Musterlösung

Wahrscheinlichkeitstheorie und Statistik (BSc, 2.VD D-ITET)

1. Sei X die Anzahl der weissen Kugeln, die aus A gezogen wurden.

a) Es gilt

$$\begin{aligned} P[Y=2] &= P[X=0,Y=2]+P[X=1,Y=2]+P[X=2,Y=2] \\ &= \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}}\cdot 0 + \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}}\cdot\frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} + \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}}\cdot\frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} \\ &= \frac{6\cdot 1+1\cdot 3}{100} = \frac{9}{100}. \end{aligned}$$

b) Nach Bayes ist für k=0,1,2

$$p_k := P[X=k\,|\,Y=2] = \frac{P[X=k]P[Y=2\,|\,X=k]}{P[Y=2]},$$

also

$$\begin{aligned} p_0 &= 0, \\ p_1 &= \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}}\cdot\frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}}\cdot\frac{1}{\frac{9}{100}} = \frac{\frac{6}{100}}{\frac{9}{100}} = \frac{2}{3}, \\ p_2 &= 1-p_0-p_1 = \frac{1}{3}. \end{aligned}$$

c) Es gilt

$$\begin{aligned} P[Y=1] &= P[X=0,Y=1]+P[X=1,Y=1]+P[X=2,Y=1] \\ &= \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}}\cdot\frac{\binom{1}{1}\binom{4}{1}}{\binom{5}{2}} + \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}}\cdot\frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} + \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}}\cdot\frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} \\ &= \frac{3\cdot 4+6\cdot 6+1\cdot 6}{100} = \frac{54}{100} \end{aligned}$$

und somit

$$E[Y] = 1\cdot P[Y=1]+2\cdot P[Y=2] = \frac{54+18}{100} = \frac{72}{100}.$$

2. a) Es muss gelten

$$1 = \int_{-\infty}^{\infty} f_Y(x)dx = c + c\left[-\frac{2}{x}\right]_1^{\infty} = c+2c,$$

also $c=\frac{1}{3}$.

Bitte wenden!

b) Die Verteilungsfunktion von V ist

$$F_V(x) = \begin{cases} 0 & (x < 0) \\ \frac{1}{3}x & (x \in [0,1]) \\ \frac{1}{3} + \frac{2}{3}\left(1 - \frac{1}{x}\right) & (x > 1), \end{cases}$$

also

$$P[L \le y] = P[V \le y^3] = \begin{cases} 0 & (y < 0) \\ \frac{1}{3}y^3 & (y \in [0,1]) \\ 1 - \frac{2}{3y^3} & (y > 1), \end{cases}$$

und somit ist die Dichte von L

$$f_L(y) = \frac{d}{dy}P[L \le y] = \begin{cases} 0 & (y < 0) \\ y^2 & (y \in [0,1]) \\ 2y^{-4} & (y > 1), \end{cases}$$

und der Erwartungswert

$$E[L] = \int_0^\infty y f_L(y) dy = \left[\frac{1}{4}y^4\right]_0^1 + 4\left[-y^{-2}\right]_1^\infty = \frac{1}{4} + 1 = \frac{5}{4}.$$

c) Es ist

$$P[Z_1 \le V \le Z_2] = P\left[\frac{1}{4}L \le L^3 \le 4L\right] = P\left[\frac{1}{4} \le L^2 \le 4\right] = P\left[\frac{1}{2} \le L \le 2\right]$$

$$= 1 - \frac{2}{3}\cdot\frac{1}{8} - \frac{1}{3}\cdot\frac{1}{8} = \frac{21}{24} = \frac{7}{8}.$$

d) Es gilt

$$E[V] = \int_0^1 x\frac{1}{3}\,dx + \int_1^\infty x\frac{2}{3x^2}\,dx = \infty,$$

also existiert der Erwartungswert von V nicht.

3. a)

$$\begin{aligned} E[X] &= 2E[V] - E[U] = 2 \times \frac{1}{3} - 2 = -\frac{4}{3} \\ Var[X] &= 2^2 Var[V] + (-1)^2 Var[U] = 4\frac{1}{3^2} + \frac{2^2}{12} = \frac{7}{9}. \end{aligned}$$

b)

$$Cov[V,Y] = E[VY] - E[V]E[Y] = E[V^2]E[U] - E[V]^2E[U] = E[U]Var[V] = 2\frac{1}{9} = \frac{2}{9}.$$

c)

$$\begin{aligned} P[Z \le 1/2] &= P[V \le U/2] = \int_1^3 \int_0^{u/2} 3e^{-3v} dv\,\frac{1}{2}du = \int_1^3 P[V \le u/2]\frac{1}{2}du \\ &= \int_1^3 (1 - e^{-\frac{3}{2}u})\frac{1}{2}du = 1 + \frac{1}{2}\left(\frac{2}{3}e^{-\frac{3}{2}u}\Big|_1^3\right) = 1 - \frac{1}{3}\left(e^{-\frac{3}{2}} - e^{-\frac{9}{2}}\right) \end{aligned}$$