A result on integral functionals with infinitely many constraints

Tahir Choulli
University of Alberta
Department of Mathematical and Statistical Sciences
632 Central Academic Building
Edmonton, AB T6G 2G1
Canada
tchoulli@ualberta.ca

Martin Schweizer *
ETH Zürich
Departement Mathematik
ETH-Zentrum, HG G 51.2
Rämistrasse 101
CH – 8092 Zürich, Switzerland
and
Swiss Finance Institute
Walchestrasse 9
CH – 8006 Zürich, Switzerland
martin.schweizer@math.ethz.ch

Abstract: A classic paper of Borwein/Lewis (1991) studies optimisation problems over L^p_+ with finitely many linear equality constraints, given by scalar products with functions from L^q. One key result shows that if some $x \in L^p_+$ satisfies the constraints and if the constraint functions are pseudo-Haar, the constraints can also be realised by another function y in the interior of L^∞_+. We establish an analogue of this result in a setting with infinitely many, measurably parametrised constraints, and we briefly sketch an application in arbitrage theory.

Key words: linear equality constraints, feasible solution, infinitely many constraints, random measure, arbitrage theory, equivalent martingale measures

MSC 2010 Classification Numbers: 46N10, 41A29, 91G99

JEL Classification Numbers: C60, C65, Z00

This version: 14.09.2015

* corresponding author

1. Introduction

In the context of an optimisation problem for an integral functional subject to finitely many linear equality constraints, Borwein/Lewis (1991) prove the following very useful result.

Theorem 1.1. [Borwein/Lewis (1991), Theorem 2.9] *Suppose (T,μ) is a finite measure space and $x \in L^p_+(T) \setminus \{0\}$, with $p \in [1,\infty]$. Suppose further that $a_1,\dots,a_n \in L^q(T)$, with q conjugate to p, are pseudo-Haar (i.e. linearly independent on every μ-nonnull subset of T). Then there exist $\varepsilon > 0$ and $y \in L^\infty(T)$ with $y \geq \varepsilon$ μ-a.e. and such that*

$$\int_T x(t)a_i(t)\mu(dt) = \int_T y(t)a_i(t)\mu(dt) \qquad \text{for } i = 1,\dots,n.$$

In words, if the n constraint functions a_i are pseudo-Haar and if there is some $x \not\equiv 0$ in $L^p_+(T)$ which satisfies the constraints $\int_T x(t)a_i(t)\mu(dt) = b_i$, $i=1,\dots,n$, then there is even some y in the (norm-)interior of $L^\infty_+(T)$ which also satisfies the same constraints. Our goal is to prove a version of this result for a setting with infinitely many, measurably parametrised constraints.

2. The result

Our original motivation for this work came from arbitrage theory, and we explain this in more detail in Section 5 below. However, for the sake of clarity, we directly present here the abstract setup. We start from a measurable space $(\bar\Omega, \mathcal{P})$ with a finite measure $\bar\pi$ on $\mathcal{P}$ and the completion $\mathcal{P}^{\bar\pi}$ of $\mathcal{P}$ with respect to $\bar\pi$. Let $\kappa = \kappa(\bar\omega, dx)$ be a $\mathcal{P}$-measurable random measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$; so $\kappa(\bar\omega, \cdot\,)$ is a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for ($\bar\pi$-almost) all $\bar\omega \in \bar\Omega$, and $\kappa(\,\cdot\,, B)$ is $\mathcal{P}$-measurable for each $B \in \mathcal{B}(\mathbb{R}^d)$. (This is sometimes also called a transition kernel from $(\bar\Omega,\mathcal{P})$ to $\mathcal{B}(\mathbb{R}^d)$.) Define $\widetilde\Omega := \bar\Omega \times \mathbb{R}^d$ and $\widetilde{\mathcal{P}} := \mathcal{P}^{\bar\pi} \otimes \mathcal{B}(\mathbb{R}^d)$ and suppose that

$$\widetilde\pi := \bar\pi \otimes \kappa \tag{2.1}$$

is a finite measure on $(\widetilde\Omega, \widetilde{\mathcal{P}})$. For later use, we also introduce the notations

$$L^p(\widetilde\pi) := L^p(\widetilde\Omega, \widetilde{\mathcal{P}}, \widetilde\pi) := L^p(\widetilde\Omega, \widetilde{\mathcal{P}}, \widetilde\pi; \mathbb{R}),$$
$$L^p_d(\bar\pi) := L^p(\bar\Omega, \mathcal{P}^{\bar\pi}, \bar\pi; \mathbb{R}^d)$$

for $p \in [0,\infty]$ and $d \in \mathbb{N}$. We write $E_{\bar\pi}$ and $E_{\widetilde\pi}$ for the integrals with respect to $\bar\pi$ and $\widetilde\pi$.

Our main result is then

Theorem 2.1. *Suppose that*

$$C_\kappa := \left\| \int_{\mathbb{R}^d} |x| \kappa(\bar\omega, dx) \right\|_{L^\infty(\bar\pi)} < \infty. \tag{2.2}$$

Let $f_0 : \widetilde\Omega \to [0,\infty)$ *be a* $\widetilde{\mathcal{P}}$*-measurable function satisfying*

$$\left\| \int_{\mathbb{R}^d} |x| f_0(\bar\omega, x) \kappa(\bar\omega, dx) \right\|_{L^\infty(\bar\pi)} < \infty, \tag{2.3}$$

$$f_0 > 0 \qquad \widetilde\pi\text{-a.e.}, \tag{2.4}$$

$$\int_{\mathbb{R}^d} x f_0(\bar\omega, x) \kappa(\bar\omega, dx) = b(\bar\omega) \qquad \bar\pi\text{-a.e.} \tag{2.5}$$

