D–MATH

Exam Probability and Statistics

401-2604-00L

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Part I: Probability Theory

Question 1

A code consists of any 3 digits chosen randomly between 0 and 4.

(a) [**1 Point**]

Let Ω be the sample space of all such codes. Write down Ω and give its cardinality $|\Omega|$.

In the following questions b)-d), we assume that the distribution of the codes follows a Laplace model.

(b) [**1 Point**]

Compute the probability that all the digits are equal.

(c) [**1 Point**]

Compute the probability that the 1^{st} and the 3^{rd} digits are equal.

(d) [**1 Point**]

Compute the probability that the digits are all different.

Question 2

In this question, we consider 4 people. We are interested in the days of the week on which they were born. For example, (Monday, Monday, Wednesday, Sunday) is a possible answer when the 4 people are asked about this day of the week. We assume that all such possible answers have the same probability to be given.

(a) [**1 Point**]

Write down Ω and give its cardinality $|\Omega|$.

(b) [**1 Point**]

Compute the probability that all the 4 people were born on Monday.

(c) [**1 Point**]

Compute the probability that all the 4 people were born on the same day of the week.

(d) [**1 Point**]

Compute the probability that the 4 people were born on different days of the week.

(e) [**2 Points**]

Conclude from d) that the probability that at least 2 people were born on the same day of the week is larger than 0.6.

Question 3

A school teacher has 2 boxes which contain books. We will call these boxes Box #1 and Box #2.
Box #1 contains: 1 English, 2 German and 2 French books.
Box #2 contains: 2 English, 3 German and 1 French books.
For her reading course, the teacher selects a box and then takes 2 books from this selected box.

In all the questions a)-c), it is assumed that each of the boxes can be selected with the same probability. Also, from each of the boxes, the books can be selected with the same probability.

(a) [**2 Points**]

Compute the probability that the selected books are French and German books.

(b) [**2 Points**]

Compute the probability that the selected books are German books.

(c) [**3 Points**]

Let $S = \{\text{The selected books are of the same language}\}$. Given S, compute the conditional probability that Box #1 was selected.

Question 4

Consider two random variables X and Y such that X and Y are independent and $X \sim \text{Pois}(\lambda)$, $Y \sim \text{Pois}(\mu)$ for $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$.

(a) [**1 Point**]

Write down the mathematical definition of independence of X and Y.

(b) [**2 Points**]

Let $S = X + Y$. Show that S has a Poisson distribution and determine its parameter.

(c) [**2 Points**]

Fix $s \in \mathbb{N}_0$. Determine the conditional distribution of X given the event $\{S = s\}$, that is, determine $\mathbb{P}(X = x | S = s)$, $x \in \mathbb{N}_0$.

(d) [**2 Points**]

Give $\mathbb{E}(X|S = s)$ and deduce $\mathbb{E}(X|S)$.

(e) [**2 Points**]

If $\lambda = \mu$, determine $\mathbb{E}(X|S)$ using only the symmetry in the problem and the properties of conditional expectation.

Question 5

Consider a sequence of random variables $(X_n)_{n\geq 1}$ such that

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^2} \quad \text{and} \quad \mathbb{P}(X_n = n^\alpha) = \frac{1}{n^2}$$

for some $\alpha > 0$.

(a) **[1 Point]**

Recall the definition of convergence in probability of some sequence of random variables $(X_n)_{n\geq 1}$ to a random variable X.

(b) **[1 Point]**

Show that the sequence $(X_n)_{n\geq 1}$ converges to 0 in probability.

(c) **[1 Point]**

For $r > 0$, show that $\lim_{n\to\infty} \mathbb{E}(X_n^r) = 0$ if and only if $r < \frac{2}{\alpha}$.

(d) **[2 Points]**

Does $(X_n)_{n\geq 1}$ converge to 0 almost surely? Justify your answer.

Question 6

The 2-dimensional movement in a unit square of some particle is random. The random position (X, Y) of the particle has a joint distribution that admits the density

$$f(x, y) = cx^2 y \mathbb{1}_{\{0 \leq x \leq 1,\, 0 \leq y \leq 1\}}$$

with respect to Lebesgue measure on $\mathcal{B}_{\mathbb{R}^2}$ (the Borel σ-algebra on $\mathbb{R}^2$), for some $c > 0$.

(a) [**1 Point**]

Determine c.

(b) [**2 Points**]

Compute the marginal densities of X and Y, respectively.

Are X and Y independent? Justify your answer.

(c) [**2 Points**]

Compute $\mathbb{P}(X \leq \frac{1}{2})$ and $\mathbb{P}(\max(X, Y) \leq \frac{1}{2})$.

