Fundamental Theorem of Asset pricing

Josef Teichmann

Autumn term 2020

Table of Contents

FTAP and its history

- The fundamental theorem of asset pricing (FTAP) is the single most important result in mathematical Finance.
- It states the equivalence of an economically convincing "absence of arbitrage" property (NFLVR) with the existence of an equivalent separating measure.
- The first complete proof has been presented by F. Delbaen and W. Schachermayer in [3].
- The proof is beautiful, impressive and tricky. We present its original version and an essential simplification obtained in 2014 in [1] based upon results of Kostas Kardaras.
- The most abstract version of FTAP elegantly worked out has been presented by Y. Kabanov in [4].

FTAP and its history

- The fundamental theorem of asset pricing (FTAP) is the single most important result in mathematical Finance.
- It states the equivalence of an economically convincing "absence of arbitrage" property (NFLVR) with the existence of an equivalent separating measure.
- The first complete proof has been presented by F. Delbaen and W. Schachermayer in [3].
- The proof is beautiful, impressive and tricky. We present its original version and an essential simplification obtained in 2014 in [1] based upon results of Kostas Kardaras.
- The most abstract version of FTAP elegantly worked out has been presented by Y. Kabanov in [4].

FTAP and its history

- The fundamental theorem of asset pricing (FTAP) is the single most important result in mathematical Finance.
- It states the equivalence of an economically convincing "absence of arbitrage" property (NFLVR) with the existence of an equivalent separating measure.
- The first complete proof has been presented by F. Delbaen and W. Schachermayer in [3].
- The proof is beautiful, impressive and tricky. We present its original version and an essential simplification obtained in 2014 in [1] based upon results of Kostas Kardaras.
- The most abstract version of FTAP elegantly worked out has been presented by Y. Kabanov in [4].

FTAP and its history

- The fundamental theorem of asset pricing (FTAP) is the single most important result in mathematical Finance.
- It states the equivalence of an economically convincing "absence of arbitrage" property (NFLVR) with the existence of an equivalent separating measure.
- The first complete proof has been presented by F. Delbaen and W. Schachermayer in [3].
- The proof is beautiful, impressive and tricky. We present its original version and an essential simplification obtained in 2014 in [1] based upon results of Kostas Kardaras.
- The most abstract version of FTAP elegantly worked out has been presented by Y. Kabanov in [4].

FTAP and its history

- The fundamental theorem of asset pricing (FTAP) is the single most important result in mathematical Finance.
- It states the equivalence of an economically convincing "absence of arbitrage" property (NFLVR) with the existence of an equivalent separating measure.
- The first complete proof has been presented by F. Delbaen and W. Schachermayer in [3].
- The proof is beautiful, impressive and tricky. We present its original version and an essential simplification obtained in 2014 in [1] based upon results of Kostas Kardaras.
- The most abstract version of FTAP elegantly worked out has been presented by Y. Kabanov in [4].

- We consider a finite time horizon $T = 1$ and a fixed probability space with usual conditions $(\Omega, \mathcal{F}, \mathbb{P})$.
- The set of semi-martingales on $[0, 1]$ starting at 0 is denoted by $\mathbb{SEM}$.
- We equip $\mathbb{SEM}$ with the Emery metric

$$\sup_{H \in b\mathcal{E}, \|H\| \leq 1} E[|(H \bullet (X - Y))|_1^* \wedge 1] = d_E(X, Y),$$

  making it a complete topological vector space.
- Pathwise uniform convergence in probability is metrized by

$$E[|X - Y|_1^* \wedge 1] = d(X, Y),$$

  which makes the set of càdlàg processes a complete topological vector space.

- We consider a finite time horizon $T = 1$ and a fixed probability space with usual conditions $(\Omega, \mathcal{F}, \mathbb{P})$.
- The set of semi-martingales on $[0, 1]$ starting at 0 is denoted by $\mathbb{SEM}$.
- We equip $\mathbb{SEM}$ with the Emery metric

$$\sup_{H \in b\mathcal{E}, \|H\| \leq 1} E[|(H \bullet (X - Y))|_1^* \wedge 1] = d_E(X, Y),$$

making it a complete topological vector space.
- Pathwise uniform convergence in probability is metrized by

$$E[|X - Y|_1^* \wedge 1] = d(X, Y),$$

which makes the set of càdlàg processes a complete topological vector space.

- We consider a finite time horizon $T = 1$ and a fixed probability space with usual conditions $(\Omega, \mathcal{F}, \mathbb{P})$.
- The set of semi-martingales on $[0, 1]$ starting at 0 is denoted by $\mathbb{SEM}$.
- We equip $\mathbb{SEM}$ with the Emery metric

$$\sup_{H \in b\mathcal{E}, \|H\| \leq 1} E[|(H \bullet (X - Y))|_1^* \wedge 1] = d_E(X, Y)\,,$$

making it a complete topological vector space.

- Pathwise uniform convergence in probability is metrized by

$$E[|X - Y|_1^* \wedge 1] = d(X, Y)\,,$$

which makes the set of càdlàg processes a complete topological vector space.

- We consider a finite time horizon $T = 1$ and a fixed probability space with usual conditions $(\Omega, \mathcal{F}, \mathbb{P})$.
- The set of semi-martingales on $[0, 1]$ starting at 0 is denoted by $\mathbb{SEM}$.
- We equip $\mathbb{SEM}$ with the Emery metric

$$\sup_{H \in b\mathcal{E}, \|H\| \leq 1} E[|(H \bullet (X - Y))|_1^* \wedge 1] = d_E(X, Y)\,,$$

making it a complete topological vector space.
- Pathwise uniform convergence in probability is metrized by

$$E[|X - Y|_1^* \wedge 1] = d(X, Y)\,,$$

which makes the set of càdlàg processes a complete topological vector space.

Definition

We consider a convex set $\mathcal{X}^1 \subset \mathbb{SEM}$ of semi-martingales starting at 0 and bounded from below by -1, which is closed in the Emery topology.
We assume that for all bounded, predictable strategies $H, G \geq 0$, $X, Y \in \mathcal{X}^1$ with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \mathcal{X}^1$.
We denote $\mathcal{X} = \cup_{\lambda > 0} \lambda \mathcal{X}^1$ and call its elements *admissible portfolio wealth processes*. We denote K_0, respectively K_0^1 the evaluations of elements of $\mathcal{X}$, respectively $\mathcal{X}^1$, at final time $T = 1$.
$\mathcal{X}$ is called (portfolio) wealth (value) process set.

Remark

The convex set $\mathcal{X}$ should be considered a set of discounted, self-financing value (wealth) processes of portfolios in a market segment.

Dynamic trading in portfolios in $\mathcal{X}$ leads again to a portfolio in $\mathcal{X}$ in the sense described in the concatenation property. When trading always a numeraire process $X^0 = 1$ is added in which we hold the predictable (sic!) amount

$$H^0 := (H \bullet X)_- + (G \bullet Y)_- - HX_- - GY_- ,$$

which is the self-financing condition.

Remark

The self-financing condition is the very reason why we consider discounted value processes as semi-martingales, since small changes in strategies should not lead to large changes in value processes.

The set $\mathcal{X}$ is chosen minimal to describe market activities, later we shall have larger sets of portfolio value processes.

Example

Let S be a semi-martigale and consider $\mathcal{X}$ to be the set of $(H \bullet S)$, for $H_0 = 0$ such that

$$(H \bullet S) \geq -\lambda$$

for some $\lambda \geq 0$. Then $\mathcal{X}$ satisfies all assumptions of a portfolio wealth process set.

Indeed by the very construction of stochastic integrals the set is closed in the Emery topology and the concatenation property holds since

$$(H_1 \bullet (G_1 \bullet S)) + (H_2 \bullet (G_2 \bullet S)) = ((H_1 G_1 + H_2 G_2) \bullet S) .$$

Notions of No Arbitrage

(NA) The set $\mathcal{X}$ is said to satisfy No Arbitrage if $K_0 \cap L^0_{\geq 0} = \{0\}$ which can be shown to be equivalent to $C \cap L^\infty_{\geq 0} = \{0\}$, with $C = C_0 \cap L^\infty$, where $C_0 = K_0 - L^0_{\geq 0}$.

(NFLVR) The set $\mathcal{X}$ is said to satisfy No free lunch with vanishing risk if

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}$ denotes the norm closure in L^∞.

(NFL) The set $\mathcal{X}$ is said to satisfy No free lunch if

$$\overline{C}^* \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}^*$ denotes the weak-$*$-closure in L^∞.

Notions of No Arbitrage

(NUPBR) The set $\mathcal{X}$ is said to satisfy No unbounded profit with bounded risk if K_0^1 is bounded in L^0. In the setting of Y. Kabanov this condition is called (BK) property.

Notions of No Arbitrage

(NUPBR) The set $\mathcal{X}$ is said to satisfy No unbounded profit with bounded risk if K_0^1 is bounded in L^0. In the setting of Y. Kabanov this condition is called (BK) property.

- (NFLVR) $\Leftrightarrow$ (NA) + (NUPBR)

Notions of No Arbitrage

(NUPBR) The set $\mathcal{X}$ is said to satisfy No unbounded profit with bounded risk if K_0^1 is bounded in L^0. In the setting of Y. Kabanov this condition is called (BK) property.

- (NFLVR) $\Leftrightarrow$ (NA) + (NUPBR)
- Both (NFLVR) and (NUPBR) are economically convincing minimal requirement for models, but only (NFL) allows to conclude relatively directly the existence of an equivalent separating measure, defined via

Notions of No Arbitrage

(NUPBR) The set $\mathcal{X}$ is said to satisfy No unbounded profit with bounded risk if K_0^1 is bounded in L^0. In the setting of Y. Kabanov this condition is called (BK) property.

- (NFLVR) $\Leftrightarrow$ (NA) + (NUPBR)
- Both (NFLVR) and (NUPBR) are economically convincing minimal requirement for models, but only (NFL) allows to conclude relatively directly the existence of an equivalent separating measure, defined via

Definition

The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property if there exists an equivalent measure $Q \sim P$ such that $\mathbb{E}_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.

Scalable versus non-scalable arbitrage

When it comes to arbitrages, i.e. non-vanishing $X_1 \in K_0 \cap L^0_{\geq 0}$, there are two cases:

- X_1 is called a *scalable arbitrage* if $X \in \mathcal{X}^0$, i.e. $X_t \geq 0$ for all $t \in [0,1]$. This contradicts of course (NUPBR).
- X_1 is called a *non-scalable arbitrage* if $X \notin \mathcal{X}^0$, but of course $X \notin \mathcal{X}^\lambda$ for some $\lambda > 0$. This does not necessarily contradict (NUPBR) and are precisely the arbitrages which can appear in the presence of (NUPBR).

(NFL) implies (ESM)

- It is a consequence of Hahn-Banach's Theorem (the Kreps-Yan Theorem) that (NFL) implies the existence of an equivalent measure $Q \sim P$ such that $E_Q[f] \leq 0$ for all $f \in C$ and hence for all $f \in K_0$.

(NFL) implies (ESM)

- It is a consequence of Hahn-Banach's Theorem (the Kreps-Yan Theorem) that (NFL) implies the existence of an equivalent measure $Q \sim P$ such that $E_Q[f] \leq 0$ for all $f \in C$ and hence for all $f \in K_0$.
- Apparently it holds that

$$(NFL) \Rightarrow (NFLVR) \Rightarrow (NA),$$

but it is a deep insight that under (NFLVR) it holds that $C = \overline{C}^*$, i.e. the cone C is already weak-$*$-closed and (NFL) holds.

(NFL) implies (ESM)

- It is a consequence of Hahn-Banach's Theorem (the Kreps-Yan Theorem) that (NFL) implies the existence of an equivalent measure $Q \sim P$ such that $E_Q[f] \leq 0$ for all $f \in C$ and hence for all $f \in K_0$.
- Apparently it holds that

$$(NFL) \Rightarrow (NFLVR) \Rightarrow (NA)\,,$$

  but it is a deep insight that under (NFLVR) it holds that $C = \overline{C}^*$, i.e. the cone C is already weak-$*$-closed and (NFL) holds.
- The goal is to show (NFLVR) $\Rightarrow C = \overline{C}^*$.
  Recall (NFLVR) $\Leftrightarrow$ (NA) + (NUPBR).

(NFLVR) $\Leftrightarrow$ (NA) + (NUPBR)

Let $\mathcal{X}$ be a value process set and assume (NA). Then for $X \in \mathcal{X}$

$$X \in \lambda \mathcal{X}^1$$

with $\lambda = \|X_1^-\|_\infty$.

This is true. Indeed, if there is $s \in [0,1]$ such that $P(X_s < -\lambda) > 0$, then $(1_{\{X_s<-\lambda\}} 1_{]s,1]} \bullet X)_1$ violates (NA).

(NFLVR) $\Leftrightarrow$ (NA) + (NUPBR)

The following conditions are equivalent:

- (NFLVR)
- For every sequence (g_n) in K_0 with $\|g_n^-\|_\infty \to 0$ we obtain that $g_n \to 0$ in probability.
- (NA) + (NUPBR)

Proof

Assume (NFLVR) and take a sequence (g_n) in K_0 with $\|g_n^-\|_\infty \to 0$ with $P(g_n \geq \alpha) \geq \alpha$ for some $\alpha > 0$ and all n. Then $f_n := g_n \wedge 1 \in C$ and by Komlos Lemma forward convex combinations $\tilde{f}_n$ converge to $\tilde{f} \geq 0$ with $P(\tilde{f} > 0) > 2\beta$. By Egorov's theorem there a set Γ with probability larger than $1 - \beta$ where $\tilde{f}_n$ converges uniformly, which contradicts (NFLVR) since

$$C \ni \tilde{f}_n 1_\Gamma - \tilde{f}_n^- 1_{\Gamma^c} \to \tilde{f} 1_\Gamma \, .$$

Assume now the second condition: (NA) follows immediately, and if (NUPBR) fails there is $g_n \in K_0^1$ with $P(g_n > n) > \epsilon > 0$, then $(1/n g_n)$ contradicts the assumption.

Proof

If (FLVR), we have a sequence (f_n) with $\|f_n - f\|_\infty \leq 1/n$ with $P(f > 0) > 0$. By definition we find $X_1^n = h_n \geq f_n$ for portfolios $X^n \in \mathcal{X}$. Apparently $nX^n \in \mathcal{X}^1$ by (NA) and the first observation and by Komlos Lemma we may assume $h_n \to h$ almost surely with $P(h > 0) > 0$. Then, however, the sequence $nX_1^n \in K_0^1$ violates (NUPBR).

Example

In the previous example with respect to one discounted price process S the existence of an equivalent separating measure means in case of locally bounded S that S is a local martingale with respect to the separating measure Q.

Indeed, we obtain that S^τ, for τ a stopping time which makes S^τ bounded, actually is a martingale by $(H \bullet S^\tau), -(H \bullet S^\tau) \in \mathcal{X}$ for all simple predictable, bounded H, whence $E_Q[(H \bullet S^\tau)_1] = 0$ for all simple predictable bounded H. This in turn means that S^τ is a martingale.

The other direction, namely that a local martingale measure for S is a separating measure, is the assertion of the Ansel-Stricker Lemma, which we shall see later.

Reminder on functional analysis (concrete)

- for $p \in [1, \infty)$, the dual space $(L^p)^*$ of L^p is L^q with $\frac{1}{p} + \frac{1}{q} = 1$. This does not hold for $p = \infty$.
- the pairing between L^p and L^q, for $p \in [1, \infty]$ is given by $(Y, Z) := \mathbb{E}[YZ]$ for $Y \in L^p$, $Z \in L^q$.
- on L^p for $p \in [1, \infty]$ we denote by $\sigma(L^p, L^q)$ the coarsest topology on L^p which makes linear functionals $Y \mapsto (Y, Z)$ continuous for all $Z \in L^q$. Hence $Y_n \to Y$ in $\sigma(L^p, L^q)$ iff $\mathbb{E}[Y_n Z] \to \mathbb{E}[YZ]$, $\forall Z \in L^q$.
- vice versa the dual space of L^p with the $\sigma(L^p, L^q)$-topology is L^q.
- for $p \in [1, \infty[$ the $\sigma(L^p, L^q)$ coincides with the so-called *weak topology*, since L^q is the dual space (with respect to the norm topology) of L^p.

Reminder on functional analysis (abstract)

- For Banach spaces X and point separating, closed subspaces of the dual space $Y \subset X^*$ we shall often speak of the weak toopology $\sigma(X, Y)$ on X which makes precisely the elements of Y continuous linear functionals. This is a locally convex topology on X and its strong dual, i.e. the set of continuous linear functionals is precisely Y.
- On dual spaces one often speaks of the *weak-$*$-topology*, i.e. the topology $\sigma(X^*, X)$ on X^*. In this topology the balls of X^* are compact by the Banach-Alaoglu theorem.

Reminder on functional analysis (abstract)

- Notice that weak topologies are metrizable if and only if X is finite dimensional, but balls in X^* are metrizable if and only if X is separable.
- The Hahn-Banach(-Helly) theorem on locally convex vector space tells that every closed convex set C can be separated by linear functionals from points outside of C.

Reminder on functional analysis (concrete)

- View L^p as the dual of L^q ($q \geq 1$), then the weak-$*$-topology is the coarsest topology on L^p which makes all linear functionals $Y \mapsto (Y, Z)$ continuous for all $Z \in L^q$. Hence, for $1 < p < \infty$, weak and weak-$*$-topology are the same. For $p = 1$, we only have the weak topology $\sigma(L^1, L^\infty)$ (since L^1 is not a dual space), and $Y_n \to Y$ in $\sigma(L^1, L^\infty)$ iff $\mathbb{E}[Y_n Z] \to \mathbb{E}[YZ]$ for all $Z \in L^\infty$. For $p = \infty$, we only have the weak-$*$-topology $\sigma(L^\infty, L^1)$ (since L^1 is not the norm-dual of L^∞); $Z_n \to Z$ in $\sigma(L^\infty, L^1)$ iff $\mathbb{E}[YZ_n] \to \mathbb{E}[YZ] \forall Y \in L^1$.
- we shall not use two many words but simply write $\sigma(L^p, L^q)$-topologies.

Reminder on functional analysis (concrete)

- The Hahn-Banach theorem for $\sigma(L^p, L^q)$-topologies reads as follows: let $C \subset L^p$ be a $\sigma(L^p, L^q)$-closed, convex cone and $x \notin C$, then there is a $l \in L^q$ such that $l(x) > 0 \geq l(C)$.
- Another important fact: for $p \in [1, \infty)$ a convex subset of L^p is weakly closed (i.e. closed in $\sigma(L^p, L^q)$) if and only if it is (strongly) closed in L^p, i.e. with respect to the norm topology. Hence the case of $\sigma(L^\infty, L^1)$ is of particular interest.

A simple proof of the Krein-Smulian theorem

Let X be a Banach space. The Krein-Smulian theorem tells that a convex subset $C \subset X^*$ is weak-$*$-closed if and only if its intersections with balls in X^* are weak-$*$-closed.

We can conclude this theorem from a separation theorem (see Conways' book on functional analysis): assume that for a convex set $C \subset X^*$ all its intersections with balls in X^* are weak-$*$-closed, and assume that the intersection of C with the unit ball (centered at 0) is empty, then there is $x \in X$ such that

$$(x, x^*) \geq 1$$

for all $x^* \in C$.

A simple proof of the Krein-Smulian theorem

From this we can conclude immediately: let $x^* \in X^*$ be in the weak-$*$=closure of C but not in C, then – due to the fact that C is norm closed (prove it!) – there is a ball of radius r around x^* which does not intersect C. Whene $r^{-1}(C - x^*)$ does not intersect the unit ball centered at 0. By the previous separation statement this however means that x^* cannot lie in the weak-$*$-closure of C.

1. The convex cone C is closed with respect to the weak-$*$-topology if and only if C_0 is Fatou-closed, i.e. for any sequence (f_n) in C_0 bounded from below and converging almost surely to f it holds that $f \in C_0$ (follows from Krein-Smulian theorem).
2. Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $f_n \leq g_n = Y_1^n$ with $Y^n \in \mathcal{X}$.
3. By (NA) it follows that $Y^n \in \mathcal{X}^1$.
4. By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \operatorname{conv}(Y^n, Y^{n+1}, \ldots)$ such that $\widetilde{Y_1^n} \to \widetilde{h}_0 \geq f$ almost surely.
5. Again by (NUPBR) it follows that we can find a sequence of semi-martingales $X^n \in \mathcal{X}^1$ such that $X_1^n \to h_0$ almost surely and h_0 is maximal above f with this property.

