Problems

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1. **(10 points)** For each of the following questions, exactly one answer is correct. Each correct answer gives 1 point, and each incorrect answer results in a 1/2 point reduction. The minimal possible total score for the full problem is 0.

   a) Let $(S_n)_{n=0,\dots,N}$ be a random walk. Which of the following is NOT a stopping time?

   1. $\inf\{n : S_n \geq 5\} \wedge N$
   2. $\inf\{n : S_{n+1} \geq 5\} \wedge N$
   3. $\inf\{n : S_{n-1} \geq 5\} \wedge N$

   b) Let $(S_n)_{n=0,..,N}$ be a random walk. Which of the following statements is TRUE?

   1. $\mathbb{E}(S_n^2)$ is increasing in n.
   2. For any stopping time T, $\mathbb{E}(S_T^2) = \mathbb{E}(S_0^2)$.
   3. $\mathrm{Var}(S_n^2) = n^2$.

   c) Let μ and μ_n, $n \in \mathbb{N}$, be distributions on $\mathbb{R}$. Let F, F_n be their respective distribution functions. Which of the following statements is equivalent to $\mu_n \to \mu$ weakly?

   1. $F_n(x) \to F(x)$ for every $x \in \mathbb{R}$ at which F is continuous.
   2. $\int_{\mathbb{R}} f(x) d\mu_n(x) \to \int_{\mathbb{R}} f(x) d\mu(x)$ for every function $f : \mathbb{R} \to \mathbb{R}$.
   3. $F_n(x) \to F(x)$ for every $x \in \mathbb{R}$ at which F is strictly positive.

   d) Which of the following statements is FALSE?

   1. Sum of independent Bernoulli random variables is still a Bernoulli random variable.
   2. Sum of independent Poisson random variables is still a Poisson random variable.
   3. Sum of independent Normal random variables is still a Normal random variable.

   e) Let X_1 and X_2 be independent random variables with characteristic function ϕ_1 and ϕ_2, respectively. What is the characteristic function ϕ of $X_1 - X_2$?

   1. $\phi(u) = \phi_1(u) - \phi_2(u)$
   2. $\phi(u) = \phi_1(u)/\phi_2(u)$
   3. $\phi(u) = \phi_1(u)\phi_2(-u)$

   f) If X_n converges in probability to X, then which of the following statements follows?

   1. X_n converges almost surely to X.
   2. $\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X)$.
   3. X_n converges weakly to X.

g) The characteristic function ϕ of a Binomial(n, p) random variable is given by:

1. $\phi(u) = (e^{ui}n + (1 - n))^p$
2. $\phi(u) = (e^{ui}p + (1 - p))^n$
3. $\phi(u) = (e^{ni}p + (1 - p))^u$

h) Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of $N(0, \sigma^2)$ random variables. Define $\overline{\sigma}_n = \frac{1}{n}\sum_{i=1}^n X_n^2$ and $\widetilde{\sigma}_n = \frac{1}{n-1}\sum_{i=1}^n X_n^2$. Which property is TRUE about these estimators?

1. $\widetilde{\sigma}_n$ is unbiased.
2. $\overline{\sigma}_n$ is consistent.
3. $\text{Var}(\overline{\sigma}_n) \geq \text{Var}(\widetilde{\sigma}_n)$.

i) Suppose a statistical test resulted in a p-value of 0.025. Which of the following statements is TRUE?

1. The null hypothesis could not be rejected at significance level 0.01.
2. The probability that the null hypothesis is true is 0.025.
3. The null hypothesis could not be rejected at significance level 0.05

j) Let A and B be two events with positive probability. Which of the following equations is Bayes' rule?

1. $\mathbb{P}(B \mid A) = \mathbb{P}(A \mid B)\mathbb{P}(B)/\mathbb{P}(A)$
2. $\mathbb{P}(B \mid A) = \mathbb{P}(A \mid B)\mathbb{P}(A)/\mathbb{P}(B)$
3. $\mathbb{P}(B \mid A) = \mathbb{P}(A \cap B)\mathbb{P}(B)/\mathbb{P}(A)$

2. (15 points) Let $X_1, X_2, \ldots$ be an i.i.d. sequence of Poisson(λ) distributed random variables for some fixed $\lambda > 0$.

a) How does $\frac{1}{n}\sum_{i=1}^{n} X_i$ behave as $n \to \infty$?
Solution: Since $\mathbb{E}(X_1^2) < \infty$, the SLLN implies that the average converges almost surely to $\mathbb{E}(X_1) = \lambda$.

b) How does $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \lambda)$ behave as $n \to \infty$?
Solution: Since $\mathbb{E}(X_1) = \lambda$ and $\mathrm{Var}(X_1) = \lambda$, the CLT implies that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \lambda)$ converges weakly to the $N(0, \lambda)$ distribution.

