Lecture on Interest Rates

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Goals

- Basic concepts of stochastic modeling in interest rate theory, in particular the notion of numéraire.
- "No arbitrage" as concept and through examples.
- Several basic implementations related to "no arbitrage" in R.
- Basic concepts of interest rate theory like yield, forward rate curve, short rate.
- Some basic trading arguments in interest rate theory.
- Spot measure, forward measures, swap measures and Black's formula.

References

As a standard reference on interest rate theory I recommend [Brigo and Mercurio(2006)].

In german language I recommend [Albrecher et al.(2009)Albrecher, Binder, and Mayer], which contains also a very readable introduction to interest rate theory

Modeling of financial markets

We are describing *models for financial products related to interest rates*, so called interest rate models. We are facing several difficulties, some of the specific for interest rates, some of them true for all models in mathematical finance:

- stochastic nature: traded prices, e.g. prices of interest rate related products, are not deterministic!
- information is increasing: every day additional information on markets appears and this stream of information should enter into our models.
- stylized facts of markets should be reflected in the model: stylized facts of time series, trading opportunities (portfolio building), etc.

Mathematical Finance 1

A financial market can be modeled by

- a filtered (discrete) probability space $(\Omega, \mathcal{F}, Q)$,
- together with price processes, namely K risky assets $(S_n^1, \ldots, S_n^K)_{0 \leq n \leq N}$ and one risk-less asset S^0 (numéraire), i.e. $S_n^0 > 0$ almost surely (no default risk for at least one asset),
- all price processes being adapted to the filtration.

This structure reflects stochasticity of prices and the stream of incoming information.

A portfolio is a predictable process $\phi = (\phi_n^0, \ldots, \phi_n^K)_{0 \le n \le N}$, where ϕ_n^i represents the number of risky assets one holds at time n. The value of the portfolio $V_n(\phi)$ is

$$V_n(\phi) = \sum_{i=0}^{K} \phi_n^i S_n^i.$$

Mathematical Finance 2

Self-financing portfolios ϕ are characterized through the condition

$$V_{n+1}(\phi) - V_n(\phi) = \sum_{i=0}^{K} \phi^i_{n+1}(S^i_{n+1} - S^i_n),$$

for $0 \leq n \leq N-1$, i.e. changes in value come from changes in prices, no additional input of capital is required and no consumption appears.

Self-financing portfolios can be characterized in *discounted terms*.

$$\widetilde{V_n}(\phi) = (S_n^0)^{-1} V_n(\phi)$$
$$\widetilde{S_n^i} = (S_n^0)^{-1} S_n^i(\phi)$$
$$\widetilde{V_n}(\phi) = \sum_{i=0}^{K} \phi_n^i \widetilde{S_n^i}$$

for $0 \leq n \leq N$, and recover

$$\widetilde{V_n}(\phi) = \widetilde{V_0}(\phi) + (\phi \cdot \widetilde{S}) = \widetilde{V_0}(\phi) + \sum_{j=1}^{n} \sum_{i=1}^{K} \phi_j^i (\widetilde{S_j^i} - \widetilde{S_{j-1}^i})$$

for self-financing predictable trading strategies ϕ and $0 \leq n \leq N$. In words: discounted wealth of a self-financing portfolio is the cumulative sum of discounted gains and losses. *Notice that we apply a generalized notion of "discounting" here, prices S^i divided by the numéraire S^0 – only these relative prices can be compared.*

Fundamental Theorem of Asset Pricing

A minimal condition for modeling financial markets is the *No-arbitrage condition*: there are no self-financing trading strategies ϕ (arbitrage strategies) with

$$V_0(\phi) = 0,\ V_N(\phi) \geq 0$$

such that $Q(V_N(\phi) \neq 0) > 0$ holds (NFLVR).

