Properly discounted asset prices are semimartingales

Dániel Ágoston Bálint and Martin Schweizer*
Department of Mathematics, ETH Zurich,
Rämistrasse 101, CH-8092 Zurich, Switzerland
*and
Swiss Finance Institute, Walchestrasse 9, CH-8006 Zürich, Switzerland
{daniel.balint, martin.schweizer}@math.ethz.ch

Abstract

We study general undiscounted asset price processes, which are only assumed to be nonnegative, adapted and RCLL (but not a priori semimartingales). Traders are allowed to use simple (piecewise constant) strategies. We prove that under a discounting-invariant condition of absence of arbitrage, the original prices discounted by the value process of any simple strategy with positive wealth must follow semimartingales. We also establish a corresponding version of the fundamental theorem of asset pricing that involves supermartingale discounters with an additional strict positivity property.

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*corresponding author

1 Introduction

In mathematical finance, it has become standard to assume that asset price processes S are semimartingales. This is mathematically important because gains from self-financing trading strategies are modelled as stochastic integrals with respect to S. But one should also ask if assuming the semimartingale property can be justified economically, and in particular if and in which sense the financial concept of absence of arbitrage (AOA) implies such a property.

The usual approach in the existing literature is to start "without loss of generality" from discounted asset prices where S is of the special form $(1, X)$ for some $\mathbb{R}^d$-valued adapted RCLL process X. Then one defines AOA properties and analyses their characterisation and consequences. In that spirit, Delbaen/Schachermayer [7, Theorem 7.2] show that the NFLVR condition for simple predictable processes implies the semimartingale property of X. A more recent paper by Kardaras/Platen [12] uses the weaker condition NA1 for simple predictable processes to again conclude that $X \geq 0$ is a semimartingale. In a similar vein, Kardaras [13] shows that if an abstract set $\mathcal{X}$ of wealth processes satisfies NA1 and contains at least one strictly positive semimartingale, then every $X \in \mathcal{X}$ is a semimartingale. An alternative proof of the classic Bichteler–Dellacherie theorem using arbitrage arguments is due to Beiglböck/Siorpaes [3] and Beiglböck/Schachermayer/Veliyev [2]; the latter paper also contains an explicit strengthening of the results of [7]. Theorems of the above type typically only hold for frictionless markets; non-semimartingale models with reasonable economic properties in markets with frictions appear for example in Cheridito [5] (trading restrictions) or Guasoni [9] and Czichowsky/Schachermayer [6] (transaction costs).

In economic terms, however, using a model of the form $(1, X)$ is a nontrivial restriction. For one thing, agents have access to a tradable numéraire with constant value and therefore can directly transfer wealth over time without risk. More importantly, there is no definition for absence of arbitrage for the original undiscounted prices, so that the above "without loss of generality" becomes questionable. In fact, simple examples show that different discountings of the same original market can lead to different and even contradictory absence-of-arbitrage properties; see Bálint/Schweizer [1, Example 1.1]. (In a nutshell, one can take any model with two assets where the ratio $X := S^2/S^1$ is a positive martingale converging to 0. Then discounting by S^1 leads to the model $(1, X)$ which satisfies NFLVR because X is a martingale; but discounting by S^2 yields the model $(1/X, 1)$ which even violates NUPBR because $1/X$ explodes to $+\infty$.) It is therefore economically important to study the general case where no a priori discounting is given, and to define absence of arbitrage for the original prices. This in turn needs an AOA concept which is discounting-invariant in the sense that for any one-dimensional strictly positive (discounter) process D, the discounted price process S/D satisfies AOA if and only if S satisfies AOA. As D need not be a semimartingale, one cannot expect that such a discounting-invariant absence-of-arbitrage concept implies the semimartingale property of S itself; indeed, this would mean that both S and S/D are semimartingales even if D is not.

Our analysis starts from a general $\mathbb{R}^N$-valued adapted RCLL process $S \geq 0$. In economic terms, this describes the prices of N traded assets, expressed in some abstract accounting unit; one can think of the latter as a perishable consumption good so that wealth in general cannot be transferred over time. Motivated by our earlier work [1], we introduce an absence-of-arbitrage condition (weaker than NA1) for simple long-only trading strategies and show that this implies that the relative price process $S/V(\vartheta, S)$ is a semimartingale for any simple self-financing strategy ϑ whose wealth $V(\vartheta, S)$ always stays positive. Moreover, we also provide a dual characterisation involving supermartingale discounters for the wealth processes of simple strategies.

As one can see from the above description, our main contribution is on the conceptual level of general economic modelling. In technical terms, our main results are mostly proved by connecting our approach to the setup in Kardaras/Platen [12] and then exploiting their results. This is fairly straightforward for the semimartingale result, whereas the details of the dual characterisation are new and need extra arguments. In comparison to our earlier work [1], the present paper starts from a more fundamental perspective by deriving, instead of assuming, a semimartingale property. So while the key concept DSV appears in both papers, we impose it here only for simple strategies, whereas [1] studies and characterises it for general strategies under a semimartingale assumption on S.

The structure of the paper is as follows. Section 2 introduces notation and absence-of-arbitrage concepts. Section 3 contains the two main results, first showing how a suitable discounting-invariant AOA property implies a semimartingale property, and then giving a dual characterisation of this AOA property. Section 4 contains the proofs and some additional results, and the final Section 5 gives a number of counterexamples and summarise the relations between semimartingale properties and absence-of-arbitrage conditions.

2 Notations and concepts

2.1 Notation

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. We assume that $\mathcal{F}_0$ is trivial and set $\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t$. Every (in)equality should be understood in the P-a.s. sense for random variables and as P-a.s. for all t for stochastic processes. There are $N \geq 2$ basic primary assets. Their nonnegative prices are expressed in some unspecified, usually nontradable accounting unit, to be thought of as a perishable consumption good, and are described by an $\mathbb{R}^N_+$-valued adapted RCLL process $S = (S_t)_{t \geq 0}$. Note that we do not assume the existence of a (separate) risk-free bank account; if there is one, it must be one component of S.

Importantly, we do not a priori assume any semimartingale property for S. Then classic stochastic integrals for S need not be well defined (see the Bichteler–Dellacherie theorem e.g. in [4]), and so we restrict our subsequent concepts to so-called simple strategies, to circumvent the need for general stochastic integrals when describing gains and wealth processes.

Definition 2.1. A *simple predictable process*, $\vartheta \in \mathcal{E}(\mathbb{R}^N)$ for short, is a process of the form $\vartheta = \sum_{j=1}^{J+1} \vartheta_j I_{\rrbracket \tau_{j-1}, \tau_j \rrbracket}$, where $J \in \mathbb{N}$, $0 \equiv \tau_0 \le \tau_1 \le \cdots \le \tau_J < \tau_{J+1} \equiv \infty$ are stopping times and each ϑ_j is $\mathbb{R}^N$-valued and $\mathcal{F}_{\tau_{j-1}}$-measurable. It is called *long-only*, $\vartheta \in \mathcal{E}(\mathbb{R}^N_+)$, if $\vartheta_j \ge 0$ for each j. Finally, ϑ is called a *simple strategy*, $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ respectively $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$, if it satisfies in addition the *self-financing condition* that $\vartheta_j \cdot S_{\tau_j} = \vartheta_{j+1} \cdot S_{\tau_j}$ for each j. We point out that $\mathcal{E}$ and $^{\mathrm{sf}}$ are mnemonic for "elementary" and "self-financing", respectively. To emphasise for which price process the self-financing condition is imposed, we sometimes write $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N, S)$ or $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+, S)$.

The meaning of the self-financing condition is financially obvious: It simply says that at any trading date τ_j of ϑ, the cost $(\vartheta_{j+1} - \vartheta_j) \cdot S_{\tau_j}$ of rebalancing the asset holdings from ϑ_j to ϑ_{j+1} is zero. For any simple strategy ϑ, it follows from the self-financing condition that the *value process* of ϑ is for all $t \ge 0$ given by

$$V_t(\vartheta, S) := \vartheta_t \cdot S_t = V_0(\vartheta, S) + \sum_{j=1}^{J+1} \vartheta_j \cdot (S_{\tau_j \wedge t} - S_{\tau_{j-1} \wedge t}) =: V_0(\vartheta, S) + (\vartheta \bullet S)_t. \tag{2.1}$$

The above sum is of course just the elementary stochastic integral of ϑ with respect to S.