(Note that b *is then* $\mathcal{P}^{\bar\pi}$*-measurable and bounded.) Then there exists a* $\widetilde{\mathcal{P}}$*-measurable function* $f_0^* : \widetilde\Omega \to [0,\infty)$ *which is bounded by a constant* $\widetilde\pi$*-a.e. and satisfies*

$$f_0^* > 0 \qquad \widetilde\pi\text{-a.e.}, \tag{2.6}$$

$$\int_{\mathbb{R}^d} x f_0^*(\bar\omega, x) \kappa(\bar\omega, dx) = b(\bar\omega) \qquad \bar\pi\text{-a.e.} \tag{2.7}$$

In comparison to Theorem 1.1, the space T from there corresponds to $\mathbb{R}^d$, and the n constraints are replaced by the $\bar\omega$-dependent function b on $\bar\Omega$. To put Theorem 1.1 into perspective, define a linear operator Φ on $\widetilde{\mathcal{P}}$-measurable functions $f : \widetilde\Omega \to \mathbb{R}$ by

$$\Phi(f)(\bar\omega) := \int_{\mathbb{R}^d} x f(\bar\omega, x) \kappa(\bar\omega, dx), \tag{2.8}$$

provided that $\int_{\mathbb{R}^d} |x f(\bar\omega, x)| \kappa(\bar\omega, dx) < \infty$ $\bar\pi$-a.e. so that $\Phi(f) : \bar\Omega \to \mathbb{R}^d$ is $\bar\pi$-a.e. well defined. Because $|x f(\bar\omega, x)| \le |x| \, \|f\|_{L^\infty(\widetilde\pi)}$ $\bar\pi$-a.e., we easily see from (2.2) that Φ maps $L^\infty(\widetilde\pi)$ into $L^\infty_d(\bar\pi)$. Moreover, (2.3) shows that $\Phi(f_0)$ is $\bar\pi$-a.e. well defined and equals $b \in L^\infty_d(\bar\pi)$ by (2.5). Finally, (2.4) imposes that f_0 is strictly positive. The statement of Theorem 2.1 is then that the value $\Phi(f_0)$ ("constraint b") can be realised by Φ in another function f_0^*, i.e. $\Phi(f_0) = \Phi(f_0^*)$, such that f_0^* is still strictly positive and in addition uniformly bounded.

With the above formulation, one can see that Theorem 2.1 is like an infinite-dimensional version of Theorem 1.1. The key difference is that the linear (integral) operator Φ now takes values in the infinite-dimensional space $L^\infty_d(\bar\pi)$, while Borwein/Lewis (1991) have only $\mathbb{R}^n$ as range space for their constraints. Because the proofs of both results ultimately rest on a separation argument, it is clear that Theorem 2.1 will need more involved techniques, and it is also not surprising that the setup and the assumptions are slightly different.

We end this section by outlining how the proof of Theorem 2.1 can be reduced to a specific property of the mapping Φ in (2.8). *Throughout the rest of this section, we assume that we are in the setup of Theorem 2.1.* We first note that Φ from (2.8) is obviously linear. Moreover, using (2.2) easily gives

$$|\Phi(f)| \leq C_\kappa \|f\|_{L^\infty(\widetilde{\pi})} \qquad \bar{\pi}\text{-a.e.,}$$

so that $\Phi : L^\infty(\widetilde{\pi}) \to L^\infty_d(\bar{\pi})$ is continuous for the norm topologies on both spaces, and also when we consider on $L^\infty(\widetilde{\pi})$ the norm topology and on $L^\infty_d(\bar{\pi})$ the weak* topology $\sigma(L^\infty, L^1)$. The key point of the proof will be to show that

$$\Phi : \big(L^\infty(\widetilde{\pi}), \|\cdot\|_{L^\infty(\widetilde{\pi})}\big) \to \big(L^\infty_d(\bar{\pi}), \sigma(L^\infty, L^1)\big) \text{ is open.} \tag{2.9}$$

Once we have this, the rest of the proof is simple. In fact, we have $\Phi(0) = 0$, and (2.9) implies that for any $\delta > 0$, the image $\Phi(U_\delta(0))$ of the open ball

$$U_\delta(0) := \big\{ f \in L^\infty(\widetilde{\pi}) \,\big|\, \|f\|_{L^\infty(\widetilde{\pi})} < \delta \big\}$$

is open in $L^\infty_d(\bar{\pi})$ for the weak* topology $\sigma(L^\infty, L^1)$. The sequence $f_m := f_0 \wedge m$, $m \in I\!N$, is in $L^\infty(\widetilde{\pi})$ and converges pointwise to f_0 as $m \to \infty$, and omitting the argument $\bar{\omega}$, we have $\bar{\pi}$-a.e.

$$\big|x\big(f_0(x) - f_m(x)\big)\big| \leq |x| f_0(x) \in L^1\big(\kappa(dx)\big)$$

due to (2.3). By dominated convergence, we therefore obtain that

$$y_m(\bar{\omega}) := \int_{I\!R^d} x\big(f_0(\bar{\omega}, x) - f_m(\bar{\omega}, x)\big)\kappa(\bar{\omega}, dx) \longrightarrow 0 \qquad \bar{\pi}\text{-a.e. as } m \to \infty.$$

Using again (2.3) gives $|y_m| \leq \int_{I\!R^d} |x| f_0(x) \kappa(dx) \in L^\infty_1(\bar{\pi})$, and multiplying with $z \in L^1_d(\bar{\pi})$ and integrating with respect to $\bar{\pi}$ shows via dominated convergence that $y_m \to 0$ in $L^\infty_d(\bar{\pi})$ for the weak* topology $\sigma(L^\infty, L^1)$. So for m large enough, we get $y_m \in \Phi(U_\delta(0))$ which means that $y_m = \Phi(\tilde{f}_m)$ for some $\tilde{f}_m \in U_\delta(0)$. But this means in turn that $\tilde{f}_m \in L^\infty(\widetilde{\pi})$ with $|\tilde{f}_m| < \delta$ $\widetilde{\pi}$-a.e. and, from the definitions of y_m and Φ, that

$$\begin{aligned} \int_{I\!R^d} x f_0(\bar{\omega}, x)\kappa(\bar{\omega}, dx) &= y_m(\bar{\omega}) + \int_{I\!R^d} x f_m(\bar{\omega}, x)\kappa(\bar{\omega}, dx) \\ &= \int_{I\!R^d} x\big(\tilde{f}_m(\bar{\omega}, x) + f_m(\bar{\omega}, x)\big)\kappa(\bar{\omega}, dx) \qquad \bar{\pi}\text{-a.e.} \end{aligned} \tag{2.10}$$