1. The convex cone C is closed with respect to the weak-$*$-topology if and only if C_0 is Fatou-closed, i.e. for any sequence (f_n) in C_0 bounded from below and converging almost surely to f it holds that $f \in C_0$ (follows from Krein-Smulian theorem).
2. Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $f_n \leq g_n = Y_1^n$ with $Y^n \in \mathcal{X}$.
3. By (NA) it follows that $Y^n \in \mathcal{X}^1$.
4. By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \operatorname{conv}(Y^n, Y^{n+1}, \dots)$ such that $\widetilde{Y_1^n} \to \widetilde{h}_0 \geq f$ almost surely.
5. Again by (NUPBR) it follows that we can find a sequence of semi-martingales $X^n \in \mathcal{X}^1$ such that $X_1^n \to h_0$ almost surely and h_0 is maximal above f with this property.

1. The convex cone C is closed with respect to the weak-$*$-topology if and only if C_0 is Fatou-closed, i.e. for any sequence (f_n) in C_0 bounded from below and converging almost surely to f it holds that $f \in C_0$ (follows from Krein-Smulian theorem).
2. Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $f_n \leq g_n = Y_1^n$ with $Y^n \in \mathcal{X}$.
3. By (NA) it follows that $Y^n \in \mathcal{X}^1$.
4. By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \mathrm{conv}(Y^n, Y^{n+1}, \ldots)$ such that $\widetilde{Y_1^n} \to \widetilde{h}_0 \geq f$ almost surely.
5. Again by (NUPBR) it follows that we can find a sequence of semi-martingales $X^n \in \mathcal{X}^1$ such that $X_1^n \to h_0$ almost surely and h_0 is maximal above f with this property.

1. The convex cone C is closed with respect to the weak-$*$-topology if and only if C_0 is Fatou-closed, i.e. for any sequence (f_n) in C_0 bounded from below and converging almost surely to f it holds that $f \in C_0$ (follows from Krein-Smulian theorem).
2. Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $f_n \leq g_n = Y_1^n$ with $Y^n \in \mathcal{X}$.
3. By (NA) it follows that $Y^n \in \mathcal{X}^1$.
4. By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \mathrm{conv}(Y^n, Y^{n+1}, \ldots)$ such that $\widetilde{Y_1^n} \to \widetilde{h}_0 \geq f$ almost surely.
5. Again by (NUPBR) it follows that we can find a sequence of semi-martingales $X^n \in \mathcal{X}^1$ such that $X_1^n \to h_0$ almost surely and h_0 is maximal above f with this property.

1. The convex cone C is closed with respect to the weak-$*$-topology if and only if C_0 is Fatou-closed, i.e. for any sequence (f_n) in C_0 bounded from below and converging almost surely to f it holds that $f \in C_0$ (follows from Krein-Smulian theorem).
2. Take now $-1 \leq f_n \in C_0$ converging almost surely to f. Then we can find $f_n \leq g_n = Y_1^n$ with $Y^n \in \mathcal{X}$.
3. By (NA) it follows that $Y^n \in \mathcal{X}^1$.
4. By (NUPBR) it follows that there are forward-convex combinations $\widetilde{Y^n} \in \operatorname{conv}(Y^n, Y^{n+1}, \ldots)$ such that $\widetilde{Y_1^n} \to \widetilde{h_0} \geq f$ almost surely.
5. Again by (NUPBR) it follows that we can find a sequence of semi-martingales $X^n \in \mathcal{X}^1$ such that $X_1^n \to h_0$ almost surely and h_0 is maximal above f with this property.

1. The previously constructed "maximal" sequence of semi-martingales $X^n \in \mathcal{X}^1$ converges in a pathwise uniform way in probability, i.e. $|X^n - X|_1^* \to 0$ in probability for some càdlàg process X.
2. It is now the goal to show that indeed $X^n \to X$ in the Emery topology, an apparently much stronger statement. Convergence in the Emery topology can be shown with respect to any equivalent measure $Q \sim P$, since this notion of convergence only depends on the equivalence class of probability measures.
3. By the basic convergence result (1) we know that $\xi := \sup_n |X^n|_1^* \in L^0$. We can therefore find a measure $Q \sim P$ (take, e.g., $dQ/dP = c \exp(-\xi)$) such that $X^n \in L^2(Q)$, hence we can continue the analysis with L^2-methods, in order to prove Emery-convergence with respect to Q. Now the proof starts!

1. The previously constructed "maximal" sequence of semi-martingales $X^n \in \mathcal{X}^1$ converges in a pathwise uniform way in probability, i.e. $|X^n - X|_1^* \to 0$ in probability for some càdlàg process X.
2. It is now the goal to show that indeed $X^n \to X$ in the Emery topology, an apparently much stronger statement. Convergence in the Emery topology can be shown with respect to any equivalent measure $Q \sim P$, since this notion of convergence only depends on the equivalence class of probability measures.
3. By the basic convergence result (1) we know that $\xi := \sup_n |X^n|_1^* \in L^0$. We can therefore find a measure $Q \sim P$ (take, e.g., $dQ/dP = c\exp(-\xi)$) such that $X^n \in L^2(Q)$, hence we can continue the analysis with L^2-methods, in order to prove Emery-convergence with respect to Q. Now the proof starts!

1. The previously constructed "maximal" sequence of semi-martingales $X^n \in \mathcal{X}^1$ converges in a pathwise uniform way in probability, i.e. $|X^n - X|_1^* \to 0$ in probability for some càdlàg process X.

2. It is now the goal to show that indeed $X^n \to X$ in the Emery topology, an apparently much stronger statement. Convergence in the Emery topology can be shown with respect to any equivalent measure $Q \sim P$, since this notion of convergence only depends on the equivalence class of probability measures.

3. By the basic convergence result (1) we know that $\xi := \sup_n |X^n|_1^* \in L^0$. We can therefore find a measure $Q \sim P$ (take, e.g., $dQ/dP = c \exp(-\xi)$) such that $X^n \in L^2(Q)$, hence we can continue the analysis with L^2-methods, in order to prove Emery-convergence with respect to Q. Now the proof starts!

Assume (NUPBR), take a sequence of (special) semi-martingales $X^n = A^n + M^n$ whose sup-processes are uniformly bounded in L^2.

1. First key Lemma: the sequence $|M^n|^*$ is bounded in L^0.
2. Second key Lemma: define $\tau_c^n := \inf\{t \mid |M^n|^* > c\}$ for some $c > 0$, $X_c^n := (1_{[\tau_c^n,\infty]} \bullet X^n)$, then for every $\epsilon > 0$ there is $c_0 > 0$ such that for all

$$\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$$

it holds that $Q[|\widetilde{M}|^* > \epsilon] \leq \epsilon$.
3. Third key Lemma: for every $\delta > 0$ there is $c_0 > 0$ such that for all $\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$ it holds that $d_E(\widetilde{M}, 0) \leq \delta$.
4. Fourth key Lemma: there exist $\widetilde{X^n} \in \operatorname{conv}(X_n, \ldots)$ such that $\widetilde{M^n} \to \widetilde{M}$ in the Emery topology.

Assume (NUPBR), take a sequence of (special) semi-martingales $X^n = A^n + M^n$ whose sup-processes are uniformly bounded in L^2.

1. First key Lemma: the sequence $|M^n|^*$ is bounded in L^0.
2. Second key Lemma: define $\tau_c^n := \inf\{t \mid |M^n|^* > c\}$ for some $c > 0$, $X_c^n := (1_{[\tau_c^n,\infty[} \bullet X^n)$, then for every $\epsilon > 0$ there is $c_0 > 0$ such that for all
$$\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \dots, X_c^n, \dots)$$
it holds that $Q[|\widetilde{M}|^* > \epsilon] \leq \epsilon$.
3. Third key Lemma: for every $\delta > 0$ there is $c_0 > 0$ such that for all $\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \dots, X_c^n, \dots)$ it holds that $d_E(\widetilde{M}, 0) \leq \delta$.
4. Fourth key Lemma: there exist $\widetilde{X^n} \in \operatorname{conv}(X_n, \dots)$ such that $\widetilde{M^n} \to \widetilde{M}$ in the Emery topology.

Assume (NUPBR), take a sequence of (special) semi-martingales $X^n = A^n + M^n$ whose sup-processes are uniformly bounded in L^2.

1. First key Lemma: the sequence $|M^n|^*$ is bounded in L^0.
2. Second key Lemma: define $\tau_c^n := \inf\{t \mid |M^n|^* > c\}$ for some $c > 0$, $X_c^n := (1_{[\tau_c^n,\infty]} \bullet X^n)$, then for every $\epsilon > 0$ there is $c_0 > 0$ such that for all
$$\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$$
it holds that $Q[|\widetilde{M}|^* > \epsilon] \leq \epsilon$.
3. Third key Lemma: for every $\delta > 0$ there is $c_0 > 0$ such that for all $\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$ it holds that $d_E(\widetilde{M}, 0) \leq \delta$.
4. Fourth key Lemma: there exist $\widetilde{X^n} \in \operatorname{conv}(X_n, \ldots)$ such that $\widetilde{M^n} \to \widetilde{M}$ in the Emery topology.

Assume (NUPBR), take a sequence of (special) semi-martingales $X^n = A^n + M^n$ whose sup-processes are uniformly bounded in L^2.

1. First key Lemma: the sequence $|M^n|^*$ is bounded in L^0.
2. Second key Lemma: define $\tau_c^n := \inf\{t \mid |M^n|^* > c\}$ for some $c > 0$, $X_c^n := (1_{[\tau_c^n,\infty]} \bullet X^n)$, then for every $\epsilon > 0$ there is $c_0 > 0$ such that for all
$$\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$$
it holds that $Q[|\widetilde{M}|^* > \epsilon] \leq \epsilon$.
3. Third key Lemma: for every $\delta > 0$ there is $c_0 > 0$ such that for all $\widetilde{X} \in \cup_{c \geq c_0} \operatorname{conv}(X_c^1, \ldots, X_c^n, \ldots)$ it holds that $d_E(\widetilde{M}, 0) \leq \delta$.
4. Fourth key Lemma: there exist $\widetilde{X^n} \in \operatorname{conv}(X_n, \ldots)$ such that $\widetilde{M^n} \to \widetilde{M}$ in the Emery topology.

Proposition on the Emery convergence of the finite variation part

Assume (NUPBR). Let $\widetilde{X^n} = \widetilde{M^n} + \widetilde{A^n} \in \mathcal{X}^1$ be a sequence of special semi-martingales converging to a maximal element h_0 such that $\widetilde{M^n} \to \widetilde{M}$ converges in the Emery topology, then $\widetilde{A^n} \to \widetilde{A}$ in the Emery topology.

From this proposition it follows by the fact that *the set $\mathcal{X}^1$ is closed in the Emery topology* that $f_0 \in C_0$.

Discussion of the proof

- the proof is beautiful but quite tricky.
- the change of measure is technical and not fully motivated from the point of view of mathematical finance.
- it remains open within the proof if the forward convex combination passing from X^n to $\widetilde{X^n}$ are really necessary or if $X^n \to X$ already in the Emery topology.
- the series of key lemmas would deserve a theorem or property on its own.
- it would be interesting to obtain proofs, which can be easier communicated from a finance point of view.

Table of Contents

We take the following important definition from Jacod/Shiryaev:

Definition

We say that a sequence $(X^n)_{n\geq 0}$ of adapted, càdlàg processes satisfies the P-UT property (predictably uniformly tight) if the family of random variables $\{(H\bullet X^n)_1 \,:\, H\in b\mathcal{E}, \|H\|\leq 1, n\geq 0\}$ is bounded in L^0, that is,

$$\sup_{H\in b\mathcal{E},\|H\|\leq 1}\ \sup_{n\geq 0} P[|(H\bullet X^n)|_t \geq c] \to 0\,.$$

as $c\to\infty$.

The heart of our considerations now consists in proving that (NUPBR) implies P-UT for sequences of semi-martingales $X^n \to X$ converging uniformly along paths in probability. From this it will be (relatively) short way towards the existence of an equivalent separating measure:

The proof is done by separating off the big jumps and providing arguments for the remaining special semi-martingales to satisfy P-UT.

Denote by $\check{X}$ the process of jumps, whose absolute values are greater than some $C > 0$, that is,

$$\check{X}_t = \sum_{s \leq t} \Delta X_s 1_{\{|\Delta X_s| > C\}} \,. \tag{1}$$

Lemma

Let $(X^n)_{n \geq 0}$ together with an adapted, càdlàg process X such that $|X^n - X|_1^* \to 0$ in probability as $n \to \infty$. Then the sequence $(\mathrm{TV}(\check{X}_1^n))_{n \geq 0}$ of total variations of $\check{X}^n$ is bounded in L^0, i.e., for every $\varepsilon > 0$ there exists some $c > 0$ such that

$$\sup_n \mathbb{P}\left[\sum_{s \leq 1} |\Delta X_s^n| 1_{\{|\Delta X_s^n| > C\}} \geq c\right] \leq \varepsilon.$$

Moreover, the sequence $(\check{X}^n)_{n \geq 0}$ satisfies the P-UT property.

Theorem

Assume (NUPBR). Let $(X^n)_{n\geq 0}$ together with an adapted, càdlàg process X such that $|X^n - X|_1^* \to 0$ in probability as $n \to \infty$ be a sequence in $\mathcal{X}^1$.

1. *Then for every $C > 0$ there exists a decomposition $X^n = M^n + B^n + \check{X}^n$ into a local martingale M^n, a predictable, finite variation process B^n and the finite variation process $\check{X}^n$, for $n \geq 0$, such that jumps of M^n and B^n are bounded by $2C$ uniformly in n.*
2. *The sequence $(|M^n|_1^*)_{n\geq 0}$ is bounded in L^0 and $(M^n)_{n\geq 0}$ satisfies P-UT (first key lemma).*
3. *The sequence $(\mathrm{TV}(B^n)_1)_{n\geq 0}$ of total variations of B^n is bounded in L^0 and $(B^n)_{n\geq 0}$ satisfies P-UT (the analogous statement on the finite variation part).*
4. *The sequence $(X^n)_{n\geq 0}$ satisfies P-UT.*

Proof

In contrast to the previous key lemmas, the proofs here have some straight forward aspect:

- (NUPBR) implies P-UT is based on the first key lemma with an additional analysis of the finite variation part.
- the P-UT property is a natural boundedness property in the Emery topology. It is therefore natural to investigate this property first.

YAP – a finance view point

Definition

A positive càdlàg adapted process D is called supermartingale deflator for $1+\mathcal{X}^1$ if D is strictly positive, $D_0 \leq 1$ and $D(1+X)$ is a supermartigale for all $X \in \mathcal{X}^1$.

YAP – a finance view point

Definition

A positive càdlàg adapted process D is called supermartingale deflator for $1+\mathcal{X}^1$ if D is strictly positive, $D_0 \leq 1$ and $D(1+X)$ is a supermartigale for all $X \in \mathcal{X}^1$.

Theorem (Karatzas and Kardaras (2007)/ Kardaras (2013))

Assume (NUPBR) for $\mathcal{X}$, then there exists a supermartingale deflator D.

(P-UT) property for supermartingales

Lemma

Let (Z^n) be a sequence of non-negative supermartingales such that $Z_0^n \leq K$ for all $n \in \mathbb{N}$ and some $K > 0$. Then (Z^n) satisfies the P-UT property.

(P-UT) property for supermartingales

Lemma

Let (Z^n) be a sequence of non-negative supermartingales such that $Z_0^n \leq K$ for all $n \in \mathbb{N}$ and some $K > 0$. Then (Z^n) satisfies the P-UT property.

Proof.

By an inequality of Burkholder for non-negative supermartingales S and processes $H \in b\mathcal{E}$ with $\|H\| \leq 1$ it holds that

$$cP[|(H \bullet S)|_1^* \geq c] \leq 9\mathbb{E}[|S_0|]$$

for all $c \geq 0$. Applying this inequality to Z^n and letting $c \to \infty$ yields the P-UT property. $\square$

(P-UT) property for sequences in $\mathcal{X}^1$

Proposition

Let $\mathcal{X}$ satisfy (NUPBR) and let $X^n \in \mathcal{X}^1$ be a sequence of semimartingales. Then (X^n) satisfies the P-UT property.

(P-UT) property for sequences in $\mathcal{X}^1$

Proposition

Let $\mathcal{X}$ satisfy (NUPBR) and let $X^n \in \mathcal{X}^1$ be a sequence of semimartingales. Then (X^n) satisfies the P-UT property.

Proof.

The (P-UT) property of the supermartingales $(Z^n) := (D(1 + X^n))$ can be easily transferred to the sequence (X^n). It relies on Itô's integration by parts formula and the fact that $(H^n_- \bullet S^n)$ satisfies (P-UT), if (S^n) is a sequence of semimartingales satisfying (P-UT) and (H^n) a sequence of adapted càdlàg processes such that $(|H^n|^*_1)_n$ is bounded in L^0. $\square$

Emery convergence for the local martingale and the big jump part under (P-UT) and up-convergence

For a sequence of semimartingales (X^n) with $X_0^n = 0$ and some $C > 0$ let us consider the following decomposition

$$X^n = B^{n,C} + M^{n,C} + \check{X}^{n,C}. \tag{2}$$

Emery convergence for the local martingale and the big jump part under (P-UT) and up-convergence

For a sequence of semimartingales (X^n) with $X_0^n = 0$ and some $C > 0$ let us consider the following decomposition

$$X^n = B^{n,C} + M^{n,C} + \check{X}^{n,C}. \tag{2}$$

Theorem (Memin and Slominski (1991))

Let (X^n) be a sequence of semimartingales with $X_0^n = 0$, which converges pathwise uniformly in probability to X and satisfies the (P-UT) property. Then there exists some $C > 0$ such that $M^{n,C} \to M^C$ and $\check{X}^{n,C} \to \check{X}^C$ in the Emery topology and $B^{n,C} \to B^C$ pathwise uniformly in probability.

Emery convergence for the finite variation part (without big jumps)

Proposition

Let $\mathcal{X}$ satisfy (NUPBR) and let $(X^n) \in \mathcal{X}^1$ be a sequence of semimartingales, which converges pathwise uniformly in probability to X such that X_1 is a maximal element in $\widehat{K_0^1}$. Assume that $M^{n,C} \to M^C$ and $\check{X}^{n,C} \to \check{X}^C$ in the Emery topology. Then $B^{n,C} \to B^C$ in the Emery topology.

Emery convergence for the finite variation part (without big jumps)

Proposition

Let $\mathcal{X}$ satisfy (NUPBR) and let $(X^n) \in \mathcal{X}^1$ be a sequence of semimartingales, which converges pathwise uniformly in probability to X such that X_1 is a maximal element in $\widehat{K_0^1}$. Assume that $M^{n,C} \to M^C$ and $\check{X}^{n,C} \to \check{X}^C$ in the Emery topology. Then $B^{n,C} \to B^C$ in the Emery topology.

Proof.

This follows essentially the proposition on Emery convergence in FTAP proof if martingale parts converge already. □

A convergence result in the Emery topology

Combining the above assertions yields...

Theorem

Let $\mathcal{X}$ satisfy (NUPBR) and let $(X^n) \in \mathcal{X}^1$ be a sequence of semimartingales, which converges pathwise uniformly in probability to X such that X_1 is a maximal element in $\widehat{K_0^1}$. Then $X^n \to X$ in the Emery topology.

A convergence result in the Emery topology

Combining the above assertions yields...

Theorem

Let $\mathcal{X}$ satisfy (NUPBR) and let $(X^n) \in \mathcal{X}^1$ be a sequence of semimartingales, which converges pathwise uniformly in probability to X such that X_1 is a maximal element in $\widehat{K_0^1}$. Then $X^n \to X$ in the Emery topology.

Proof.

This follows from ((NUPBR) $\Rightarrow$ (P-UT)), Memin and Slominski's theorem together with a modification of Part 3 of Y. Kabanov's proof. $\square$

Table of Contents

In this section we present the essential parts of the construction of super-martingale deflators for general markets. This is a beautiful almost self-contained part of mathematical finance which sheds light on the proof of FTAP and also provides an alternative foundational point of view. In particular it relieves us from the numeraire dependence of (NFLVR).

Probabilistic notation and definitions

All stochastic elements in the sequel are defined on a probability space $(\Omega, \mathcal{G}, P)$, where $\mathcal{G}$ is a σ-field over Ω and P is a probability measure on $(\Omega, \mathcal{G})$. Fix some terminal time $T \in \mathbb{R}_{>0}$. We consider a right-continuous filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ such that $\mathcal{F}_t \subseteq \mathcal{G}$ holds for all $t \in [0, T]$ and $\mathcal{F}_0$ is trivial modulo P and contains all nullsets.