For a more compact exposition, we denote by $\mathcal{R}$ the space of all adapted RCLL processes $Y = (Y_t)_{t \ge 0}$. We set $\mathcal{R}_+ := \{Y \in \mathcal{R} : Y \ge 0\}$ and $\mathcal{R}_{++} := \{Y \in \mathcal{R} : Y > 0 \text{ and } Y_- > 0\}$. We define the set of *wealth processes* corresponding to long-only simple strategies as

$$\mathcal{X}^+_{\mathrm{s}}(S) := \big\{ V^\vartheta = \big(V_t(\vartheta, S)\big)_{t \ge 0} : \vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+) \big\} \tag{2.2}$$

and note that each such $V(\vartheta, S)$ is in $\mathcal{R}_+$ because $\vartheta \ge 0$ and each S^i is in $\mathcal{R}_+$. We also need later $\mathcal{X}^{++}_{\mathrm{s}}(S) := \mathcal{X}^+_{\mathrm{s}}(S) \cap \mathcal{R}_{++}$. If S is more generally $\mathbb{R}^N$-valued and $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$, we sometimes also consider the analogous set $\mathcal{X}_{\mathrm{s}}(S) \subseteq \mathcal{R}$.

Note that the self-financing concept is *discounting-invariant* in the following sense: If we take some abstract discounting process $D \in \mathcal{R}_{++}$ and define new prices $S' := S/D$, then $\vartheta \in \mathcal{E}(\mathbb{R}^N)$ is self-financing for S if and only if it is self-financing for S'; more compactly, $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N, S) = \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N, S')$. Moreover, we have the *change-of-numéraire formula*

$$V_0(\vartheta, S/D) + \vartheta \bullet (S/D) = V(\vartheta, S/D) = V(\vartheta, S)/D = V_0(\vartheta, S)/D_0 + (\vartheta \bullet S)/D \tag{2.3}$$

for $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$. We note that this is not valid for general $\vartheta \in \mathcal{E}(\mathbb{R}^N)$.

Definition 2.2. A *simple reference strategy* is a long-only simple strategy $\eta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$ with $V(\eta, S) = \eta \cdot S \in \mathcal{R}_{++}$.

The interpretation of a reference strategy η is that this is a *desirable strategy*, because its property $V(\eta, S) \in \mathcal{R}_{++}$ means that it keeps us forever from complete starvation. Note that actually $V(\eta, S) \in \mathcal{X}^{++}_{\mathrm{s}}(S)$.

Lemma 2.3. *Assume that $S \geq 0$. Then there exists a simple reference strategy η if and only if the sum $\sum_{i=1}^N S^i = \mathbb{1} \cdot S$ is in $\mathcal{R}_{++}$, where $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^N$ is the market portfolio of holding one share of each asset.*

Proof. The "if" direction is clear because $\mathbb{1} \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^N)$. Conversely, for any simple reference strategy η, we have $\eta \cdot S \leq (\max_{i,j} \eta_j^i)(\mathbb{1} \cdot S)$, where $\max_{i,j} \eta_j^i$ is a nonnegative random variable as η is long-only. Thus $\eta \cdot S > 0$ implies $\mathbb{1} \cdot S > 0$ and $\eta \cdot S_- > 0$ implies $\mathbb{1} \cdot S_- > 0$. $\square$

In the sequel, we always assume that there exists a simple reference strategy. Lemma 2.3 shows that this is a very weak assumption for a nonnegative asset price process S.

2.2 Absence of arbitrage

The following two definitions are versions of the absence-of-arbitrage concept DSV introduced in Bálint/Schweizer [1, Definition 2.7] when we consider long-only *simple* strategies only.

Definition 2.4. Fix a simple strategy η. We say that *simple dynamic share viability* ($\mathrm{DSV_s}$) for η holds if there is no $[0,1]$-valued adapted process ψ converging P-a.s. as $t \to \infty$ to some $\psi_\infty \in L_+^0(\mathcal{F}_\infty) \setminus \{0\}$ and such that for every $\varepsilon > 0$, there is a long-only simple strategy ϑ with $V_0(\vartheta, S) \leq \varepsilon$ and $\liminf_{t \to \infty}(\vartheta_t - \psi_t \eta_t) \geq 0$. We say that *local simple dynamic share viability* ($\mathrm{DSV_s^{loc}}$) for η holds if there is no $T \in (0, \infty)$ and then no $[0,1]$-valued $\psi_T \in L_+^0(\mathcal{F}_T) \setminus \{0\}$ such that for every $\varepsilon > 0$, there is a long-only simple strategy ϑ with $V_0(\vartheta, S) \leq \varepsilon$ and $\vartheta_T - \psi_T \eta_T \geq 0$.

As already explained in [1], the interpretation of the DSV concept is economically very natural: It says that *the zero strategy* of doing nothing at all *cannot be improved by dynamic trading* in the financial market. This explains "dynamic" and "viability". The (attempted but impossible) improvement is measured not in terms of value, but in terms of *shares* — we fix a strategy η and ask for a strategy ϑ which improves on 0 by a nontrivial multiple ψ of η, until the given time horizon (∞ or T) and with the allowed strategies (general for DSV, simple for $\mathrm{DSV_s}$). Later, η is taken as a simple reference strategy so that the improvement is significant as it is a multiple of a desirable investment.

Just like the original DSV, the concepts $\mathrm{DSV_s}$ and $\mathrm{DSV_s^{loc}}$ are for any fixed η clearly *discounting-invariant*, meaning that $\mathrm{DSV_s}$ (respectively $\mathrm{DSV_s^{loc}}$) holds for S if and only if it holds for $S' = S/D$, for any discounter $D \in \mathcal{R}_{++}$.

For a dual characterisation of $\mathrm{DSV_s}$, we need one more concept.

Definition 2.5. A *supermartingale discounter for long-only simple strategies*, SMD for $\mathcal{X}_\mathrm{s}^+(S)$, is a $D \in \mathcal{R}_{++}$ with $D_0 = 1$ and such that X/D is a supermartingale for each $X \in \mathcal{X}_\mathrm{s}^+(S)$. For a simple reference strategy η, we say that D is an $\mathrm{SMD}^{\eta+}$ for $\mathcal{X}_\mathrm{s}^+(S)$ if it satisfies in addition $\lim_{t \to \infty}(\eta_t \cdot S_t)/D_t > 0$.

Remark 2.6. As $S \geq 0$, every $X \in \mathcal{X}_\mathrm{s}^+(S)$ from (2.2) is nonnegative so that the supermartingale X/D in Definition 2.5 converges P-a.s. as $t \to \infty$. In particular, $\lim_{t \to \infty}(\eta_t \cdot S_t)/D_t$ always exists in $[0, \infty)$ P-a.s., and the only extra requirement above is that the limit is > 0.

3 Main results

We are now ready for our main results. The first one says that if the original prices S satisfy the discounting-invariant absence-of-arbitrage property $\mathrm{DSV_s}$ for a simple reference strategy η, then relative prices must be semimartingales. More precisely:

Theorem 3.1. *Assume that $S \geq 0$ and there exists a simple reference strategy η. If S satisfies DSV_{s} for η, then $S/V(\vartheta, S)$ is a semimartingale for any simple strategy $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ such that $V(\vartheta, S) \in \mathcal{R}_{++}$. In particular, S/Y is a semimartingale for any $Y \in \mathcal{X}_{\mathrm{s}}^{++}(S)$.*

The proof of Theorem 3.1 is given in Section 4. However, we state here a number of immediate consequences.