Now suppose first that instead of (2.4), we even have

$$f_0 \geq 2\delta > 0 \qquad \widetilde{\pi}\text{-a.e. for some } \delta > 0. \tag{2.4'}$$

Because $|\tilde{f}_m| \le \delta$ $\widetilde{\pi}$-a.e. and $f_m = f_0 \wedge m \ge 2\delta$ $\widetilde{\pi}$-a.e. by (2.4′), the function $f^*_\delta := \tilde{f}_m + f_m$ then satisfies $\delta = -\delta + 2\delta \le \tilde{f}_m + f_m = f^*_\delta \le \delta + f_m \le \delta + m$ $\widetilde{\pi}$-a.e., and so f^*_δ is in $L^\infty(\widetilde{\pi})$ and satisfies (2.6) and (2.7), due to (2.10) and (2.5). We even have instead of (2.6) the stronger property

$$(2.6') \qquad f^*_\delta \ge \delta > 0 \qquad \widetilde{\pi}\text{-a.e.}$$

In general, if we only have (2.4), we set $\hat{f}_0 := f_0 I_{\{f_0 \ge 2\delta\}}$ and define the random measure $\hat{\kappa}$ by $\hat{\kappa}(\bar{\omega}, dx) := I_{\{f_0(\bar{\omega},x) \ge 2\delta\}} \kappa(\bar{\omega}, dx)$. Then $\hat{\kappa}$ satisfies (2.2) like κ, and $\hat{f}_0$ satisfies (2.3) and (2.4′) with $\widetilde{\pi} = \bar{\pi} \otimes \kappa$ replaced by $\bar{\pi} \otimes \hat{\kappa}$. Moreover, we have

$$(2.5') \qquad \int_{I\!R^d} x \hat{f}_0(\bar{\omega}, x) \hat{\kappa}(\bar{\omega}, dx) = \int_{I\!R^d} x \hat{f}_0(\bar{\omega}, x) \kappa(\bar{\omega}, dx)$$
$$=: \hat{b}(\bar{\omega}) = b(\bar{\omega}) - \int_{I\!R^d} x f_0(\bar{\omega}, x) I_{\{f_0(\bar{\omega},x) < 2\delta\}} \kappa(\bar{\omega}, dx).$$

So the above argument gives by (2.6′) an $\hat{f}^*_\delta \ge \delta$ $\bar{\pi} \otimes \hat{\kappa}$-a.e. with

$$(2.7') \qquad \int_{I\!R^d} x \hat{f}^*_\delta(\bar{\omega}, x) \hat{\kappa}(\bar{\omega}, dx) = \hat{b}(\bar{\omega})$$

and which is bounded by a constant $\bar{\pi} \otimes \hat{\kappa}$-a.e. From (2.7′), the definition of $\hat{b}$ in (2.5′) and the definition of $\hat{\kappa}$, we then see that

$$f^*_0 := \hat{f}^*_\delta + f_0 I_{\{f_0 < 2\delta\}}$$

satisfies both (2.6) and (2.7). Hence we have the assertion of Theorem 2.1. **q.e.d.**

3. The proof that Φ is open

In this section, we prove the key result (2.9) that Φ is open under an extra condition, (3.3) below, on the random measure κ. We then show in the next section how to remove this assumption and prove Theorem 2.1. We first need some concepts. Define, using $\widetilde{\pi} = \bar{\pi} \otimes \kappa$,

$$(3.1) \qquad E := \big\{ y \in L^0(\bar{\Omega}, \mathcal{P}^{\bar{\pi}}, \bar{\pi}; I\!R^d) \,\big|\, y^\top x \in L^1(\widetilde{\pi}) \big\},$$
$$\|y\|_E := \|y^\top x\|_{L^1(\widetilde{\pi})} = E_{\bar{\pi}}\left[\int_{I\!R^d} |y^\top(\bar{\omega}) x| \kappa(\bar{\omega}, dx) \right].$$

Because $|y^\top x| \le |y||x|$, using (2.2) gives $\int_{I\!R^d} |y^\top(\bar{\omega}) x| \kappa(\bar{\omega}, dx) \le C_\kappa |y|$ $\bar{\pi}$-a.e. and therefore

$$(3.2) \qquad \|y\|_E = \|y^\top x\|_{L^1(\widetilde{\pi})} \le C_\kappa \|y\|_{L^1_d(\bar{\pi})}.$$

A major step in the proof of (2.9) will be to show that $\|\cdot\|_E$ and $\|\cdot\|_{L^1_d(\bar\pi)}$ are actually equivalent and that $E = L^1_d(\bar\pi)$. We start with a preliminary result.

Lemma 3.1. *Suppose that*

(3.3) *for $\bar\pi$-almost all $\bar\omega$, the measure $\kappa(\bar\omega,\,\cdot\,)$ on $(I\!R^d, \mathcal{B}(I\!R^d))$ has full support.*

Then $(E, \|\cdot\|_E)$ is a Banach space.

Proof. First of all, $\|\cdot\|_E$ from (3.1) clearly satisfies positive homogeneity and the triangle inequality. If $\|y\|_E = 0$, then $\|y^\top x\|_{L^1(\widetilde\pi)} = 0$ so that $y^\top x = 0$ $\widetilde\pi$-a.e. This means that for $\bar\pi$-a.a. $\bar\omega$, we have $y(\bar\omega)^\top x = 0$ $\kappa(\bar\omega, dx)$-a.e. on $I\!R^d$, and so the full support property (3.3) implies that $y = 0$ $\bar\pi$-a.e. So $\|\cdot\|_E$ is a norm on E.