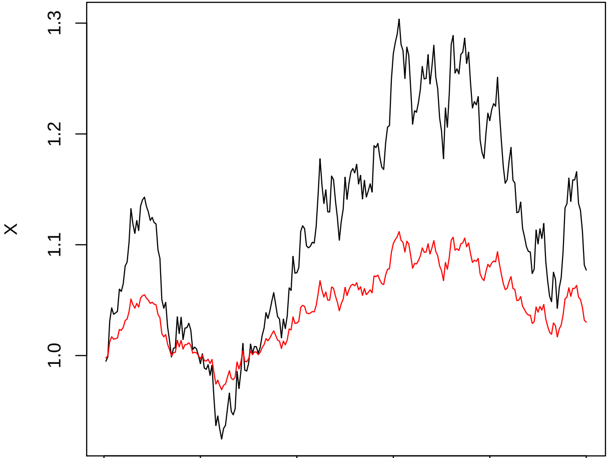
In the sequel we generate two sets random numbers (normalized log-returns) and introduce two examples of markets with constant bank account and two assets. The first market allows for arbitrage, then second one not. In both cases we run the same portfolio:

```
> Delta = 250
> Z = rnorm(Delta, 0, 1/sqrt(Delta))
> Z1 = rnorm(Delta, 0, 1/sqrt(Delta))
```

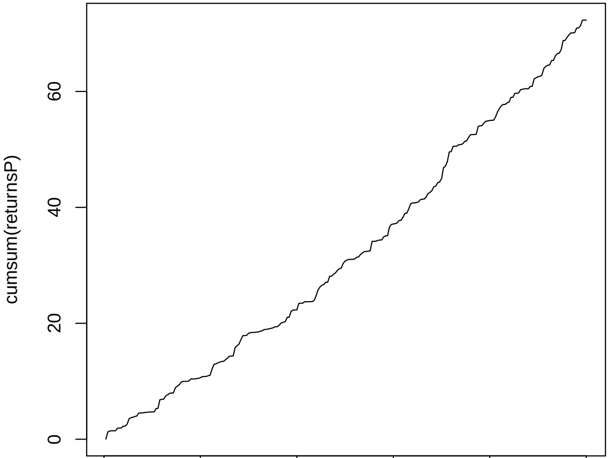
Incorrect modelling with arbitrage

```
> S = 10000
> rho = 0
> sigmaX = 0.25
> sigmaY = 0.1
> X = exp(sigmaX * cumsum(Z))
> Y = exp(sigmaY * cumsum(sqrt(1 - rho^2) * Z + rho * Z1))
> returnsX = c(diff(X, lag = 1, differences = 1), 0)
> returnsY = c(diff(Y, lag = 1, differences = 1), 0)
> returnsP = ((sigmaY * Y)/(sigmaX * X) * returnsX - returnsY) *
+     S
```