Corollary 3.2. *Assume that $S \geq 0$, there exists a simple reference strategy η and S satisfies DSV_{s} for η. Then:*

1) *The process $S/(\mathbb{1} \cdot S) = S/\sum_{i=1}^N S^i$ of "market capitalisations" is a semimartingale.*
2) *If there is an $i_0 \in \{1, \dots, N\}$ such that $S^{i_0} \in \mathcal{R}_{++}$, then S/S^{i_0} is a semimartingale.*
3) *If there is an $i_0 \in \{1, \dots, N\}$ with $S^{i_0} \equiv 1$, then S itself is a semimartingale.*

Proof. All results follow from Theorem 3.1. Indeed, 1) is obtained by taking $\vartheta \equiv \mathbb{1}$ and recalling Lemma 2.3, and 2) and 3) use $\vartheta \equiv \mathrm{e}^{i_0}$ (the i_0-th unit vector in $\mathbb{R}^N$). $\square$

Remark 3.3. 1) Without the existence of an asset price process $S^{i_0} \equiv 1$, S need not be a semimartingale. Indeed, take any $\mathbb{R}_+^{N-1}$-valued semimartingale X such that $(1, X)$ satisfies $\mathrm{DSV_s}$ for $\eta \equiv \mathbb{1}$. For any $D \in \mathcal{R}_{++}$, $S := D(1, X) = (D, DX)$ then also satisfies $\mathrm{DSV_s}$ for η by discounting-invariance. But if D is not a semimartingale, neither is S.

To formulate this more pointedly, take $X_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$, $t \geq 0$, for a Brownian motion W and $\sigma > 0$. Then $(1, X)$ is the risk-neutral discounted Black–Scholes model. This satisfies NFLVR, hence NUPBR and thus DSV for $\mathbb{1}$ by [1, Proposition 5.6], and a fortiori $\mathrm{DSV_s}$. If we multiply both 1 and X by a geometric fractional Brownian motion D (with Hurst parameter $H \neq \frac{1}{2}$), we are only rescaling prices and therefore not changing the fundamental economic properties of the model. But the rescaled price processes are no longer semimartingales.

2) As discussed in the introduction, our absence-of-arbitrage property $\mathrm{DSV_s}$ is formulated for the original, undiscounted prices S, and in such a way that it remains invariant under discounting with arbitrary $D \in \mathcal{R}_{++}$. In contrast, the classic literature defines absence of arbitrage only for S^{i_0}-discounted prices S/S^{i_0}. Our approach is not only more general, but also brings extra conceptual clarity, because it makes it evident that only *relative* prices $S/V(\vartheta, S)$ can be expected to be semimartingales. (This is of course no contradiction to the classic results because there S/S^{i_0} are already relative prices.)

3) Once we have as in Theorem 3.1 a semimartingale property for some discounted prices S/D, we can without loss of generality assume for further arbitrage analysis that S itself is a semimartingale. Indeed, as our absence-of-arbitrage condition DSV is designed to be discounting-invariant, we can equivalently impose it on S/D or on S. Moreover, to define stochastic integrals of self-financing strategies with respect to S, we can use the change-of-numéraire formula (2.3).

Another consequence of Theorem 3.1 is the following result about wealth processes from long-only simple strategies.

Proposition 3.4. *Assume that $S \geq 0$, there exists a simple reference strategy η and S satisfies DSV_{s} for η. Then:*

1) For any $Y \in \mathcal{X}_{\mathrm{s}}^{++}(S) \neq \emptyset$, the ratio V/Y is a semimartingale for each $V \in \mathcal{X}_{\mathrm{s}}^{+}(S)$.

2) If there is a $Y \in \mathcal{X}_{\mathrm{s}}^{++}(S)$ which is in addition a semimartingale, then each $V \in \mathcal{X}_{\mathrm{s}}^{+}(S)$ is a semimartingale.

Proof. 2) follows from 1) by writing $V = \frac{V}{Y} Y$ because products of semimartingales are semimartingales. For 1), note that $\tilde{S} := S/Y$ is a semimartingale by Theorem 3.1. For any $V \in \mathcal{X}_{\mathrm{s}}^{+}(S)$, there is a long-only simple strategy $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}_{+}^{N})$ such that $V = V(\vartheta, S)$. Then $\tilde{V} := \vartheta_0 \cdot \tilde{S}_0 + \vartheta \bullet \tilde{S}$ is a semimartingale like $\tilde{S}$, and by (2.3) and (2.1),

$$V/Y = V(\vartheta, S)/Y = V(\vartheta, S/Y) = V(\vartheta, \tilde{S}) = \tilde{V}$$

because ϑ is self-financing. So $V/Y = \tilde{V}$ is a semimartingale as claimed. □

Remark 3.5. In Kardaras [13], one can find similar results in a slightly different framework. There one considers an abstract set $\mathcal{X}$ of wealth processes which must contain a positive element which is in addition a semimartingale. The set $\mathcal{X}_{\mathrm{s}}^{+}(S)$ defined in (2.2) is almost an example of such an $\mathcal{X}$; it contains a process in $\mathcal{R}_{++}$ (hence in $\mathcal{X}_{\mathrm{s}}^{++}(S)$) if there exists a simple reference strategy, but it need not contain a semimartingale in general. The first main result in [13, Theorem 1.3] is then an abstract version of 2) in Proposition 3.4. However, 1) in Proposition 3.4 is more general — it shows again as in Remark 3.3 that relative quantities (here, ratios V/Y of wealth processes) will be semimartingales, but absolute quantities (here, wealth processes) need not be.

Our second main result is a dual characterisation of DSV$_{\mathrm{s}}$. To the best of our knowledge, there is no comparable result in the literature because there is not even any classic analogue of DSV$_{\mathrm{s}}$ so far; see Remark 4.6 below for a more detailed discussion.

Theorem 3.6. *Assume that $S \geq 0$ and there exists a simple reference strategy η. Then S satisfies DSV_{s} for η if and only if there exists an $SMD^{\eta+}$ for $\mathcal{X}_{\mathrm{s}}^{+}(S)$.*

The proof is given in Section 4.

4 Proofs and ramifications

This section mostly provides the proofs of our main results, but also adds some complements. The key idea and technique is to reduce things to a situation where we can exploit the results from Kardaras/Platen [12]. While this is enough to prove Theorem 3.1, the proof of Theorem 3.6 needs additional arguments.

Because being a semimartingale is a local property, it should be no surprise that already a local form of an absence-of-arbitrage condition will be sufficient. So the following first step is natural.

Lemma 4.1. *Assume that $S \geq 0$ and there exists a simple reference strategy η. Then DSV_{s} for η implies $DSV_{\mathrm{s}}^{\mathrm{loc}}$ for η.*

Proof. If $\mathrm{DSV}_{\mathrm{s}}^{\mathrm{loc}}$ for η does not hold, take $T \in (0, \infty)$ and a $[0,1]$-valued $\psi_T \in L_+^0(\mathcal{F}_T) \setminus \{0\}$ such that for any $\varepsilon > 0$, there exists a $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^N)$ with $V_0(\vartheta, S) \leq \varepsilon$ and $\vartheta_T - \psi_T \eta_T \geq 0$. Then $\tilde{\vartheta} := \vartheta I_{[\![0,T]\!]} + V_T(\vartheta, S/(\eta \cdot S))\eta I_{]\!]T,\infty[\![}$ is well defined because η is a (simple) reference strategy, and $\tilde{\vartheta}_j = \vartheta_j$ for $j = 1, \dots, J+1$. Now we add one more time step by setting $\tilde{J} := J+1$, $\tilde{\tau}_j := \tau_j$ for $j = 0, 1, \dots, J$, $\tilde{\tau}_{\tilde{J}} := T$ and $\tilde{\tau}_{\tilde{J}+1} := \infty$ as in Definition 2.1. Then we see that $\tilde{\vartheta}$ is in $\mathcal{E}(\mathbb{R}_+^N)$ with $\tilde{\vartheta}_{\tilde{J}+1} := V_T(\vartheta, S/(\eta \cdot S))\eta_T$. Moreover,

