Constrained Optimal Transport

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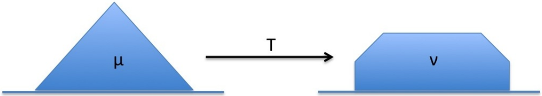
Outline

1. Classical Optimal Transport
2. Martingale Optimal Transport
3. Supermartingale Optimal Transport

Monge Optimal Transport

Given:

- Probabilities μ, ν on $\mathbb{R}$.
- Reward (cost) function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.



Objective:

- Find a map $T : \mathbb{R} \to \mathbb{R}$ satisfying $\nu = \mu \circ T^{-1}$ such as to maximize the total reward,

$$\max_T \int f(x, T(x))\, \mu(dx).$$

Monge–Kantorovich Optimal Transport

Relaxation:

- Find a probability P on $\mathbb{R} \times \mathbb{R}$ with marginals μ, ν such as to maximize the reward:

$$\max_{P \in \Pi(\mu,\nu)} E^P[f(X,Y)], \quad \text{where} \quad \Pi(\mu,\nu) := \{P : \; P_1 = \mu, \; P_2 = \nu\}$$

  and $(X, Y) = \mathrm{Id}_{\mathbb{R}\times\mathbb{R}}$.

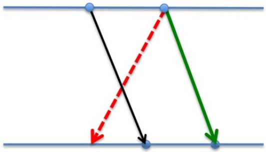
- $P \in \Pi(\mu,\nu)$ is a Monge transport if of the form $P = \mu \otimes \delta_{T(x)}$.

Example: Hoeffding–Frechet Coupling

Theorem: Let f satisfy the Spence–Mirrlees condition $f_{xy} > 0$. Then the optimal P is unique and given by the Hoeffding–Frechet Coupling:

- P is the law of $((F_\mu)^{-1}, (F_\nu)^{-1})$ under the uniform measure on $[0, 1]$.
- If μ is diffuse, P is of Monge type with $T = (F_\nu)^{-1} \circ F_\mu$.
- P is characterized by monotonicity:

$$\text{if } (x, y), (x', y') \in \text{supp}(P) \text{ and if } x < x', \text{ then } y \leq y'.$$



Kantorovich Duality

- Buy $\varphi(X)$ at price $\mu(\varphi) := E^\mu[\varphi]$ and $\psi(Y)$ at $\nu(\psi)$ to superhedge,

$$f(X,Y) \leq \varphi(X) + \psi(Y).$$

- Then for all $P \in \Pi(\mu,\nu)$,

$$E^P[f(X,Y)] \leq E^P[\varphi(X)+\psi(Y)] = \mu(\varphi) + \nu(\psi).$$

- **Theorem** (Kantorovich, Kellerer): For any measurable $f \geq 0$,

$$\sup_{P\in\Pi(\mu,\nu)} E^P[f(X,Y)] = \inf_{\varphi,\psi} \mu(\varphi)+\nu(\psi)$$

and dual optimizers $\hat\varphi$, $\hat\psi$ exist.

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Application: Fundamental Theorem of Optimal Transport

Let $\Gamma = \{(x, y) : \hat{\varphi}(x) + \hat{\psi}(y) = f(x, y)\}$ and $P \in \Pi(\mu, \nu)$. TFAE:

(1) P is optimal.

(2) $P(\Gamma) = 1$.

(3) supp(P) is f-cyclically monotone P-a.s.; i.e.,

$$\sum_{i=1}^{n} f(x_i, y_i) \geq \sum_{i=1}^{n} f(x_i, y_{\sigma(i)}) \quad \forall\, (x_i, y_i) \in \text{supp}(P), \quad \sigma \in \text{Perm}(n).$$

(1)(2) If $P(\Gamma) < 1$, then P charges $\{(x, y) : \hat{\varphi}(x) + \hat{\psi}(y) > f(x, y)\}$ and thus $\mu(\hat{\varphi}) + \nu(\hat{\psi}) > E^P[f(X, Y)]$.

(2)(1) If $P(\Gamma) = 1$, then $\mu(\hat{\varphi}) + \nu(\hat{\psi}) = E^P[f(X, Y)]$, hence $P, \hat{\varphi}, \hat{\psi}$ are optimal.