Processes will in general not be adapted but just a collection of $\mathcal{G}$-measurable random variables.

Stochastic Processes

A stochastic process X will be called *nonnegative* if $X_t \in L^0_{\geq 0}$ for all $t \in [0, T]$; X will be called *strictly positive* if $X_t \in L^0_{++}$ for all $t \in [0, T]$, i.e. $P[X_t > 0] = 1$. A nonnegative stochastic process X will be called cad if the mapping $[0, T] \ni t \mapsto X_t$ is right-continuous. Further, a nonnegative process X will be called cadlag if the mapping $[0, T] \ni t \mapsto X_t$ is right-continuous and admits left-hand limits.

The notions of process-continuity in the definition are weaker than the corresponding pathwise notions.

Generalized supermartingales

We now introduce a "supermartingale" property with respect to $\mathcal{F}$ for nonnegative processes, when these processes are not necessarily $\mathcal{F}$-adapted.

A *nonnegative* stochastic process Z will be called a generalized supermartingale with respect to $\mathcal{F}$ if $E[Z_t/Z_s \mid \mathcal{F}_s] \leq 1$ holds whenever $s \in [0, T]$ and $t \in [s, T]$.

We use the following convention: on $\{Z_s = 0,\ Z_t > 0\}$ we set $Z_t/Z_s = \infty$, while on $\{Z_s = 0,\ Z_t = 0\}$ we set $Z_t/Z_s = 1$. In particular, if Z is a nonnegative generalized supermartingale with respect to $\mathcal{F}$, then $P[Z_s = 0,\ Z_t > 0] = 0$ holds whenever $s \in [0, T]$ and $t \in [s, T]$.

Main Theorem

Let $\mathcal{X}$ be a set of stochastic processes such that:

(a) Each $X \in \mathcal{X}$ is nonnegative, cad, and satisfies $X_0 = 1$.

(b) There exists a strictly positive process $\overline{X} \in \mathcal{X}$.

(c) $\mathcal{X}$ is convex.

(d) $\mathcal{X}$ shares the switching property: for all $\tau \in [0, T]$ and $A \in \mathcal{F}_\tau$, all $X \in \mathcal{X}$, and all strictly positive $X' \in \mathcal{X}$, the process

$$1_{\Omega \setminus A} X_{\cdot} + 1_A \frac{X'_{\tau \vee \cdot}}{X'_\tau} X_{\tau \wedge \cdot} = \begin{cases} X_t(\omega), & \text{if } t \in [0, \tau[, \text{ or } \omega \notin A; \\ \frac{X_\tau(\omega)}{X'_\tau(\omega)} X'_t(\omega), & \text{if } t \in [\tau, T] \text{ and } \omega \in A \end{cases}$$

is also an element of $\mathcal{X}$.

Main Theorem

Then, the following statements are equivalent:

1. The set $\{X_T$ such that $X \in \mathcal{X}\}$ is bounded in probability: $\lim_{\ell \to \infty} \sup_{X \in \mathcal{X}} P[X_T > \ell] = 0$.
2. There exists a cadlag and strictly positive process Y such that YX is a generalized supermartingale with respect to $\mathcal{F}$ for all $X \in \mathcal{X}$.

Under any of the above equivalent conditions, each $X \in \mathcal{X}$ is cadlag.

Main Theorem

If $\mathcal{X}$ is such that *(a)* through *(d)* are satisfied and furthermore $\{X_T \text{ such that } X \in \mathcal{X}\}$ is closed in probability, conditions *(1)* and *(2)* above are also equivalent to:

(3) There exists a strictly positive wealth process $\hat{X} \in \mathcal{X}$ such that $X/\hat{X}$ is a generalized supermartingale with respect to $\mathcal{F}$ for all $X \in \mathcal{X}$.

Numeraire

When $\{X_T \text{ such that } X \in \mathcal{X}\}$ is bounded and closed in probability, it is natural to call a process $\hat{X}$ that satisfies condition (3) of the previous theorem above a *numeraire* in $\mathcal{X}$. Note that numeraires, if they exist, are unique up to modification, since $E[\hat{X}'_t/\hat{X}_t] \leq 1$ and $E[\hat{X}_t/\hat{X}'_t] \leq 1$ holds for all $t \in [0, T]$. Jensen's inequality gives that $P[\hat{X}_t = \hat{X}'_t] = 1$ for all $t \in [0, T]$.

Komlos Lemma

Let (f^n) be a sequence of non-negative random variables and define $\mathcal{C}^n$ as the convex hull of the set $\{f^n, f^{n+1}, \ldots\}$, for each n. Assume that $\mathcal{C}^1$ is bounded in probability. Then, there exists a non-negative random variable g and a sequence (g^n) such that $g^n \in \mathcal{C}^n$ for all n and $\lim g^n = g$ in probability.

Static version of the main theorem

Let $\mathcal{C} \subseteq L^0_{\geq 0}$ with $\mathcal{C} \cap L^0_{++} \neq \emptyset$. Assume that $\mathcal{C}$ is convex and closed in probability. Then, the following statements are equivalent:

1. $\mathcal{C}$ is bounded in probability.
2. There exists a strictly positive $g \in L^0_{++}$ such that $E[gf] \leq 1$ holds for all $f \in \mathcal{C}$.
3. There exists $\hat{f} \in \mathcal{C} \cap L^0_{++}$ such that $E[f/\hat{f}] \leq 1$ holds for all $f \in \mathcal{C}$.

Proof

Implication (3) $\Rightarrow$ (2) and Implication (2) $\Rightarrow$ (1) are immediate. We concentrate on (1) $\Rightarrow$ (3).

First we understand that we can without any restriction assume that C is solid, convex, closed and bounded in probability, as well as that $1 \in \mathcal{C}$.

Proof

Since $\mathcal{C}$ contains a strictly positive element g, we may assume that $1 \in \mathcal{C}$. Indeed, otherwise, we consider $\widetilde{\mathcal{C}} := (1/g)\mathcal{C}$. Then, $1 \in \widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ is still convex, closed and bounded in probability. Furthermore, if $E[f/\widetilde{f}] \leq 1$ holds for all $f \in \widetilde{\mathcal{C}}$, then, with $\hat{f} := g\widetilde{f}$, $E[f/\hat{f}] \leq 1$ holds for all $f \in \mathcal{C}$. Therefore, in the sequel we assume that $1 \in \mathcal{C}$.

Proof

We may also assume that $\mathcal{C}$ is solid. Indeed, let $\mathcal{C}'$ be the *solid hull* of $\mathcal{C}$, i.e., $\mathcal{C}' := \{f \in L^0_{\geq 0} \mid f \leq h$ holds for some $h \in \mathcal{C}\}$. Of course $1 \in \mathcal{C}'$, as well as that $\mathcal{C}'$ is still convex and bounded in probability. It is also true that $\mathcal{C}'$ is still closed in probability: indeed, pick a $\mathcal{C}'$-valued sequence (f^n) that converges P almost surely to f. Let (h^n) be a $\mathcal{C}$-valued sequence with $f^n \leq h^n$ for all n. By Komlos lemma we can extract a sequence of forward convex combinations converging to h, which satisfies by closedness $h \in \mathcal{C}$ and by construction $f \leq h$. Whence $f \in \mathcal{C}'$.

Clearly a numerarire for $\mathcal{C}'$ is also a numeraire for $\mathcal{C}$ since $\mathcal{C} \subset \mathcal{C}'$.

Proof

For all n, let $\mathcal{C}^n := \{f \in \mathcal{C} \text{ such that } f \leq n\}$, which is convex, closed and bounded in probability and satisfies $\mathcal{C}^n \subseteq \mathcal{C}$. Consider the following optimization problem:

$$\text{find } f_*^n \in \mathcal{C}^n \text{ such that } E[\log(f_*^n)] = \sup_{f \in \mathcal{C}^n} E[\log(f)].$$

Proof

Since $1 \in \mathcal{C}^n$ the solution of the above problem is not $-\infty$. Further, since $f \leq n$ for all $f \in \mathcal{C}^n$, one can use Komlos lemma and the inverse Fatou lemma to obtain the existence of the optimizer f_*^n.

For all $f \in \mathcal{C}^n$ and $\epsilon \in]0, 1/2]$, one has

$$E[\Delta_\epsilon(f|f_*^n)] \leq 0, \text{ where } \Delta_\epsilon(f|f_*^n) := \frac{\log((1-\epsilon)f_*^n + \epsilon f) - \log(f_*^n)}{\epsilon}.$$

Proof

Observe that $\Delta_\epsilon(f|f_*^n) \geq 0$ on the event $\{f > f_*^n\}$. Also, the inequality $\log(y) - \log(x) \leq (y-x)/x$, valid for $0 < x < y$, gives that, on $\{f \leq f_*^n\}$, the following lower bound holds (remember that $\epsilon \leq 1/2$):

$$\Delta_\epsilon(f|f_*^n) \geq -\frac{f_*^n - f}{f_*^n - \epsilon(f_*^n - f)} \geq -\frac{f_*^n - f}{f_*^n - (f_*^n - f)/2} = -2\frac{f_*^n - f}{f_*^n + f} \geq -2.$$

Using Fatou's Lemma on gives $E[(f - f_*^n)/f_*^n] \leq 0$ for all $f \in \mathcal{C}^n$.

Proof

Komlos Lemma again gives the existence of a sequence $(\hat{f}^n)$ of forward convex combinations of (f_*^n) and $\hat{f}$. Since $\mathcal{C}$ is convex, $\hat{f}^n \in \mathcal{C}$ for all n; therefore, since $\mathcal{C}$ is closed, $\hat{f} \in \mathcal{C}$ as well. Fix n and some $f \in \mathcal{C}^n$. For all $k \geq n$, we have $f \in \mathcal{C}^k$. Therefore, $E[f/f_*^k] \leq 1$, for all $k \geq n$. Since $\hat{f}^n$ is a finite convex combination of $f_*^n, f_*^{n+1}, \ldots$, an easy application of Jensen's inequality for the convex function $]0,\infty[\ni x \mapsto 1/x \in]0,\infty[$ gives that $E[f/\hat{f}^n] \leq 1$. Then, Fatou's lemma implies that for all $f \in \bigcup_k \mathcal{C}^k$ one has $E[f/\hat{f}] \leq 1$. The extension of the last inequality to all $f \in \mathcal{C}$ follows from the solidity of $\mathcal{C}$ by an application of the monotone convergence theorem.

Numeraire

By Jensen's inequality, an element $\hat{f} \in \mathcal{C}$ satisfying condition (3) of the previous theorem is necessarily unique. Therefore the next definition makes sense.

Let $\mathcal{C}$ be a convex, closed and bounded in probability set of nonnegative random variables containing a strictly positive element. The (unique) $\hat{f} \in \mathcal{C}$ satisfying condition (3) of the previous theorem is called numeraire.

Helpful lemma 1

Consider two sequences (g^n), (h^n) of non-negative random variables with $E[g^n] \leq 1$ and $E[h^n] \leq 1$ for all n, as well as $\lim(g^n h^n) = 1$ in probability. Then, $\lim g^n = 1 = \lim h^n$.

Proof

$\lim(g^n h^n) = 1$ implies that $\lim \sqrt{g^n h^n} = 1$, so

$$\limsup_{n\to\infty}(1 - E[\sqrt{g^n h^n}]) = 1 - \liminf_{n\to\infty} E[\sqrt{g^n h^n}] \leq 0,$$

from Fatou's Lemma, whence

$$E[(\sqrt{g^n} - \sqrt{h^n})^2] = E[g^n] + E[h^n] - 2E[\sqrt{g^n h^n}] \leq 2(1 - E[\sqrt{g^n h^n}]),$$

we obtain that $\lim[\sqrt{g^n} - \sqrt{h^n}] = 0$. In view of $g^n - h^n = \left(\sqrt{g^n} - \sqrt{h^n}\right)\left(\sqrt{g^n} + \sqrt{h^n}\right)$ and the fact that both sequences (g^n), (h^n) are bounded in probability (because $E[g^n] \leq 1$ and $E[h^n] \leq 1$ for all n), we also have $\lim(g^n - h^n) = 0$. Furthermore, the equality $g^n + h^n = \left(\sqrt{g^n} - \sqrt{h^n}\right)^2 + 2\sqrt{g^n h^n}$ gives $\lim(g^n + h^n) = 2$. Finally, combining $\lim(g^n - h^n) = 0$ and $\lim(g^n + h^n) = 2$ gives $\lim g^n = 1 = \lim h^n$.

Helpful lemma 2

For each $n \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{C}^n$ be a convex, closed and bounded subset of nonnegative random variables containing a strictly positive element, and let $\hat{f}^n \in \mathcal{C}^n$ be its numeraire. Then, $\lim \hat{f}^n = \hat{f}^\infty$ holds in either of the following cases:

1. $(\mathcal{C}^n)$ is nondecreasing and $\mathcal{C}^\infty$ is the closure in probability of $\bigcup_n \mathcal{C}^n$ and $\mathcal{C}^\infty$ is bounded.
2. $(\mathcal{C}^n)$ is nonincreasing and $\mathcal{C}^\infty = \bigcap_n \mathcal{C}^n$.

Proof

We shall drop all superscripts "∞" to ease the readability. To establish both statements (1) and (2) below, we shall just show the existence of a subsequence $(\hat{f}^{m_n})$ of $(\hat{f}^n)$ such that $\lim \hat{f}^{m_n} = \hat{f}$. By the same argument, it follows that *any* subsequence of $(\hat{f}^n)$ has a *further* subsequence that converges to $\hat{f}$. Since $L^0_{\geq 0}$ is equipped with a metric topology, this will imply that the whole sequence $(\hat{f}^n)$ converges to $\hat{f}$.

Proof of (1)

Komlos lemma gives the existence of a sequence $(\tilde{f}^n)$ such that each $\tilde{f}^n$ is a convex combination of $(\hat{f}^k)_{k=n,\dots,m_n}$ for some $n \le m_n$, and such that $\tilde{f} := \lim \tilde{f}^n$ exists. Of course, $\tilde{f} \in \mathcal{C}$ and we can also also assume that (m_n) is a strictly increasing sequence.

Since $E[f/\hat{f}^k] \le 1$ holds for all $f \in \mathcal{C}^n$ and $n \le k$, Jensen's inequality applied by using the convex function $]0,\infty[\ni x \mapsto 1/x \in]0,\infty[$ implies that $E[f/\tilde{f}^k] \le 1$ holds for all $f \in \mathcal{C}^n$ and $n \le k$. By Fatou's lemma, $E[f/\tilde{f}] \le 1$ holds for all n and $f \in \mathcal{C}^n$. In particular, $\tilde{f} \in \mathcal{C} \cap L^0_{++}$. As $(\mathcal{C}^n)$ is nondecreasing and $\mathcal{C}$ is the closure in probability of $\bigcup_n \mathcal{C}^n$, Fatou's lemma applied once again will give $E[f/\tilde{f}] \le 1$ for all $f \in \mathcal{C}$. By uniqueness of the numeraire, we get $\tilde{f} = \hat{f}$. Since $\hat{f} \in L^0_{++}$, it follows that $\lim(\tilde{f}^n/\hat{f}) = 1$.

Proof of (1)

Since $\hat{f}^{m_n}$ is the numeraire in $\mathcal{C}^{m_n}$ and $\tilde{f}^n \in \mathcal{C}^{m_n}$ for all n, $E[\tilde{f}^n/\hat{f}^{m_n}] \leq 1$ holds for all n. Also, $E[\hat{f}^{m_n}/\hat{f}] \leq 1$ is obvious because $\hat{f}$ is the numeraire in $\mathcal{C}$. Letting $g^n := \tilde{f}^n/\hat{f}^{m_n}$ and $h^n := \hat{f}^{m_n}/\hat{f}$ for all n, the conditions of the statement of the first helpful lemma are satisfied. Therefore, $\lim h^n = 1$, which exactly translates to $\lim \hat{f}^{m_n} = \hat{f}$.

Proof of (2)

One applies again Komlos lemma to get the existence of a sequence $(\tilde{f}^n)$ such that each $\tilde{f}^n$ is a convex combination of $(\hat{f}^k)_{k=n,\dots,\ell_n}$ for some $n \leq \ell_n$, and such that $\tilde{f} := \lim \tilde{f}^n$ exists. We can assume that (ℓ_n) is a strictly increasing sequence.

Define $m_0 = 1$ and a strictly increasing sequence (m_n) recursively via $m_n = \ell_{m_{n-1}}$ for all n. Then, it is straightforward to check that $E[\hat{f}^{m_n}/\tilde{f}^{m_{n-1}}] \leq 1$ and $E[\tilde{f}^{m_n}/\hat{f}^{m_n}] \leq 1$ hold for all n. Letting $g^n := \hat{f}^{m_n}/\tilde{f}^{m_{n-1}}$ and $h^n := \tilde{f}^{m_n}/\hat{f}^{m_n}$ for all n, the conditions of the statement of the first helpful lemma are satisfied. Therefore, $\lim h^n = 1$, which, in view of $\lim \tilde{f}^{m_n} = \hat{f}$ gives $\lim \hat{f}^{m_n} = \hat{f}$.

Regularization in probability (definition)

Fix a nonnegative process $X \in \mathcal{X}$. For $s \in [0, T[$, if $\lim X_{t^n}$ exists and is the *same* for any *strictly* decreasing $[0, T]$-valued sequence (t^n) such that $\lim t^n = s$, we shall be denoting this common limit by $\lim_{t \downarrow\downarrow s} X_t$. By definition, we set $\lim_{t \downarrow\downarrow T} X_t = X_T$.

Similarly, if $t \in]0, T]$ and $\lim X_{s^n}$ exists and is the same for any *strictly* increasing $[0, T]$-valued sequence (s^n) such that $\lim s^n = t$, we shall be denoting this latter limit by $\lim_{s \uparrow\uparrow t} X_s$.

Regularization in probability (theorem)

Let Z be a strictly positive generalized supermartingale with respect to $\mathcal{F}$. Then, for all $t \in [0, T]$, $Z_{t+} := \lim_{\tau \downarrow\downarrow t} Z_\tau$ exists. If $\tau \in]0, T]$, $Z_{\tau-} := \lim_{t \uparrow\uparrow \tau} Z_t$ exists as well. Furthermore, $(Z_{t+})_{t \in [0,T]}$ is a strictly positive generalized supermartingale with respect to $\mathcal{F}$, and $\lim_{t \uparrow\uparrow \tau} Z_{t+}$ exists and is equal to $Z_{\tau-}$ for all $\tau \in]0, T]$.

Proof

For $t \in [0, T]$, let $\mathcal{C}_t$ be the closed (in probability) convex hull of $\{Z_\tau$ such that $\tau \in [t, T]\}$. It follows that $\mathcal{C}_t \subseteq \mathcal{C}_s$ whenever $s \in [0, T]$ and $t \in [s, T]$. Also, Z_t is the numerarire in $\mathcal{C}_t$, since $E[Z_\tau / Z_t] \leq 1$ whenever $t \in [0, T]$ and $\tau \in [t, T]$. In particular $\mathcal{C}_t$ is bounded in probability for all $t \in [0, T]$.

For all $t \in [0, T[$, let $\mathcal{C}_{t+} := \bigcup_{\tau \in]t, T]} \mathcal{C}_\tau$, as well as $\mathcal{C}_{T+} := \mathcal{C}_T$. For all $t \in [0, T]$, $\mathcal{C}_{t+} \subseteq \mathcal{C}_t$, and $\mathcal{C}_{t+} = \bigcup_n \mathcal{C}_{\tau^n}$ holds for any *strictly* decreasing $[0, T]$-valued sequence (τ^n) with $\lim \tau^n = t$ whenever $t \in [0, T[$. Whence by the second helpful lemma $Z_{t+} := \lim_{\tau \downarrow\downarrow t} Z_\tau$ exists for all $t \in [0, T]$ and it is actually equal to the numeraire in $\overline{\mathcal{C}_{t+}}$, where $\overline{\mathcal{C}_{t+}}$ denotes the closure in probability of $\mathcal{C}_{t+}$. Observe that the numeraire in $\overline{\mathcal{C}_{t+}}$ always exists, as $\mathcal{C}_{t+} \cap L^0_{++} \neq \emptyset$ and $\overline{\mathcal{C}_{t+}}$ is convex and bounded in probability.