$$\tilde{\vartheta}_{\tilde{J}} \cdot S_{\tau_{\tilde{J}}} = V_T(\vartheta, S) = V_T\big(\vartheta, S/(\eta \cdot S)\big)\eta_T \cdot S_T = \tilde{\vartheta}_{\tilde{J}+1} \cdot S_{\tau_{\tilde{J}}}$$

shows that $\tilde{\vartheta}$ is even in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^N)$. The process $\psi = (\psi_t)_{t \geq 0}$ defined by $\psi_t := E[\psi_T \,|\, \mathcal{F}_t]$ is $[0,1]$-valued, adapted and P-a.s. convergent to $\psi_\infty = \psi_T \in L_+^0(\mathcal{F}_T) \setminus \{0\}$. Finally, $\tilde{\vartheta}$ satisfies $V_0(\tilde{\vartheta}, S) = V_0(\vartheta, S) \leq \varepsilon$ and

$$\begin{aligned}\liminf_{t\to\infty}(\tilde{\vartheta}_t - \psi_t \eta_t) &= \liminf_{t\to\infty}\Big(V_T\big(\vartheta, S/(\eta \cdot S)\big)\eta_t - \psi_T \eta_t\Big) = \liminf_{t\to\infty}\Big(V_T\big(\vartheta, S/(\eta \cdot S)\big) - \psi_T\Big)\eta_t \\ &= \liminf_{t\to\infty}\big((\vartheta_T - \psi_T \eta_T) \cdot S_T/(\eta_T \cdot S_T)\big)\eta_t \geq 0.\end{aligned}$$

So $\mathrm{DSV}_{\mathrm{s}}$ for η does not hold and this proves our claim. □

The converse of Lemma 4.1 is not true; a counterexample is given in Example 5.2.

To make the connection to Kardaras/Platen [12], we next recall their condition of *"no opportunities for arbitrage of the first kind with simple, no-short-sales trading"*.

Definition 4.2. For the special case where $S = (1, X)$ for some $(N-1)$-dimensional adapted RCLL process $X \geq 0$, define

$$\mathcal{X}_{\mathrm{s},1}(1, X) := \Bigg\{ V^{x,\varphi} := x + \sum_{j=1}^{J+1} \varphi_j \cdot (X_{\tau_j \wedge \cdot} - X_{\tau_{j-1} \wedge \cdot}) = x + \varphi \bullet X : \\ x \in \mathbb{R}_+,\ \varphi \in \mathcal{E}(\mathbb{R}_+^{N-1}) \text{ with } \varphi_j \cdot X_{\tau_{j-1}} \leq V_{\tau_{j-1}}^{x,\varphi} \text{ for all } j = 1, \dots, J+1 \Bigg\}.$$

We say that $S = (1, X)$ satisfies $\mathrm{NA1}_{\mathrm{s},1}^{\mathrm{loc}}$ if there is no $T \in (0, \infty)$ and then no $\xi \in L_+^0(\mathcal{F}_T) \setminus \{0\}$ such that for any $\varepsilon > 0$, there exists $V \in \mathcal{X}_{\mathrm{s},1}(1, X)$ with $V_0 \leq \varepsilon$ and $V_T \geq \xi$.

The above definition calls for some comments:

1) We do not call the condition $\mathrm{NA1_s}$ as in [12], but use the more explicit notation $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s},1}$ to emphasise that $\mathrm{NA1_s}$ is like $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ a local property. The subscript 1 in ${}_{\mathrm{s},1}$ serves to remind that we consider a model containing one asset with constant price 1.

2) Any pair $(x,\varphi) \in \mathbb{R} \times \mathcal{E}(\mathbb{R}^{N-1})$ corresponds bijectively to a $\vartheta = (\vartheta^0, \varphi) \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$; this is classic and shown explicitly in the proof of Lemma 4.4. The condition $\varphi_j \cdot X_{\tau_{j-1}} \le V^{x,\varphi}_{\tau_{j-1}}$ is needed to ensure that ϑ^0 is nonnegative if φ is $\mathbb{R}^{N-1}_+$-valued.

In contrast to [12], we do not assume the existence of an asset with constant price 1. Therefore we need to generalise the condition $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s},1}$ to our setting. Recall from (2.2) the set $\mathcal{X}^+_{\mathrm{s}}(X)$ of wealth processes corresponding to long-only simple strategies.

Definition 4.3. For $S \ge 0$, we say that *no arbitrage of the first kind for simple strategies* ($\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$) holds if there is no $T \in (0,\infty)$ and then no $\xi \in L^0_+(\mathcal{F}_T) \setminus \{0\}$ such that for any $\varepsilon > 0$, there exists $V \in \mathcal{X}^+_{\mathrm{s}}(S)$ with $V_0 \le \varepsilon$ and $V_T \ge \xi$.

The next simple but important lemma collects some useful results.

Lemma 4.4. *Suppose S is $\mathbb{R}^N$-valued and X is $\mathbb{R}^{N-1}$-valued, both adapted and RCLL. Then:*

1) $\mathcal{X}_{\mathrm{s},1}(1,X) = \mathcal{X}^+_{\mathrm{s}}(1,X)$.

2) *$NA1^{\mathrm{loc}}_{\mathrm{s},1}$ for $(1,X)$ and $NA1^{\mathrm{loc}}_{\mathrm{s}}$ for $(1,X)$ are equivalent.*

3) *If $\bar\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ and $S' := (S, \bar\vartheta \cdot S)$, then $\mathcal{X}_{\mathrm{s}}(S) = \mathcal{X}_{\mathrm{s}}(S')$. If in addition $\bar\vartheta \ge 0$, then also $\mathcal{X}^+_{\mathrm{s}}(S) = \mathcal{X}^+_{\mathrm{s}}(S')$.*

4) *If η is a simple reference strategy for S and $\tilde S := S/(\eta \cdot S)$, then $\mathcal{X}^+_{\mathrm{s}}(\tilde S) = \mathcal{X}_{\mathrm{s},1}(1,\tilde S)$.*

Proof. 1) This proof is essentially standard, but we give the details for completeness. For "$\supseteq$", take $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ for $(1,X)$. Then $\varphi := (\vartheta^2, \dots, \vartheta^N)$ is in $\mathcal{E}(\mathbb{R}^{N-1})$, and setting $x := V_0(\vartheta, S)$ yields $V^{x,\varphi} = V^\vartheta$ due to the self-financing condition (2.1) for $S = (1,X)$. Moreover, it is clear that $\vartheta \ge 0$ implies $\varphi \ge 0$. For "$\subseteq$", fix $x > 0$ and let φ be in $\mathcal{E}(\mathbb{R}^{N-1})$ with corresponding stopping times $(\tau_j)_{j=0,\dots,J+1}$. Define $\vartheta^1_1 := x - \varphi_1 \cdot X_0$ and $\vartheta^1_j := \vartheta^1_{j-1} + \varphi_{j-1} \cdot X_{\tau_{j-1}} - \varphi_j \cdot X_{\tau_{j-1}}$ for $j = 2, \dots, J+1$, then $\vartheta_j := (\vartheta^1_j, \varphi^1_j, \dots, \varphi^{N-1}_j)$ and $\vartheta := \sum_{j=1}^{J+1} \vartheta_j I_{\rrbracket \tau_{j-1}, \tau_j \rrbracket}$. It is straightforward to verify that $\vartheta := (\vartheta^1, \varphi^1, \dots, \varphi^{N-1})$ is in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ for $(1,X)$ and satisfies $V^\vartheta = V^{x,\varphi}$. If also $\varphi_j \cdot X_{\tau_{j-1}} \le V^{x,\varphi}_{\tau_{j-1}}$ for $j = 1, \dots, J+1$ as in Definition 4.2, then $\vartheta^1_j \ge 0$ for each $j = 1, \dots, J+1$ from the construction. So for both directions, ϑ is long-only if and only if φ is long-only.

2) This follows immediately from 1).