(2)(3) This argument even shows: if $\tilde{P}(\Gamma) = 1$, then $\tilde{P}$ is an optimal transport between its own marginals. Apply this with discrete $\tilde{P}$ $\Rightarrow \Gamma$ is cyclically monotone.

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Outline

Dynamic Hedging

- Dynamically tradable underlying $S = (S_0, S_1, S_2)$.
- Semi-static superhedge:

$$f((S_t)_t) \leq \varphi(S_1) + \psi(S_2) + H_0(S_1 - S_0) + H_1(S_2 - S_1).$$

- With $S_0 = 0$, $S_1 = X \sim \mu$, $S_2 = Y \sim \nu$ and normalization $H_0 = 0$:

$$f(X, Y) \leq \varphi(X) + \psi(Y) + h(X)(Y - X).$$

- Formally, duality with $P \in \Pi(\mu, \nu)$ satisfying the constraint that

$$E^P[h(X)(Y - X)] = 0 \quad \forall h; \quad \text{i.e.} \quad E^P[Y|X] = X.$$

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Martingale Transport

- Set of martingale transports:

$$\mathcal{M}(\mu,\nu) = \{P \in \Pi(\mu,\nu) : E^P[Y|X] = X\}.$$

- **Theorem** (Strassen): $\mathcal{M}(\mu,\nu)$ is nonempty iff $\mu \leq_c \nu$; i.e.,

$$\mu(\phi) \leq \nu(\phi) \quad \forall\, \phi \text{ convex}.$$

- Martingale Optimal Transport problem: Given $\mu \leq_c \nu$,

$$\sup_{P \in \mathcal{M}(\mu,\nu)} E^P[f(X,Y)].$$

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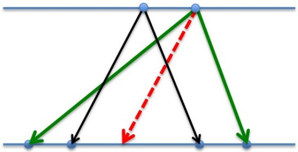
Example: Left-Curtain Coupling

- **Theorem** (Beiglböck, Juillet): Let f satisfy the martingale Spence–Mirrlees condition $f_{xyy} > 0$. Then the optimal P is given by the Left-Curtain Coupling:



Left-Curtain Coupling for Uniform Marginals

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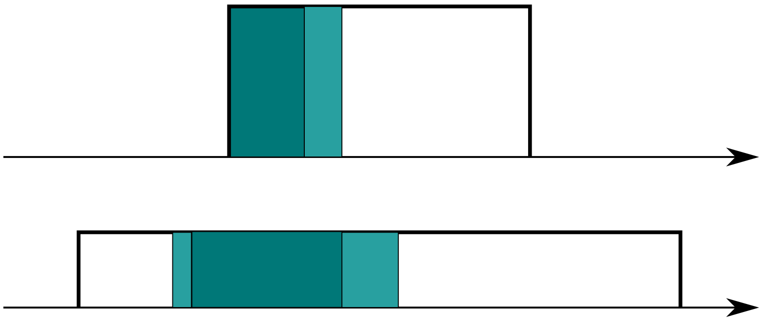

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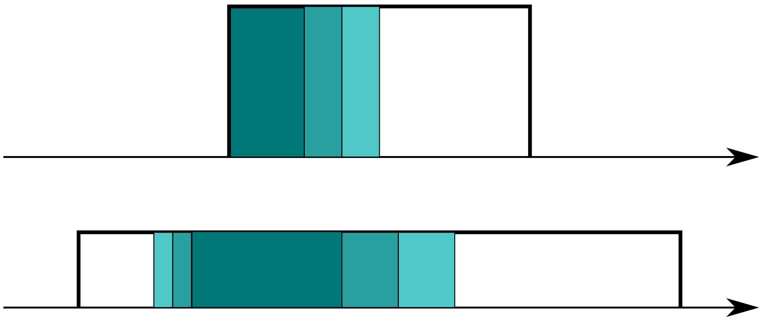

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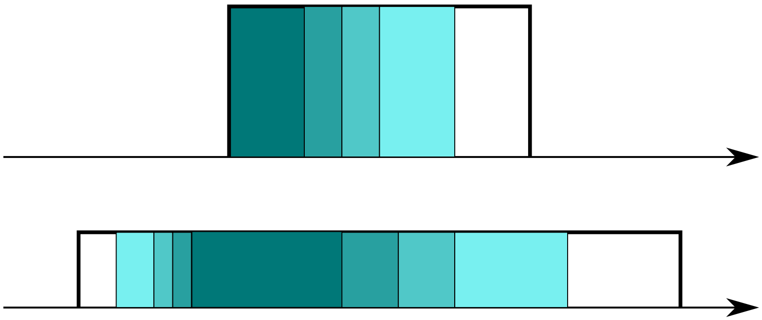