Proof

Consider $Z_{\cdot+} := (Z_{t+})_{t\in[0,T]}$. Since $\mathcal{C}_{t+} \cap L^0_{++} \neq \emptyset$ for all $t \in [0,T]$ and Z_{t+} is numeraire in $\overline{\mathcal{C}_{t+}}$, it follows that $Z_{t+} \in Ł^0_{++}$, i.e., $Z_{\cdot+}$ is strictly positive. We claim that $Z_{\cdot+}$ is cadlag in probability; indeed, for $t \in [0,T[$, and as $\overline{\mathcal{C}_{t+}}$ coincides with the closure in probability of $\bigcup_{\tau\in\,]t,T]} \overline{\mathcal{C}_{\tau+}}$, the second helpful lemma gives again that $Z_{t+} = \lim_{\tau\downarrow\downarrow t} Z_{\tau+}$. Now, for all $\tau \in]0,T]$ we have $\bigcap_{t\in[0,\tau[} \overline{\mathcal{C}_{t+}} = \bigcap_{t\in[0,\tau[} \mathcal{C}_t$. Again we obtain that $\lim_{t\uparrow\uparrow\tau} Z_{t+}$ and $\lim_{t\uparrow\uparrow\tau} Z_t$ exist, and they are actually equal.

Proof

It only remains to show that $E[Z_{t+}/Z_{s+} \mid \mathcal{F}_s] \leq 1$ holds whenever $s \in [0, T]$ and $t \in [s, T]$. Fix $s \in [0, T]$ and $t \in [s, T]$, as well as $A \in \mathcal{F}_s$. For all $n \in \mathbb{N}$, with $s^n := (1 - 1/n)s + T/n$ and $t^n := (1 - 1/n)t + T/n$, the generalized supermartingale property of Z with respect to $\mathcal{F}$ and the fact that $A \in \mathcal{F}_s \subseteq \mathcal{F}_{s^n}$ give $E[(Z_{t^n}/Z_{s^n})1_A] \leq P[A]$. Then, Fatou's lemma gives $E[(Z_{t+}/Z_{s+})1_A] \leq P[A]$. Since $A \in \mathcal{F}_s$ was arbitrary we get $E[Z_{t+}/Z_{s+} \mid \mathcal{F}_s] \leq 1$.

Proof of the main theorem

We show the implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) below. Without any restriction we shall assume that $1 \in \mathcal{X}$.

$(1) \Rightarrow (2)$

For all $t \in [0, T]$, let $\mathcal{C}_t := \{X_t \text{ such that } X \in \mathcal{X}\}$. The convexity of $\mathcal{X}$ implies that $\mathcal{C}_t$ is convex for all $t \in [0, T]$. Let $X \in \mathcal{X}$. The switching property of $\mathcal{X}$, combined with $1 \in \mathcal{X}$ gives that the stopped at t process $\tilde{X} := X_{t\wedge\cdot}$ is also in $\mathcal{X}$; since $\tilde{X}_T = X_t$, we obtain that $\{X_t \text{ such that } X \in \mathcal{X}\} \subseteq \{X_T \text{ such that } X \in \mathcal{X}\}$. Therefore, $\mathcal{C}_t$ is bounded in probability for all $t \in [0, T]$. Whence for all $t \in [0, T]$, there exists a numeraire $\hat{f}_t$ in the closure in probability of $\mathcal{C}_t$ such that $E[f/\hat{f}_t] \leq 1$ holds for all $f \in \mathcal{C}_t$.

$(1) \Rightarrow (2)$

Now, let (ξ^n) be a sequence in $\mathcal{X}$ such that $\xi^n_t \in L^0_{++}$ for all n, t and $\lim \xi^n_T = \hat{f}_T$. We shall show that $\lim \xi^n_t = \hat{f}_t$ actually holds for all $t \in [0, T]$.

Fix $t \in [0, T]$ and let (χ^n) be another sequence in $\mathcal{X}$ such that $\chi^n_t \in L^0_{++}$ for all n and $\lim \chi^n_t = \hat{f}_t$. We can assume without loss of generality that $E[\xi^n_t/\chi^n_t] \leq 1$ for all n. (Otherwise replace χ^n with ψ^n, an appropriate convex combination of χ^n and ξ^n, such that $E[\xi^n_t/\psi^n_t] \leq 1$ and $E[\chi^n_t/\psi^n_t] \leq 1$ hold for all n; where ψ^n_t is the numeraire of $\{(1-\alpha)\chi^n_t + \alpha\xi^n_t$ such that $\alpha \in [0,1]\}$. The first helpful lemma with $g^n := \chi^n_t/\psi^n_t$ and $h^n := \psi^n_t/\hat{f}_t$ for all n implies that this new $\mathcal{C}_t$-valued sequence (ψ^n_t) will still converge to $\hat{f}_t$.)

$(1) \Rightarrow (2)$

Now define $\zeta^n := \chi^n_{t\wedge\cdot}(\xi^n_{t\vee\cdot}/\xi^n_t)$. We have $\zeta^n \in \mathcal{X}$ by the switching property, and $\zeta^n_T = (\chi^n_t/\xi^n_t)\xi^n_T$. Then, $E[\xi^n_T/\zeta^n_T] = E[\xi^n_t/\chi^n_t] \leq 1$ for all n, so with $g^n := \xi^n_T/\zeta^n_T$ and $f^n := \zeta^n_T/\hat{f}_T$ we obtain by the first helpful lemma that $\lim \zeta^n_T = \hat{f}_T$. Combining this with $\lim \chi^n_t = \hat{f}_t$, we get $\lim(\xi^n_t/\xi^n_T) = \hat{f}_t/\hat{f}_T$, and, therefore, $\lim \xi^n_t = \hat{f}_t$, which is the first claim, namely sequencing of portfolio wealth processes approximating the numeraire at T also approximate it before t.

$(1) \Rightarrow (2)$

Define $\hat{Y}_t := 1/\hat{f}_t$ for all $t \in [0, T]$; as $\hat{f}_t \in Ł^0_{++}$, $\hat{Y}$ is a well-defined and strictly positive process. We claim that $\lim E\big[|\hat{Y}_t \xi^n_t - 1|\big] = 0$ holds for each $t \in [0, T]$. Indeed, since $\lim(\hat{Y}_t \xi^n_t) = 1$ and $(\hat{Y}_t \xi^n_t) \in L^0_{\geq 0}$ for all n, one only needs to establish that $\lim E[\hat{Y}_t \xi^n_t] = 1$, which follows from

$$1 = E\big[\liminf_{n\to\infty} \hat{Y}_t \xi^n_t\big] \leq \liminf_{n\to\infty} E[\hat{Y}_t \xi^n_t] \leq \limsup_{n\to\infty} E[\hat{Y}_t \xi^n_t] \leq 1\,.$$

In particular, for all $A \in \mathcal{G}$ we have $\lim E[\hat{Y}_t \xi^n_t 1_A] = P[A]$.

$(1) \Rightarrow (2)$

Fix $s \in [0, T]$, $t \in [s, T]$, $A \in \mathcal{F}_s$ and a strictly positive $X \in \mathcal{X}$. For n, let $\tilde{X}^n := 1_{\Omega \setminus A}\, \xi^n_{\cdot} + 1_A (\xi^n_{s \wedge \cdot}/X_s) X_{s \vee \cdot}$. The switching property of $\mathcal{X}$ implies that $\tilde{X}^n \in \mathcal{X}$. Furthermore, $\tilde{X}^n_t = 1_{\Omega \setminus A}\, \xi^n_t + 1_A(\xi^n_s/X_s)X_t$. Then, $E[\tilde{X}^n_t \hat{Y}_t] \leq 1$ translates to the inequality $E\big[(X_t/X_s)\hat{Y}_t \xi^n_s 1_A\big] \leq 1 - E[1_{\Omega \setminus A} \hat{Y}_t \xi^n_t]$.

Using Fatou's lemma on the left-hand side of this inequality and the fact that $\lim E[1_{\Omega \setminus A} \hat{Y}_t \xi^n_t] = 1 - P[A]$ on the right-hand-side, we obtain

$$E[\frac{X_t \hat{Y}_t}{X_s \hat{Y}_s} 1_A] \leq P[A].$$

Since $A \in \mathcal{F}_s$ was arbitrary, it follows that $E\big[X_t \hat{Y}_t/(X_s \hat{Y}_s) \,|\, \mathcal{F}_s\big] \leq 1$ for all strictly positive $X \in \mathcal{X}$.

$(1) \Rightarrow (2)$

We can regularize in probability and obtain a strictly positive generalized supermartingale Y with respect to $\mathcal{F}$, such that $Y_0 = 1$ and $Y_t = \lim_{\tau \downarrow\downarrow t} \hat{Y}_\tau$ holds for all $t \in [0, T]$. Fix $s \in [0, T]$, $t \in [s, T]$, $A \in \mathcal{F}_s$ and a strictly positive $X \in \mathcal{X}$. For all n, let $s^n := (1 - 1/n)s + T/n$ and $t^n := (1 - 1/n)t + T/n$. For all n, and since $A \in \mathcal{F}_s$, we have $E[(\hat{Y}_{t^n} X_{t^n} / (\hat{Y}_{s^n} X_{s^n}))1_A] \leq P[A]$ by construction. As X is cad, Fatou's lemma gives $E[(Y_t X_t / (Y_s X_s))1_A] \leq P[A]$ for all strictly positive $X \in \mathcal{X}$.

$(1) \Rightarrow (2)$

Fix now $X \in \mathcal{X}$ nonnegative, and define $X^n := (1/n) + (1 - 1/n)X$ for all n; then, $X^n \in \mathcal{X}$ and X^n is strictly positive. It follows that $E[Y_t X_t^n / (Y_s X_s^n) \mid \mathcal{F}_s] \leq 1$ for all n. Now, $\liminf_{n\to\infty}(X_t^n / X_s^n) = (X_t/X_s)1_{\{X_s>0\}} + 1_{\{X_s=0,\, X_t=0\}} + \infty 1_{\{X_s=0,\, X_t>0\}}$. As $E[\liminf_{n\to\infty}(Y_t X_t^n/(Y_s X_s^n)) \mid \mathcal{F}_s] \leq 1$ holds by the conditional version of Fatou's lemma, and $P[Y_s > 0,\ Y_t > 0] = 1$, we obtain $P[X_s = 0,\ X_t > 0] = 0$. Then, using the division conventions we get $E[Y_t X_t/(Y_s X_s) \mid \mathcal{F}_s] \leq 1$ for all $X \in \mathcal{X}$. In other words, YX is a nonnegative generalized supermartingale with respect to $\mathcal{F}$ for all $X \in \mathcal{X}$.

$(1) \Rightarrow (3)$

We apply the previous reasoning $(1) \Rightarrow (3)$ to the set $\mathcal{C} := \{X_T \text{ such that } X \in \mathcal{X}\}$, which is now assume to be closed. Then actually $\hat{X} := 1/\hat{Y}$ lies in $\mathcal{X}$, is strictly positive and cad, and therefore its regularization coincides with itself. This yields the result.

$(3) \Rightarrow (2)$

Assume that $\hat{X}$ exists and set $\hat{Y} := 1/\hat{X}$. A priori, $\hat{Y}$ is not necessarily cadlag. However, passing to Y as in the proof of implication $(1) \Rightarrow (2)$ above and following the rest of the argument, we can conclude the existence of a generalized supermartingale deflator. Note that this implies that $\hat{X}$ is necessarily cadlag – therefore, the original $\hat{Y}$ is necessarily cadlag as well.

$(2) \Rightarrow (1)$

Pick Y with the properties of statement (2). For all $\ell \in \mathbb{R}_{>0}$, we have the inequality $\ell \sup_{X \in \mathcal{X}} P[Y_T X_T > \ell] \leq \sup_{X \in \mathcal{X}} E[Y_T X_T] \leq 1$. Therefore, the set $\{Y_T X_T \mid X \in \mathcal{X}\}$ is bounded in probability. Since $Y_T \in L^0_{++}$, $\{X_T \mid X \in \mathcal{X}\}$ is bounded in probability.

The cadlag property in probability

Finally, we establish that if Y is a process satisfying condition (2), then all wealth processes in $\mathcal{X}$ are cadlag. Pick $X \in \mathcal{X}$. Let $X' = (1 + X)/2$; then $X' \in \mathcal{X}$ and X' is strictly positive. It follows that YX' is a strictly positive generalized supermartingale with respect to $\mathcal{F}$. According to regularization in probability $\lim_{t\uparrow\uparrow\tau}(Y_t X'_t)$ exists for all $\tau \in]0, T]$; as $\lim_{t\uparrow\uparrow\tau} Y_t$ also exists and is an element of L^0_{++}, we obtain that $\lim_{t\uparrow\uparrow\tau} X'_t$ exists for all $\tau \in]0, T]$. This is equivalent to saying that $\lim_{t\uparrow\uparrow\tau} X_t$ exists for all $\tau \in]0, T]$. Since X is already cad, we conclude that X is cadlag.

Table of Contents

In this section we shall introduce different models classes (discrete time, Ito process models, Lévy process models, etc) and analyze their properties. We shall rely more on concrete trades and introduce new notions related to it.

Basic definitions

The main **ingredients** building blocks of model for financial markets are:

- $T \in (0, \infty)$: time horizon,
- $t \in [0, T]$: trading dates,
- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space,
- $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$: filtration which satisfies the usual conditions (right continuous and complete) w.r.t. $\mathbb{P}$,
- $\mathcal{F}_t$: information up to and including time t.

Basic definitions

- $d+1$ assets, where $d \geq 1$, composed of an asset $S^0 = B$, called numeraire, used as denomination basis, and d price processes $S^i = (S^i_t)_{0 \leq t \leq T}, i = 1, \ldots d$. From discrete model considerations we learned that it is reasonable to express all prices/values with respect to this numeraire. Whence the assumption: $B_t \equiv 1$. This means that prices S are already expressed in units of the numeraire.
- We assume that prices processes are adapted and càdlàg processes.

Black-Scholes model

The Bank account has instantaneous interest rate r, so $\tilde{B}_t = e^{rt}$ (in undiscounted values). We also have a stock price for $t \in [0, T]$

$$\tilde{S}_t = S_0 \exp\left\{\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right\}$$

where W is a Brownian motion. Switching to discounted values we get.

$$\begin{aligned} B_t &= \frac{\tilde{B}_t}{\tilde{B}_t} = 1 \\ S_t &= \frac{\tilde{S}_t}{\tilde{B}_t} = S_0 \exp\left\{\sigma W_t + \left(\mu - r - \frac{1}{2}\sigma^2\right) t\right\} \end{aligned}$$

Furthermore, applying Itô's formula gives us that $dS_t = S_t((\mu - r)dt + \sigma dW_t)$.

General Itô process model

We have

$$dS_t^i = S_t^i \left(b_t^i dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j \right)$$

where the processes b and σ are $\mathbb{R}^d$ and $\mathbb{R}^{d \times n}$ dimensional respectively, predictable and integrable processes.

Cox-Ross-Rubinstein binomial model

$\tilde{B}_k = (1+r)^k$ and $\frac{\tilde{S}_k}{\tilde{S}_{k-1}}$ do only have two possible values $1+u$, $1+d$ with positive probability respectively (usually $u > r > d > -1$).

Trading strategies

We call a predictable process $\varphi = (\vartheta^1, \ldots, \vartheta^d, \eta)$ with $\vartheta := (\vartheta^1, \ldots, \vartheta^d)$ a trading strategy with value process

$$V(\varphi) = (V_t(\varphi))_{0 \le t \le T}\,, \tag{3}$$

where

$$V_t(\varphi) = \sum_{i=1}^{d} \vartheta_t^i S_t^i + \eta_t \cdot 1 = \vartheta_t^{\mathrm{tr}} S_t + \eta_t$$

is the time t value of the current portfolio. The cost of the trading strategy is defined as

$$C_t(\varphi) := V_t(\varphi) - \int_0^t \sum_{i=1}^{d} \vartheta_u^i dS_u^i, \quad 0 \le t \le T\,,$$

meaning the total cost/expense, on $[0, t]$, from using strategy φ. Notice that we need $\vartheta \in L(S)$ here.

Self-financing trading strategies

Strategy $\varphi = (\eta, \vartheta)$ is self-financing if $C(\varphi) \equiv C_0(\varphi)$, i.e. $C_t(\varphi) = C_0(\varphi)$ $\mathbb{P}$-a.s. for all t.

Lemma

The following holds true:

1. $\varphi = (\vartheta, \eta)$ *is self-financing iff* $V(\varphi) = V_0(\varphi) + \int \vartheta dS$.
2. *There is a bijection between self-financing strategies* $\varphi = (\vartheta, \eta)$ *and pairs* (V_0, ϑ), *where* $V_0 \in L^0(\mathcal{F}_0)$ *and* ϑ *is predictable and* S*-integrable. Explicitly:* $V_0 = V_0(\varphi)$ *and* $\eta = V_0 + \int \vartheta dS - \vartheta^{\mathrm{tr}} S$.
3. *If we have* $\varphi = (\vartheta, \eta)$ *self-financing, then also* η *is predictable.*

Proof

The first assertion is immediate from definition of $C(\varphi)$. The second assertion follows from teh first and $V(\varphi) = \vartheta^{\mathrm{tr}} S + \eta$. For the third assertion we consider a càdlàg process $Y = (Y_t)_{0 \leq t \leq T}$, write $\Delta Y_t := Y_t - Y_{t-}$ for the jump of Y at time t. From stochastic integration theory, $\Delta(\int \vartheta dS)_t = \vartheta_t^{\mathrm{tr}} \Delta S_t = \vartheta_t^{\mathrm{tr}} S_t - \vartheta_t^{\mathrm{tr}} S_{t-}$. So then the second assertion gives $\eta_t = V_0 + \int_0^t \vartheta_u dS_u - \vartheta_t^{\mathrm{tr}} S_t = V_0 + \int_0^{t-} \vartheta_u dS_u - \vartheta_t^{\mathrm{tr}} S_{t-}$, where the last three terms are all predictable.

Model assumptions

Building up the model as we have, we have some *implicit assumptions* in our setup:

- we can trade continuously in time,
- prices for buying and selling shares are given by S: there are no transaction costs and we have frictionless trading,
- ϑ is $\mathbb{R}^d$-valued, so ϑ_t^i can be positive or negative. η is $\mathbb{R}$-valued, so η_t can be negative. So, short sales and borrowing are allowed; more generally: no constraints on strategies,
- asset prices S are given a priori and exogenously, and do not react to trading activities. Our agents are small investors or price takers. Consequence: the "book value" $V(\varphi)$ agrees with the liquidation value.

A counterexample

Allowing too many self-financing strategies may be bad. Let $d = 1$, $S = \exp(W_t - t/2)$ be an exponential Brownian motion on $[0, \infty]$ (with the understanding that $S_\infty = 0$), and the time horizon be $T = \infty$. Going short in S, i.e. choosing a trading strategy with $\vartheta = -1$ yields $V_\infty = -(S_\infty - S_0) = 1$ with zero initial investment. The problem is that its wealth $V = \int \vartheta dS$ is not bounded from below and so we may experience huge losses before realizing profit. If we had $S_0 - S_t \geq -a$ for some constant, then S_t would be bounded from above which is apparently not the case.

Notations

We shall always write

$$G_t(\vartheta) = (\vartheta \bullet S) = \int_0^{\cdot} \vartheta_s dS_s$$

for the value process of a self-financing portfolio.

We call a trading strategy ϑ *admissible* if $G(\vartheta) \geq -\lambda$ for some $\lambda \geq 0$.

Simple strategies

$\vartheta \in b\mathcal{E}$: $\vartheta = \sum_{i=1}^{n} h_i 1_{((\tau_{i-1},\tau_i]]}$, with $n \in \mathbb{N}$, stopping times $0 \leq \tau_0 < \tau_1 < \cdots < \tau_n < T$ and h^i that is $\mathbb{R}^d$-valued, bounded and $\mathcal{F}_{\tau_{i-1}}$-measurable. We write $\vartheta \in b\mathcal{E}_{det}$ if in addition the τ_i (but not the h_i) are deterministic.

Simple Arbitrage Opportunity

Let $\vartheta \in b\mathcal{E}$ be admissible, with $G_T(\vartheta) \in L^0_+ \backslash \{0\}$, i.e. $G_T(\vartheta) \geq 0$ P-a.s. and $P[G_T(\vartheta) > 0] > 0$. Then we call ϑ a *simple arbitrage opportunity*.

Arbitrage Opportunity

Suppose S is a semimartingale; then an arbitrage opportunity is a strategy ϑ that is predictable, $\mathbb{R}^d$-valued, S-integrable, admissible and with $G_T(\vartheta) \in L^0_{\geq 0} \setminus \{0\}$.