Suppose now you observe a noisy version of the X_i's. Specifically, let $Y_i = X_i + Z_i$, where $Z_i \sim N(0, \sigma^2)$ for some $\sigma \geq 0$. Assume all random variables are mutually independent.

c) How does $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - \lambda)$ behave as $n \to \infty$?
Solution: Since $\mathbb{E}(Y_i) = \lambda$ and $\mathrm{Var}(Y_i) = \mathrm{Var}(X_i) + \mathrm{Var}(Z_i) = \lambda + \sigma^2$, the CLT implies that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - \lambda)$ converges to the $N(0, \lambda + \sigma^2)$ distribution.

d) Suppose you observe a sample $Y_1, \ldots, Y_n$, and would like to test for the presence of noise. Specifically, you consider the hypotheses

$$\begin{cases} \text{Null:} & \sigma^2 = 0; \\ \text{Alternative:} & \sigma^2 > 0. \end{cases}$$

Design a test at significance level $\alpha = 0$, with power 1 at any $\sigma^2 > 0$.
Solution: If $\sigma^2 = 0$ then all the Y_i take integer values almost surely. If $\sigma^2 > 0$ then all the Y_i take non-integer values almost surely. Thus, the test

$$\text{Reject the Null} \quad \Longleftrightarrow \quad Y_i \text{ is non-integer for all } i = 1, \ldots, n$$

has level $\alpha = 0$ (since the Null is almost surely accepted if it is true) and power 1 at any $\sigma^2 > 0$ (since the Null is almost surely rejected whenever $\sigma^2 > 0$).

3. (15 points) Consider the probability density function

$$f_X(x) = \begin{cases} (\alpha+\alpha^2)x^{\alpha-1}(1-x) & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$ is a parameter. The corresponding distribution is called the Beta$(\alpha, 2)$ distribution.

a) Show that f_X is indeed a probability density function.

Solution: Clearly f_X is nonnegative. To show it integrates to one, compute:

$$\int_0^1 x^{\alpha-1}(1-x)dx = \int_0^1 x^{\alpha-1}dx - \int_0^1 x^{\alpha}dx = \frac{1}{\alpha} - \frac{1}{1+\alpha} = \frac{1}{\alpha+\alpha^2}.$$

Thus $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

b) Show that $\mathbb{E}(-\log X) = \dfrac{1+2\alpha}{\alpha+\alpha^2}$, where $X \sim f_X$.

Solution: Write

$$\mathbb{E}(\log X_1) = (\alpha+\alpha^2)\int_0^1 \log(x)x^{\alpha-1}(1-x)dx.$$

Integration by parts yields

$$\int_0^1 \log(x)x^{\alpha-1}dx = [\alpha^{-1}\log(x)x^{\alpha}]_0^1 - \alpha^{-1}\int_0^1 x^{\alpha-1}dx = -\alpha^{-2},$$

and similarly $\int_0^1 \log(x)x^{\alpha}dx = -(1+\alpha)^{-2}$. Thus

$$\mathbb{E}(-\log X_1) = (\alpha+\alpha^2)\left(\frac{1}{\alpha^2} - \frac{1}{(1+\alpha)^2}\right) = \frac{1+2\alpha}{\alpha+\alpha^2}.$$

c) Let $\widehat{\alpha}_n$ denote the maximum likelihood estimator of α based on an i.i.d. sample $X_1, \ldots, X_n$ from f_X. Show that $\widehat{\alpha}_n$ exists, is unique, and is the solution of the equation

$$\frac{1+2\widehat{\alpha}_n}{\widehat{\alpha}_n+\widehat{\alpha}_n^2} = -\frac{1}{n}\sum_{i=1}^n \log X_i.$$

Solution: Since the sample is iid, the log-likelihood function is

$$\log L(X_1,\ldots,X_n;\alpha) = \log\prod_{i=1}^n f_X(X_i) = n\log(\alpha+\alpha^2)+(\alpha-1)\sum_{i=1}^n \log X_i + \sum_{i=1}^n \log(1-X_i).$$

Differentiate the log-likelihood to get

$$\frac{d}{d\alpha}\log L(X_1,\ldots,X_n;\alpha) = n\frac{1+2\alpha}{\alpha+\alpha^2} + \sum_{i=1}^n \log X_i.$$

Now $\widehat{\alpha}_n$ is the zero of the right-hand side, provided it exists, is unique, and is a maximizer. To check this, just observe that $h(\alpha) := \frac{1+2\alpha}{\alpha+\alpha^2} = 1/\alpha + 1/(\alpha+1)$ satisfies $\lim_{\alpha\downarrow 0} h(\alpha) = \infty$, $\lim_{\alpha\to\infty} h(\alpha) = 0$, and is decreasing. So there exists only one value such that $h(\alpha) = -1/n \sum \log X_i$ and this is a maximizer because the log-likehood is strictly concave (its second derivative is $h'(\alpha) < 0$).

d) Show that the maximum likelihood estimator is consistent.