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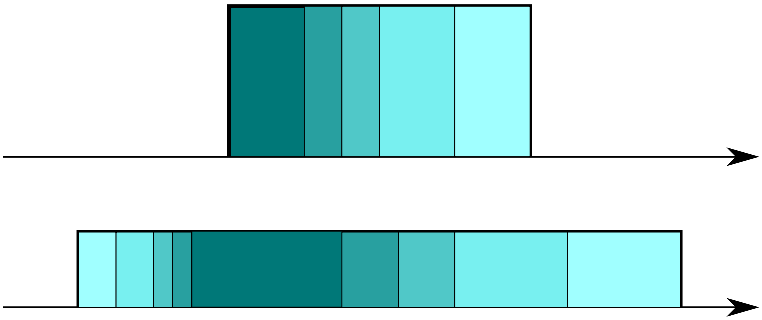

Duality for Martingale Optimal Transport

In analogy to Monge–Kantorovich duality we want:

(1) No duality gap:

$$\sup_{P \in \mathcal{M}(\mu,\nu)} E^P[f(X,Y)] = \inf_{\varphi,\psi,h} \mu(\varphi) + \nu(\psi).$$

(2) Dual existence: $\hat{\varphi}$, $\hat{\psi}$, $\hat{h}$.

Theorem (Beiglböck, Henry-Labordère, Penkner):

- For upper semicontinuous $f \leq 0$, there is no duality gap.
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An Example with Duality Gap

- Let f be the bounded, lower semicontinuous function

$$f(x,y) = \mathbf{1}_{x \neq y} = \begin{cases} 0 & \text{on the diagonal,} \\ 1 & \text{off the diagonal.} \end{cases}$$

- Let $\mu = \nu =$ Lebesgue measure on $[0,1]$.
- There exists a unique martingale transport P, concentrated on the diagonal ($T(x) = x$).
- Primal value: $\sup_{P \in \mathcal{M}(\mu,\nu)} E^P[f(X,Y)] = 0$.
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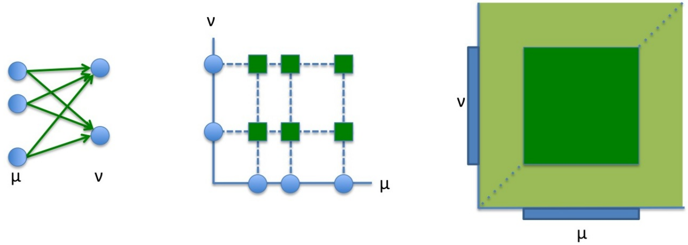
Ordinary and Martingale OT: What is the Difference?

- In ordinary OT, all roads $x \to y$ can be used.
- E.g., in the discrete case, $\mu \times \nu$ already has full support.



- **Theorem** (Kellerer): $A \subseteq \mathbb{R} \times \mathbb{R}$ is $\Pi(\mu, \nu)$-polar if and only if

$$A \subseteq (N_1 \times \mathbb{R}) \cup (\mathbb{R} \times N_2), \quad \text{where} \quad \mu(N_1) = \nu(N_2) = 0.$$

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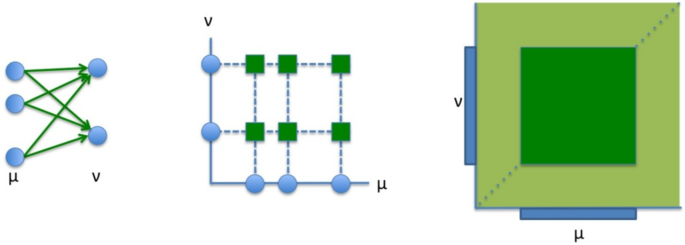