3) The inclusion "$\subseteq$" is clear because $\vartheta' := (\vartheta, 0) \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^{N+1}, S')$ for any $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N, S)$. For the converse, take $\vartheta' \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^{N+1}, S')$ and write $\vartheta' = (\vartheta, H)$ where H is the last coordinate of ϑ'. Then $\vartheta \in \mathcal{E}(\mathbb{R}^N)$ and $H \in \mathcal{E}(\mathbb{R})$, and clearly

$$V(\vartheta', S') = \vartheta' \cdot (S, \bar\vartheta \cdot S) = (\vartheta + H\bar\vartheta) \cdot S = V(\vartheta + H\bar\vartheta, S),$$

where $\vartheta+H\bar{\vartheta}$ is again in $\mathcal{E}(\mathbb{R}^N)$. But we also have that ϑ' is self-financing for S' and therefore

$$\begin{aligned}V(\vartheta+H\bar{\vartheta},S)&=V(\vartheta',S')\\&=V_0(\vartheta',S')+\vartheta'\bullet S'\\&=V_0(\vartheta+H\bar{\vartheta},S)+\vartheta\bullet S+H\bullet(\bar{\vartheta}\bullet S)\\&=V_0(\vartheta+H\bar{\vartheta},S)+(\vartheta+H\bar{\vartheta})\bullet S,\end{aligned}$$

which shows that $\vartheta+H\bar{\vartheta}\in\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N,S)$. This yields "$\supseteq$". Finally, note that if $\vartheta\geq 0$, then ϑ' is long-only if ϑ is respectively only if $\vartheta+H\bar{\vartheta}$ is.

4) Because $\eta\in\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+,\tilde{S})$ and $\eta\cdot\tilde{S}\equiv 1$, using 3) and then 1) yields

$$\mathcal{X}^+_{\mathrm{s}}(\tilde{S})=\mathcal{X}^+_{\mathrm{s}}(\tilde{S},1)=\mathcal{X}^+_{\mathrm{s}}(1,\tilde{S})=\mathcal{X}_{\mathrm{s},1}(1,\tilde{S}).$$

□

We next connect our approach to the classic framework.

Lemma 4.5. *Assume that $S\geq 0$ and there exists a simple reference strategy η. Then S satisfies NA1$^{\mathrm{loc}}_{\mathrm{s}}$ if and only if S satisfies DSV$^{\mathrm{loc}}_{\mathrm{s}}$ for η.*

Proof. If S does not satisfy DSV$^{\mathrm{loc}}_{\mathrm{s}}$ for η, take $T\in(0,\infty)$ and $\psi_T\in L^0_+(\mathcal{F}_T)\setminus\{0\}$ such that for every $\varepsilon>0$, there is a $\vartheta\in\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$ with $V_0(\vartheta,S)\leq\varepsilon$ and $\vartheta_T-\psi_T\eta_T\geq 0$. Then due to $S\geq 0$, also $V_T(\vartheta,S)=\vartheta_T\cdot S_T\geq\psi_T\eta_T\cdot S_T$, and hence $\eta_T\cdot S_T>0$ implies that $\xi:=\psi_T\eta_T\cdot S_T\in L^0_+(\mathcal{F}_T)\setminus\{0\}$. Therefore S does not satisfy NA1$^{\mathrm{loc}}_{\mathrm{s}}$.

If S does not satisfy NA1$^{\mathrm{loc}}_{\mathrm{s}}$, take $T\in(0,\infty)$ and $\xi\in L^0_+(\mathcal{F}_T)\setminus\{0\}$ such that for every $\varepsilon>0$, there is a $V\in\mathcal{X}^+_{\mathrm{s}}(S)$ with $V_0\leq\varepsilon$ and $V_T\geq\xi$. Using that $V\in\mathcal{X}^+_{\mathrm{s}}(S)$, take $\vartheta\in\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$ with $V^\vartheta=V$ and define $\tilde{\vartheta}:=\vartheta I_{[\![0,T]\!]}+(V_T/(\eta_T\cdot S_T))\eta I_{]\!]T,\infty[\![}$. As in the proof of Lemma 4.1, $\tilde{\vartheta}$ is well defined and in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$, and $V_0(\tilde{\vartheta},S)=V_0\leq\varepsilon$ and $\tilde{\vartheta}_{T+1}=(V_T/(\eta_T\cdot S_T))\,\eta_{T+1}\geq\xi/(\eta_T\cdot S_T)\eta_{T+1}$. Then $\psi:=(\xi/(\eta_T\cdot S_T))\wedge 1\in L^0_+(\mathcal{F}_T)\setminus\{0\}$ is $[0,1]$-valued, and it follows that S does not satisfy DSV$^{\mathrm{loc}}_{\mathrm{s}}$ for η. □

Remark 4.6. The global analogue on $[0,\infty)$ of DSV$^{\mathrm{loc}}_{\mathrm{s}}$ is obviously DSV$_{\mathrm{s}}$. A global analogue of NA1$^{\mathrm{loc}}_{\mathrm{s},1}$ for $(1,X)$ has not appeared in the literature so far, and in consequence, there is no classic analogue to Theorem 3.6, the dual characterisation of DSV$_{\mathrm{s}}$. If S or $(1,X)$ is already a semimartingale, the global properties DSV and NA1 have been defined for *general* instead of simple strategies, and NA1 is then better known as NUPBR. Even for the classic model $S=(1,X)$, there is no equivalence between NUPBR and DSV (for $\eta\equiv\mathbb{1}$), but only the implication NUPBR $\Rightarrow$ DSV; see [1, Proposition 5.6 and Example 6.8]. In the same way and with an analogous counterexample, one could also show that NA1$^{\mathrm{glob}}_{\mathrm{s},1}$ (suitably defined) implies DSV$_{\mathrm{s}}$ and that the converse does not hold.

Two very useful consequences of Lemma 4.5 are collected in the next result.

Corollary 4.7. *Assume that $S \geq 0$ and there exists a simple reference strategy η.*

1) *If η' is any simple reference strategy, then DSV$^{\mathrm{loc}}_{\mathrm{s}}$ for η and for η' are equivalent.*

2) *NA1$^{\mathrm{loc}}_{\mathrm{s}}$ is discounting-invariant in the sense that NA1$^{\mathrm{loc}}_{\mathrm{s}}$ holds for S if and only if it holds for S/D, for any discounter $D \in \mathcal{R}_{++}$.*

Proof. Both statements follow directly from Lemma 4.5 — 1) because $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$ does not involve any reference strategy, and 2) because $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ for η is discounting-invariant. □

Now we are ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. If $S \geq 0$ and S satisfies $\mathrm{DSV}_{\mathrm{s}}$ for η, then $\tilde{S} := S/(\eta \cdot S)$ satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$ by Lemmas 4.1 and 4.5 and Corollary 4.7, 2). From part 4) of Lemma 4.4, we have $\mathcal{X}^{+}_{\mathrm{s}}(\tilde{S}) = \mathcal{X}_{\mathrm{s},1}(1,\tilde{S})$, and therefore $\tilde{S}$ satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$ if and only if $(1,\tilde{S})$ satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s},1}$. Further, as $\eta \cdot S = V(\eta, S)$ is in $\mathcal{R}_{++}$ because η is a simple reference strategy, $(1,\tilde{S})$ is adapted RCLL like S. Hence the conditions of [12, Theorem 1.3] are satisfied for $(1,\tilde{S})$ and it follows that $\tilde{S}$ is a semimartingale. Now take any $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ with $\vartheta \cdot S = V(\vartheta, S) \in \mathcal{R}_{++}$. By refining the partitions of η and ϑ if necessary, we can assume that $\vartheta = \sum_{j=1}^{J+1} \vartheta_j I_{\rrbracket \tau_{j-1},\tau_j \rrbracket}$ and $\eta = \sum_{j=1}^{J+1} \eta_j I_{\rrbracket \tau_{j-1},\tau_j \rrbracket}$ with the same stopping times $\tau_0, \tau_1, \dots, \tau_{J+1}$. But then we have on each interval $\rrbracket \tau_{j-1}, \tau_j \rrbracket$ that $\vartheta \cdot S = \vartheta_j \cdot S$ and $\eta \cdot S = \eta_j \cdot S$ as well as $\vartheta \cdot \tilde{S} = \vartheta_j \cdot \tilde{S} = \vartheta_j \cdot (S/(\eta \cdot S)) = (\vartheta_j \cdot S)/(\eta_j \cdot S)$ and therefore also $S/(\vartheta \cdot S) = \tilde{S}/(\vartheta \cdot \tilde{S})$. This also holds at $t = 0$ and hence on $\llbracket 0, \infty \rrbracket$, and $\vartheta \cdot \tilde{S} = (\vartheta \cdot S)/(\eta \cdot S)$ is in $\mathcal{R}_{++}$. Because ϑ is in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$, $\vartheta \cdot \tilde{S} = \vartheta_0 \cdot \tilde{S}_0 + \vartheta \bullet \tilde{S}$ is a semimartingale like $\tilde{S}$, and then so is $S/(\vartheta \cdot S) = \tilde{S}/(\vartheta \cdot \tilde{S})$ because fractions of semimartingales with denominators in $\mathcal{R}_{++}$ are semimartingales. □