Solution: By the WLLN and part b), we have

$$\lim_{n\to\infty} -\frac{1}{n}\sum_{i=1}^{n} \log X_i = \mathbb{E}(-\log X_1) = \frac{1+2\alpha}{\alpha+\alpha^2},$$

where the limit is in probability. Let

$$g(u) := \frac{2-u+\sqrt{4+u^2}}{2u}, \qquad u > 0,$$

be the inverse of $h(\alpha) := \frac{1+2\alpha}{\alpha+\alpha^2}$, $\alpha > 0$. Then, in view of part c),

$$\lim_{n\to\infty} \widehat{\alpha}_n = \lim_{n\to\infty} g\left(-\frac{1}{n}\sum_{i=1}^{n} \log X_i\right) = g\left(\frac{1+2\alpha}{\alpha+\alpha^2}\right) = \alpha,$$

where the limits are in probability. Thus the estimator is consistent. *Remark: The SLLN, and hence WLLN, is applicable since $\mathbb{E}(-\log X_1) < \infty$ by part b), as is mentioned after the proof of Satz 4.2 in the Script. In order to apply Satz 4.2 itself, or some other result in the Script like Equation (4.2), one would have to show that $\mathbb{E}((\log X_1)^2) < \infty$. This is however not required for full credit on this problem.*

4. (15 points) Let $X_1, X_2, \ldots$ be i.i.d. Exponential(λ) for some fixed $\lambda > 0$, and consider the maxima $M_n = \max(X_1, \ldots, X_n)$ for each n.

a) Show that $\mathbb{P}(M_n \le x) = (1 - e^{-\lambda x})^n$ for $x \ge 0$.

Solution: Note that

$$\{M_n \le x\} = \{X_1 \le x\} \cap \{X_2 \le x\} \cap \cdots \cap \{X_n \le x\}.$$

Thus by the i.i.d. property,

$$\mathbb{P}(M_n \le x) = \mathbb{P}(X_1 \le x)^n = (1 - e^{-\lambda x})^n.$$

b) Show that the distribution of $\lambda M_n - \log n$ converges weakly to the *Gumbel distribution*, whose distribution function is $F(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$.

Solution: For any $x \in \mathbb{R}$ and any n such that $\log n \ge x$,

$$\mathbb{P}(\lambda M_n - \log n \le x) = \mathbb{P}\left(M_n \le \frac{x + \log n}{\lambda}\right) = \left(1 - \frac{1}{n} e^{-x}\right)^n \to e^{-e^{-x}} = F(x).$$

as $n \to \infty$. This is equivalent to the desired statement.

c) Let $(a_n)_{n\in\mathbb{N}}$ be a positive increasing sequence with $\lim_{n\to\infty} \frac{\log n}{a_n} = c$ for some $c \in [0, \infty]$. Show that $\lim_{n\to\infty} a_n^{-1} M_n = \lambda^{-1} c$ in probability.

Solution: We rely on part b). In preparation for this, write

$$\frac{M_n}{a_n} = \lambda^{-1} \frac{\log n}{a_n} + \lambda^{-1} \frac{1}{a_n} (\lambda M_n - \log n).$$

The first term on the right-hand side converges to $\lambda^{-1} c$ by assumption. Thus, we must show that the second term converges to zero in probability. Pick any $\varepsilon > 0$ and any $\delta > 0$. Let $K > 0$ be large enough that $1 - F(K) - F(-K) \le \delta$. Then, for all $n \ge K/(\lambda\varepsilon)$,

$$\begin{aligned}\mathbb{P}\left(\left|\lambda^{-1} \frac{1}{a_n} (\lambda M_n - \log n)\right| > \varepsilon\right) &= \mathbb{P}(|\lambda M_n - \log n| > \lambda \varepsilon a_n) \\ &\le \mathbb{P}(|\lambda M_n - \log n| > K).\end{aligned}$$

Thus, sending n to infinity and using part b),

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\lambda^{-1} \frac{1}{a_n} (\lambda M_n - \log n)\right| > \varepsilon\right) \le 1 - F(K) - F(-K) \le \delta,$$

as required.

d) Show that $\lim_{n\to\infty} M_n = \infty$ almost surely.