Plot of the two Asset prices



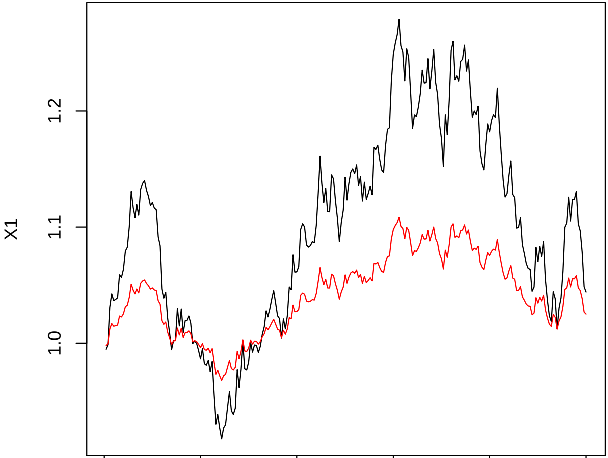
Plot of the value process: an arbitrage



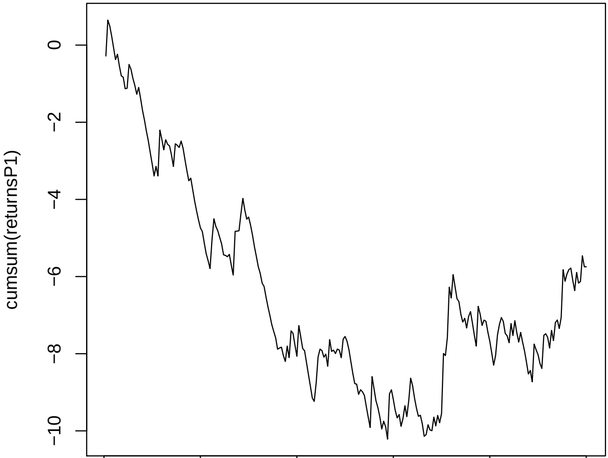
Correct arbitrage-free modelling

```
> Delta = 250
> S = 10000
> rho = 0
> sigmaX = 0.25
> sigmaY = 0.1
> time = seq(1/Delta, 1, by = 1/Delta)
> X1 = exp(-sigmaX^2 * 0.5 * time + sigmaX * cumsum(Z))
> Y1 = exp(-sigmaY^2 * 0.5 * time + sigmaY * cumsum(sqrt(1 - rho^2) *
+     Z + rho * Z1))
> returnsX1 = c(diff(X1, lag = 1, differences = 1), 0)
> returnsY1 = c(diff(Y1, lag = 1, differences = 1), 0)
> returnsP1 = ((sigmaY * Y1)/(sigmaX * X1) * returnsX1 - returnsY1) *
+     S
```

Plot of two asset prices



Plot of the value process: no arbitrage



Theorem

Given a financial market, then the following assertions are equivalent:

1. *(NFLVR) holds.*
2. *There exists an equivalent measure $P \sim Q$ such that the discounted price processes are P-martingales, i.e.*

$$E_P(\frac{1}{S_N^0} S_N^i | \mathcal{F}_n) = \frac{1}{S_n^0} S_n^i$$

for $0 \leq n \leq N$.

Main message: Discounted (relative to the numéraire) prices behave like martingales with respect to one martingale measure.

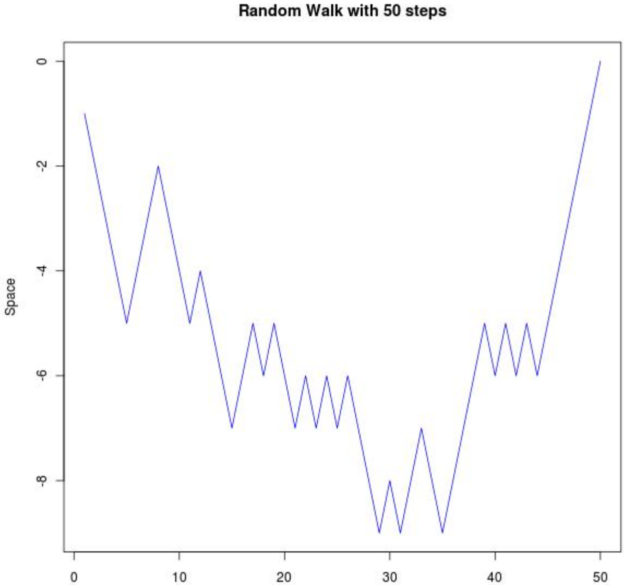
What is a martingale?

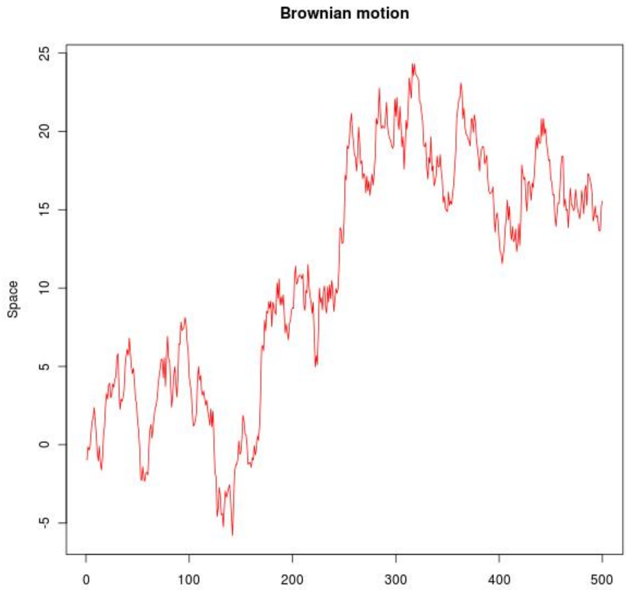
Formally a martingale is a stochastic process such that today's best prediction of a future value of the process is today's value, i.e.