(d) [**2 Points**]

Suppose that the particle can only move in the lower triangle $\{0 \leq y \leq x \leq 1\}$ and the density of the random position is

$$f(x, y) = c' x^2 y \mathbb{1}_{\{0 \leq y \leq x \leq 1\}}$$

for some $c' > 0$. Determine c'.

Part II: Statistics
Question 7

The number of people coming to a restaurant between 12:00 and 14:00 is assumed to have a Poisson distribution with some (unknown) rate $\lambda_0 > 0$. We observe $X_1, \ldots, X_n$ i.i.d. random variables from this distribution.

(a) [**3 Points**]

Write down the log-likelihood function based on the random sample $\mathbb{X} = (X_1, \ldots, X_n)$ and find the MLE of λ_0.

(b) [**1 Point**]

Write down the CLT for $\bar{X}_n$.

(c) [**3 Points**]

Using the relevant theorems, show that

$$\frac{\sqrt{n}(\bar{X}_n - \lambda_0)}{h(\bar{X}_n)} \xrightarrow{d} \mathcal{N}(0,1)$$

for some function h on $(0, \infty)$ and specify the function h.

(d) [**2 Points**]

Deduce from c) a two-sided and symmetric confidence interval for λ_0 with asymptotic level α, $\alpha \in (0,1)$.

Question 8

The delay of some train is a random variable which we denote here by T. We assume that T admits an absolutely continuous distribution with density which belongs to the parametric family

$$\Big\{ p_\theta(t) = \frac{2(\theta - t)}{\theta^2} \mathbb{1}_{t \in [0,\theta]}, \quad \theta \in (0, \infty) \Big\}.$$

(a) [**2 Points**]

For a positive integer $k \in \mathbb{N}$, show that

$$\mathbb{E}_\theta(T^k) = \frac{2\theta^k}{(k+1)(k+2)}.$$

(b) [**2 Points**]

Deduce from a) the expectation and variance of T when $T \sim p_\theta$.

(c) [**2 Points**]

Let $T_1, \ldots, T_n$ be i.i.d. delays of this train. We denote by θ_0 the true unknown parameter. Determine $\hat{\theta}_n$ the moment estimator of θ_0 based on the observed delays.

(d) [**2 Points**]

Recall the CLT for $\bar{T}_n = \frac{1}{n} \sum_{i=1}^n T_i$ and show that it implies that $\forall z \in \mathbb{R}$

$$\mathbb{P}_{\theta_0}\Big(\sum_{i=1}^n T_i > \frac{\theta_0 z}{3\sqrt{2}} \sqrt{n} + \frac{\theta_0}{3} n \Big) \xrightarrow[n \to \infty]{} \frac{1}{\sqrt{2\pi}} \int_z^{+\infty} e^{-\frac{x^2}{2}} dx.$$

(e) [**2 Points**]

In this question, we assume that $\theta_0 = 9$ (minutes), and that the number of working days in a month of an employee who takes this train is 20.
Show that the probability that the employee loses in a month more than 1 hour because of the train delay is approximately $\frac{1}{2}$. (We assume that $n = 20$ is big enough for the convergence in d) to hold).

Question 9

Let X denote either a random variable or a random sample. We assume that X admits a distribution that has a density p with respect to a σ-finite dominating measure μ.

Consider the problem of testing

$$H_0 : p = p_0 \quad \text{versus} \quad H_1 : p = p_1 \quad (\star)$$

for some given densities p_0 and p_1 such that $p_0 \neq p_1$.

(a) [**1 Point**]

Recall the definition of a UMP test of level $\alpha \in (0, 1)$ for the testing problem in $(\star)$.

(b) [**2 Points**]

Give the Neyman-Pearson test of level α for the problem in $(\star)$ by specifying all the quantities on which it depends.

(c) [**2 Points**]

Show that the Neyman-Pearson test is UMP of level α.

Question 10

Let $X_1, \ldots, X_n$ be i.i.d. $\sim \mathcal{N}(\theta, \sigma^2)$ for $(\theta, \sigma^2) \in \Theta = \mathbb{R} \times (0, \infty)$.

We want to test

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta \neq 0.$$

(a) [**1 Point**]

If $\sigma = \sigma_0$ is known, construct a suitable test of level α.

(b) [**1 Point**]

If σ is not known, construct a suitable test of level α.

(c) [**1 Point**]

If $H_1 : \theta > 0$ and $\sigma = \sigma_0$ is known, construct a suitable test of level α.