To check completeness of $(E, \|\cdot\|_E)$, let $(y_n)_{n\in I\!N}$ be a Cauchy sequence. Then $(y_n^\top x)_{n\in I\!N}$ is by (3.1) a Cauchy sequence in $L^1(\widetilde\pi)$ and hence convergent in $L^1(\widetilde\pi)$ to some $f \in L^1(\widetilde\pi)$. Along a subsequence still denoted by $(y_n^\top x)_{n\in I\!N}$, we then have $\widetilde\pi$-a.e. convergence so that

$$f(\bar\omega, x) = \lim_{n\to\infty} y_n(\bar\omega)^\top x \qquad \kappa(\bar\omega, dx)\text{-a.e., for } \bar\pi\text{-a.a. } \bar\omega.$$

Due to the full support condition (3.3), we therefore must have $f(\bar\omega, x) = y_\infty(\bar\omega)^\top x$ for some $I\!R^d$-valued function y_∞ on $\bar\Omega$, and choosing for x the unit vectors $e_i \in I\!R^d$, $i = 1, \dots, d$, shows that $y_\infty(\bar\omega) = \lim_{n\to\infty} y_n(\bar\omega)$ $\bar\pi$-a.e. Hence y_∞ is $\mathcal{P}^{\bar\pi}$-measurable and $y_\infty^\top x = f \in L^1(\widetilde\pi)$, so that the limit of the sequence (y_n) is $y_\infty \in E$. So E is complete. **q.e.d.**

We know from (3.2) that $\|\cdot\|_E \le C_\kappa \|\cdot\|_{L^1_d(\bar\pi)}$ so that clearly $L^1_d(\bar\pi) \subseteq E$. We shall see eventually that the two spaces coincide, but before then, we have to distinguish the dual spaces $L^\infty_d(\bar\pi)$ and E' to avoid confusion and misleading statements. Let us write $\langle\,\cdot\,,\,\cdot\,\rangle_{E,E'}$ for the dual pairing on (E, E') and

$$\langle y, z\rangle_{L^1,L^\infty} = E_{\bar\pi}[y^\top z] \qquad \text{for } y \in L^1_d(\bar\pi), z \in L^\infty_d(\bar\pi).$$

Any $h \in E'$ is a continuous linear functional on $E \supseteq L^1_d(\bar\pi)$; so its restriction to $L^1_d(\bar\pi)$ can be identified with some $z = z_h \in L^\infty_d(\bar\pi)$ and we have

$$\langle y, h\rangle_{E,E'} = \langle y, z_h\rangle_{L^1,L^\infty} = E_{\bar\pi}[y^\top z_h] \qquad \text{for all } y \in L^1_d(\bar\pi) \subseteq E. \tag{3.4}$$

Conversely, if we start from $z \in L^\infty_d(\bar\pi)$, then $y \mapsto E_{\bar\pi}[y^\top z] = \langle y, z\rangle_{L^1,L^\infty}$ is well defined for $y \in L^1_d(\bar\pi)$, but not necessarily for y from the larger space E. However, we can say more if $z \in \Phi(L^\infty(\widetilde\pi)) \subseteq L^\infty_d(\bar\pi)$, due to (2.2). Indeed, we then have $z = \Phi(f) = \int_{I\!R^d} x f(x)\kappa(dx)$ for some $f \in L^\infty(\widetilde\pi)$ and therefore

$$|E_{\bar\pi}[y^\top z]| = \left| E_{\bar\pi}\left[\int_{I\!R^d} y^\top x f(x)\kappa(dx)\right]\right| \le \|f\|_{L^\infty(\widetilde\pi)} E_{\bar\pi}\left[\int_{I\!R^d} |y^\top x|\kappa(dx)\right] = \|f\|_{L^\infty(\widetilde\pi)}\|y\|_E.$$

So $y \mapsto E_{\bar{\pi}}[y^\top z] = E_{\bar{\pi}}[y^\top \Phi(f)]$ is then well defined for $y \in E$ and a continuous linear functional on E, hence in E'. So we can identify $\Phi(L^\infty(\widetilde{\pi}))$ with a subset of E' by setting $h_f : E \to I\!R$, $y \mapsto E_{\bar{\pi}}[y^\top \Phi(f)] = \langle y, h_f \rangle_{E,E'}$, and then we have that

$$\langle y, h_f \rangle_{E,E'} = E_{\bar{\pi}}[y^\top \Phi(f)] = \langle y, \Phi(f) \rangle_{L^1,L^\infty} \qquad \text{for } y \in L^1_d(\bar{\pi}) \subseteq E. \tag{3.5}$$

The above arguments make it clear that "$\Phi(L^\infty(\widetilde{\pi})) \subseteq E' \subseteq L^\infty_d(\bar{\pi})$" up to suitable identifications. The next result shows that the differences in the "inclusions" cannot be big.

Lemma 3.2. *Suppose that κ satisfies the full support condition (3.3). Then $\Phi(L^\infty(\widetilde{\pi}))$ is dense in $L^\infty_d(\bar{\pi})$ for the weak* topology $\sigma(L^\infty, L^1)$.*

Proof. For brevity, write $\mathcal{X} := L^\infty(\widetilde{\pi})$, $\mathcal{Z} := L^\infty_d(\bar{\pi})$ and denote by $\overline{\cdot}$ the closure for $\sigma(L^\infty, L^1)$. Then we want to show that $\overline{\Phi(\mathcal{X})} = \mathcal{Z}$. If there exists some $z \in \mathcal{Z} \setminus \overline{\Phi(\mathcal{X})}$, the Hahn–Banach theorem yields the existence of some $\alpha \in I\!R$ and $y \in \mathcal{Z}' = L^1_d(\bar{\pi})$, the dual of $\mathcal{Z}$ for the weak* topology $\sigma(L^\infty, L^1)$ we work with here, such that

$$E_{\bar{\pi}}[y^\top z] = \langle y, z \rangle_{L^1,L^\infty} < \alpha \le \langle y, \Phi(f) \rangle_{L^1,L^\infty} = E_{\bar{\pi}}[y^\top \Phi(f)] \qquad \text{for all } f \in \mathcal{X}.$$