A closer look at the proof of Theorem 3.1 shows that already the local version $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ of $\mathrm{DSV}_{\mathrm{s}}$ is sufficient to imply the same semimartingale property. As mentioned before, this is not surprising, because being a semimartingale is itself a local property.

As an extra result, we next show that the converse of Theorem 3.1 holds under an additional assumption. This result is analogous to "(iv) ⇒ (i)" in Kardaras/Platen [12, Theorem 1.3]. We call a process X an *exponential semimartingale* if for each coordinate i, we have $X^i = X^i_0 + \int X^i_- \, dR^i$ for some semimartingale R^i with $R^i_0 = 0$. (We could write this as $X^i = X^i_0 \, \mathcal{E}(R^i)$, where $\mathcal{E}$ denotes the stochastic exponential operator, but we do not use this notation to avoid confusion with the space $\mathcal{E}$ of elementary predictable processes.)

Proposition 4.8. *Assume that $S \geq 0$ and there exists a simple reference strategy η. Let $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ be a simple strategy with $V(\vartheta, S) \in \mathcal{R}_{++}$. If $S/V(\vartheta, S)$ is an exponential semimartingale, then S satisfies DSV$^{\mathrm{loc}}_{\mathrm{s}}$ for η.*

Proof. Due to Lemma 2.3, $\mathbb{1}$ is a simple reference strategy and hence $\mathbb{1} \cdot S \in \mathcal{R}_{++}$. By Corollary 4.7, 1), it suffices to show that S satisfies $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ for $\mathbb{1}$. Because $S/V(\vartheta, S)$ is a semimartingale, so is $Y := (\mathbb{1} \cdot S)/V(\vartheta, S)$, and Y is in $\mathcal{R}_{++}$ like both $\mathbb{1} \cdot S$ and $V(\vartheta, S)$. But then $Y' := 1/Y = V(\vartheta, S)/(\mathbb{1} \cdot S)$ is also a semimartingale and in $\mathcal{R}_{++}$, and hence an exponential semimartingale by Jacod [10, Proposition 6.5 and Exercise 6.1] (see also [12, Theorem 1.3, (2)]). By Yor's formula, $\tilde{S} := S/(\mathbb{1} \cdot S) = Y'S/V(\vartheta, S)$ is then an exponential semimartingale as well. Applying now [12, Theorem 1.3] for $(1, \tilde{S})$ implies that $(1, \tilde{S})$ satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s},1}$. But $\mathcal{X}^+_{\mathrm{s}}(\tilde{S}) = \mathcal{X}_{\mathrm{s},1}(1, \tilde{S})$ by Lemma 4.4, 4), and thus $\tilde{S}$ satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$. Corollary 4.7, 2) implies in turn that S satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$, and hence also $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ for $\mathbb{1}$ by Lemma 4.5. □

The converse of Proposition 4.8 is not true; a counterexample is given in Example 5.1.

Before we can prove our second main result, we need the following proposition. Its proof follows Kardaras/Platen [12, Theorem 1.3]. Note that the appearing supermartingale discounter has no particular property at ∞.

Proposition 4.9. *Assume that $S \geq 0$ and there exists a simple reference strategy η. Then S satisfies $DSV^{\mathrm{loc}}_{\mathrm{s}}$ for η if and only if there exists an SMD for $\mathcal{X}^+_{\mathrm{s}}(S)$.*

Proof. Due to Lemma 4.5 and Corollary 4.7, 2), S satisfies $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ for η if and only if $\tilde{S} := S/(\eta \cdot S)$ satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$. Because $\mathcal{X}^+_{\mathrm{s}}(\tilde{S}) = \mathcal{X}_{\mathrm{s},1}(1, \tilde{S})$ by Lemma 4.4, 4), this is further equivalent to $(1, \tilde{S})$ satisfying $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s},1}$. But now $(1, \tilde{S})$ satisfies the conditions in [12, Theorem 1.3], and so $(1, \tilde{S})$ satisfies $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s},1}$ if and only if there exists a process $Y \in \mathcal{R}_+$ with $Y > 0$, $Y_0 = 1$ and such that $\tilde{Z}Y$ is a supermartingale for all $\tilde{Z} \in \mathcal{X}_{\mathrm{s},1}(1, \tilde{S}) = \mathcal{X}^+_{\mathrm{s}}(\tilde{S})$. As $\tilde{Z} \equiv 1$ is in $\mathcal{X}_{\mathrm{s},1}(1, \tilde{S})$, Y itself is also a supermartingale, and because $Y > 0$, it is even in $\mathcal{R}_{++}$ by the minimum principle for supermartingales; see Dellacherie/Meyer [8, Theorem VI.17].

For the "if" direction, suppose D is an SMD for $\mathcal{X}^+_{\mathrm{s}}(S)$. By the above reasoning, it suffices to show that there exists a Y as above. Define $Y := (\eta \cdot S)/(D\eta_0 \cdot S_0)$; then Y is in $\mathcal{R}_{++}$ like $\eta \cdot S$ with $Y_0 = 1/D_0 = 1$. Fix any $\tilde{Z} \in \mathcal{X}^+_{\mathrm{s}}(\tilde{S})$ so that $\tilde{Z} = \vartheta \cdot \tilde{S} = (\vartheta \cdot S)/(\eta \cdot S)$ for a $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$ and observe that $Z := \tilde{Z}(\eta \cdot S) = \vartheta \cdot S$ is then in $\mathcal{X}^+_{\mathrm{s}}(S)$ by the change-of-numéraire formula (2.3). Therefore we have $\tilde{Z}Y = (\tilde{Z}\eta \cdot S)/(D\eta_0 \cdot S_0) = (Z/D)/(\eta_0 \cdot S_0)$, which is a supermartingale because D is an SMD for $\mathcal{X}^+_{\mathrm{s}}(S)$.

For the "only if" direction, we can assume by the above reasoning that there exists a Y as above. We claim that $D := (\eta \cdot S)/(Y\eta_0 \cdot S_0)$ is then an SMD for $\mathcal{X}^+_{\mathrm{s}}(S)$. Clearly D is in $\mathcal{R}_{++}$ like $\eta \cdot S$ and Y, and $D_0 = 1/Y_0 = 1$. Now fix any $Z \in \mathcal{X}^+_{\mathrm{s}}(S)$ so that $Z = \vartheta \cdot S$ for a $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$ and observe that $\tilde{Z} := Z/(\eta \cdot S) = \vartheta \cdot (S/(\eta \cdot S)) = \vartheta \cdot \tilde{S}$ is then in $\mathcal{X}^+_{\mathrm{s}}(\tilde{S})$ by the change-of-numéraire formula (2.3). So we have $Z/D = (ZY\eta_0 \cdot S_0)/(\eta \cdot S) = \tilde{Z}Y(\eta_0 \cdot S_0)$, which is a supermartingale by the properties of Y. This concludes the proof. □

Finally, we prove the dual characterisation of $\mathrm{DSV}_{\mathrm{s}}$ in Theorem 3.6. In addition to Proposition 4.9, this uses upcrossing arguments similarly as in Bálint/Schweizer [1, Theorem 2.11].

