aus a) berechnen wir die gesuchte Wahrscheinlichkeit als

$$P(C|W) = \frac{P(W|C)P(C)}{P(W)} = \frac{0.7 \cdot 0.6}{0.78} = 0.538.$$

c) We get by the pairwise independence assumption

$$P[C \cup V] = P[C] + P[V] - P[C \cap V] = P[C] + P[V] - P[C]P[V] = 2p - p^2 = p(2-p).$$

Now we calculate the desired probability as

$$P[X = k] = (1 - P[C \cup V])^{k-1} P[C \cup V] = (1 - 2p + p^2)^{k-1} p(2-p)$$

where X is defined in d).

d) Clearly the distribution of X calculated in part **c)** is a geometric distribution with success parameter $P[C \cup V]$. Therefore, its expectation is given by $E[X] = 1/P[C \cup V] = 1/[p(2-p)]$. Knowing that $E[X] = \frac{4}{3}$ amounts to solving $p^2 - 2p + \frac{3}{4} = 0$ for p, which we can rewrite as $(2p-1)(2p-3) = 0$ to read off $p = \frac{1}{2}$ as the only solution in $(0, 1)$.

3. a) As a density function $f_{X,Y}$ must satisfy $\iint f_{X,Y}(x,y)\,dxdy = 1$. Direct calculation then gives us

$$\iint f_{X,Y}(x,y)\,dxdy = \int_0^1 \left(\int_0^{1-y} cxy\,dx\right) dy = c\int_0^1 \frac{y(1-y)^2}{2}\,dy$$
$$= \frac{c}{2}\int_0^1 y - 2y^2 + y^3\,dy = \frac{c}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{c}{24}$$

We thus obtain $c = 24$.

b) A simple symmetry argument shows that the wanted probability is 1/2. However the calculations go as follows:

$$P[Y > X] = \int_0^{1/2}\left(\int_x^{1-x} cxy\,dy\right) dx = \int_0^{1/2} \frac{cx}{2}((1-x)^2 - x^2)\,dx$$
$$= \frac{c}{2}\int_0^{1/2} x - 2x^2\,dx = \frac{c}{2}\left(\frac{1}{8} - \frac{1}{12}\right) = \frac{c}{48} = \frac{1}{2}.$$

c) We first calculate f_X. Suppose $0 \le x \le 1$, we have

$$f_X(x) = \int_0^{1-x} cxy\,dy = \frac{c(x - 2x^2 + x^3)}{2} = 12(x - 2x^2 + x^3).$$

Clearly otherwise $f_X(x) = 0$. By a symmetry argument, we have $f_Y(y) = 12(y - 2y^2 + y^3)\mathbb{1}_{[0,1]}(y)$.

d) Since clearly $f_{X,Y} \neq f_X f_Y$ given the calculations of part **c)**, X and Y are not independent.

e) Für $0 \geq x$, $0 \geq y$, $x + y \leq 1$ ist die bedingte Dichte von Y gegeben $X = x$ definiert als

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

(und für alle anderen x, y ist sie 0).

Für fixes $x \in (0,1)$ kann der bedingte Erwartungswert von Y gegeben $X = x$ somit berechnet werden als

$$E[Y \mid X = x] = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy = \frac{cx}{f_X(x)}\int_0^{1-x} y^2 dy$$
$$= \frac{cx(1-x)^3}{3f_X(x)} = \frac{2}{3}(1-x).$$

Somit erhalten wir $E[Y \mid X = \frac{1}{3}] = \frac{4}{9}$.

4. a) By assumptions X_i, $i = 1, \dots, n$ has a binomial distribution $\mathrm{Bin}(m,p)$ with parameters $m = 3$, and p. Therefore, the Likelihood function is given by

$$L(p; x_i) = \prod_{i=1}^{n} \binom{3}{x_i} p^{x_i}(1-p)^{3-x_i}.$$

This yields the following log-Likelihood,

$$\ell(p; x_i) = \sum_{i=1}^{n} \left[\log \binom{3}{x_i} + x_i \log(p) + (3 - x_i) \log(1 - p) \right].$$

Differentiating it with respect to p and setting it equal to 0, we have

$$\frac{\partial \ell}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{3n - \sum_{i=1}^{n} x_i}{1 - p} = 0,$$

which leads to

$$\widehat{p} = \frac{\sum_{i=1}^{n} X_i}{3n}.$$

The data implies the following values for $n = 5$ and $\sum_{i=1}^{5} x_i = 7$. Therefore, the MLE estimate for p is $7/15$.

b) Given the true value of p, the probability of Super Mario being able to throw fire balls is given by

$$P[X_i \geq 2] = P[X_i = 2] + P[X_i = 3] = 3p^2(1 - p) + p^3.$$

Assuming then $p = 7/15$ as calculated in a), we have $P[X_i \geq 2] \approx 0.450$.

c) We note that given the independence assumption between the mushroom counts of the different games, the test statistics $Y = \sum_{i=1}^{5} X_i$ has a binomial distribution $\text{Bin}(m, p)$ with parameters $m = 15$, and p. Then

- the setup of the test is given by the null hypothesis $H_0 : p = p_0$ and the alternate hypothesis $H_A : p > p_0$.
- As on page 78 in the Script, to do a one-sided Binomial-test, we would like to find the smallest c such that $P_{p=p_0}(Y \geq c) \leq \alpha$. The table shows that $P_{p_0}(Y \geq 8) = 0.018$ and $P_{p_0}(Y \geq 7) = 0.057$ so that we select $c = 8$ and Verwerfungsbereich $K = \{8, 9, 10, 11, 12, 13, 14, 15\}$.
- As the realized value for Y is $7 \notin K$, the test does not reject the null hypothesis, i.e. we cannot conclude that the true value of p should be larger than what Nintendo claims.

d) Die Macht des Tests ist definiert als $P_{p=0.5}(Y \in K)$ und kann damit direkt aus der Tabelle abgelesen werden:

$$P_{p=0.5}(Y \in K) = P_{p=0.5}(Y \geq 8) = \sum_{i=8}^{15} P(X = i) = 0.5$$

für $X \sim \text{Bin}(15, 0.5)$ oder direkt als $1 - P(X \leq 7) = 1 - \frac{1}{2} = \frac{1}{2}$.