Since $\mathcal{X}$ and $\mathcal{Z}$ are both linear, we can choose $\alpha = 0$ and obtain equality in "$\le$"; so we get

$$E_{\bar{\pi}}[y^\top z] < 0 = E_{\bar{\pi}}[y^\top \Phi(f)] \qquad \text{for all } f \in L^\infty(\widetilde{\pi}).$$

For $f := I_{\{y^\top x > 0\}}$, this yields by (2.8) and (2.1) that

$$0 = E_{\bar{\pi}}\left[y^\top(\bar{\omega}) \int_{I\!R^d} x f(\bar{\omega}, x) \kappa(\bar{\omega}, dx) \right] = E_{\widetilde{\pi}}\left[(y^\top x) I_{\{y^\top x > 0\}} \right]$$

and therefore $y^\top x \le 0$ $\widetilde{\pi}$-a.e. In the same way, $f := I_{\{y^\top x < 0\}}$ gives $y^\top x \ge 0$ $\widetilde{\pi}$-a.e., and again using (2.1), we thus obtain that

$$y(\bar{\omega})^\top x = 0 \qquad \text{for } \kappa(\bar{\omega}, \cdot)\text{-a.a. } x \in I\!R^d, \text{ for } \bar{\pi}\text{-a.a. } \bar{\omega}.$$

But now the full support condition (3.3) implies that $y = 0$ $\bar{\pi}$-a.e., which contradicts the fact that $E_{\bar{\pi}}[y^\top z] < 0$. So $\overline{\Phi(\mathcal{X})} = \mathcal{Z}$. **q.e.d.**

The next result is central in our proof.

Proposition 3.3. *Suppose that κ satisfies the full support condition (3.3). Then the norms $\| \cdot \|_E$ and $\| \cdot \|_{L^1_d(\bar{\pi})}$ are equivalent on $L^1_d(\bar{\pi}) \subseteq E$, and we have $E = L^1_d(\bar{\pi})$.*

Proof. For brevity, we write L^p for $L^p_d(\bar{\pi})$ in this proof, for $p \in \{1, \infty\}$.

1) We already know from (3.2) that $\|\cdot\|_E \le C_\kappa \|\cdot\|_{L^1}$ and hence $L^1 \subseteq E$. Suppose we know that both norms are equivalent on L^1. For $y \in E$, we then define $y_n := y I_{\{\|y| \le n\}} \in L^1$ to get first $y_n \to y$ pointwise and hence by dominated convergence $y_n \to y$ in E since $y \in E$. So (y_n) is a sequence in L^1 and Cauchy for $\|\cdot\|_E$, hence also for $\|\cdot\|_{L^1}$ by the equivalence of norms, and so $y_n \to y_\infty$ in L^1 for some $y_\infty \in L^1$. But we already know that $y_n \to y$ pointwise; so we must have $y_\infty = y$ $\bar{\pi}$-a.e., and so $y = y_\infty \in L^1$, giving $E \subseteq L^1$.

2) We claim (and prove in Step 4) below) that

$$\text{(3.6)} \qquad \begin{gathered} \text{for all } z \in L^\infty, \text{ there exist } \alpha > 1 > \varepsilon > 0 \text{ such that} \\ \big\{y \in L^1 \,\big|\, \|y\|_E \le \varepsilon\big\} \subseteq \big\{y \in L^1 \,\big|\, |\langle y, z\rangle_{L^1,L^\infty}| \le \alpha\big\}. \end{gathered}$$

We then claim (and prove in Step 3) below) that $(L^1, \|\cdot\|_E)$ is a Banach space. Once we have that, we look at the identity map from $(L^1, \|\cdot\|_{L^1})$ to $(L^1, \|\cdot\|_E)$. This is linear, surjective, continuous due to (3.2), and injective because $\|\cdot\|_E$ is a norm by Lemma 3.1. So part (c) of Corollary 2.12 in Rudin (1991) implies that the norms $\|\cdot\|_E$ and $\|\cdot\|_{L^1}$ are equivalent.

3) We next argue, using (3.6), the claim in 2) that $(L^1, \|\cdot\|_E)$ is a Banach space. For that, it is enough to show that L^1 is complete for the norm $\|\cdot\|_E$ on $L^1 \subseteq E$. Let $(y_n)_{n\in\mathbb{N}} \subseteq L^1 \subseteq E$ be a Cauchy sequence for $\|\cdot\|_E$. Then $y_n \to y$ in E for some $y \in E$ since $(E, \|\cdot\|_E)$ is a Banach space by Lemma 3.1, and (y_n) is bounded in E in the usual sense that $\sup_{n\in\mathbb{N}} \|y_n\|_E < \infty$. We need to show that y is even in L^1. By (3.6), we can fix any $z \in L^\infty$ and find α, ε (depending on z) such that

$$|\langle y', z\rangle_{L^1,L^\infty}| \le \frac{\alpha}{\varepsilon}\|y'\|_E \qquad \text{for all } y' \in E.$$

For every $z \in L^\infty = (L^1)'$, we thus obtain

$$\sup_{n\in\mathbb{N}} |\langle y_n, z\rangle_{L^1,L^\infty}| \le \frac{\alpha(z)}{\varepsilon(z)} \sup_{n\in\mathbb{N}} \|y_n\|_E < \infty,$$

and so the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded in L^1 with respect to the weak topology $\sigma(L^1, L^\infty)$. As $(L^1, \|\cdot\|_{L^1})$ is locally convex, Theorem 3.18 in Rudin (1991) implies that $(y_n)_{n\in\mathbb{N}}$ is also bounded in L^1 for the norm topology so that we have

$$\text{(3.7)} \qquad \sup_{n\in\mathbb{N}} \|y_n\|_{L^1} < \infty.$$

Now $y_n \to y$ in E means that $y_n^\top x \to y^\top x$ in $L^1(\widetilde{\pi})$, hence $\widetilde{\pi}$-a.e. along a subsequence $(y_{n_k})_{k\in\mathbb{N}}$. Since $\widetilde{\pi} = \bar{\pi} \otimes \kappa$, we thus have

$$y_{n_k}(\bar{\omega})^\top x \longrightarrow y(\bar{\omega})^\top x \qquad \text{as } k \to \infty,\ \kappa(\bar{\omega}, dx)\text{-a.e. on } \mathbb{R}^d, \text{ for } \bar{\pi}\text{-a.a. } \bar{\omega}.$$