$$E[M_n|\mathcal{F}_m] = M_m$$

for $m \leq n$, where $E[M_n|\mathcal{F}_m]$ calculates the best prediction with knowledge up to time m of the future value M_n.

Random walks and Brownian motions are well-known examples of martingales. Martingales are particularly suited to describe (discounted) price movements on financial markets, since the prediction of future returns is 0. This is not the most general approach, but already contains the most important features. Two implementations in R are provided here, which produce the following graphs.


Random Walk with 50 steps
Space
0
-2
-4
-6
-8
0
10
20
30
40
50


Brownian motion
Space
25
20
15
10
5
0
-5
0
100
200
300
400
500

Pricing rules

(NFLVR) also leads to arbitrage-free pricing rules. Let X be the payoff of a claim X paying at time N, then an adapted stochastic process $\pi(X)$ is called pricing rule for X if

- $\pi_N(X) = X$.
- $(S^0, \dots, S^N, \pi(X))$ is free of arbitrage.

This is obviously equivalent to the existence of one equivalent martingale measure P such that

$$E_P\left(\frac{X}{S_N^0}|\mathcal{F}_n\right) = \frac{\pi_n(X)}{S_n^0}$$

holds true for $0 \leq n \leq N$.

The previous framework for stochastic models of financial markets is not bound to a "discrete" setting even though one can perfectly well motivate the theory there. We shall see two examples and several remarks in the sequel

- the one-step binomial model.
- the Black-Merton-Scholes model.
- Hedging.

One step binomial model

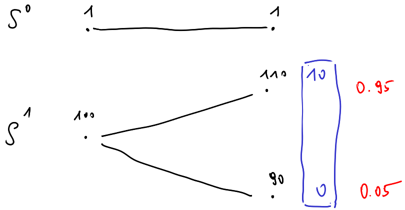
We model one asset in a zero-interest rate environment just before the next tick. We assume two states of the world: up, down. The riskless asset is given by $S^0 = 1$. The risky asset is modeled by

$$S_0^1 = S_0,\ S_1^1 = S_0 * u > S_0 \text{ or } S_1^1 = S_0 * d$$

where the events at time one appear with probability q and $1 - q$ ("physical measure"). The martingale measure is apparently given through $u * p + (1 - p)d = 1$, i.e. $p = \frac{1-d}{u-d}$.

Pricing a European call option at time one in this setting leads to fair price

$$E[(S_1^1 - K)_+] = p * (S_0 u - K)_+ + (1 - p) * (S_0 d - K)_+.$$



"bank account"

"stock"

What is the price of a call contract?

$X = (S^1_1 - 100)_+$

"payoff"

Answer: $\pi_0(X) = 5$

"fair price"

trading argument

	S^0	S^1
0	-45	$\frac{1}{2}$
1	-45	$\frac{1}{2} S^1 < \begin{matrix} 55 \\ 45 \end{matrix}$

"hedging portfolio"

Black-Merton-Scholes model 1

We model one asset with respect to some numeraire by an exponential Brownian motion. If the numeraire is a bank account with constant rate we usually speak of the Black-Merton-Scholes model, if the numeraire some other traded asset, for instance a zero-coupon bond, we speak of Black's model. Let us assume that $S_0 = 1$, then

$$S_t^1 = S_0 \exp(\sigma B_t - \frac{\sigma^2 t}{2})$$

with respect to the martingale measure P. In the physical measure Q a drift term can be added in the exponent, i.e.

