Probability and Statistics
FS 2019
Session Exam
16.08.2019
Time Limit: 180 Minutes

Name: ______________________________

Student ID: ______________________________

This exam contains 11 pages (including this cover page) and 10 problems. A formulae sheet is provided with the exam.

Grade Table (for grading use only, please leave empty)

Question	Points	Score
1	8	
2	10	
3	8	
4	10	
5	12	
6	10	
7	10	
8	9	
9	11	
10	12	
Total:	100	

1. (8 points) An urn contains 6 distinct balls, of which 2 are red, 2 are white and 2 are black. You take 3 balls out of the urn, at random and without replacement.

   (a) (3 points) Compute the probability that the selected balls have three different colors.

   (b) (3 points) Compute the probability that the selected balls are of exactly 2 different colors.

   (c) (2 points) Compute the probability that none of the selected balls is white.

2. (10 points) 3 friends are sitting at a café. Each of them has a coin, with the following properties: for some $p \in (0,1)$,

   - The coin of friend 1 shows heads with probability p.
   - The coin of friend 2 also shows heads with probability p.
   - The coin of friend 3 shows heads with probability $1-p$.

   Each of the friends throws his or her coin. We assume that the outcomes of the tosses are independent.

   (a) (3 points) What is the probability that one of the friends is the "odd one out", i.e. his/her coin shows a different outcome than the other two?

   (b) (3 points) For what value of p is the probability you computed in (a) minimal? Does this result make sense?

   (c) (4 points) Take $p = \frac{1}{2}$. The three friends play the following game: the odd one out will pay for the drinks. If nobody is the odd one out, then they will toss their coins again and again until a decision can be made.

   What is the probability that they need at least 4 rounds of tosses to know who is going to pay?

3. (8 points) Among students who attended a Probability and Statistics course, some were happy and some were not. Moreover, some dropped out of the course before the end.

   For a randomly chosen student S, consider the events:

$$\begin{aligned} L &= \{\text{the student } S \text{ liked the course}\} \\ L^c &= \{\text{the student } S \text{ did not like the course}\} \\ D &= \{\text{the student } S \text{ dropped the course early}\} \\ D^c &= \{\text{the student } S \text{ attended the course until the end of the semester}\}. \end{aligned}$$

   You are given the following information:

$$\begin{aligned} P(D) &= 0.20, \\ P(L \mid D^c) &= 0.90, \\ P(L^c \mid D) &= 0.80. \end{aligned}$$

   (a) (3 points) What is the probability that the student S liked the course?

   (b) (3 points) You meet Maria in the hall and ask her whether she liked the course. Given that the answer is "yes", what is the conditional probability that she attended until the end of the semester?

   (c) (2 points) Simon did not like the course. What is the conditional probability that he dropped the course early?

4. (10 points) You are browsing mathematics textbooks by two publishers, publisher A and publisher B.

   (a) (2 points) You pick up a book from publisher A. You know that books from publisher A have, on average, 1 typographical error per 50 pages.

   What is an appropriate model for the number of errors in the book you picked up? Specify the parameter of this model.

   (b) (3 points) If this book has 200 pages, give the probability that it contains at least 2 errors.

   (c) (2 points) Give the probability that it contains no errors in its last 40 pages.

   (d) (3 points) Maths books from publisher B contain on average 1 typographical error per 80 pages. Along with your 200 page book from publisher A, you also pick up a 400 page book from publisher B. The numbers of errors in these two books are independent.

   What is the probability that these two books (combined) have strictly fewer than 4 errors?

5. (12 points) A random vector $(X, Y)^T$ takes values in $\mathbb{N}_0 \times \mathbb{N}_0$, and has joint pmf given by

$$p(x, y) = \frac{c}{2^{\max(x,y)}}, \quad (x, y) \in \mathbb{N}_0 \times \mathbb{N}_0$$

for some $c > 0$.

(a) (3 points) Show that

$$\sum_{x \in \mathbb{N}_0, y \in \mathbb{N}_0} \frac{1}{2^{\max(x,y)}} = 2\left(1 + \sum_{y=1}^{\infty} \frac{y}{2^y}\right).$$

Hint. Consider the cases $x \neq y$, $x = y$ separately.

(b) (3 points) Let $Z \sim \text{Geo}(p)$ for some $p \in (0, 1)$. Using standard results on convergence and differentiation of power series, show that

$$E(Z) = 1/p.$$

Deduce the value of the normalising constant c, by taking $p = 1/2$.

Remark. You **may not** quote from the formulae sheet for this part!

(c) (3 points) Find the marginal pmf of X and that of Y. Are X and Y independent random variables? Justify your answer.

(d) (3 points) Taking again $Z \sim \text{Geo}(p)$, and again using standard results on convergence and differentiation of power series, show that

$$E[Z(Z - 1)] = 2(1 - p)/p^2.$$

Deduce E(X) and E(Y) by taking $p = 1/2$.