The full support condition (3.3) therefore yields $|y_{n_k}| \longrightarrow |y|$ $\bar{\pi}$-a.e. as $k \to \infty$, and Fatou's lemma together with (3.7) gives $\|y\|_{L^1} \le \liminf_{k\to\infty} \|y_{n_k}\|_{L^1} \le \sup_{n\in\mathbb{N}} \|y_n\|_{L^1} < \infty$. So y is in L^1 and $(L^1, \|\cdot\|_E)$ is complete.

4) To finish the proof, we now argue (3.6). More precisely, we show that for every $z \in L^\infty$, we can find $\alpha > 1 > \varepsilon > 0$ with

$$\{y \in L^1 \,|\, \|y\|_E \le \varepsilon\} \subseteq L^1 \cap V_{w,E} = L^1 \cap \overline{V_{w,L^1}}^{w,E} \subseteq \{y' \in L^1 \,|\, |\langle y', z\rangle_{L^1,L^\infty}| \le \alpha\}, \tag{3.8}$$

where $V_{w,E}$ is a neighbourhood of 0 in E for the weak topology of E, and $\overline{V_{w,L^1}}^{w,E}$ the closure, in the same topology, of a neighbourhood of 0 for the weak topology of L^1. The first inclusion in (3.8) is clear since any weak neighbourhood contains a strong one. For the equality, we look at the identity from $L^1 \subseteq E$ to E, both with their respective weak topologies. By the result (7) in Section 8 of Köthe (1965), this mapping is nearly open which means that $\overline{V_{w,L^1}}^{w,E}$ is a weak neighbourhood $V_{w,E}$ of 0, for any weak neighbourhood V_{w,L^1} of 0.

For the second inclusion in (3.8), we now fix $z \in L^\infty$ and take any $y \in L^1 \cap \overline{V_{w,L^1}}^{w,E}$. Take a net (y_λ) in $V_{w,L^1} \subseteq L^1$ converging to y in $\sigma(E, E')$ so that we have

$$\langle y_\lambda, h\rangle_{E,E'} \xrightarrow{\lambda} \langle y, h\rangle_{E,E'} \qquad \text{for every } h \in E'. \tag{3.9}$$

By Lemma 3.2, $\Phi(L^\infty(\widetilde{\pi}))$ is dense in L^∞ for $\sigma(L^\infty, L^1)$; we take a net (f_γ) in $L^\infty(\widetilde{\pi})$ with

$$\langle y', \Phi(f_\gamma)\rangle_{L^1,L^\infty} \xrightarrow{\gamma} \langle y', z\rangle_{L^1,L^\infty} \qquad \text{for every } y' \in L^1. \tag{3.10}$$

Finally, as seen before Lemma 3.2, we can identify $\Phi(L^\infty(\widetilde{\pi}))$ with a subset of E' and hence find as in (3.5) for each γ an $h_\gamma \in E'$ with

$$\langle y', h_\gamma\rangle_{E,E'} = \langle y', \Phi(f_\gamma)\rangle_{L^1,L^\infty} \qquad \text{for every } y' \in L^1 \subseteq E. \tag{3.11}$$

Now $|\langle y, h\rangle_{E,E'}| \le$ const. for all $h \in E'$ because $y \in \overline{V_{w,L^1}}^{w,E}$. So (3.9) implies that also $|\langle y_\lambda, h\rangle_{E,E'}| \le$ const. for all $h \in E'$ and all λ. By (3.11), then, $|\langle y_\lambda, \Phi(f_\gamma)\rangle_{L^1,L^\infty}| \le$ const. for all λ and all γ, and therefore by (3.10), applied for a fixed λ, also $|\langle y_\lambda, z\rangle_{L^1,L^\infty}| \le$ const. for all λ. But $\langle y_\lambda, z\rangle_{L^1,L^\infty} = \langle y_\lambda, h_z\rangle_{E,E'}$ by (3.4) and so (3.9) yields $|\langle y, h_z\rangle_{E,E'}| \le$ const. So if $y \in L^1 \cap \overline{V_{w,L^1}}^{w,E}$, using again (3.4) gives $|\langle y, z\rangle_{L^1,L^\infty}| = |\langle y, h_z\rangle_{E,E'}| \le$ const., and so we have the third inclusion in (3.8) and hence (3.6). **q.e.d.**

Now define the mapping $\Psi : L^0(\bar{\Omega}, \mathcal{P}^{\bar{\pi}}, \bar{\pi}; I\!R^d) \to L^0(\widetilde{\Omega}, \widetilde{\mathcal{P}}, \widetilde{\pi}; I\!R)$ by $\Psi(y) := y^\top x$. Then as in the argument for (3.2), we obtain from (2.2) that

$$\|\Psi(y)\|_{L^1(\widetilde{\pi})} = \|y^\top x\|_{L^1(\widetilde{\pi})} = \|y\|_E \le C_\kappa \|y\|_{L^1_d(\bar{\pi})}.$$

So $\Psi : L^1_d(\bar{\pi}) \to L^1(\widetilde{\pi})$ is well defined, linear and continuous for the respective norm topologies.

Lemma 3.4. *Suppose that κ satisfies the full support condition (3.3). Then $\Psi(L^1_d(\bar{\pi}))$ is closed in $L^1(\widetilde{\pi})$ for the norm topology.*

Proof. If $\Psi(y_n) = y_n^\top x$, $n \in \mathbb{N}$, converges in $L^1(\widetilde{\pi})$, this means that $(y_n)_{n\in\mathbb{N}}$ converges in E. By Lemma 3.1, using the full support property (3.3), we then know that the limit y is in E, and $E = L^1_d(\bar{\pi})$ by Proposition 3.3. So $\lim_{n\to\infty} \Psi(y_n) = y^\top x = \Psi(y) \in \Psi(L^1_d(\bar{\pi}))$. **q.e.d.**

With all the above preparations, we can now almost prove (2.9).