$$S_t^1 = S_0 \exp(\sigma B_t - \frac{\sigma^2 t}{2} + \mu t).$$

Black-Merton-Scholes model 2

Our theory tells that the price of a European call option on S^1 at time T is priced via

$$E[(S^1_T - K)_+] = S_0\Phi(d_1) - K\Phi(d_2)$$

yielding the Black-Scholes formula, where Φ is the cumulative distribution function of the standard normal distribution and

$$d_{1,2} = \frac{\log\frac{S_0}{K} \pm \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}.$$

Notice that this price corresponds to the value of a portfolio mimicking the European option at time T.

Hedging

Having calculated prices of derivatives we can ask if it is possible to hedge as seller against the risks of the product. By the very construction of prices we expect that we should be able to build – at the price of the premium which we receive – a portfolio which hedges against some (all) risks. In the Black-Scholes model this hedging is perfect.

A time series of yields

- AAA yield curve of the euro area from ECB webpage.
- Yield curves exist in all major economies and are calculated from different products like deposit rates, swap rates, zero coupon bonds, coupon bearing bonds.
- Interest rates express the time value of money quantitatively.

Interest Rate mechanics 1

Prices of zero-coupon bonds (ZCB) with maturity T are denoted by $P(t, T)$. Interest rates are governed by a market of (default free) zero-coupon bonds modeled by stochastic processes $(P(t, T))_{0 \leq t \leq T}$ for $T \geq 0$. We assume the normalization $P(T, T) = 1$.

- T denotes the *maturity* of the bond, $P(t, T)$ its price at a time t before maturity T.
- The yield
$$Y(t, T) = -\frac{1}{T - t} \log P(t, T)$$
describes the compound interest rate p. a. for maturity T.
- f is called the *forward rate curve* of the bond market
$$P(t, T) = \exp(-\int_t^T f(t, s) ds)$$

Interest Rate mechanics 2

- The *short rate process* is given through $R_t = f(t,t)$ for $t \geq 0$ defining the "bank account process"

$$(B(t))_{t\geq 0} := (\exp(\int_0^t R_s ds))_{t\geq 0}.$$

- No arbitrage is guaranteed if on the filtered probability space $(\Omega, \mathcal{F}, Q)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$,

$$E(\exp(-\int_t^T R_s ds)|\mathcal{F}_t) = P(t,T)$$

holds true for $0 \leq t \leq T$ for some equivalent (martingale) measure P.

Simple forward rates

Consider a bond market $(P(t,T))_{t\leq T}$ with $P(T,T)=1$ and $P(t,T)>0$. Let $t\leq T\leq T^*$. We define the simple forward rate through

$$F(t;T,T^*):=\frac{1}{T^*-T}\left(\frac{P(t,T)}{P(t,T^*)}-1\right).$$

and the simple spot rate through

$$F(t,T):=F(t;t,T).$$

Apparently $P(t, T^*)F(t; T, T^*)$ is the fair value at time t of a contract paying $F(T, T^*)$ at time T^*.

Indeed, note that

$$P(t, T^*)F(t; T, T^*) = \frac{P(t, T) - P(t, T^*)}{T^* - T},$$
$$F(T, T^*) = \frac{1}{T^* - T}\left(\frac{1}{P(T, T^*)} - 1\right).$$

Fair value means that we can build a portfolio at time t and at price $\frac{P(t,T)-P(t,T^*)}{T^*-T}$ yielding $F(T, T^*)$ at time T^*:

- Holding a ZCB with maturity T at time t has value $P(t, T)$, being additionally short in a ZCB with maturity T^* amounts all together to $P(t, T) - P(t, T^*)$.
- at time T we have to rebalance the portfolio by buying with the maturing ZCB another bond with maturity T^*, precisely an amount $1/P(T, T^*)$.

Caps

In the sequel, we fix a number of future dates

$$T_0 < T_1 < \ldots < T_n$$

with $T_i - T_{i-1} \equiv \delta$.