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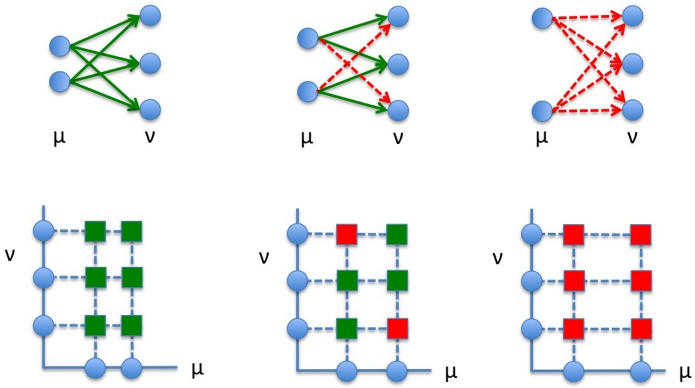
Obstructions for Martingale Transport

- In martingale OT, some roads $x \to y$ can be blocked.



Potential Functions

Potential $u_\mu(x) := \int |t-x|\,\mu(dt) = E[|X-x|]$ under any $P \in \mathcal{M}(\mu,\nu)$.

- $\mu \leq_c \nu \iff u_\mu \leq u_\nu$.
- If

$$u_\mu(x) = u_\nu(x); \quad \text{i.e.} \quad E[|X-x|] = E[|Y-x|] \qquad (*),$$

then x is a barrier for any martingale transport:

1. Jensen: $|X-x| = |E[Y|X]-x| = |E[Y-x|X]| \leq E[|Y-x|\,|X]$
2. Under $(*)$, it follows that $|X-x| = E[|Y-x|\,|X]$ a.s. Hence,

$$E[|Y-x|\mathbf{1}_{X\geq x}] = E[|X-x|\mathbf{1}_{X\geq x}] = E[(X-x)\mathbf{1}_{X\geq x}] = E[(Y-x)\mathbf{1}_{X\geq x}]$$

so that $Y \geq x$ a.s. on $\{X \geq x\}$.

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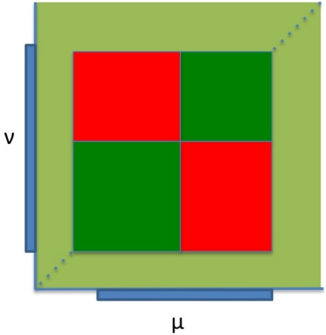
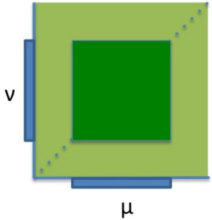
Structure of $\mathcal{M}(\mu,\nu)$-polar Sets





Theorem: "These are precisely the $\mathcal{M}(\mu,\nu)$-polar sets."

Duality Result

Theorem

Let $f \geq 0$ be measurable and consider the quasi-sure relaxation of the dual problem:

$$f(X, Y) \leq \varphi(X) + \psi(Y) + h(X)(Y - X) \quad \mathcal{M}(\mu, \nu)\text{-q.s.}$$

Then,

(1) *there is no duality gap,*

(2) *dual optimizers $\hat{\varphi}$, $\hat{\psi}$, $\hat{h}$ exist.*

- The superhedge is pointwise on each component (e.g., $\mu = \delta_{x_0}$).
- Dual existence in the pointwise formulation typically fails as soon as there is more than one component.
- Application as in the FTOT.

Outline

Supermartingale Optimal Transport

- Set of supermartingale transports:

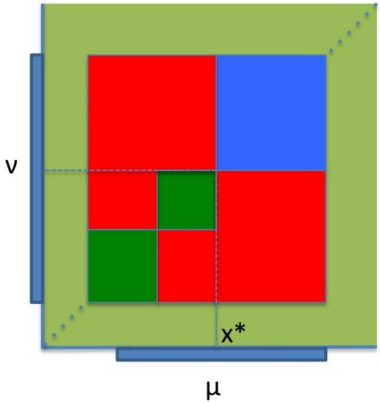
$$\mathcal{S}(\mu,\nu) = \{P \in \Pi(\mu,\nu) : \; E^P[Y|X] \leq X\}.$$

- $\mathcal{S}(\mu,\nu)$ is nonempty iff $\mu \leq_{cd} \nu$; i.e.,

$$\mu(\phi) \leq \nu(\phi) \quad \forall\, \phi \text{ convex decreasing}.$$

- Coincides with MOT if μ, ν have same mean, and with OT if supports are ordered.