Absence of Arbitrage

We define the following conditions:

$$(\mathrm{NA}_{elem}) \quad G_T(b\mathcal{E}) \cap L^0_{\geq 0} = \{0\}$$

$$(\mathrm{NA}^{adm}_{elem}) \quad G_T(b\mathcal{E}_{adm}) \cap L^0_{\geq 0} = \{0\}$$

$$(\mathrm{NA}) \quad G_T(\Theta_{adm}) \cap L^0_{\geq 0} = \{0\}$$

Lemma (simple direction FTAP)

If there exists a probability measure $Q \approx P$ such that S is a local Q-martingale, then (NA) and (NA^{adm}_{elem}) hold (and by extension also (NA_{elem}) holds).

Ansel-Stricker Lemma

The proof of this lemma requires a result known as the Ansel-Stricker lemma, which we now state.

Lemma

Suppose S is a local martingale. If ϑ is predictable and S-integrable, then the stochastic integral $\int \vartheta dS$ is well defined and again a semimartingale. If in addition we require that $\int \vartheta dS$ to be uniformly bounded from below, $\int \vartheta dS$ is again a local martingale (and then, since it is bounded from below, it is a supermartingale by Fatou's Lemma).

Proof (Ansel-Stricker)

We are following a short proof presented by de Donno and Pratelli in [2]. We first prove a more general statement: let X be an adapted, càdlàg process and let $(M^n)_{n\geq 0}$ be a sequence of martingales converging uniformly pathwise in probability to X together with a localizing sequence of stopping times $(\eta^k)_{k\geq 0}$ (notice here again so called "stationarity", i.e. $P[\eta_k=\infty]\to 1$ as $k\to\infty$) and integrable random variables $(\theta^k)_{k\geq 0}$. Assume that $X_t^{\eta_k}\geq\theta^k$ for all $k\geq 0$ and that for all stopping times τ the $(\Delta M^n_\tau)^+\leq(\Delta X_\tau)^+$ and $(\Delta M^n_\tau)^-\leq(\Delta X_\tau)^-$ holds true, then X is a local martingale. For the proof define stopping times

$$\tau_n:=\inf\{t>0\,|\,X_t>n \text{ or } M^n_t>X_t+1 \text{ or } M^n_t<X_t-1\}\wedge T$$

for $n\geq 0$.

Proof (Ansel-Stricker)

We can assume, by possibly passing to a subsequence, that $\sum \mathbb{P}[\tau_n < 1] < \infty$. We define $\sigma_m := \inf_{n \geq m} \tau_n \wedge \eta^m$ and show now that X^{σ_m} is a martingale. The sequence $(\sigma_m)_{m \geq 0}$ is additionally localizing by the previous construction, since $\sum 1_{\{\tau_n < 1\}}$ is integrable and hence $\mathbb{P}[\inf_{n \geq m} \tau_n = 1] \to 1$ as $m \to \infty$.

At σ_m we can make assertions about the jumps of X by our two further assumptions: let $m \geq 0$ be given, then

$$(\Delta M^n_{t \wedge \sigma_m})^- \leq (\Delta X_{t \wedge \sigma_m})^- \leq m - \theta^m$$

for $n \geq m$ by the second assumption. Since $M^n_t \geq X_t - 1$ for $n \geq m$ (notice that the jumps of M^n are bounded by the jumps of X), we arrive at

$$M^n_{t \wedge \sigma_m} \geq \theta_m - 1 - (m - \theta_m) = 2\theta_m - m - 1\,.$$

Proof (Ansel-Stricker)

This yields by Fatou's Lemma that $X_{t\wedge\sigma_m}$ is integrable since $M^n_{t\wedge\sigma_m} \to X_{t\wedge\sigma_m}$ in probability as $n \to \infty$. For $t = T$ we obtain in particular $X_{t\wedge\sigma_m}$ is integrable, and hence also $\Delta X_{t\wedge\sigma_m}$ by $X_{t\wedge\sigma}$ being bounded from below by an integrable random variable. Again by

$$M^n_{t\wedge\sigma_m} \leq m + 1 + (\Delta M^n_{t\wedge\sigma_m})^+ \leq m + 1 + (\Delta X_{t\wedge\sigma_m})^+$$

for $n \geq m$, hence $M^n_{t\wedge\sigma_m} \to X_{t\wedge\sigma_m}$ in $L^1(\mathbb{P})$ for $0 \leq t \leq T$ yielding that X^{σ_m} is a martingale.

Proof (Ansel-Stricker)

We return now to the proof of the Ansel-Stricker Lemma: we assume by stopping that S lies in $\mathcal{H}^1$ and let $\vartheta \in L(S)$ be given and define $\vartheta_n := \vartheta 1_{\{\|\vartheta\| \leq n\}}$. Then by definition of the stochastic integral $(\vartheta_n \bullet S) \to (\vartheta \bullet S)$ in the Emery topology, in particular $(\vartheta_n \bullet S) \in \mathcal{H}^1$ for $n \geq 1$. All assumptions of the previous statement are fulfilled due to $(\vartheta \bullet S)$ being bounded from below and jumps of approximations $(\vartheta_n \bullet S)$ being bounded by jumps of $(\vartheta \bullet S)$.

Proof (simple direction of FTAP)

$S \in \mathcal{M}_{loc}(Q)$ and $Q \approx P$ give us via Bichteler-Dellacherie that S is a P-semimartingale. We also have that $b\mathcal{E}_{adm} \subseteq \Theta_{adm}$. So it is enough to prove (NA) since this implies (NA^{adm}_{elem}). Now, $S \in \mathcal{M}_{loc}(Q)$, take $\vartheta \in \Theta^{adm}$, so ϑ is S-integrable and predictable, so $\int \vartheta dS$ is well defined. Moreover, since ϑ is admissible, $\int \vartheta dS$ is by Ansel-Stricker again in $\mathcal{M}_{loc}(Q)$, hence Q-supermartingale. So $\mathbb{E}_Q[G_T(\vartheta)] \leq \mathbb{E}_Q[G_0(\vartheta)] = 0$.

Whence, if $G_T(\vartheta) \geq 0$ P-a.s., then also (since $Q \approx P$) $G_T(\vartheta) \geq 0$ Q-a.s.; but $\mathbb{E}_Q[G_T(\vartheta)] \leq 0$, so $G_T(\vartheta) = 0$ Q-a.s. and also P-a.s. (since $P \approx Q$).

To prove (NA_{elem}) we use that in discrete time, $G(\vartheta) = \int \vartheta dS$ is always a local martingale if S is a local martingale and ϑ is predictable.

Equivalent local martingale measure

An equivalent (local) martingale measure for S is a probability measure $Q \approx P$ such that S is a (local) Q-martingale.

With this definition, the previous lemma says that if (ELMM) holds for S, then we have (NA).

For the case of finite discrete time, $S = (S_k)_{k=0,1,\ldots T}$, the converse holds, as we will shall see later. In general however, the converse is not true.

Why does this happen? The key point is that if one can trade infinitely often, one can do "doubling strategies".

To exclude such phenomena, we must forbid not only arbitrage opportunities, but also “limit arbitrage opportunities”. For that, we look first at

$$G_T(b\mathcal{E}) - L^\infty_{\geq 0}(P) = \{Y = G_T(\vartheta) - b \,|\, \vartheta \in b\mathcal{E}, b \in L^\infty_{\geq 0}(P)\},$$

the set of all payoffs starting with wealth 0, doing elementary bounded self-financing trading and discarting a bounded amount b. Intuitively, nothing of that type should be non-negative (except 0), otherwise we again have a “money pump”.

Kreps-Yan Theorem

We can quite easily prove the following theorem on the existence of equivalent separating measures.

Theorem

Fix $p \in [1, \infty]$ and set q conjugate to p. Suppose $C \subseteq L^p$ is a convex cone with $C \supseteq -L^p_{\geq 0}$ and $C \cap L^p_{\geq 0} = \{0\}$. If C is closed in $\sigma(L^p, L^q)$, then there exists $Q \approx P$ with $\frac{dQ}{dP} \in L^q(P)$ and $\mathbb{E}_Q[Y] \leq 0$ for all $Y \in C$.

Proof (Kreps-Yan)

Any $x \in L^p_{\geq 0} \backslash \{0\}$ is disjoint from C, so we can by the Hahn-Banach-theorem strictly separate x from C by some $z_x \in L^q$. Then the cone property gives us $\mathbb{E}[z_x Y] \leq 0$, $\forall Y \in C$ and $C \supseteq -L^p_{\geq 0}$ gives $z_x \geq 0$. The strict separation implies $z_x \not\equiv 0$, so that we can normalise to $\mathbb{E}[z_x] = 1$.

We next form the family of sets $\{\Gamma_x := \{z_x > 0\} | x \in L^p_{\geq 0} \backslash \{0\}\}$. Then one can find a countable subfamily $(\Gamma_{x_i})_{i \in \mathbb{N}}$ with $P[\cup_i \Gamma_{x_i}] = 1$. For suitably chosen weights $\gamma_i > 0$, $i \in \mathbb{N}$, one gets that $z := \sum_{i=1}^{\infty} \gamma_i z_{x_i}$ is $z > 0$ P-a.s., $z \in L^q$ and $\mathbb{E}[zY] \leq 0$, for all $Y \in C$. Normalize to get $\mathbb{E}[z] = 1$ and then $dQ := zdP$ does the job.

Stricker's theorem

Fix $p \in [1, \infty]$, q conjugate to p and suppose S is an adapted, càdlàg process and that $S_t \in L^p(P)$ for all $t \in [0, T]$. Denote by $\overline{\cdots}$ the closure in L^p for $1 \leq p < \infty$, or the weak-$*$-closure, i.e. the closure in the $\sigma(L^\infty, L^1)$-topology for $p = \infty$. Then are equivalent:

1. $\overline{G_T(b\mathcal{E}_{det}) - L^\infty_+(P)} \cap L^p_+(P) = \{0\}$
2. The propery (EMM) holds for S, i.e. there exists $Q \approx P$ for S with density $\frac{dQ}{dP} \in L^q(P)$

Proof (Stricker's Theorem)

As for direction "2) $\Rightarrow$ 1)": S is a Q-martingale and $\sigma \in b\mathcal{E}_{det}$ is bounded, so $G_\bullet(\vartheta) = \sum_{i=1}^n h^i(S_{t_i \wedge \bullet} - S_{t_{i-1} \wedge \bullet})$ is again a Q-martingale. This gives us that $\mathbb{E}_Q[G_T(\vartheta)] = 0$ and $\mathbb{E}_Q[G_T(\vartheta) - b] \leq 0$ if $b \geq 0$ and bounded. But then, since $\frac{dQ}{dP} \in L^q(P)$, we also get $\mathbb{E}_Q[Y] \leq 0$ for all $Y \in \overline{G_T(b\mathcal{E}_{det}) - L^\infty_{\geq 0}(P)}$. So if also $Y \in L^p_{\geq 0}(P)$, we get $Y = 0$ almost surely.

Proof (Stricker's Theorem)

For "1) $\Rightarrow$ 2)" we have the following consideration: the set $G_T(b\mathcal{E}_{det})$ is a convex cone in $L^p(P)$, so

$$C := \overline{G_T(b\mathcal{E}_{det}) - L^\infty_{\geq 0}(P)}$$

is again a convex cone, C contains $-L^p_{\geq 0}(P)$ and C is closed in $\sigma(L^p, L^q)$. But also $C \cap L^p_{\geq 0}(P) = \{0\}$. Then the *Kreps-Yan* Theorem gives the existence of the probability measure $Q \approx P$ with $\mathbb{E}_Q[Y] \leq 0$ for all $Y \in C$ and hence $\mathbb{E}[G_T(\vartheta)] \leq 0$ for all $\vartheta \in b\mathcal{E}_{det}$.

We can now take $\vartheta := \pm I_{A_s} I_{(s,t]}$ with $s \leq t$, $A_s \in \mathcal{F}_s$ to get

$$\mathbb{E}_Q[\pm I_{A_s}(S_t - S_s)] \leq 0$$

for all $A_s \in \mathcal{F}_s$. This gives us $\mathbb{E}_Q[S_t - S_s | \mathcal{F}_s] = 0$, which is the martingale property of S under Q. Also, $S_t \in L^1(Q)$ by Hölder, as $S_t \in L^p(P)$, $\frac{dQ}{dP} \in L^q(P)$.

Remark

Looking back at Stricker's Theorem we see that it has the following pros and cons:

+ works for any adapted, càdlàg process S, proof is nice and simple, strategies from $b\mathcal{E}$ are reasonably realistic.
− need integrability for S ($S_t \in L^p(P)$), strategies in $b\mathcal{E}$ are not admissible in general. The closure with respect to $\sigma(L^\infty, L^1)$ is quite weak and therefore it might be very reasonable to look for alternative hypotheses on the price process.

Counterexample

We now show that (NA_{elem}) does not imply (EMM) by giving a counterexample. Start with $(Y_n)_{n\in\mathbb{N}}$ under P that are independent, taking values in ± 1, with $P[Y_n = +1] = \frac{1}{2}(1+\alpha_n)$. Set $S_0 := 1$ and $\Delta S_n := S_n - S_{n-1} = \beta_n Y_n$. Choose $\mathbb{F} = \mathbb{F}^S = \mathbb{F}^Y$.

The only way to get S to be a $(Q,\mathbb{F})$-martingale is to have $Q[Y_n = +1|\mathcal{F}_{n-1}] = \frac{1}{2}$. So all (Y_n) must be under Q independent and symmetric around 0, i.e. iid under Q with $Q[Y_n = +1] = \frac{1}{2}$. Kakutani's dichotomy theorem (see Williams) then gives us that $Q \approx P$ if and only if

$$\sum_{n=1}^{\infty} \alpha_n^2 < \infty.$$

Counterexample

Otherwise, we must have $Q \perp P$. So if we take $\sum \alpha_n^2 = +\infty$, then (EMM) does not hold.

What is the role of β_n? It has not been important so far, we just note that $\sum |\beta_n| < \infty$ implies that S is bounded. (Exercise: Show that there exists an arbitrage opportunity in $b\mathcal{E}$ if and only if there exists an arbitrage opportunity with ϑ of the form $\vartheta = h1_{((\sigma,\tau]]}$ for stopping times $\sigma \leq \tau$ and h bounded $\mathcal{F}_\sigma$-measurable). We now choose

$$\beta_n = 3^{-n}$$

so that for each n, we get that $\sum_{k>n}^{\infty} \beta_k < \beta_n$. A simple consequence of this is that for $m > n$, $\text{sign}(S_m - S_n) = \text{sign}(Y_{n+1})$.

Counterexample

We now claim that there does not exist an arbitrage opportunity in $b\mathcal{E}$. Take $\vartheta = h1_{((\sigma,\tau]]}$ and consider $A_n = \{\sigma = n, \tau > n\} \in \mathcal{F}_n$. Then $G_\infty(\vartheta) = \int_0^\infty \vartheta_u dS_u = h(S_\tau - S_\sigma)$ has $\mathrm{sign}(h(S_\tau - S_\sigma)) = \mathrm{sign}(hY_{n+1})$ on A_n. So if $G_\infty(\vartheta) \geq 0$ P-a.s., we have for all n $\mathrm{sign}(hY_{n+1}) \geq 0$.

But this is not possible: $A_n \in \mathcal{F}_n$, h is $\mathcal{F}_\sigma$-measurable, so hI_{A_n} is $\mathcal{F}_n$-measurable; and Y_{n+1} is independent of $\mathcal{F}_n$ with values ± 1. So we can only have

$$\mathrm{sign}(hI_{A_n} Y_{n+1}) I_{A_n} \geq 0$$

if $hI_{A_n} = 0$. This is for all n, so we get that $h \equiv 0$, $\vartheta \equiv 0$ and as a result $G_\infty(\vartheta) \equiv 0$.

Suppose S is a semimartingale (with no integrability conditions) and recall the space Θ_{adm} of admissible strategies. Condition

$$\text{(NA)} \quad G_T(\Theta_{adm}) \cap L^0_{\geq 0} = \{0\}$$

can be easily shown to be equivalent to $(G_T(\Theta_{adm}) - L^0_{\geq 0}) \cap L^\infty \cap L^0_{\geq 0} = \{0\}$ or equivalently

$$\text{(NA)} \quad C \cap L^\infty_{\geq 0} = \{0\}, \text{ with } C := (G_T(\Theta_{adm}) - L^0_{\geq 0}) \cap L^\infty.$$

Defined as above, C consists of bounded payoffs one can be dominated by final wealth of an admissible, self-financing strategy with 0 investment capital.

Instead of the hypothesis on $\sigma(L^p, L^q)$-closedness in the Kreps-Yan-Theorem we only speak of intuitive norm closures: notice that for $1 \leq p < \infty$ norm and $\sigma(L^p, L^q)$ closures coincide for convex sets, whereas only in the case $p = \infty$ a (big) gap appears.

Definition

A semimartingale $S = (S_t)_{0 \leq t \leq T}$ satisfies (NFLVR) (no free lunch with vanishing risk) if

$$\overline{C}^{L^\infty(\mathbb{P})} \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{\cdots}^{L^\infty(\mathbb{P})}$ denotes the norm closure in $L^\infty(\mathbb{P})$.

Proposition

For semimartingale S are equivalent:

1. *(NFLVR)*
2. *Any sequence $g_n = G_T(\vartheta^n)$ in $G_T(\Theta_{adm})$ with $G_T^-(\vartheta^n) = g_n^- \to 0$ in L^∞ converges to 0 in L^0.*
3. *S satisfies (NA) plus one of the following:*
   3.1 *(NUBPR) (no unbounded profit with bounded risk) The set*

   $$\mathcal{G}^1 := \{G_T(\vartheta) \,|\, \vartheta \in \Theta_{adm} \textit{ is 1-admissible}\}$$

   is bounded in L^0.

   3.2 *For every sequence $\epsilon_n \searrow 0$ and every sequence (ϑ^n) of strategies with $G_\bullet(\vartheta^n) \geq -\epsilon_n$, we have $G_T(\vartheta^n) \to 0$ in L^0.*

Making a short overview of notation, we have for $C := (G_T(\Theta_{adm}) - L^0_{\geq 0}) \cap L^\infty$ and S semimartingale:

(NFLVR) $\overline{C}^{L^\infty(P)} \cap L^0_{\geq 0} = \{0\}$,

(NA) $C \cap L^0_{\geq 0} = \{0\}$,

(NUBPR) The set

$$\mathcal{G}^1 := \{G_T(\vartheta) \mid \vartheta \in \Theta_{adm} \text{ is 1-admissible}\}$$

is bounded in L^0.

σ-martingale

The formulation of the next result needs the concept of a *σ-martingale*, which is already familiar to us since it is related to the fact that not every stochastic integral $(\varphi \bullet S)$ along a local martingale is a local martingale.

Definition (σ-martingale)

An $\mathbb{R}^d$-valued process X is a σ-martingale (under $\mathbb{P}$) if $X = \int \psi dM = \psi \bullet M$ for an $\mathbb{R}^d$-valued local martingale (under $\mathbb{P}$) and an $\mathbb{R}$-valued predictable M-integrable ψ with $\psi > 0$.

σ-martingale

Clearly, X being a martingale implies it is a local martingale, which implies it is a σ-martingale. The converse does not hold in general, see Michel Emery's famous example: σ-martingales come with the generality of stochastic integration – it can be seen a cumulative effect of re-scaling of infinitesimal increments of martingales.

σ-martingale

However, we have the following important remark: suppose X is a σ-martingale and bounded below. Then the Ansel-Stricker theorem gives that X is also a local martingale (and even supermartingale).

Consider a probability space carrying one Bernoulli random variable B and an independent, exponentially distributed random time T with $\mathbb{P}[T \geq x] = \exp(-x)$. Then we can define a stochastic process M via

$$M_t := 1_{\{t \geq T\}} B$$

for $t \geq 0$.

σ-martingale

We equip the probability space with the natural filtration generated by M. Apparently M is a martingale with respect to its natural filtration, since

$$
\begin{aligned}
& E[(M_t - M_s) g((M_u)_{u \leq s})] \\
& = E[\int_0^\infty B 1_{\{s \leq x < t\}} g((1_{\{u \geq x\}} B)_{u \leq s}) \exp(-x) dx] \\
& = E[B(\exp(-s) - \exp(-t)] = 0
\end{aligned}
$$

for $0 \leq s \leq t$.

σ-martingale

Define now $H_t := \frac{1}{t}$, then this deterministic process is M-integrable since the process

$$(H 1_{\{\|H\| \leq n\}} \bullet M) \to X$$

in the semimartingale topology, where

$$X_t = 1_{\{t \geq T\}} \frac{B}{T},$$

for $t \geq 0$.