Fix a rate $\kappa > 0$. At time T_i the holder of the cap receives

$$\delta(F(T_{i-1}, T_i) - \kappa)^+.$$

Let $t \leq T_0$. We write

$$\mathrm{Cpl}(t; T_{i-1}, T_i), \quad i = 1, \ldots, n$$

for the time t price of the ith caplet, and

$$\mathrm{Cp}(t) = \sum_{i=1}^{n} \mathrm{Cpl}(t; T_{i-1}, T_i)$$

for the time t price of the cap.

Floors

At time T_i the holder of the floor receives

$$\delta(\kappa - F(T_{i-1}, T_i))^+.$$

Let $t \leq T_0$. We write

$$\mathrm{Fll}(t; T_{i-1}, T_i), \quad i = 1, \ldots, n$$

for the time t price of the ith floorlet, and

$$\mathrm{Fl}(t) = \sum_{i=1}^{n} \mathrm{Fll}(t; T_{i-1}, T_i)$$

for the time t price of the floor.

Swaps

Fix a rate K and a nominal N. The cash flow of a payer swap at T_i is

$$(F(T_{i-1}, T_i) - K)\delta N.$$

The total value $\Pi_p(t)$ of the payer swap at time $t \leq T_0$ is

$$\Pi_p(t) = N\left(P(t, T_0) - P(t, T_n) - K\delta \sum_{i=1}^{n} P(t, T_i)\right).$$

The value of a receiver swap at $t \leq T_0$ is

$$\Pi_r(t) = -\Pi_p(t).$$

The swap rate $R_{\mathrm{swap}}(t)$ is the fixed rate K which gives $\Pi_p(t) = \Pi_r(t) = 0$. Hence

$$R_{\mathrm{swap}}(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}, \quad t \in [0, T_0].$$

Swaptions

A *payer (receiver) swaption* is an option to enter a payer (receiver) swap at T_0. The payoff of a payer swaption at T_0 is

$$N\delta(R_{\text{swap}}(T_0) - K)^+ \sum_{i=1}^{n} P(T_0, T_i),$$

and of a receiver swaption

$$N\delta(K - R_{\text{swap}}(T_0))^+ \sum_{i=1}^{n} P(T_0, T_i).$$

Spot measure

From now on, let P be a martingale measure in the bond market $(P(t,T))_{t \leq T}$, i.e. for each $T > 0$ the discounted bond price process

$$\frac{P(t,T)}{B(t)}$$

is a martingale. This leads to the following fundamental formula of interest rate theory

$$P(t,T) = E(\exp(-\int_t^T R_s ds))|\mathcal{F}_t)$$

for $0 \leq t \leq T$.

Forward measures

For $T^* > 0$ define the T^*-*forward measure* P^{T^*} such that for any $T > 0$ the discounted bond price process

$$\frac{P(t,T)}{P(t,T^*)}, \quad t \in [0,T]$$

is a P^{T^*}-martingale.

Forward measures

For any $T < T^*$ the simple forward rate

$$F(t; T, T^*) = \frac{1}{T^* - T}\left(\frac{P(t, T)}{P(t, T^*)} - 1\right)$$

is a $\mathbb{P}^{T^*}$-martingale.

For any time derivative $X \in \mathcal{F}_{T^*}$ paid at T^* we have that the fair value via "martingale pricing" is given through

$$P(t, T^*)\mathbb{E}^{T^*}[X|\mathcal{F}_t].$$

The fair price of the ith caplet is therefore given by

$$\mathrm{Cpl}(t; T_{i-1}, T_i) = \delta P(t, T_i)\mathbb{E}^{T_i}[(F(T_{i-1}, T_i) - \kappa)^+|\mathcal{F}_t].$$

By the martingale property we obtain therefore

$$\mathbb{E}^{T_i}[F(T_{i-1}, T_i)|\mathcal{F}_t] = F(t; T_{i-1}, T_i),$$

what was proved by trading arguments before.