Proposition 3.5. *Suppose that κ satisfies the full support condition (3.3). Then*

$$\Phi : \big(L^\infty(\widetilde{\pi}), \|\cdot\|_{L^\infty(\widetilde{\pi})}\big) \to \big(L^\infty_d(\bar{\pi}), \sigma(L^\infty, L^1)\big) \text{ is open.} \tag{2.9}$$

Proof. The mapping $\Psi : L^1_d(\bar{\pi}) \to L^1(\widetilde{\pi})$, $y \mapsto y^\top x$ is linear and continuous. Its adjoint $\Psi^* : L^\infty(\widetilde{\pi}) \to L^\infty_d(\bar{\pi})$ is by Theorem 4.10 of Rudin (1991) uniquely characterised by the property that

$$\langle \Psi(y), f\rangle_{L^1(\widetilde{\pi}),L^\infty(\widetilde{\pi})} = \langle y, \Psi^*(f)\rangle_{L^1_d(\bar{\pi}),L^\infty_d(\bar{\pi})}$$

for all $y \in L^1(\widetilde{\pi})$ and $f \in L^\infty(\widetilde{\pi})$. But due to (2.1) and (2.8),

$$\begin{aligned}
\langle \Psi(y), f\rangle_{L^1(\widetilde{\pi}),L^\infty(\widetilde{\pi})} &= E_{\widetilde{\pi}}[\Psi(y)f] \\
&= E_{\bar{\pi}}\left[\int_{\mathbb{R}^d} y^\top(\bar{\omega}) x f(\bar{\omega}, x)\kappa(\bar{\omega}, dx)\right] \\
&= E_{\bar{\pi}}[y^\top \Phi(f)] \\
&= \langle y, \Phi(f)\rangle_{L^1_d(\bar{\pi}),L^\infty_d(\bar{\pi})}
\end{aligned}$$

shows that Ψ^* equals Φ. By Lemma 3.4, the range of Ψ is closed in $L^1(\widetilde{\pi})$, and so Theorem 4.14 of Rudin (1991) implies that the range of $\Psi^* = \Phi$ is closed for the weak* topology $\sigma(L^\infty, L^1)$ on $L^\infty_d(\bar{\pi})$. But by Lemma 3.2, $\Phi(L^\infty(\widetilde{\pi}))$ is dense in $L^\infty_d(\bar{\pi})$ for $\sigma(L^\infty, L^1)$, and so we must have $\Phi(L^\infty(\widetilde{\pi})) = L^\infty_d(\bar{\pi})$. This means that

$$\Phi : \big(L^\infty(\widetilde{\pi}), \|\cdot\|_{L^\infty(\widetilde{\pi})}\big) \to \big(L^\infty_d(\bar{\pi}), \sigma(L^\infty, L^1)\big)$$

is linear, continuous (as seen before (2.9)) and surjective, i.e. onto. So part (a) of Corollary 2.12 of Rudin (1991) implies that Φ is open, and the proof is complete. **q.e.d.**

4. Proof of Theorem 2.1

We have seen at the end of Section 2 that Theorem 2.1 follows if we can show the result (2.9) that Φ is open. Proposition 3.5 has proved this openness under the extra full support condition (3.3) on κ. It remains to get rid of this.

Proof of Theorem 2.1. Let us introduce

$$\mathcal{Y} := \big\{ y \in L^0(\bar{\Omega}, \mathcal{P}^{\bar{\pi}}, \bar{\pi}; I\!R^d) \,\big|\, \text{for } \bar{\pi}\text{-a.a. } \bar{\omega},\ y(\bar{\omega}) \in \operatorname{lin}\big(\operatorname{supp} \kappa(\bar{\omega}, \,\cdot\,)\big) \big\}.$$

Then $\mathcal{Y}$ is a vector space, so is then $\mathcal{Y} \cap L^\infty_d(\bar{\pi})$, and the weak* topology $\sigma(L^\infty, L^1)$ makes $\mathcal{Y} \cap L^\infty_d(\bar{\pi})$ a topological vector space which is locally convex by Theorem 3.10 of Rudin (1991). Moreover, $y \in L^0(\bar{\Omega}, \mathcal{P}^{\bar{\pi}}, \bar{\pi}; I\!R^d)$ is 0 in $\mathcal{Y}$ if and only if $y^\top x = 0$ $\kappa(dx)$-a.e. $\bar{\pi}$-a.e., i.e., $\widetilde{\pi}$-a.e. This allows us to show with the same argument as for Lemma 3.1 that $(\mathcal{Y} \cap E, \|\cdot\|_E)$ is a Banach space, without needing (3.3). Also without (3.3), we can then show that $\Phi(L^\infty(\widetilde{\pi}))$ is dense in $\mathcal{Y} \cap L^\infty_d(\bar{\pi}) =: \mathcal{Z}$ as in the proof of Lemma 3.2. Next we have $\mathcal{Y} \cap L^1_d(\bar{\pi}) \subseteq \mathcal{Y} \cap E$, and going through the proof of Proposition 3.3 allows us to argue that we even have equality because the norms $\|\cdot\|_E$ and $\|\cdot\|_{L^1_d(\bar{\pi})}$ are equivalent on $\mathcal{Y} \cap L^1_d(\bar{\pi})$. To see this, we replace E by $\mathcal{Y} \cap E$ and $L^p_d(\bar{\pi})$ by $\mathcal{Y} \cap L^p_d(\bar{\pi})$ for $p \in \{1, \infty\}$ throughout the proof of Proposition 3.3, and we note in Step 3) there that also $(\mathcal{Y} \cap L^1, \|\cdot\|_{L^1})$ is locally convex and that $y_n \to y$ in $\mathcal{Y} \cap E$ implies even without (3.3) that $|y_{n_k}| \to |y|$ $\bar{\pi}$-a.e. along a subsequence. Finally, we can also prove like in Lemma 3.4, but without using (3.3), that $\Psi(\mathcal{Y} \cap L^1_d(\bar{\pi}))$ is closed in $L^1(\widetilde{\pi})$. The argument in the proof of Proposition 3.5, with $L^p_d(\bar{\pi})$ replaced by $\mathcal{Y} \cap L^p_d(\bar{\pi})$ for $p \in \{1, \infty\}$, then goes through as before and yields (2.9), without needing (3.3). This completes the proof. **q.e.d.**