σ-martingale

This is true since X is a finite variation process, hence a semimartingale, and the process $(H1_{\{\|H\|>n\}} \bullet M)$ converges to 0 in the semimartingale topology, since

$$E[|\frac{B}{T}|1_{\{T \leq 1/n\}}] \to 0$$

as $n \to \infty$. The process X looks a bit like a martingale having again jumps as multiples of B, but there are some integrability issues: first we observe that

$$E[|X_t|] = \int_0^t \frac{1}{x} \exp(-x) dx = \infty \, .$$

This can be easily strengthened since for every stopping time $\tau \neq 0$ with respect to the natural filtration it even holds that $E[|X_\tau|] = \infty$. Hence it also cannot be a local martingale, but X apparently is a σ-martingale.

FTAP

Theorem (Fundamental theorem of asset pricing)

For semimartingales $S = (S_t)_{0 \le t \le T}$ the following statements are equivalent:

1. *S satisfies (NFLVR),*
2. *S admits an equivalent separating measure, i.e. the property (ESM) holds for S: there exists $Q \approx P$ with $\mathbb{E}_Q[G_T(\vartheta)] \le 0$, for all $\vartheta \in \Theta_{adm}$,*
3. *S admits an equivalent σ-martingale measure (EσMM), i.e. the property (EσMM) holds for S: there exists $Q \approx P$ such that S is a Q-σ-martingale.*

Remark

(NFLVR) implies the existence of an EσMM. Any process S satisfying (EMM) or (ELMM) satisfies (ESM) as shown by using the Ansel-Stricker theorem. Conversely, if S is (locally) bounded, then any equivalent separating measure is an equivalent (local) martingale measure, as seen in the proof of Stricker's theorem. But if S is unbounded (i.e. has unbounded jumps, so that it can't be made bounded, even by localizing), an equivalent separating measure need not be an equivalent σ-martingale measure. However, one can show that the set of equivalent σ-martingale measures is dense in the set of all equivalent separating measures.

The main mathematical ingredient of FTAP is the following important and surprising fact:

Theorem

If the semimartingale $S = (S_t)_{0 \leq t \leq T}$ satisfies (NFLVR), then the set

$$C = (G_T(\Theta_{adm}) - L^0_{\geq 0}) \cap L^\infty$$

is weak*-closed in L^∞, i.e. closed in the $\sigma(L^\infty, L^1)$-topology.

Proof (sketch)

The direction "3) $\Rightarrow$ 1)" is proven in the same way as Stricker's theorem by means of the Ansel-Stricker Lemma.

The direction "1) $\Rightarrow$ 2)" can be seen as follows: by FTAP, (NFLVR) implies that C is closed in $\sigma(L^\infty, L^1)$. As C is also a convex subset of L^∞, and $C \supseteq -L^\infty_{\geq 0}$, and $C \cap L^\infty_{\geq 0} = \{0\}$, we conclude by the Kreps-Yan theorem, that there exists $Q \approx P$ such that $\mathbb{E}_Q[Y] \leq 0$ for all $Y \in C$, i.e. an equivalent separating measure. This easily implies $\mathbb{E}_Q[G_T(\vartheta)] \leq 0$, for all $\vartheta \in \Theta_{adm}$ (use: $G_T(\vartheta) \wedge n \in C$, $n \to \infty$).

Direction "2) $\Rightarrow$ 3)" follows by the previous remark.

No arbitrage in finite discrete time

For the case of finite discrete time, results are easier. Let us denote in this section in a discrete way: $S = (S_k)_{k=0,1,\dots,T}$ be an $\mathbb{R}^d$-valued process adapted to $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,T}$ and recall that $\mathbb{F}$-predictable processes are simply $\vartheta = (\vartheta_k)_{k=1,\dots,T}$ (or set $\vartheta_0 := 0$) with ϑ_k $\mathcal{F}_{k-1}$-measurable for all k. Then $G_k(\vartheta) = \sum_{j=1}^k \vartheta_j^{\mathrm{tr}}(S_j - S_{j-1}) = \sum_{j=1}^k \vartheta_j^{\mathrm{tr}} \Delta S_j$, $k = 0, 1, \dots, T$. Here:

$$\Theta = \left\{ \text{all predictable } \mathbb{R}^d\text{-valued } \vartheta \right\},$$

$$\Theta_{adm} = \{\vartheta \in \Theta | G_\bullet(\vartheta) \geq -a \text{ for some } a \geq 0\}.$$

With the above notation, the classical no arbitrage (NA) condition then becomes

$$\text{(NA)}\quad G_T(\Theta_{adm}) \cap L^0_{\geq 0} = \{0\}$$

Lemma

In finite discrete time: (NA) $\Leftrightarrow G_T(\Theta) \cap L^0_{\geq 0} = \{0\}$.

Proof

The direction "$\Leftarrow$" is clear, since $G_T(\Theta) \supseteq G_T(\Theta_{adm})$. For "$\Rightarrow$" we need to show that any arbitrage from a general $\vartheta \in \Theta$ can also be realized by an admissible $\vartheta' \in \Theta_{adm}$. So we suppose that $\vartheta \in \Theta$ with $G_T(\vartheta) \cap L^0_{\geq 0} \backslash \{0\}$ is not empty. Assume that $G_{\cdot}(\vartheta) \ngeq 0$, since otherwise we can take $\vartheta = \vartheta'$.

Proof

Let $n_0 := \max\{k \in \{0,1,\ldots,T\} | P[G_k(\vartheta) < 0] > 0\}$ be the "last time when ϑ violates 0-admissibility". Then $0 < n_0 < T$ and $A := \{G_{n_0}(\vartheta) < 0\} \in \mathcal{F}_{n_0}$ has $P[A] > 0$. Take $\vartheta' := I_A I_{\{n_0+1,\ldots,T\}}\vartheta$, i.e. on A, after n_0, we trade with ϑ. This gives us that $G_k(\vartheta') = I_A I_{\{k>n_0\}} \sum_{j=n_0+1}^{k} \vartheta_j^{\mathrm{tr}} \Delta S_j = I_A I_{\{k>n_0\}}(G_k(\vartheta) - G_{n_0}(\vartheta)) \geq 0$ by definition of n_0, A; so ϑ' is 0-admissible.

Moreover, $G_T(\vartheta') = I_A(G_T(\vartheta) - G_{n_0}(\vartheta))$ is in $L^0_{\geq 0}$ like $G_T(\vartheta)$ and greater than 0 on A with $P[A] > 0$, so $G_T(\vartheta') \in L^0_{\geq 0} \setminus \{0\}$.

The key mathematical result in this section is

Theorem

In finite discrete time, if S satisfies (NA), the set $C' := G_T(\Theta) - L^0_{\geq 0}$ is closed in L^0.

In finite discrete time, this translates to the Dalang-Morton-Willinger theorem

Theorem (Dalang/Morton/Willinger)

For an $\mathbb{R}^d$-valued adapted process $S = (S_k)_{k=0,\dots,T}$ in finite discrete time, are equivalent:

1. *S satisfies (NA), i.e. $G_T(\Theta_{adm}) \cap L^0_+ = \{0\}$,*
2. *There exists and equivalent measure $Q \approx P$ such that S is a Q-martingale, i.e. (EMM) holds for S.*

Proof

For the direction "2) $\Rightarrow$ 1)" see previous lemma. As for direction "1) $\Rightarrow$ 2)": (NA) is invariant under a change to an equivalent probability measure, so change to $R \approx P$, such that $S_k \in L^1(R)$ for all k. We then drop the R notation and work without loss of generality under the assumption that S is P-integrable. But (NA) is equivalent to $G_T(\Theta) \cap L^0_{\geq 0} = \{0\}$.

Proof

Setting $C' := G_T(\Theta) - L^0_{\geq 0}$, (NA) is equivalent to $C' \cap L^0_{\geq 0} = \{0\}$. Set $C := C' \cap L^1$. This set is convex, $\subseteq L^1$, $\supseteq -L^1_{\geq 0}$ and $C \cap L^1_{\geq 0} = \{0\}$. By (NA) and Theorem 19, C is closed in L^1 (notice that C' is closed in L^0, which is an even weaker topology), hence also in $\sigma(L^1, L^\infty)$ since it is convex. So the Kreps-Yan theorem gives $Q \approx P$ such that $\mathbb{E}[Y] \leq 0$, for all $Y \in C$. Choose $\vartheta := \pm I_{A_k \times \{k,\dots,l\}}$ with $A_k \in \mathcal{F}_k$ and $k \leq l$ to get $G_T(\vartheta) = \pm I_{A_k}(S_l - S_k)$. As in proof of Stricker's theorem, this shows that S is a Q-martingale.

Remark

In the proof, we could choose for instance

$$dR = \text{const} \cdot \exp\{-\sum_{k=0}^{T} |S_k|\} dP \,.$$

Then $R \approx P$, $\mathbb{E}_R[|S_k|] < \infty$, for all k and $\frac{dR}{dP} \in L^\infty$. Then the Kreps-Yan theorem gives and equivalent martingale measure Q for S with $\frac{dQ}{dR} \in L^\infty$, and so we even have even have an equivalent martingale measure with $\frac{dQ}{dP} \in L^\infty$.

In finite discrete time, we have:

1. The space $G_T(\Theta) = \{\sum_{j=1}^T \vartheta_j^{\mathrm{tr}} \Delta S_j \,|\, \vartheta \text{ predictable } \mathbb{R}^d\text{-valued}\}$ of all final values of stochastic integrals with respect to S is always closed in L^0.
2. If S satisfies (NA), then $G_T(\Theta) - L^0_{\geq 0}$ is also closed in L^0.

Proofs are not difficult, but are notationally involved; use induction over time and dimension of S (when doing induction over dimension, we want to exclude 0 integrals for non-0 strategies).

An instructive example

The single period case with trivial $\mathcal{F}_0$ is instructive: in this case

$$C = \langle \mathbb{R}^d, (S_1 - S_0) \rangle - L^0_{\geq 0}$$

which is closed in probability under (NA) (use Komlos lemma). Whence there exists an equivalent martingale measure under (NA).

One can also argue as follows: (NA) holds if and only if the 0 is contained in the relative interior of the convex hull of the support of $S_1 - S_0$, since only then no $\varphi \in \mathbb{R}^d$ exists with $0 \neq \langle \varphi, S_1 - S_0 \rangle \geq 0$. The relative inteior of the convex hull of the support of the law of $S_1 - S_0$ corresponds, however, with the set of expectations $E_Q[S_1 - S_0]$ for all $Q \sim P$ (such that the expecation exists).

Ito processes

We start with a general probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{R}^n$-valued Brownian motion W. Consider the *undiscounted model* with bank account B and d stocks $\tilde{S} = (\tilde{S}_i)_{i=1,\dots,d}$, given by

$$dB_t = B_t r_t dt, \quad B_0 = 1$$

$$d\tilde{S}^i_t = \tilde{S}^i_t \mu^i_t dt + \tilde{S}^i_t \sum_{j=1}^{n} \sigma^{ij}_t dW^j_t, \quad \tilde{S}^i_0 = s^i_0 > 0$$

We assume r, μ, σ all predictable and suitably integrable processes. Pass to discounted prices $B := \frac{\tilde{B}}{\tilde{B}} \equiv 1$ and $S := \frac{\tilde{S}}{\tilde{B}}$. These then satisfy

$$dS^i_t = S^i_t(b^i_t dt + \sum_{j=1}^{n} \sigma^{ij}_t dW^j_t),\ S^i_0 = s^i_0 > 0, \text{with } b^i_t = \mu^i_t - r_t$$

Ito processes

Compactly we write $dS_t = S_t(b_t dt + \sigma_t dW_t)$ with $b_t \in \mathbb{R}^d$, $\sigma_t \in \mathbb{R}^{d \times n}$, $S_t \in \mathbb{R}^d$ or $S_t = \text{diag}(S_t^i)$.

Ito processes

Assume $d \leq n$ (so we have more sources of uncertainty than risky assets available for trading) and $\operatorname{rank}(\sigma_t) = d$ P-a.s. for all t. Introduce now $\overline{\lambda}_t := \sigma_t^{\mathrm{tr}}(\sigma_t \sigma_t^{\mathrm{tr}})^{-1} b_t \in \mathbb{R}^n$ to get

$$dS_t = S_t \sigma_t (\overline{\lambda}_t \, dt + dW_t) \, .$$

We call $\overline{\lambda}$ the multi-dimensional instantaneous market price of risk.

Ito processes

What is the structure of martingale measures? We start with some probability measure $Q \approx P$. The density process is defined as $Z^Q = (Z_t^Q)_{0 \le t \le T}$ with $Z_t^Q = \frac{dQ}{dP}\big|_{\mathcal{F}_t}$, choosing a càdlàg version. Introduce the stochastic logarithm

$$L^Q := \int \frac{1}{Z_-^Q} dZ^Q \in \mathcal{M}_{0,loc}(P)$$

to get $Z^Q = Z_0^Q \mathcal{E}(L^Q)$, $dZ_t^Q = Z_{t-}^Q dL_t^Q$ (which could be discontinuous since we did not assume $\mathbb{F}$ generated by a Brownian motion).

Remark

Notice that Z^Q is a strictly positive martingale by equivalence of $P \approx Q$, hence $Z^Q_- > 0$ by the Absorption Theorem. Therefore the stochastic integral is well-defined along the càglàd process Z_- and leads to a local martingale by local boundedness of the integrand.

S is a continuous semimartingale with canonical decomposition $S = S_0 + M + A$ with $M = \int_0^\cdot S_s \sigma_s \, dW_s$ and $A = \int_0^\cdot S_s \sigma_s \overline{\lambda}_s \, ds$. This gives us $\langle M, M \rangle = \langle M^i, M^k \rangle_{i,k=1,\dots,d}$ as $\langle M, M \rangle = \int_0^\cdot S_s \sigma_s \sigma_s^{\mathrm{tr}} S_s \, ds$ and so we see that $A \ll \langle M, M \rangle$ in the sense that $dA_t = d\langle M, M \rangle_t \lambda_t$ with $\lambda_t \in \mathbb{R}^d$:

$$dA_t = S_t b_t \, dt = S_t \sigma_t \sigma_t^{\mathrm{tr}} S_t S_t^{-1} (\sigma_t \sigma_t^{\mathrm{tr}})^{-1} b_t \, dt = d\langle M, M \rangle \lambda_t$$

with

$$\lambda_t := S_t^{-1} (\sigma_t \sigma_t^{\mathrm{tr}})^{-1} b_t \,.$$

The process

$$K = \int \lambda^{\mathrm{tr}} d\langle M, M\rangle \lambda = \int b^{\mathrm{tr}}(\sigma\sigma^{\mathrm{tr}})^{-1} b dt$$

is often called the *mean-variance tradeoff* process. We also have that $K = \int \overline{\lambda}^{\mathrm{tr}} \overline{\lambda} dt = \int |\overline{\lambda}_t|^2 dt$.

S defined as above is called an Itô process model with coefficients b (or μ and r), σ.

Continuous model: $S = S_0 + M + A$ is a continuous semimartingale with its canonical decomposition into a continuous local martingale M and a predictable process A. We say S satisfies the stucture condition (SC') if $A \ll \langle M \rangle$ in the sense that $dA = d\langle M \rangle \lambda$ for some predictable λ. We say that S satisfies (SC) if it (SC') is true and if λ is in $L^2_{loc}(M)$. The last condition means that $\int \lambda^{\mathrm{tr}} d\langle M, M \rangle \lambda$ is finite-valued (i.e. K is finite valued).

Suppose S is a continuous semimartingale. Then S satisfies the structure condition (SC) if and only if S satisfies (NUPBR).

Suppose we have a continuous model and that (SC) holds. If $Q \approx P$ is an equivalent local martingale measure for S, what can be said about L^Q?

Since $M \in \mathcal{M}^2_{0,loc}(P)$ (after all a continuous local martingale), we can use the Kunita-Watanabe decomposition to write $L^Q = \int \gamma^Q dM + N^Q$ with $N^Q \in \mathcal{M}_{0,loc}(P)$ and $N^Q \perp M$ (so again, because M is continuous, $\langle N^Q, M \rangle \equiv 0$).

Lemma

$Q \approx P$ is an equivalent local martingale measure for S iff $\gamma^Q = -\lambda$. In the Itô process case we have $\gamma^Q = -S^{-1}(\sigma\sigma^{\mathrm{tr}})^{-1}b$.

Proof

By Bayes' rule, we have that $Q \approx P$ an equivalent martingale measure for S iff $Z^Q S$ is in $\mathcal{M}_{loc}(P)$. Using Itô's formula, we compute

$$\begin{aligned} d(Z^Q S) &= Z^Q_- dS + S dZ^Q + d\langle Z^Q, S\rangle \\ &= Z^Q_- dM + S dZ^Q + Z^Q_- dA + Z^Q_- d\langle L^Q, S\rangle \end{aligned}$$

The first two terms of the right hand side are local martingales, so for $Z^Q S$ to be a martingale in $\mathcal{M}_{loc}(P)$, $A + \langle L^Q, M\rangle$ must be in $\mathcal{M}_{loc}(P)$. Since A and $\langle \bullet \rangle$ are predictable and of finite variation, this is equivalent to saying $A + \langle L^Q, M\rangle \equiv 0$, or
$0 \equiv \int d\langle M, M\rangle \lambda + \int d\langle M, M\rangle \gamma^Q = \int d\langle M, M\rangle (\lambda + \gamma^Q)$.

Corollary

Equivalent local martingale measures Q for S are parametrized via

$$\frac{Z^Q}{Z_0^Q} = \mathcal{E}\left(-\int \lambda dM + N^Q\right)$$

with $N^Q \in \mathcal{M}_{loc,0}(P)$, $N^Q \perp M$ under P as long as the right hand side is a strictly positive martingale.

More precisely, if Q is an equivalent local martingale measure, then Z^Q has the above form with some such N^Q. We also have the converse, so if N^Q is as above, then the corresponding $Z^Q := Z_0^Q \mathcal{E}(-\int \lambda dM + N^Q)$ gives an equivalent local martingale measure, if $Z^Q > 0$ and if we also have that Z^Q is a true P-martingale on $[0, T]$.

Remark

The simplest choice of N^Q is $N^Q \equiv 0$. The corresponding process is then (taking $Z_0^Q := 1$)
$\hat{Z} := \mathcal{E}(-\int \lambda dM) = \exp\{-\int \lambda dM - \frac{1}{2}K\}$. If this is a true P-martingale, then the corresponding equivalent local martingale measure $\hat{P}$ is called the *minimal martingale measure*.

Since $N^Q \perp M$, Yor's formula gives
$\frac{Z^Q}{Z_0^Q} = \mathcal{E}(-\int \lambda dM + N^Q) = \hat{Z}\mathcal{E}(N^Q)$.

What can we say if S is in addition also an Itô process model?

Lemma

Suppose S is an Itô process model with b, σ. Suppose $\mathbb{F} = \mathbb{F}^W$ and $N \in \mathcal{M}_{0,loc}(P)$. Then $N \perp M$ under P iff $N = \int \gamma dW$ with γ predictable, $\mathbb{R}^n$-valued and $\sigma\gamma \equiv 0$.

Proof.

$N = \int \gamma dW$ by Itô's representation theorem. $N \perp M$ under P if and only if $\langle N, M \rangle \equiv 0$, i.e. if and only if $0 \equiv \langle \int \gamma dW, \int S\sigma dW \rangle = \int S\sigma\gamma dt$. $\square$

Corollary

Suppose S is an Itô process model with b, σ. If $\mathbb{F} = \mathbb{F}^W$, then equivalent local martingale measures Q are parametrized via processes γ from the kernel of σ by

$$Z^Q = \mathcal{E}\left(- \int (\sigma\sigma^{\mathrm{tr}})^{-1} b\sigma dW + \int \gamma dW \right)$$

with $\sigma\gamma \equiv 0$ as long as the right hand side is a strictly positive martingale.

Remark

If $d = n$, then there is at most one equivalent local martingale measure for S, since $\sigma\gamma \equiv 0$ implies $\gamma \equiv 0$, since σ is now invertible.

A special case of the above is the Black-Scholes model:
$d = n = 1$, $\mu, r, \sigma > 0$ are all constants, so we have a unique candidate for the density process of the equivalent local martingale measure: $\hat{Z} = \mathcal{E}(-\int \frac{\mu - r}{\sigma} dW) = \mathcal{E}(-\frac{\mu - r}{\sigma} W)$. Since all coefficients are constant, $\hat{Z}$ is a true P-martingale, so $\hat{P}$ is an equivalent local martingale measure, and $dS_t = S_t \sigma d\hat{W}_t$ is even a true $\hat{P}$-martingale; so $\hat{P}$ is even an equivalent martingale measure.

A Lévy process L is a stochastically continuous $\mathbb{R}^d$-valued with stationary, independent increments. Following [5] we can choose a càdlàg version of a Lévy process. Additionally we know that the logarithm of the characteristic function of L is of Lévy-Khintchine form.