Remark. You **may not** quote from the formulae sheet for this part!

6. (10 points) Let X and Y be independent random variables such that $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$, for some $\lambda > 0$ and $\mu > 0$.

   (a) (1 point) Write down the joint density of the random vector $(X, Y)^T$ with respect to Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$.

   (b) (3 points) Find $E(\max(X, Y))$.

   (c) (1 point) Find a simple expression for $\max(X, Y) + \min(X, Y)$ and deduce $E(\min(X, Y))$.

   (d) (3 points) Assuming that $\mu = \lambda = 1$, calculate $P(Y < X + 1)$ and deduce $P(|X - Y| < 1)$.

   (e) (2 points) Taking general $\lambda, \mu > 0$ again, let $Z = X + Y$. Using well-known properties of conditional expectation, find $E(Z \mid X)$ (specifying what properties you use).

7. (10 points) Suppose that the heights (in cm) of individuals of some population are distributed according to a normal distribution. You are given the following:

   - The average height is 170cm.
   - The proportion of individuals who are taller than 190cm is 5%.

   In the following, Φ denotes the cdf of $\mathcal{N}(0,1)$, and $z_\alpha = \Phi^{-1}(\alpha)$ its α-quantile for $\alpha \in (0,1)$.

   (a) (3 points) Find the standard deviation, σ, of the distribution of the heights.

   (b) (2 points) Let p be the probability that a random individual is taller than 160cm. Write p in terms of some value of Φ which you must determine.

   (c) (2 points) 100 individuals are selected at random. Find an expression for the probability that at least half of them are taller than 160cm.

   (d) (3 points) Assuming that 100 is a large enough sample size, give an approximation for the probability in (c). Justify your answer.

8. (9 points) Consider a Gaussian vector $(X_1, X_2, X_3)^T$ with expectation $\mu = (0, 1, -1)^T$ and covariance matrix

$$\Sigma = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 9 \end{pmatrix}.$$

(a) (2 points) Give the marginal densities of X_1 and X_2 with respect to Lebesgue measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

(b) (2 points) Are X_1 and X_2 independent? Are X_2 and X_3 independent?

(c) (2 points) What is the distribution of $X_1 + X_2$? Give its density with respect to Lebesgue measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

(d) (3 points) Find all the possible values of $\alpha_0 \in \mathbb{R}$ such that X_2 and $X_2 + \alpha_0 X_3$ satisfy

$$\operatorname{cov}(X_2, X_2 + \alpha_0 X_3) = 0.$$

What can you conclude about X_2 and $X_2 + \alpha_0 X_3$?

9. (11 points) Let $X_1, ..., X_n$ be i.i.d $\sim \text{Geo}(\theta_0)$ for some unknown $\theta_0 \in \Theta = (0, 1)$.

   (a) (2 points) Find a moment estimator, $\tilde{\theta}_n$, for θ_0.

   (b) (3 points) Justifying your answer, find the MLE, $\hat{\theta}_n$, for θ_0.

   (c) (3 points) Assuming that sufficient regularity conditions hold for this model, compute the Cramér-Rao Lower Bound.

   (d) (1 point) Again assuming sufficient regularity conditions, explicitly give the result on asymptotic normality for $\hat{\theta}_n$.

   (e) (2 points) Deduce from (d) an asymptotic confidence interval for θ_0 of level 0.95.

10. (12 points) Let $X_1, ..., X_n$ be i.i.d. $\sim \mathcal{N}(\theta, 1)$ for some $\theta \in \mathbb{R}$. We want to test

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = \theta_1,$$

for some fixed $\theta_1 > 1$.

Fix $\alpha \in (0, 1)$. We denote once again by Φ the cdf of $\mathcal{N}(0, 1)$ and $z_\gamma = \Phi^{-1}(\gamma)$ for $\gamma \in (0, 1)$.

(a) (3 points) Give the NP-test of level α for this hypothesis testing problem.

(b) (2 points) Give the expression for the power of this test. Show that the power converges to 1 as $n \to +\infty$.

Now, let us exchange the roles of the null and alternative hypotheses. That is, we want to test

$$H_0 : \theta = \theta_1 \quad \text{versus} \quad H_1 : \theta = 1.$$

We still assume that $\theta_1 > 1$.

(c) (3 points) Find the NP-test of level α for this new problem as well as its power.

(d) (2 points) Show that for any $x \in \mathbb{R}$, $\Phi(-x) = 1 - \Phi(x)$. Conclude that $z_\gamma = -z_{1-\gamma}$ for $\gamma \in (0, 1)$.

Remark. You **may not** quote from the lectures for this part!

(e) (2 points) What do you conclude about the powers of the tests obtained in (b) and (c)?

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