Solution: For any $x \ge 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n \le x) = \sum_{n=1}^{\infty} (1 - e^{-\lambda x})^n = \frac{1}{1 - (1 - e^{-\lambda x})} - 1 = e^{\lambda x} - 1 < \infty.$$

Thus by the Borel-Cantelli lemma, $M_n \le x$ holds for at most finitely many n almost surely. Thus $\lim_{n\to\infty} M_n \ge x$ almost surely. Since x was arbitrary, $\lim_{n\to\infty} M_n = \infty$.

5. **(15 points)** Define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$, where $X_1, X_2, \ldots$ are i.i.d. random variables with $\mathbb{P}(X_i = +1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote the probability space on which these objects are defined.

a) Fix any $N \in \mathbb{N}$ and consider $(S_n)_{n=0,\ldots,N}$, which is a random walk of length N. For each $n = 0, \ldots, N$, let $\mathcal{A}_n$ denote the set of all events of the form $\{\omega \colon (S_0(\omega), \ldots, S_n(\omega)) \in D\}$ with $D \subseteq \mathbb{R}^n$ measurable. State the definition of a stopping time.

Solution: A map $T : \Omega \to \{0, \ldots, N\}$ is called a stopping time if for all $n = 0, \ldots, N$,

$$\{\omega \colon T(\omega) = n\} \in \mathcal{A}_n.$$

Recall the following two facts, which you may use later on:

(i) For any stopping time T, one has $\mathbb{E}(S_T) = 0$. Here it is crucial that $T \leq N$ almost surely for some deterministic number N.

(ii) For any $a \in \mathbb{Z}$, letting $T_a = \inf\{n > 0 : S_n = a\}$ one has $\lim_{N\to\infty} \mathbb{P}(T_a > N) = 0$.

Fix integers $a > 0 > b$ and let $T_{a,b} = \inf\{n > 0 : S_n = a \text{ or } S_n = b\}$ denote the first time S_n hits either a or b. If this never happens, set $T_{a,b} = \infty$.

b) Show that $\mathbb{P}(T_{a,b} < \infty) = 1$.

Solution: Since $T_a \leq T_{a,b}$ and using Fact (ii), one has

$$\mathbb{P}(T_{a,b} = \infty) = \lim_{N\to\infty} \mathbb{P}(T_{a,b} > N) \leq \lim_{N\to\infty} \mathbb{P}(T_a > N) = 0.$$

c) Show that $\mathbb{E}(S_{T_{a,b}}) = 0$.

Hint: Consider the stopping time $T = \min(T_{a,b}, N)$, apply the above facts about stopping times, and take limits.

Solution: Since $T_{a,b}$ is not bounded by any deterministic constant, one cannot directly use Fact (i). Instead, proceed as follows. For any $N \in \mathbb{N}$, note that $T_{a,b} \wedge N = T_a \wedge T_b \wedge N$ is a stopping time. Thus

$$0 = \mathbb{E}(S_{T_{a,b}\wedge N}) = \mathbb{E}(S_{T_{a,b}} \mathbf{1}_{\{T_{a,b} \leq N\}}) + \mathbb{E}(S_N \mathbf{1}_{\{T_{a,b} > N\}}).$$

On the event $\{T_{a,b} > N\}$ we have $|S_N| < a \wedge |b|$. Hence

$$\left|\mathbb{E}(S_N \mathbf{1}_{\{T_{a,b} > N\}})\right| \leq (a \wedge |b|)\mathbb{P}(T_{a,b} > N) \to 0 \qquad (N \to \infty).$$

Also, since $S_{T_{a,b}} \in \{a, b\}$ and $T_{a,b} < \infty$ almost surely, the dominated convergence theorem yields

$$\mathbb{E}(S_{T_{a,b}} \mathbf{1}_{\{T_{a,b} \leq N\}}) \to \mathbb{E}(S_{T_{a,b}}) \qquad (N \to \infty).$$

Putting these facts together yields $\mathbb{E}(S_{T_{a,b}}) = 0$.

d) What is the probability that S_n reaches a before it reaches b?

Solution: Due to part b) one has $T_{a,b} = T_a \wedge T_b < \infty$. Using also part c) one gets

$$\begin{aligned}
0 &= \mathbb{E}(S_{T_{a,b}}) \\
&= \mathbb{E}(S_{T_a}\mathbf{1}_{\{T_a<T_b\}}) + \mathbb{E}(S_{T_b}\mathbf{1}_{\{T_a>T_b\}}) \\
&= a\mathbb{P}(T_a < T_b) + b\mathbb{P}(T_a > T_b) \\
&= (a-b)\mathbb{P}(T_a < T_b) + b.
\end{aligned}$$

Thus

$$\mathbb{P}(T_a < T_b) = \frac{-b}{a-b}.$$