Black's formula

Let $X \sim N(\mu, \sigma^2)$ and $K \in \mathbb{R}$. Then we have

$$\mathbb{E}[(e^X - K)^+] = e^{\mu + \frac{\sigma^2}{2}} \Phi\left(-\frac{\log K - (\mu + \sigma^2)}{\sigma} \right) - K\Phi\left(-\frac{\log K - \mu}{\sigma} \right),$$

$$\mathbb{E}[(K - e^X)^+] = K\Phi\left(\frac{\log K - \mu}{\sigma} \right) - e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\log K - (\mu + \sigma^2)}{\sigma} \right).$$

Black's formula for caps and floors

Let $t \leq T_0$. From our previous results we know that

$$\mathrm{Cpl}(t; T_{i-1}, T_i) = \delta P(t, T_i) \mathbb{E}_t^{T_i}[(F(T_{i-1}, T_i) - \kappa)^+],$$
$$\mathrm{Fll}(t; T_{i-1}, T_i) = \delta P(t, T_i) \mathbb{E}_t^{T_i}[(\kappa - F(T_{i-1}, T_i))^+],$$

and that $F(t; T_{i-1}, T_i)$ is an P^{T_i}-martingale.

We assume that under P^{T_i} the forward rate $F(t; T_{i-1}, T_i)$ is an exponential Brownian motion

$$F(t; T_{i-1}, T_i) = F(s; T_{i-1}, T_i) \exp\left(-\frac{1}{2}\int_s^t \lambda(u, T_{i-1})^2 du + \int_s^t \lambda(u, T_{i-1}) dW_u^{T_i} \right)$$

for $s \leq t \leq T_{i-1}$, with a function $\lambda(u, T_{i-1})$.

We define the volatility $\sigma^2(t)$ as

$$\sigma^2(t) := \frac{1}{T_{i-1} - t} \int_t^{T_{i-1}} \lambda(s, T_{i-1})^2 ds.$$

The P^{T_i}-distribution of $\log F(T_{i-1}, T_i)$ conditional on $\mathcal{F}_t$ is $N(\mu, \sigma^2)$ with

$$\begin{aligned} \mu &= \log F(t; T_{i-1}, T_i) - \frac{\sigma^2(t)}{2}(T_{i-1} - t), \\ \sigma^2 &= \sigma^2(t)(T_{i-1} - t). \end{aligned}$$

In particular

$$\begin{aligned} \mu + \frac{\sigma^2}{2} &= \log F(t; T_{i-1}, T_i), \\ \mu + \sigma^2 &= \log F(t; T_{i-1}, T_i) + \frac{\sigma^2(t)}{2}(T_{i-1} - t). \end{aligned}$$

We have

$$\begin{aligned}\mathrm{Cpl}(t;T_{i-1},T_i) &= \delta P(t,T_i)\big(F(t;T_{i-1},T_i)\Phi(d_1(i;t)) - \kappa\Phi(d_2(i;t))\big),\\ \mathrm{Fll}(t;T_{i-1},T_i) &= \delta P(t,T_i)\big(\kappa\Phi(-d_2(i;t)) - F(t;T_{i-1},T_i)\Phi(-d_1(i;t))\big),\end{aligned}$$

where

$$d_{1,2}(i;t) = \frac{\log\left(\frac{F(t;T_{i-1},T_i)}{\kappa}\right) \pm \frac{1}{2}\sigma(t)^2(T_{i-1}-t)}{\sigma(t)\sqrt{T_{i-1}-t}}.$$

Concrete calculation of caplet price

Consider the setting $t = 0$, $T_0 = 0.25y$ and $T_1 = 0.5y$. Market data give us $P(0, T_1) = 0.9753$, $F(0, T_0, T_1) = 0.0503$ and $\lambda(u, T_0) = 0.2$ is constant, hence we can calculate

$$\sigma(t)^2(T_0 - t) = 0.2 * 0.25,$$

and therefore by Black's formula gives the caplet price for $\kappa = 0.03$

$$0.25 * 0.9753 * (0.0503 * \Phi(\frac{\log(0.0503) - \log(0.03) + 0.5 * 0.2 * 0.25}{\sqrt{0.2 * 0.25}}) -$$
$$- 0.03 * \Phi(\frac{\log(0.0503) - \log(0.03) - 0.5 * 0.2 * 0.25}{\sqrt{0.2 * 0.25}})),$$

where Φ is the cumulative distribution function of a standard normal random variable, which yields 0.004957.