We analyze how a Lévy process L looks like with respect to an equivalent σ-martingale measure:

Theorem

Let L be a one dimensional Lévy process and assume that $S = \exp(L)$ is a σ-martingale, then S is already a martingale

Proof

By the Ansel-Stricker Lemma a bounded from below σ-martingale is in fact a local martingale, and hence a super-martingale. We therefore have that

$$E[\exp(L_t)] \leq 1\,,$$

for $t \geq 0$, by the super-martingale property. Since L is a Lévy process we know that the Lévy exponent κ is at least well defined on the strip in $\mathbb{C}$ of complex numbers u with real part $0 \leq \Re(u) < 1$ and has Lévy-Khintchine form there. We are interested in showing that $\kappa(u) \to 0$ as $u \nearrow 1$, which then yields the martingale property.

Proof

Due to κ's Lévy-Khintchine form there are numbers $b \in \mathbb{R}$, $c \geq 0$ and a Radon measure ν on $\mathbb{R} \setminus \{0\}$ such that

$$\begin{aligned}\kappa(u) =& bu + \frac{c^2}{2}u^2 + \int_{\|\xi\|\geq 1}(\exp(u\xi) - 1)\nu(d\,\xi) + \\ &+ \int_{\|\xi\|\leq 1}(\exp(u\xi) - 1 - u\xi)\nu(d\xi)\end{aligned}$$

for $0 \leq u < 1$.

Proof

The first, second and fourth summand are continuous in u as $u \nearrow 1$ by continuity of polynomials and dominated convergence. The third summand can be split in two parts (on the positive and negative real line, respectively), where we can conclude by dominated convergence on the negative real line and by monontone convergence on the positive real line by the fact that $\kappa(u) \leq 0$ as $u \in [0, 1[$ by convexity of the moment generating function.

A slightly more complicated situation is given when we look at Lévy processes themselves. We can conclude the same result, however, we cannot use the Ansel-Stricker Lemma.

Theorem

Let L be a Lévy process. Assume that L is a σ-martingale, then it is a martingale.

Proof

A σ-martingale L is a semi-martingale such that there is an increasing sequence of predictable sets $D_n \nearrow \Omega \times [0,1]$ with $(1_{D_n} \bullet L)$ is a local martingale (take the definition of σ-martingales as limit of stochastic integrals of the form $(H1_{\{\|H\| \leq n\}} \bullet M)$ for some predictable strategy $H \in L(M)$). For every n we can hence choose a localizing sequence of stopping times τ_{nm} such that $(1_{D_n} \bullet L^{\tau_{nm}})$ actually are martingales. The compensator (i.e. the predictable process $\tilde{A}$ uniquely associated by the Doob-Meyer decomposition to an increasing, locally integrable finite variation process A making the difference $A - \tilde{A}$ a local martingale) $\tilde{A} = t\nu$ of $A = \sum_{s \leq t} 1_{\{\|\Delta L_s\| \geq 1\}}$ is always well-defined and deterministic due to independent increments and linear in time due to stationarity of increments.

Proof

We do additionally have that

$$\int_0^{\tau_{nm}} 1_{D_n}(s)\, ds\; 1_{\{\|\xi\|\geq 1\}}\nu$$

is the compensator of $\sum_{s\leq t} 1_{\{\|(\Delta(1_{D_n}\bullet L^{\tau_{nm}})_s\|\geq 1\}}$. If we integrate now $s \mapsto \Delta(1_{D_n} \bullet L)_s$ with respect to this counting measure of the jumps we obtain

$$\sum_{s\leq t} 1_{\{\|(\Delta(1_{D_n}\bullet L^{\tau_{nm}})_s\|\geq 1\}} \Delta(1_{D_n} \bullet L^{\tau_{nm}})_s$$

which in turn is integrable by martingality. Hence we obtain that $\int_{\|\xi\|\geq 1} \xi \nu(d\,\xi)$ is finite, which proves the martingale property of L.

Table of Contents

Bipolar Theorem

Let X be locally convex space, $C \subset X$ a convex cone (i.e. for all $x, y \in C$ and all $\lambda, \mu \geq 0$ the cone combination $\lambda x + \mu y \in C$), then the polar cone

$$C^0 := \left\{ f \in X^* \text{ such that } \langle f, x \rangle \leq 0 \text{ for all } x \in C \right\}$$

is again a convex cone, and it holds that $C^{00} = \overline{C}$. We consider the weak-$*$-topology on X^*.

Obviously the polar cone is closed, and therefore also the bipolar cone. Additionally for $x \in C$ it holds that $\langle f, x \rangle \leq 0$ for all $f \in C^0$, so $x \in C^{00} \cap X$. Therefore $\overline{C} \subset C^{00}$. Now, take $y \in C^{00} \setminus \overline{C}$, then there exists $l \in X^*$ such that $l(x) \leq 0$ for all $x \in C$ and $l(y) > 0$ by Hahn-Banach, i.e. $l \in C^0$. But then $\langle l, y \rangle \leq 0$, a contradiction.

Super-replication of contracts

Let $\mathcal{X}$ be a set of (self-financing) portfolio wealth processes, and assume (NFLVR), therefore the cone C of bounded outcomes minus consumption is weak-$*$-closed and we can apply the Bipolar Theorem.

Let us first calculate the polar of C: every $0 \neq h \in C^0$ actually satisfies $E[hf] \leq 0$ for all $f \in C$, whence $h \geq 0$ and there is $Q \ll P$ and $\lambda > 0$ such that $\lambda \frac{dQ}{dP} = h$. Note that $E_Q[f] \leq 0$, whence Q is a separating measure. Therefore

$$C^0 = \bigcup_{Q \text{ separating measures}} \mathbb{R}_{\geq 0} \frac{dQ}{dP} .$$

The Bipolar theorem tells in this case that $C^{00} = C$.

Super-replication of contracts

Let $g \in L^\infty$ be a contract (derivative, claim), then we can define the super-replication price

$$\pi(g) := \inf \left\{ \pi \in \mathbb{R} \text{ such that there exists } f \in C \, : \, \pi + f \geq g \right\}.$$

By the previous considerations it holds that

$$\pi(g) = \sup_{Q \text{ (ESM)}} E_Q[g] \, .$$

One direction is obvious. Assume now – possibly after translation – that $\sup_{Q \text{ (ESM)}} E_Q[g] = 0$, then $g \in C^{00} = C$, so $\pi(g) = 0$. In particular the infimum is a minimum.

Replication of contracts

We also understand what happens when for a contract $g \in L^\infty$ it holds that

$$E_Q[g] = \sup_{Q\ (\text{ESM})} E_Q[g]$$

for all (ESM) Q.

Under this assumption and an appropriate translation we achieve $E_Q[g] = 0$ and we obtain that $g, -g \in C$ by the Bipolar theorem. Therefore there are $f, \tilde{f} \in K^0$ and $h, \tilde{h} \in L^0_{\geq 0}$ such that $g = f - h, -g = \tilde{f} - \tilde{h}$ by the definition of C. So

$$0 = f + \tilde{f} - h - \tilde{h}$$

which – by the absence of arbitrages – yields $f + \tilde{f} = 0$ and $h + \tilde{h} = 0$. Whence $g, -g \in K^0$. We say that g as well as $-g$ can be replicated.

Incomplete markets

A market is called *incomplete* if the set of equivalent separating measures contains more than one measure.

In this case there are claims $g \in L^\infty$ where the set of all $E_Q[g]$ has non-empty interior. Every such value is considered an arbitrage free price of g, since all prices strictly above leave the seller with an arbitrage and the supremum allows for a super-replication, all prices strictly below leave the seller of $-g$ with an arbitrage and minus the infimum allows for super-replication of $-g$.

Complete Markets

A market is called *complete* if the set of equivalent separating measures is a singleton.

In this case every claim $g \in L^\infty$ can be replicated. We can consider the (super-)replication price as the only arbitrage-free price of g (pricing by absence of arbitrage).

Standard models

Assume that we have a standard model of a financial market $(\Omega, \mathcal{F}, \mathbb{F}, P)$ over $[0, T]$ with $B \equiv 1$ and S a $\mathbb{R}^d$-valued semimartingale.

The basic question is: given $H \in L^0(\mathcal{F}_T)$, viewed as a random payoff of a contract at time T, what is its value at $t \leq T$?

Replicating Strategy

Definition (Replicating strategy)

A replicating strategy for H is a self-financing φ with $V_T(\varphi) = H$ P-a.s.; we then call H replicable or attainable by φ.

Valuation by replication

Theorem (Valuation of attainable payoffs I)

If $H \in L^0(\mathcal{F}_T)$ is replicable by φ, then the value at time 0 is dominated by $t \leq T$ is $V_0(\varphi)$.

Proof

Take $\varphi = (\vartheta, \eta)$. For every equivalent σ-martingale measure we obtain the super-martingale property for $V(\varphi)$ and therefore $V_0^H \geq \mathbb{E}_Q[H]$.

Remark

How can we compute $V_t(\varphi)$ more easily? Note: H is attainable $\Leftrightarrow$. then there exists self financing φ with $V_T(\varphi) = H$ P-a.s. $\Leftrightarrow$ $H = V_0 + \int_0^T \vartheta_u dS_u$ P-a.s., i.e. H is up to V_0 representable as a stochastic integral of S.

Moreover, φ self-financing implies that $V_t(\varphi) = V_0 + \int_0^t \vartheta_u dS_u$, so if Q is an EσMM for S, then $\int \vartheta dS$ is (for sufficiently integrable ϑ) a Q-martingale, and so $V_t(\varphi) = \mathbb{E}_Q[H|\mathcal{F}_t]$, $0 \leq t \leq T$.

Valuation of attainable payoffs, for all times

If $H \in L^0(\mathcal{F}_T)$ is attainable by "reasonable" strategy φ, the value of H at any time $t \leq T$, if there is no arbitrage, is given by $V_t(\varphi) = V_t^H := \mathbb{E}_Q[H|\mathcal{F}_t]$ for any EσMM Q for S.

Non-attainable claims

We use the standard model of $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and S on $[0, T]$. Denote by $\mathbb{P}$ the set of all equivalent σ martingale measures for S and assume $\mathbb{P} \neq \emptyset$; by the fundamental theorem of asset pricing, this guarantees (NFLVR).

Non-attainable claims

Fix a payoff $H \in L^0_{\geq 0}(\mathcal{F}_T)$. Everything would work for $H \geq -\mathrm{const.}$ as well. We assume H is not attainable, so there is no self-financing strategy φ with $V_T(\varphi) = H$ P-a.s. How do we hedge such an H? Idea: look at strategies that produce at least H and try to find the cheapest one.

Super-replicating strategies

Definition (Superreplication price)

The super-replication price of $H \in L^0_{\geq 0}(\mathcal{F}_T)$ is

$$\begin{aligned}\Pi_s(H) &= \inf\{V_0 \in \mathbb{R} \,|\, \exists\, \vartheta \in \Theta_{adm} : \; V_0 + \int_0^T \vartheta_u dS_u \geq H \; P\text{-a.s.}\} \\ &= \inf\{V_0 \in \mathbb{R} \,|\, H - V_0 \in G_T(\Theta_{adm}) - L^0_{\geq 0}\}\end{aligned}$$

The intuition behind this definition is that we can sell H for $\Pi_s(H)$ without risk, because $(\Pi_s(H), \vartheta)$ is a self-financing admissible strategy which produces at least H by time T. We have to be careful, however, since $\Pi_s(H)$ is an infimum; we do not know if it is attained. So we do not know if there exists a $\vartheta \in \Theta_{adm}$ for $V_0 := \Pi_s(H)$.

An easy lemma

Lemma

Assume that $\mathbb{P} \neq 0$. Then for any payoff $H \in L^0_{\geq 0}(\mathcal{F}_T)$

$$\Pi_s(H) \geq \sup_{Q \in \mathbb{P}} \mathbb{E}_Q[H].$$

Proof

Without loss of generality suppose that

$$B := \left\{ V_0 \in \mathbb{R} \,|\, \exists\, \vartheta \in \Theta_{adm} : \; V_0 + \int_0^T \vartheta_u dS_u \geq H \; P\text{-a.s.} \right\} \neq \emptyset,$$

else $\Pi_s(H) = \infty$. So let $V_0 \in B$ and take some $\vartheta \in \Theta_{adm}$ such that $V_0 + \int_0^T \vartheta_u dS_u \geq H$ P-a.s. Let $Q \in \mathbb{P}$, then $S \in \mathcal{M}_\sigma(Q)$ is a σ-martingale under Q and $G(\vartheta) = \int \vartheta dS$ is bounded below; The Ansel-Stricker Lemma gives us that $G(\vartheta) \in \mathcal{M}_{loc}(Q)$ is a local martingale under Q and in particular a super-martingale. So we get

$$\mathbb{E}_Q[H] \leq V_0 + \mathbb{E}_Q[G_T(\vartheta)] \leq V_0$$

Hence, taking the supremum over Q, infimum over V_0 we get

$$\sup_{Q \in \mathbb{P}} \mathbb{E}_Q[H] \leq \inf B = \Pi_s(H)\,.$$

Optional decomposition

Our goal now is to prove the equality and also that the infimum for $\Pi_s(H)$ is attained. We fix $H \in L^0_{\geq 0}(\mathcal{F}_T)$ and define the adapted process

$$U_t := \text{ess-sup}_{Q \in \mathbb{P}} \mathbb{E}_Q[H|\mathcal{F}_t], \; 0 \leq t \leq T$$

which is the smallest random variable that dominates the set of random variables for any $t \in [0, T]$, i.e. the measurable version of the "supremum". If $\mathcal{F}_0$ is trivial, then $U_0 = \sup_{Q \in \mathbb{P}} \mathbb{E}_Q[H]$.

Optional decomposition

Proposition

Assume $\mathbb{P} \neq \emptyset$ and $H \in L^0_{\geq 0}(\mathcal{F}_T)$. If $\sup_{Q \in \mathbb{P}} \mathbb{E}_Q[H] < \infty$ then U is a Q-supermartingale for every $Q \in \mathbb{P}$, which allows for a càdlàg version.

Proof

We argue that U has the supermartingale property: let $s \le t$, we want to show that $\mathbb{E}_Q[U_t|\mathcal{F}_s] \le U_s$ for any $Q \in \mathbb{P}$. We fix $Q \in \mathbb{P}$ and introduce for $t \in [0, T]$

$$\begin{aligned}\zeta_t := &\{Z \,|\, Z \text{ density process of } R \in \mathbb{P}, \text{ and } Z_s = 1 \text{ for } s \le t\} \\ = &\{Z \,|\, Z \text{ density process of } R \in \mathbb{P}, \text{ with } R = Q \text{ on } \mathcal{F}_t\}\end{aligned}$$

Taking $R = Q$ shows that $1 \in \zeta_t$, so it is not empty; and $\zeta_t \subseteq \zeta_s$ for $0 \le s \le t \le T$. Moreover we claim
$$\zeta_t = \left\{ \frac{Z^R_{t\vee\bullet}}{Z^R_t} \,|\, Z^R \text{ is density process of } R \in \mathbb{P} \right\} \ .$$

Proof

"$\subseteq$" Take $Z \in \zeta_t$ with corresponding $R \in \mathbb{P}$. Then $Z_t = 1$ and so:

$$Z_\bullet = I_{\{\bullet \le t\}} + Z_\bullet I_{\{\bullet > t\}} = \frac{Z_{t \vee \bullet}}{Z_t}.$$

"$\supseteq$" Take $R \in \mathbb{P}$ with Q density process Z^R. Let $Z_\bullet = Z^R_{t \vee \bullet} / Z^R_t$. Then $Z > 0$, $Z_s = 1$ for $s \le t$ and Z is like Z^R a Q-martingale. Moreover, both S and SZ^R are both local Q-martingales (the first one since $Q \in \mathbb{P}$, the second by the Bayes rule because $R \in \mathbb{P}$). So

$$S_\bullet Z_\bullet$$

$$= S_\bullet I_{\{\bullet \le t\}} + \frac{S_\bullet Z^R_\bullet}{Z^R_t} I_{\{\bullet > t\}}$$

Proof

Now we use Bayes rule again to write

$$\begin{aligned} U_t &= \text{ess-sup}_{R\in\mathbb{P}}\, \mathbb{E}_R\left[H|\mathcal{F}_t\right] \\ &= \text{ess-sup}_{R\in\mathbb{P}}\, \mathbb{E}_Q\left[\frac{HZ_T^R}{Z_t^R}\Big|\mathcal{F}_t\right] = \\ &= \text{ess-sup}_{Z\in\zeta_t}\, \underbrace{\mathbb{E}_Q[HZ_T|\mathcal{F}_t]}_{=:\Gamma_t(Z)}. \end{aligned}$$

Proof

We claim that the family $\{\Gamma_t(Z)|Z \in \zeta_t\}$ is an *upwards directed set*: if Z and Z' are in ζ_t and $A \in \mathcal{F}_t$, then apparently $ZI_A + Z'I_{A^c}$ is again in ζ_t. So with $A := \{\Gamma_t(Z) \geq \Gamma_t(Z')\} \in \mathcal{F}_t$, we get

$$\begin{aligned}\max\{\Gamma_t(Z), \Gamma_t(Z')\} &= \Gamma_t(Z)I_A + \Gamma_t(Z')I_{A^c} \\ &= \mathbb{E}_Q[H(Z_t I_A + Z'_t I_{A^c})|\mathcal{F}_t] = \Gamma_t(\overline{Z})\end{aligned}$$

with $\overline{Z} := ZI_A + Z'I_{A^c} \in \zeta_t$. This is useful because the essential supremum of an upward directed family of random variables can be obtained as a monotone increasing limit of a sequence in that family.

Proof

So for each $t \in [0, T]$ there is an increasing sequence $\left(Z^{(n)}\right)_{n\in\mathbb{N}} \subset \zeta_t$ with

$$U_t = \lim_{n\to\infty} \mathbb{E}_Q[HZ_T^{(n)}|\mathcal{F}_t]\,,$$

hence we obtain

$$\begin{aligned}&\mathbb{E}_Q[U_t|\mathcal{F}_s]\\ &= \lim \mathbb{E}_Q\left[\mathbb{E}_Q[HZ_T^{(n)}|\mathcal{F}_t]|\mathcal{F}_s\right] \leq \text{ess-sup}_{Z\in\zeta_s} \mathbb{E}_Q[HZ_T|\mathcal{F}_s] = U_s\,,\end{aligned}$$

where the inequality follows from $Z^{(n)} \in \zeta_t \subseteq \zeta_s$. By a similar argument we obtain that $t \mapsto E_Q[U_t]$ is càdlàg, hence there is a càdlàg version of U by martingale regularisation.

An example

So we have that $U = (U_t)_{0 \leq t \leq T}$ is a Q-supermartingale for any $Q \in \mathbb{P}$. One concrete example of such a process is as follows: take $x \in \mathbb{R}$, ϑ an $\mathbb{R}^d$-valued, predictable, S-integrable process and C an increasing càdlàg, adapted process with $C_0 = 0$. Define

$$V^{x,\vartheta,C} := x + \int \vartheta dS - C$$

and interpret this as the value process of a generalised strategy (x, ϑ, C); x is the initial value, ϑ describes the trading and C_t is the amount spent for consumption on $[0, t]$. Note that $C \geq 0$ and

$$V^{x,\vartheta,C} + C = x + \int \vartheta dS \,,$$

so if $V^{x,\vartheta,C}$ is bounded below, then $\vartheta \in \Theta_{adm}$.

An example

Whenever $\vartheta \in \Theta_{adm}$, $\int \vartheta dS$ is by Ansel-Stricker a Q-supermartingale for all $Q \in \mathbb{P}$. The same is then true for $V^{x,\vartheta,C}$ if this process is uniformly (in t, ω) bounded below; note that

$$0 \leq C \leq \mathrm{const} + \int \vartheta dS$$

shows that C is Q-integrable. Hence each $V^{x,\vartheta,C}$ with $V^{x,\vartheta,C} \geq \mathrm{const.}$ is a Q-supermartingale, for all $Q \in \mathbb{P}$. This is the *only* such example.

Optional decomposition theorem

Theorem (Optional decomposition, Dima Kramkov)

Suppose $\mathbb{P} \neq \emptyset$. Suppose $U = (U_t)_{0 \leq t \leq T}$ is an adapted, càdlàg process $U_t \geq 0$ with the property that U is a Q-supermartingale for all $Q \in \mathbb{P}$. Suppose $\mathcal{F}_0$ is trivial. Then there is some $x \in \mathbb{R}$, $\vartheta \in \Theta_{adm}$ and an adapted, increasing, càdlàg process C with $C_0 = 0$ such that

$$U = V^{x,\vartheta,C} = x + \int \vartheta dS - C$$

(In fact, $x = U_0$.)