Structure of $\mathcal{S}(\mu, \nu)$-polar Sets



Theorem: There exist a maximal barrier x^* such that:

- martingale transport on $(-\infty, x^*]$,
- single component of strict supermartingale transport on $[x^*, \infty)$.

Duality for Supermartingale Transport

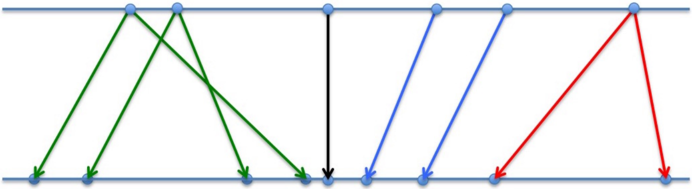
- Duality results similar to martingale case,
- with additional constraint $h \geq 0$ (long-only hedging).
- Duality leads to a version of the Fundamental Theorem with an additional condition of complementary slackness:

$$E^P[h(X)(Y - X)] = 0.$$

Decomposition of Optimal Supermartingale Couplings

Let $P \in \mathcal{S}(\mu, \nu)$ be optimal. Then $J_0 := \{h = 0\}$ and $J_1 := \{h > 0\}$ yield a (non-unique) decomposition:

- $\mathbb{R} = J_0 \cup J_1, \quad \mu = \mu_0 + \mu_1 := \mu|_{J_0} + \mu|_{J_1}$,
- $P|_{J_0 \times \mathbb{R}}$ is an optimal Monge–Kantorovich transport from μ_0 to $P(\mu_0)$,
- $P|_{J_1 \times \mathbb{R}}$ is an optimal martingale transport from μ_1 and $P(\mu_1)$.



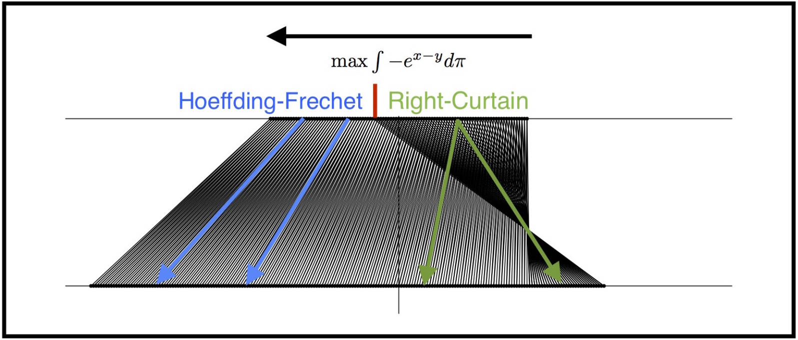
First Canonical Coupling

Theorem: Let f satisfy

1) $f_{xy} > 0$ and $f_{xyy} < 0$ e.g., $f(x, y) = -\exp(x - y)$;

Then the optimal P is exists, is unique and independent of f.

- Obtained by sending each bit of mass to the minimal destination relative to the convex-decreasing order.
- Here we work from right to left.



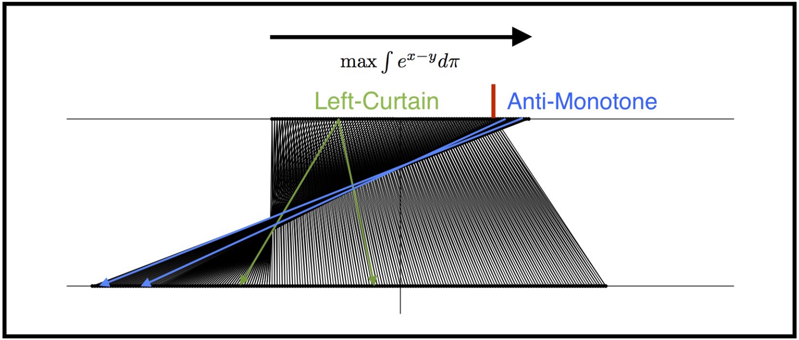
Second Canonical Coupling

Theorem: Let f satisfy

2) $f_{xy} < 0$ and $f_{xyy} > 0$ e.g., $f(x, y) = \exp(x - y)$;

Then the optimal P is exists, is unique and independent of f.

- Here we work from left to right.
- (No) symmetry?



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- Duality in a quasi-sure sense is useful for their analysis.
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