Proof of Theorem 3.6. For the "if" direction, suppose by way of contradiction that there is a $[0, 1]$-valued adapted process ψ converging P-a.s. as $t \to \infty$ to some $\psi_\infty \in L^0_+(\mathcal{F}_\infty) \setminus \{0\}$ and such that for every $\varepsilon > 0$, there is a $\vartheta^\varepsilon \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N_+)$ with $V_0(\vartheta^\varepsilon, S) \leq \varepsilon$ and

$$\liminf_{t\to\infty}(\vartheta^\varepsilon_t - \psi_t\eta_t) \geq 0. \tag{4.1}$$

Let D be an SMD$^{\eta+}$ for $\mathcal{X}_{\mathrm{s}}^{+}(S)$. Note that $S' := S/D$ is then also a supermartingale because for all i, the i-th unit vector e^i is in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^N)$. Then we have for each $\varepsilon > 0$ that

$$
\begin{aligned}
E\left[\liminf_{t\to\infty}(\vartheta_t^\varepsilon \cdot S_t')\right] &\geq E\left[\left(\liminf_{t\to\infty}\vartheta_t^\varepsilon\right)\cdot\left(\liminf_{t\to\infty}S_t'\right)\right] \\
&\geq E\left[\left(\liminf_{t\to\infty}\psi_t\eta_t\right)\cdot\left(\liminf_{t\to\infty}S_t'\right)\right] = E\left[\psi_\infty \lim_{t\to\infty}(\eta_t\cdot S_t')\right] =: c > 0,
\end{aligned}
$$

where the first inequality is due to [1, Lemma A.1], the second to (4.1) and superadditivity of the $\liminf$, the equality holds because ψ, the simple strategy η and the nonnegative supermartingale S' all converge, and the final (strict) inequality uses $\psi_\infty \in L_+^0(\mathcal{F}_\infty)\setminus\{0\}$ and $\lim_{t\to\infty}(\eta_t\cdot S_t)/D_t > 0$; see Remark 2.6. On the other hand, $\vartheta^\varepsilon\cdot S$ is in $\mathcal{X}_{\mathrm{s}}^{+}(S)$; therefore $\vartheta^\varepsilon \cdot S' = (\vartheta^\varepsilon\cdot S)/D$ is a supermartingale and so $E[\liminf_{t\to\infty}(\vartheta_t^\varepsilon\cdot S_t')] \leq E[V_0(\vartheta^\varepsilon, S)/D_0] \leq \varepsilon$, which is a contradiction because $c > 0$ above does not depend on ε.

For the "only" if direction, assume that S satisfies DSV$_{\mathrm{s}}$ for η and define $\tilde{S} := S/(\eta\cdot S)$. We first show via upcrossing arguments that $\tilde{S}_\infty := \lim_{t\to\infty}\tilde{S}_t$ exists and is finite, P-a.s. Suppose to the contrary that some $\tilde{S}^i$ does not converge. Following the exact steps of the proof in Delbaen/Schachermayer [7, Theorem 3.3] by considering $(1,\tilde{S})$ instead of S there, the $(i+1)$-th unit vector e^{i+1} instead of H and hence $\int \mathrm{e}^{i+1}\,d(1,\tilde{S}) = \tilde{S}^i$ instead of $\int H\,dS$ there, we obtain for each $n\in\mathbb{N}$ an $L^n \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^{N+1})$ for $(1,\tilde{S})$ with $P[\int_0^\infty L^n\,d(1,\tilde{S}) > nb] > a$ for some constants $a, b > 0$. Because Lemma 4.4, 4) gives $\mathcal{X}_{\mathrm{s},1}(1,\tilde{S}) = \mathcal{X}_{\mathrm{s}}^{+}(\tilde{S})$, this yields a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^N)$ for $\tilde{S}$ with $P[\int_0^\infty \vartheta^n\,d\tilde{S} > nb] > a$ for all n. It follows by [1, Lemma A.2] that $\mathcal{X}_{\mathrm{s}}^{+}(\tilde{S})$ is not bounded in L^0; so $\tilde{S}$ does not satisfy NA1$_{\mathrm{s}}^{\mathrm{loc}}$ (see [12, Proposition 1.1]), and hence $\tilde{S}$ does not satisfy DSV$_{\mathrm{s}}$ either, by Lemmas 4.5 and 4.1. By discounting-invariance, also S then does not satisfy DSV$_{\mathrm{s}}$, which is a contradiction. This shows that $\tilde{S}_\infty := \lim_{t\to\infty}\tilde{S}_t$ is well defined and finite.

Consider now the time-transformed process $\hat{S}$ defined by $\hat{S}_u := \tilde{S}_{u/(1-u)}$ for $u\in[0,1)$ and $\hat{S}_u := \tilde{S}_\infty$ for $u\in[1,\infty)$, and similarly $\hat{\eta}_u := \eta_{u/(1-u)}$ for $u\in[0,1)$ and $\hat{\eta}_u := \lim_{t\to\infty}\eta_t = \eta_{J+1}$ for $u\in[1,\infty)$. The filtration $\hat{\mathbb{F}}$ is defined analogously from $\mathbb{F}$. Note that $\hat{\eta}\cdot\hat{S}\equiv 1$ on $[0,1)$ due to $\eta\cdot\tilde{S}\equiv 1$ on $[0,\infty)$. Because S and hence also $\tilde{S}$ satisfy DSV$_{\mathrm{s}}$ for η, so does $\hat{S}$ for $\hat{\eta}$ and we can apply Proposition 4.9 via Lemma 4.1 to obtain an SMD $\hat{D}$ for $\mathcal{X}_{\mathrm{s}}^{+}(\hat{S})$. In particular, $\lim_{u\nearrow 1}(\hat{\eta}_u\cdot\hat{S}_u)/\hat{D}_u = \lim_{u\nearrow 1} 1/\hat{D}_u > 0$, because $\hat{D} > 0$ and $\hat{D}$ does not explode to ∞ as it is RCLL on $[0,\infty)$. Transforming back shows that $\tilde{D}_t := \hat{D}_{t/(1+t)}$ for $t\in[0,\infty)$ is well defined and an SMD for $\mathcal{X}_{\mathrm{s}}^{+}(\tilde{S})$. Moreover, $\lim_{t\to\infty}(\eta_t\cdot\tilde{S}_t)/\tilde{D}_t = \lim_{t\to\infty}1/\tilde{D}_t = \lim_{u\nearrow 1}1/\hat{D}_u > 0$ and hence $\tilde{D}$ is even an SMD$^{\eta+}$ for $\mathcal{X}_{\mathrm{s}}^{+}(\tilde{S})$. Using the change-of-numéraire formula (2.3), it is straightforward to verify that $D := \tilde{D}(\eta\cdot S)$ is then an SMD$^{\eta+}$ for $\mathcal{X}_{\mathrm{s}}^{+}(S)$. □

5 Overview and counterexamples

This last section gives an overview of the relations between different kinds of semimartingale properties and discounting-invariant absence-of-arbitrage conditions, and links all this to the special case where $S = (1, X)$. We start by presenting a number of counterexamples.

Example 5.1. *The converse of Proposition 4.8 does not hold in general.* We give an example for an S which is a semimartingale satisfying DSV for $\mathbb{1}$ as in Bálint/Schweizer [1, Definition 2.7] and hence $\mathrm{DSV}_{\mathrm{s}}^{\mathrm{loc}}$ for $\eta \equiv \mathbb{1}$, but which is not an exponential semimartingale. This is similar to the example in Kardaras/Platen [12, Section 2.5] and can be regarded as a (rescaled) special case of [1, Example 1.1]. Let $Y_t := \exp(W_t - t/2)$ for a one-dimensional Brownian motion W on a probability space $(\Omega, \mathcal{F}, P)$ and set $X_t := Y_{t/(1-t)}$ for $t \in [0,1)$, $X_t := 0$ for $t \in [1,\infty)$ and $S := (1, X)$. Define the filtration $\mathbb{F}$ to be the augmentation of the natural filtration of X. Because $\lim_{t\to\infty} Y_t = 0$, X is a nonnegative continuous martingale, but not an exponential semimartingale as $X_1 = 0$. On the other hand, being a martingale like X, S satisfies NFLVR and a fortiori NUPBR = NA1; see [11, Proposition 1]. It then follows from [1, Proposition 5.6] that S satisfies DSV for $\mathbb{1}$.