Remark 4.1. Intuitively, the full support condition (3.3) is the analogue of the assumption in Theorem 1.1 that the constraint functions a_i are pseudo-Haar. The above argument exploits the measurable structure of our constraints in Theorem 2.1 to get rid of this condition, after having exploited its consequences in Section 3. ◇

5. An application in arbitrage theory

In this section, we briefly sketch (without going into details) how Theorem 2.1 can be used in arbitrage theory. We start with a filtered probability space $(\Omega, \mathcal{F}, I\!F, P)$, where $I\!F = (\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions of right-continuity and P-completeness. We are given an $I\!R^d$-valued stochastic process $S = (S_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, I\!F, P)$, and we want to construct an equivalent σ-martingale measure (EσMM) Q for S (i.e., a probability measure

Q equivalent to P such that S becomes under Q a σ-martingale). More precisely, we assume that there exists some EσMM Q_0 for S, and we want to obtain another EσMM Q_0^* such that the density $\frac{dQ_0^*}{dP}$ has some a priori specified integrability (actually boundedness, in that particular application).

The existence of Q_0 implies that S is under P a semimartingale, and so we can choose a good version A for the process dominating the semimartingale characteristics of S. In more detail, $A = (A_t)_{t \geq 0}$ is an increasing predictable process with RCLL (right-continuous with left limits) trajectories null at 0, and we denote by $\bar{\pi} := P \otimes A$ the induced measure on the product space $\bar{\Omega} := \Omega \times [0, \infty)$ with the product-σ-field $\bar{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}([0, \infty))$. We call $\mathcal{P}$ the predictable σ-field on $\bar{\Omega}$ and denote by $\mathcal{P}^{\bar{\pi}}$ its completion with respect to $\bar{\pi}$. We can and do also assume that A_∞ is integrable so that $\bar{\pi}$ is a finite measure, and we write $\bar{\omega} = (\omega, t)$. All this fits perfectly into the abstract setup from Section 2.

In the application, our process S had one single random jump at some random time, and this allowed us to parametrise our martingale measures by functions f on $\widetilde{\Omega}$. Moreover, the σ-martingale condition for S under a general $Q \approx P$ translated into a drift condition which (expressed in terms of the semimartingale characteristics of S) had exactly the form of the $\bar{\omega}$-dependent constraints (2.5). Theorem 2.1 then (essentially) gave us that the existence of some EσMM Q_0 even implies the existence of an EσMM Q_0^* whose density $\frac{dQ_0^*}{dP}$ is bounded. We say here "essentially" because the precise formulation and result still involve quite a number of additional technicalities; see Section 7 of Choulli/Schweizer (2015) for more details.

Remark 5.1. Experts in arbitrage theory will no doubt have noticed the similarity between Theorem 2.1 and the main result (Theorem 2.4 there) behind the version of the fundamental theorem of asset pricing due to Dalang/Morton/Willinger (1990). Another very similar result can be found in Delbaen/Schachermayer (1998); see Lemma 3.5 and the subsequent remarks on pages 226 and 228. Hence the experts may wonder if we could not use one of these results directly to obtain Theorem 2.1. However, this is not possible, and the reason is a combination of two circumstances. One is that in contrast to the above results, the random measure κ is a transition kernel, but not a transition probability — $\kappa(\bar{\omega}, \cdot\,)$ is for each $\bar{\omega}$ a finite measure, but we have on its mass $\kappa(\bar{\omega}, I\!R^d)$ no control which is uniform in $\bar{\omega}$. (We remark that we genuinely need this generality for our application.) The second is that b in (2.5) is not 0 in general, which reflects the fact that a single-jump process in continuous time can – in contrast to discrete time – have a nonvanishing drift prior to the jump. Any attempt to reduce our problem to the classic results via a normalisation will therefore first destroy the boundedness of b, and in the final un-normalisation also destroy in general any boundedness property of the auxiliary integrand $\widetilde{f}_0^*$, say, that one might have obtained from the classic results. These issues are also discussed in Choulli/Schweizer (2015) in Sections 7 and 9.1. So Theorem 2.1 is really a new result. ◇

Acknowledgements. Tahir Choulli gratefully acknowledges financial support by NSERC (the Natural Sciences and Engineering Research Council of Canada, Grant G121210818). Martin Schweizer gratefully acknowledges financial support by the Swiss Finance Institute (SFI).

References

J. M. Borwein and A. S. Lewis (1991), "Duality relationships for entropy-like minimization problems", *SIAM Journal on Control and Optimization 29*, 325–338

T. Choulli and M. Schweizer (2015), "Locally Φ-integrable σ-martingale densities for general semimartingales", *to appear in Stochastics,*
(future doi: `http://dx.doi.org/10.1080/17442508.2015.1052809`*)*

R. C. Dalang, A. Morton and W. Willinger (1990), "Equivalent martingale measures and no-arbitrage in stochastic securities market models", *Stochastics 29*, 185–201

F. Delbaen and W. Schachermayer (1998), "The fundamental theorem of asset pricing for unbounded stochastic processes", *Mathematische Annalen 312*, 215–250

G. Köthe (1965), "General linear transformations of locally convex spaces", *Mathematische Annalen 159*, 309–328

C. Léonard (2008), "Minimization of entropy functionals", *Journal of Mathematical Analysis and Applications 346*, 183–204

W. Rudin (1991), "Functional Analysis", *second edition, McGraw-Hill*