FTAP for LFM –

a new perspective

joint work with Christa CUCHIERO & Irene KLEIN (2015)

Insights for small Markets

$(X^1, \dots, X^d)$ discounted asset prices on $[0,1]$ (finite interval)

$$\mathcal{X}_1 = \{ (\phi \cdot X) \mid \phi \text{ 1-admissible},\ \phi \text{ predictable} \}$$

portfolio value processes with initial capital 0

$$X := \bigcup_{\lambda \geq 0} \lambda \cdot X_1$$

$$K_1^0 = \left\{ (\phi \cdot X)_1 \;\middle|\; \begin{array}{l} \phi \quad 1\text{-admissible} \\ \phi \quad \text{predictable} \end{array} \right\}$$

$$K^0 = \bigcup_{\lambda \geq 0} \lambda K_1^0$$

$$(K^0 - L^0_{\geq 0}) \cap L^\infty = C$$

$$\overline{C}^{w-*} \cap L^0_{\geq 0} \iff (\text{NFL})$$

$$\overline{C}^{\|\cdot\|_\infty} \cap L^0_{\geq 0} \iff (\text{NFLVR})$$

$$C \cap L^\infty_{\geq 0} = \{0\} \iff (NA)$$

$$K_1^0 \text{ bounded in probability} \iff (NUPBR)$$

Theorem (DS '94):

$$(NFLVR) \iff (NUPBR) + (NA)$$

$$(NFLVR) \iff C = \overline{C}^{w-*}$$

$$\iff (NFL)$$

$$\iff (ESM) \qquad \exists\, Q \sim P \;\forall\, V \in \mathcal{X}: \; E_Q[V] \leq 0$$

Application for SPT:

... from the point of view of dominating measures

$(X^1, \dots, X^d)$ satisfies (NFLVR) w.r.t. measure P

Choose $\tilde{P} \ll P$ such that (NFLVR)

does NOT hold. Since of course (NUPBR) still

holds, there is ϕ admissible s.t. $(\phi \cdot S)_1 \gneq 0$.

Example:

$(S^1, \ldots, S^d)$ capitalizations $\sum_{i=1}^d S^i > 0$ P-a.s.

$$X^i := \frac{S^i}{\sum_{i=1}^d S^i}$$ market weights

$(X^1, \ldots, X^d)$ satisfy (NFLVR) w.r.t. P

$(X^1, \ldots, X^d)$ do not satisfy (NFLVR) w.r.t. $\tilde{P} \approx P$

but $\tilde{P}$ is economically justified, then there exist portfolios beating the market portfolio, i.e. $\exists$ ϕ such that

$$(\phi \cdot X)_1 \geq 0 \quad \tilde{P}\text{-a.s.}, \quad \neq 0$$

that is there are self-financing portfolios π

$$\sum_{i=1}^{d} \pi_i S_1^i \geq \sum_{i=1}^{d} S_1^i$$

("beating the market").

Problem :

(1) Similar insights for (LFM) ?

(2) Asymptotic relative entropies ?

(3) Rates of convergence ?

How to define admissible portfolios?

$$\mathcal{X}_1 := \left\{ \lim_{n\to\infty} (\phi^n \cdot X^n) \;\middle|\; \begin{array}{l} \phi^n \text{ 1-admissible} \\ \phi^n \text{ predictable} \end{array} \right\}$$

$$\mathcal{X} := \bigcup_{\lambda \geq 0} \lambda \mathcal{X}_1$$

$$K^0_{(1)} := \{ V_1 \mid V \in \mathcal{X}_{(1)} \}$$

$$C := (K^0 - L^0_{\geq 0}) \cap L^\infty$$

Now we can transfer all notions.

$$(\mathrm{NAFL}) \iff \overline{C}^{\,w\text{-}*} \cap L^\infty_{\geq 0} = \{0\}$$

$$(\mathrm{NAFLVR}) \iff \overline{C}^{\,\|.\|_\infty} \cap L^\infty_{\geq 0} = \{0\}$$

$$(\mathrm{NA}) \iff C \cap L^\infty_{\geq 0} = \{0\}$$

$$(\mathrm{NUPBR}) \iff K_1^0 \text{ is bounded in probability}$$

Theorem (CC, IK, JT 2017):

$$(\mathrm{NAFLVR}) \iff (\mathrm{NUPBR}) + (\mathrm{NA})$$

$$\iff (\mathrm{ESM})$$

Remarks:

(1) Originally definitions were made with a smaller set

$$\tilde{C} := \left(\tilde{K}^0 - L^0_{\geq 0} \right) \cap L^\infty$$

$$\tilde{K}^0_{(1)} := \left\{ \left(\phi^u \cdot X^u \right)_1 \;\middle|\; \phi^u \text{ } 1\text{-admissible} \right\}$$

This cone is too small, i.e.

$$\overline{\tilde{C}}^{\|\cdot\|} \cap L^\infty_{\geq 0} = \{0\} \iff (NUPBR)$$

$$\overline{\tilde{C}}^{w^*\text{-}*} \cap L^\infty_{\geq 0} = \{0\} \iff (NAFL).$$

(2) Of course one can formulate everything in the realm of Kabanov's portfolio value process setting.

(3) The proof works well even though X_1 does NOT satisfy Kabanov's concatenation property. This is due to a simplification of the proof obtained by Christa Cuchiero and myself (F&S 2014).

(4) Also in finite discrete time (NUPBR) and (NA) different!

Applications for SPT:

Consider a market with countably many assets $(S^1, S^2, \dots)$ and that

$$\sum_{i \geq 1} S^i < \infty \quad P\text{-almost surely.}$$

$$X^i := \frac{S^i}{\sum_i S^i},$$ then we can

impose (NAFLVR) on the discounted market $(X^1, X^2, \ldots)$

$\tilde{P} \ll P$ for economically reasonable absolutely cont. measure such that (NAFLVR) does not hold any more

$$\Rightarrow \exists (\phi^n) : \left(\lim_{n\to\infty} (\phi^n \cdot X^n)\right)_1 \not\geq 0 .$$

"asymptotically beating the market".