Example 5.2. *The converse of Lemma 4.1 does not hold in general.* We give a simple example for an exponential semimartingale S which satisfies $\mathrm{DSV}_{\mathrm{s}}^{\mathrm{loc}}$ for $\mathbb{1}$, but not $\mathrm{DSV}_{\mathrm{s}}$ for $\mathbb{1}$. Let $X_t := 1 + 2t - \lfloor 2t \rfloor$ and $S := (1, X)$. Then S is strictly positive, of finite variation and deterministic, hence an exponential semimartingale, and taking $\vartheta \equiv \mathrm{e}^1$ in Proposition 4.8, we conclude that S satisfies $\mathrm{DSV}_{\mathrm{s}}^{\mathrm{loc}}$ for $\mathbb{1}$. On the other hand, fix $n \in \mathbb{N}$ and consider

$$\vartheta^n := \sum_{j=0}^{n} \big(I_{(j,j+1/2]}(0, V_j(\vartheta^n, S)) + I_{(j+1/2,j+1]}(V_{j+1/2}(\vartheta^n, S), 0) \big) + I_{]\!](n+1,\infty[\![}(V_{n+1}(\vartheta^n, S)/2)\mathbb{1}$$

with $\vartheta_0^n := (0, 1/n)$ so that $V_0(\vartheta^n, S) = 1/n$. In words, ϑ^n starts by buying $1/n$ units of the second (risky) asset and subsequently, after every time step of length $1/2$, shifts all its wealth from one asset to the other and back. These reallocations continue until time $n+1$; then ϑ^n moves all its wealth in equal parts to the market portfolio $\mathbb{1}$ and keeps this constant position. It is straightforward to verify that ϑ^n is in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^2)$. Moreover, for any $n \in \mathbb{N}$, $V_0(\vartheta^n, S) = 1/n$ and $\liminf_{t\to\infty} \vartheta_t^n \geq (2^n/(2n))\mathbb{1}$. Hence S does not satisfy $\mathrm{DSV}_{\mathrm{s}}$ for $\mathbb{1}$.

The next example illustrates that $\mathrm{DSV}_{\mathrm{s}}$ is a very weak assumption; in particular, it does not pick up fairly obvious arbitrage opportunities due to its long-only restriction. So Theorem 3.1 is really quite a strong conclusion.

Example 5.3. *$\mathrm{DSV}_{\mathrm{s}}$ for $\mathbb{1}$ (as in Definition 2.4) does not imply DSV for $\mathbb{1}$ (as in [1, Definition 2.7]) for simple (but not necessarily long-only) strategies.* Let $X := I_{[\![0,1[\![} + 2I_{[\![1,\infty]\!]}$ and $S := (1, X)$. Note that every $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^2)$ has $V_0(\vartheta, S) = \vartheta_1^1 + \vartheta_1^2$. Because $\vartheta \geq 0$ and S^2 is constant except for $\Delta S_1^2 = 1$, it follows that $V_t(\vartheta, S) \leq V_0(\vartheta, S) + \vartheta_1^2 \leq 2V_0(\vartheta, S)$ for any $t > 0$. Moreover, as $\mathbb{1} \cdot S \geq 2$, any $\vartheta \in \mathcal{E}^{\mathrm{sf}}(\mathbb{R}_+^2)$ with $V_0(\vartheta, S) \leq \varepsilon$ cannot satisfy $\vartheta_t - \psi_\infty \mathbb{1} \geq 0$ for any $t \geq 0$ and any $\psi_\infty \in L_+^1(\mathcal{F}_\infty)$ with $E[\psi_\infty] > \varepsilon > 0$, because otherwise $V_t(\vartheta, S) = \vartheta_t \cdot S_t \geq \psi_\infty \mathbb{1} \cdot S_t \geq 2\psi_\infty$ would lead to a contradiction. This shows that $\mathrm{DSV}_{\mathrm{s}}$ for $\mathbb{1}$ is satisfied. On the other hand, $\vartheta' := (-1, 1)I_{[\![0,1]\!]} + (1/3, 1/3)I_{]\!]1,\infty]\!]}$ is in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^2)$ and satisfies $V_0(\vartheta', S) = 0$, $V(\vartheta', S) \geq 0$ and $\vartheta_2' = (1/3)\mathbb{1}$, so that DSV for $\mathbb{1}$ for simple strategies is violated. Note that ϑ' exploits the arbitrage in S by going short in the first asset.

Remark 5.4. Example 5.3 suggests that any absence-of-arbitrage condition implied by some semimartingale property should restrict itself to long-only strategies. In particular, Proposition 4.8 cannot be strengthened in general by allowing simple (but not necessarily long-only) strategies in the definition of $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ for $\mathbb{1}$.

We summarise our results and counterexamples in the following diagram.

$$\begin{array}{ccccc} & & \text{DSV for } \eta & & \\ & \not\Nearrow & \Downarrow \not\Uparrow & & \\ \begin{array}{c} S/V(\vartheta,S) \text{ is exponential} \\ \text{semimartingale} \end{array} & \not\Rightarrow & \mathrm{DSV}_{\mathrm{s}} \text{ for } \eta & & \\ & \Searrow & \Downarrow \not\Uparrow & & \\ & & \mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}} \text{ for } \eta & \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} & \begin{array}{c} S/V(\vartheta,S) \text{ is} \\ \text{semimartingale} \end{array} \end{array}$$

Figure 1: Overview of relations between semimartingale properties and absence-of-arbitrage conditions for a general adapted RCLL S. DSV for η is as in [1, Definition 2.7], η is any simple reference strategy, and ϑ is any strategy in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ with $V(\vartheta,S) \in \mathcal{R}_{++}$.

In Figure 1, the arrow "$\not\Nearrow$" is due to Example 5.1, both "$\Downarrow$" are trivial, the upper "$\not\Uparrow$" is due to Example 5.3, "$\not\Rightarrow$" and the lower "$\not\Uparrow$" are due to Example 5.2, "$\Searrow$" is Proposition 4.8, "$\Rightarrow$" is from (the proof of) Theorem 3.1, and "$\not\Leftarrow$" is due to the counterexample in Kardaras/Platen [12, Section 2.5] and Lemmas 4.5 and 4.4, 2).

One can translate the above results for the special case $S = (1, X)$ and formulate them in classic terms. This yields the following diagram.

$$\begin{array}{ccccc} \begin{array}{c} S/V(\vartheta,S) \text{ is exponential} \\ \text{semimartingale} \end{array} & \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} & \mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s},1} = \mathrm{NA1}_{\mathrm{s}} & \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} & \begin{array}{c} S/V(\vartheta,S) \text{ is} \\ \text{semimartingale} \end{array} \end{array}$$

Figure 2: Overview of relations between semimartingale properties and absence-of-arbitrage conditions for $S = (1, X)$, where $X \geq 0$ is adapted and RCLL, and ϑ is any strategy in $\mathcal{E}^{\mathrm{sf}}(\mathbb{R}^N)$ with $V(\vartheta,S) \in \mathcal{R}_{++}$.

The implications in Figure 2 follow from those in Figure 1 by noting that by Lemma 4.5, $\mathrm{NA1}^{\mathrm{loc}}_{\mathrm{s}}$ is equivalent to $\mathrm{DSV}^{\mathrm{loc}}_{\mathrm{s}}$ for any simple reference strategy η. To connect things to Kardaras/Platen [12], note that the two arrows "$\Rightarrow$" constitute a slight generalisation of the statement of [12, Theorem 1.3, (1)], whereas the second "$\not\Leftarrow$" recovers the counterexample in [12, Section 2.5].

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