Hedging Duality

An immediate consequence of the above theorem is the hedging duality:

Theorem

Suppose $\mathbb{P} \neq \emptyset$ and $\mathcal{F}_0$ is trivial. For any $H \in L^0_{\geq 0}(\mathcal{F}_T)$ we then have

$$\Pi_s(H) = \inf\{V_0 \in \mathbb{R} | H - V_0 \in G_T(\Theta_{adm}) - L^0_{\geq 0}\} = \sup_{Q \in \mathbb{P}} \mathbb{E}_Q[H].$$

Moreover, the infimum is attained as a minimum if $\sup_{Q \in \mathbb{Q}} \mathbb{E}[H] < \infty$.

Proof

"$\geq$" Follows from the easy lemma.

"$\leq$" is trivial if $RHS = +\infty$. So suppose that $\sup_{Q\in\mathbb{P}} \mathbb{E}[H] < \infty$; with $U_0 := \text{ess-sup}_{Q\in\mathbb{P}} \mathbb{E}_Q[H|\mathcal{F}_0]$, this means by the above proposition that U is a Q-supermartingale, $\forall Q \in \mathbb{P}$, so $U = U_0 + \int \vartheta dS - C$ by optional decomposition with $\vartheta \in \Theta_{adm}$, $C \nearrow$, null at 0. So $C_T \geq 0$ and so $H - U_0 = U_T - U_0 = \int_0^T \vartheta dS - C_T \in G_T(\Theta_{adm}) - L^0_{\geq 0}$ shows (by using the definition of $\Pi_s(H)$) that $V_0 \leq U_0 = \sup_{Q\in\mathbb{P}} \mathbb{E}_Q[H]$; the argument also shows that the infimum is attained by $V_0 = U_0$.

American Options

With a European option, the time of the payoff is fixed (usually T). With an American option, the owner/holder can also choose the exercise time. How can we model, value and hedge such a product?

We use the usual model of $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, $B \equiv 1$ and $S = (S_t)_{0 \leq t \leq T}$ an $\mathbb{R}^d$-valued semimartingale. We impose absence of arbitrage via $\mathbb{P} \neq \emptyset$.

American Options

An *American option* is described by its payoff process $U = (U_t)_{0 \leq t \leq T}$ (discounted as usual); U is $\mathbb{F}$-adapted, càdlàg, ≥ 0. Then U_τ is the payoff due at time τ if the owner decides to exercise the option at τ. The owner/holder chooses τ, but it must be a stopping time to exclude prophets and clairvoyance, with values $\tau \in [0, T]$.

Notation: $\mathcal{S}_{t,T}$ is the set of all stopping times τ with values in $[t, T]$.

American Options

Consider the seller/writer of an American option at time $t \in [0, T]$. What can she do?

- If option has already been exercised: nothing.
- Otherwise: suppose the owner chooses to exercise at τ. Then the seller faces a payoff (at τ) of U_τ. To be safe, the seller would like to be able to super-replicate this, from t on; so he needs $\text{ess-sup}_{Q \in \mathbb{P}} \mathbb{E}_Q[U_\tau | \mathcal{F}_t]$. But the seller does not know τ, so to be safe, he will also need to maximise over $\tau \in \mathcal{S}_{t,T}$. This prepares him for the worst case.

The Snell Envelop

So, the natural selling price at t is:

$$\overline{V}_t := \text{ess-sup}_{Q\in\mathbb{P},\tau\in\mathcal{S}_{0,T}} I_{\{\tau\geq t\}}\mathbb{E}_Q[U_\tau|\mathcal{F}_t] = \text{ess-sup}_{Q\in\mathbb{P},\tau\in\mathcal{S}_{t,T}} \mathbb{E}_Q[U_\tau|\mathcal{F}_t] .$$

Proposition

Suppose $\mathbb{P}\neq\emptyset$ and $\mathcal{F}_0$ is trivial. If

$$\overline{V}_0 = \sup_{Q\in\mathbb{P},\tau\in\mathcal{S}_{0,T}} \mathbb{E}[U_\tau] < \infty ,$$

then $\overline{V}$ is a Q-super-martingale for all $Q\in\mathbb{P}$. Moreover, it is the smallest of all càdlàg processes $V'\geq U$ such that V' is Q-super-martingale, for all $Q\in\mathbb{P}$.

Proof

We shall use actually the essential supremum can be expressed by an upwards directed family. Let us fix $Q \in \mathbb{P}$ and set

$$\zeta_t := \{\text{density process } Z \text{ w.r.t. } Q \text{ of } R \in \mathbb{P}, \text{ with } R = Q \text{ on } \mathcal{F}_t\}$$

Then we obtain that $\overline{V}_t = \text{ess-sup}_{Z \in \zeta_t, \tau \in \mathcal{S}_{t,T}} \underbrace{\mathbb{E}_Q[Z_\tau U_\tau | \mathcal{F}_t]}_{=:\Gamma_t(Z,\tau)}$.

Proof

Moreover, the family $\{\Gamma_t(Z,\tau)\,|\,Z\in\zeta_t,\tau\in\mathcal{S}_{t,T}\}$ is upwards directed: For $\Gamma_t(Z^i,\tau_i)$, set $A:=\{\Gamma_t(Z^1,\tau_1)\geq\Gamma_t(Z^2,\tau_2)\}\in\mathcal{F}_t$, so $\overline{Z}:Z^1I_A+Z^2I_{A^C}$ is in ζ_t and $\overline{\tau}:=\tau_1I_A+\tau_2I_{A^C}$ is in $\mathcal{S}_{t,T}$, and then $\max(\Gamma_t(Z^1,\tau_1),\Gamma_t(Z^2,\tau_2))=\Gamma_t(\overline{Z},\overline{\tau})$.

So for $s\leq t$, we get:

$$\overline{V}_t=\text{ess-sup}_{Z\in\zeta_t,\tau\in\mathcal{S}_{t,T}}\Gamma_t(Z,\tau)=\lim_{n\to\infty}\mathbb{E}_Q[Z^n_{\tau_n}U_{\tau_n}|\mathcal{F}_t],$$

and so (by using, in the first equality, monotone convergence due to the set being upwards directed)

$$\mathbb{E}_Q[\overline{V}_t|\mathcal{F}_s]=\lim_{n\to\infty}\mathbb{E}_Q\left[\mathbb{E}_Q[Z^n_{\tau_n}U_{\tau_n}|\mathcal{F}_t]|\mathcal{F}_s\right]\leq$$
$$\text{ess-sup}_{Z\in\zeta_t,\tau\in\mathcal{S}_{s,T}}\mathbb{E}_Q[Z_\tau U_\tau|\mathcal{F}_s]=\overline{V}_s\,,$$

which gives us the super-martingale property, and also $\overline{V}\geq0$ and then $\mathbb{E}_Q[\overline{V}_t]\leq\overline{V}_0<\infty$.

Proof

We now prove the minimality of $\overline{V}$: Since $t \in \mathcal{S}_{t,T}$, we get $\overline{V} \geq U$ in the sense that $\overline{V}_t \geq U_t$ P-a.s., for all $t \in [0, T]$. If V' satisfies this as well and is a Q-super-martingale, for all $Q \in P$, and càdlàg, then $V'_t \geq \mathbb{E}_Q[V'_\tau|\mathcal{F}_t] \geq \mathbb{E}_Q[U_\tau|\mathcal{F}_t]$, for all $Q \in \mathbb{P}$, for all $\tau \in \mathcal{S}_{t,T}$, where the first inequality follows from the stopping theorem and the second one since $V' \geq U$ and both are càdlàg. So we get that $V'_t \geq \text{ess-sup}_{Q\in\mathbb{P},\tau\in\mathcal{S}_{t,T}} \mathbb{E}_Q[U_\tau|\mathcal{F}_t] = \overline{V}_t$ P-a.s., for all t.

Remark

One has to show that $\overline{V}$ has version which is càdlàg. This is important for the comparison between V' and $\overline{V}$. This is also important since we want $\overline{V}_\tau \geq U_\tau$, for all $\tau \in \mathcal{S}_{0,T}$.

Superreplication Price of American Options (definition)

We now look at generalised strategies with consumption, $x \in \mathbb{R}, \vartheta \in \Theta_{adm}, C$ adapted, increasing càdlàg, null at 0, with $V^{x,\vartheta,C} = x + \int \vartheta dS - C$. We also introduce for the American option the *super-replication price* at 0 as:

$$\Pi_s(U) := \inf \{ V_0 \in \mathbb{R} \,|\, \exists \vartheta \in \Theta_{adm} \text{ with } V_0 + G(\vartheta) \geq U \} .$$

Note that we want $V_0 + G_\tau(\vartheta) \geq U_\tau$ a.s. for all stopping times; which is well defined as $G(\vartheta)$, U are both càdlàg.

Superreplication Price of American Options (theorem)

Theorem

Suppose $\mathbb{P} \neq \emptyset$ *and* $\mathcal{F}_0$ *trivial. If*

$$\overline{V}_0 = \sup_{Q \in \mathbb{P}, \tau \in \mathcal{S}_{0,T}} \mathbb{E}_Q[U_\tau] < \infty \,,$$

then it holds that

1. *there exists a generalized strategy with consumption* (x, ϑ, C) *with* $V^{x,\vartheta,C} \geq U$ *and* (x, ϑ, C) *is minimal in the sense that for any* (x', ϑ', C') *with* $V^{x',\vartheta',C'} \geq U$, *we have* $V^{x,\vartheta,C} \leq V^{x',\vartheta',C'}$. *Moreover, we can take* $x = \overline{V}_0 = \sup_{Q \in \mathbb{P}, \tau \in \mathcal{S}_{0,T}} \mathbb{E}_Q[U_\tau]$.
2. *the super-replication price is* $\Pi_s(U) = \overline{V}_0 = \sup\{\mathbb{E}_Q[U_\tau] | Q \in \mathbb{P}, \tau \in \mathcal{S}_{0,T}\}$.

Proof

By the previous consideration $\overline{V} \geq U$ is a Q-supermartingale, for all $Q \in \mathbb{P}$. So existence of (x, ϑ, C) is immediate from the optional decomposition Theorem, and also $x = \overline{V}_0$. The minimality: $V' := V^{x', \vartheta', C'}$ is a Q-supermartingale for all $Q \in \mathbb{P}$. So if also $V' \geq U$, then $\overline{V} \leq V'$.

If $V^{x,\vartheta,0} = x + G(\vartheta) \geq U$, then for any $Q \in \mathbb{P}$: $\mathbb{E}_Q[U_\tau] \leq x + \mathbb{E}_Q[G_\tau(\vartheta)] \leq x$, for all $\tau \in \mathcal{S}_{0,T}$, as $G(\vartheta)$ is Q-supermartingale. So $\Pi_s(U) \geq \overline{V}_0$. For the "$\leq$" part, take (x, ϑ, C) from part 1) with $x = \overline{V}_0$ to get $\vartheta \in \Theta_{adm}$ with $x + G(\vartheta) = V^{x,\vartheta,0} \geq V^{x,\vartheta,C} \geq U$ by 1), and so $\Pi_s(U) \leq x = \overline{V}_0$.

Interpretation

The initial capital $x = \overline{V}_0 = \sup_{Q\in\mathbb{P},\tau\in\mathcal{S}_{0,T}} \mathbb{E}_Q[U_\tau]$ allows construction of self-financing strategy (x, ϑ) whose value process $V(x, \vartheta) = x + \int \vartheta dS \geq V^{x,\vartheta,C} \geq U$ always lies above U, so following (x, ϑ) keeps the option seller safe and allows him to make the payoff U_τ, no matter which τ is chosen by the option holder. Depending on the τ, the option seller might make a profit of: $x + G_\tau(\vartheta) - U_\tau = V_\tau^{x,\vartheta,C} - U_\tau + C_\tau \geq C_\tau$.

The same reasoning holds at any time t instead of 0; then starting with $\overline{V}_t$ at t leads to profit of $C_\tau - C_t \geq 0$ for $\tau \in \mathcal{S}_{t,T}$, since C is decreasing.

If $\mathbb{P} = \{Q^*\}$ is a singleton (so that, as we know from finite discrete time, we have a complete market), then

$$\overline{V}_t = \text{ess-sup}_{\tau\in\mathcal{S}_{t,T}} \mathbb{E}_{Q^*}[U_\tau|\mathcal{F}_t],\ 0 \leq t \leq T.$$

Optimal Stopping Problems

Finding this is the classical *optimal stopping problem*. If one has a Markov structure, this further reduces to the *free boundary problem*, which is a PDE problem with an unknown boundary.

For general $\mathbb{P}$, finding $\overline{V}$ is usually difficult. A frequent approach, especially in the Lévy setting is to start with a P-Lévy model for S and then look for a $Q \in \mathbb{P}$ such that S (or $\log S$) is also Q-Lévy. Then we try to work out
$V_t^Q := \text{ess-sup}_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}_Q[U_\tau | \mathcal{F}_t]$, $0 \leq t \leq T$.

The next step is to use V^Q as the price process of U. This is partly all right, since it gives no arbitrage; usually, however, there is no hedging strategy to guarantee that one can stay above U in a self-financing way.

Backwards Induction

For *finite discrete time*, the results are more explicit, since we can construct $\overline{V}$ by backward recursion. For $Q \in \mathbb{P}$, denote by $Z^Q = \left(Z_k^Q\right)_{k=0,\dots,T}$ the density process of Q w.r.t. P. Define the process J recursively backward by $J_T = U_T$ and for $k = 0, 1, \dots, T-1$:

$$J_k = \max\left\{U_k, \text{ess-sup}_{Q\in\mathbb{P}} \mathbb{E}_Q[J_{k+1}|\mathcal{F}_k]\right\}$$

Note: by Bayes' rule we obtain $\mathbb{E}_Q[J_{k+1}|\mathcal{F}_k] = \mathbb{E}_P\left[J_{k+1}\frac{Z_{k+1}^Q}{Z_k^Q}|\mathcal{F}_k\right]$ and this needs only the one-step transition probabilities of Q between k and $k+1$.

Assume $\mathbb{P} \neq \emptyset$ and final discrete time. Then $J = \overline{V}$, so that $\overline{V}$ has a recursive representation.

Backwards Induction

All the conditional expectations are well defined in $[0, \infty]$, and we get from $\overline{V}$ the supermartingale property (for each Q) and the above minimality, even without integrability.

"$\geq$" By construction, $J \geq U$ and for each $Q \in \mathbb{P}$, $J_k \geq \mathbb{E}_Q[J_{k+1}|\mathcal{F}_k]$, i.e., J has the Q-supermartingale property for all $Q \in \mathbb{P}$. But $\overline{V}$ is minimal, so $J \geq \overline{V}$.

"$=$" Induction: $J_T = U_T = \overline{V}_T$, and if $J_{k+1} \leq \overline{V}_{k+1}$, we get for all $Q \in \mathbb{P}$ that $\mathbb{E}_Q[J_{k+1}|\mathcal{F}_k] \leq \mathbb{E}_Q[\overline{V}_{k+1}|\mathcal{F}_k] \leq \overline{V}_k$ by the previous consideration; so $J_k = \max\left\{U_k, \text{ess-sup}_{Q\in\mathbb{P}}\,\mathbb{E}_Q[J_{k+1}|\mathcal{F}_k]\right\} \leq \max(U_k, \overline{V}_k) = \overline{V}_k$.

Backwards Induction

If the market is complete, so we have $\mathbb{P} = \{Q^*\}$, the recursion becomes

$$\overline{V}_k = \max\{U_k, \mathbb{E}_{Q^*}[\overline{V}_{k+1}|\mathcal{F}_k]\}\,.$$

Backwards Induction

At time k, the option holder can either exercise the option (and get U_k) or he can continue to hold the option for at least one time step. Then the value at time $k+1$ will be $\overline{V}_{k+1}$, and viewed as a time $k+1$ payoff, that has a time k value of $\text{ess-sup}_{Q \in \mathbb{P}} \mathbb{E}_Q[\overline{V}_{k+1} | \mathcal{F}_k]$. As the option holder is free to choose his decision at k, the value of the contract for him at k is the maximum of the two possibilities.

In the complete market case $\mathbb{P} = \{Q^*\}$, the optional decomposition of $\overline{V}$ is given by the Doob-Meyer decomposition of the Q^*-supermartingale $\overline{V}$. Indeed, $\overline{V}$ is a Q^*-supermartingale, so by Doob-Meyer $\overline{V} =$"Q^*-(local) martingale" $-$ "increasing predictable process"; and since $\mathbb{P} = \{Q^*\}$, S has the martingale representation property, so the above Q^*-martingale is a stochastic integral of S, which gives us the optional decomposition and even C predictable.

The Black Scholes Case

Suppose $\mathbb{P} = \{Q^*\}$ and $S = \frac{\tilde{S}}{\tilde{B}}$ is a true Q^*-martingale. Consider $\tilde{U}_t = (\tilde{S}_t - K)^+$, $0 \leq t \leq T$. Then: if $\tilde{B}$ is increasing (i.e. the interest rates are non-negative), then

$$\tilde{V}_t = \tilde{B}_t \mathbb{E}_{Q^*}\left[\frac{(\tilde{S}_T - K)^+}{\tilde{B}_T} \Big| \mathcal{F}_t\right] = \tilde{B}_t \mathbb{E}_{Q^*}\left[\frac{\tilde{U}_T}{\tilde{B}_T} \Big| \mathcal{F}_t\right].$$

So: American call option has the same value as a European call option.

The Black Scholes Case

An important point is that S is a Q^*-martingale and $x \mapsto (x-K)^+$ is convex; so we get a submartingale, and it is never optimal to stop a submartingale early. More precisely:

$$\frac{\tilde{U}_\tau}{\tilde{B}_\tau} = \left(S_\tau - \frac{K}{\tilde{B}_\tau}\right)^+ \geq \left(S_\tau - \frac{K}{\tilde{B}_t}\right)^+$$

and so

$$\mathbb{E}_{Q^*}[\frac{\tilde{U}_\tau}{\tilde{B}_\tau}|\mathcal{F}_t] \geq \mathbb{E}_{Q^*}[(S_\tau - \frac{K}{\tilde{B}_t})^+|\mathcal{F}_t] \geq$$

$$(\mathbb{E}_{Q^*}[S_\tau - \frac{K}{\tilde{B}_t}|\mathcal{F}_t])^+ = (S_t + \frac{K}{\tilde{B}_t})^+ = \frac{\tilde{U}_t}{\tilde{B}_t}.$$

The Black Scholes Case

So $\frac{\tilde{U}}{\tilde{B}}$ is a Q^*-submartingale, so that $\mathbb{E}_{Q^*}\left[\frac{\tilde{U}_T}{\tilde{B}_T}|\mathcal{F}_\tau\right] \geq \frac{\tilde{U}_\tau}{\tilde{B}_\tau}$, for all $\tau \in \mathcal{S}_{t,T}$, hence by conditioning on $\mathcal{F}_t$, $\mathbb{E}_{Q^*}\left[\frac{\tilde{U}_T}{\tilde{B}_T}|\mathcal{F}_t\right] \geq \text{ess-sup}_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}_{Q^*}\left[\frac{\tilde{U}_\tau}{\tilde{B}_\tau}|\mathcal{F}_t\right]$, whence we get the desired equality.

The Black Scholes Case

Now we replace the call by the put, i.e. $(x - K)^+$ by $(K - x)^+$. Then one might naively expect (since we again have a convex function) that the same result holds for the American put as well, but this is not so (the problem is that $\tilde{B}$ being increasing no longer helps us in the proof). One can even show (e.g. for the binomial tree): if the interest rate r is positive, then for some K the American put has a strictly higher value than a European put. However, if we model dividends by negative rates, we end up with the same phenomenon in the case of the American put.

[1] C. Cuchiero and J. Teichmann.

A convergence result for the emery topology and a variant of the proof of the fundamental theorem of asset pricing.

Finance and Stochastics, 4(19), 2015.

[2] Marzia De Donno and Maurizio Pratelli.

On a lemma by Ansel and Stricker.

In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 411–414. Springer, Berlin, 2007.

[3] Freddy Delbaen and Walter Schachermayer.

The mathematics of arbitrage.

Springer Finance. Springer-Verlag, Berlin, 2006.

[4] Yu. M. Kabanov.

On the FTAP of Kreps-Delbaen-Schachermayer.

In *Statistics and control of stochastic processes (Moscow, 1995/1996)*, pages 191–203. World Sci. Publ., River Edge, NJ, 1997.

[5] Philip E. Protter.

Stochastic integration and differential equations, volume 21 of *Applications of Mathematics (New York)*.