Exercises

Simulate a simple interest rate model:

- We choose a simple interest rate model of Vasiček type, i.e.

$$R_t = \exp(-0.2t)0.05 + 0.03 \int_0^t \exp(-0.2(t-s))dB_s.$$

- First we simulate the bank account, i.e. we calculate the value $B(t)$ for different trajectories of Brownian motion. Write an R-function called vasicek with input parameter t and discretization parameter n which provides the value of $B(t)$. Use the following iteration for this: $B(0) = 1$, $R(0) = 0.05$ and

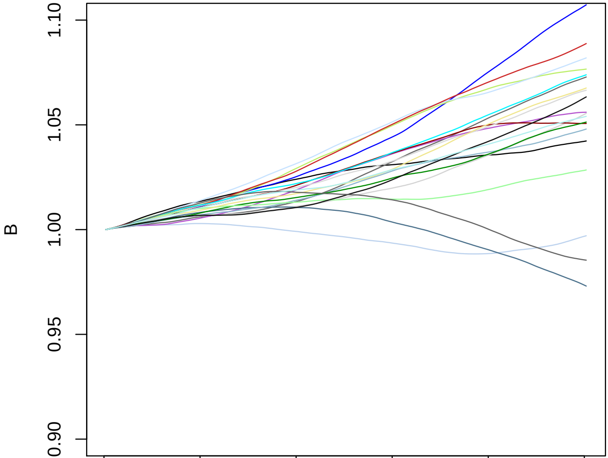
$$B(t\frac{i+1}{n}) = B(t\frac{i}{n})\left(1+(R(t\frac{i}{n})-0.2R(t\frac{i}{n})\frac{t}{n}+0.03\sqrt{\frac{t}{n}}N)\frac{t}{n}\right),$$

where N is a standard normal random variable.

Bank account in the Vasiček model

```
> B0 = 1
> X0 = 0.05
> b = 0
> beta = -0.2
> alpha = 0.03
> time = 1
> n = 250
> m = 20
> x <- (1:657)
> y <- sample(x)
> for (j in (1:m)) {
+     B = vector(length = n + 1)
+     X = vector(length = n + 1)
+     X[1] <- X0
+     B[1] <- 1
+     for (i in (1:n)) {
+         W <- rnorm(1)
+         X[i + 1] <- (X[i] + W * (sqrt(time/n)) * alpha * 2) *
+             exp(beta * time/n) + b * time/n
+         B[i + 1] <- B[i] * (1 + X[i + 1] * time/n)
+     }
+     if (j == 1)
+         plot(B, type = "l", ylim = c(0.9, 1.1))
+     else lines(B, col = colors()[y[j]])
+ }
```

Bank account Scenarios with Vasiček-short-rate



- Second we take the simulation results and calculate the bond price (or any other derivative price) by the law of large numbers

$$P(0,t) = E(1/B(t)) \sim \frac{1}{m}\sum_{i=1}^{m} 1/B(t)(\omega_i).$$

- Collect the result again in a function called vasicekZCB with input parameters t, n and m. For large m we should obtain nice yield curves.

Exam

- No arbitrage theory: discounting, numéraire, martingale measure for discounted prices, arbitrage.
- Notions of interest rate theory: yield, forward rate, short rate, simple forward rate, zero coupon bond, cap, floor.
- one calculation with Black's formula in the forward measure.

[Albrecher et al.(2009)Albrecher, Binder, and Mayer] Hansjörg Albrecher, Andreas Binder, and Philipp Mayer.

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