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DYNAMIC MEAN-VARIANCE OPTIMISATION PROBLEMS WITH DETERMINISTIC INFORMATION

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We solve the problems of mean-variance hedging (MVH) and mean-variance portfolio selection (MVPS) under restricted information. We work in a setting where the underlying price process S is a semimartingale, but not adapted to the filtration $\mathbb{G}$ which models the information available for constructing trading strategies. We choose as $\mathbb{G} = \mathbb{F}^{\mathrm{det}}$ the zero-information filtration and assume that S is a time-dependent affine transformation of a square-integrable martingale. This class of processes includes in particular arithmetic and exponential Lévy models with suitable integrability. We give explicit solutions to the MVH and MVPS problems in this setting, and we show for the Lévy case how they can be expressed in terms of the Lévy triplet. Explicit formulas are obtained for hedging European call options in the Bachelier and Black–Scholes models.

Keywords: mean-variance hedging; mean-variance portfolio selection; restricted information; partial information; deterministic strategies; quadratic optimisation problems; financial markets; type (A) semimartingales.

1. Introduction

This paper is a case study on solving dynamic quadratic optimisation problems in financial markets under restricted information. We start on $[0, T]$ with a discounted price process S adapted to a filtration $\mathbb{F}$. For an initial wealth c and a strategy ϑ

from a set Θ, the final wealth from self-financing trading according to (c, ϑ) is then

$$c + \int_0^T \vartheta_t \, \mathrm{d}S_t = c + \vartheta \bullet S_T = c + G_T(\vartheta).$$

We can then study, for a time-T payoff H, the *mean-variance hedging (MVH)* problem,

$$\text{minimise } E\big[\big(H - c - G_T(\vartheta)\big)^2\big] \text{ over } (c, \vartheta) \in \mathbb{R} \times \Theta, \tag{1.1}$$

and we can also consider the *mean-variance portfolio selection (MVPS)* problem,

$$\text{maximise } E[G_T(\vartheta)] - \alpha \mathrm{Var}[G_T(\vartheta)] \text{ over } \vartheta \in \Theta, \tag{1.2}$$

for a fixed risk-aversion parameter $\alpha > 0$. Both S and ϑ should satisfy integrability conditions to ensure that $G_T(\Theta) = \{G_T(\vartheta) : \vartheta \in \Theta\}$ is a subset of L^2. In addition, ϑ should be predictable, to avoid obvious issues with insiders or prophets and to ensure that the stochastic integral $\vartheta \bullet S = \int \vartheta \, \mathrm{d}S$ is well defined. (This also motivates why S is assumed to be a semimartingale.) Usually, there is only one filtration $\mathbb{F}$, and S is a semimartingale in $\mathbb{F}$ while strategies are chosen $\mathbb{F}$-predictable. Then there is a vast literature on (1.1) and (1.2); see for instance Schweizer (2010) for a first impression of the scope and extent of it.

If we think of $\mathbb{F}$ as describing all the information in the market, $\mathbb{F}$-predictability of ϑ means that investors can and do use all available information to construct their trading strategies. But in many situations, one naturally uses only a smaller information set; this can be due to delays, cost aspects, practicality, or even personal choice. It therefore makes sense to study (1.1) and (1.2), or more generally questions from mathematical finance, in a setting where $\vartheta \in \Theta$ is only allowed to be $\mathbb{G}$-predictable for a subfiltration $\mathbb{G} \subseteq \mathbb{F}$.

When we study the problem (1.1) for $\mathbb{G}$-predictable ϑ, the connection between $\mathbb{G}$ and S plays a crucial role. If $\mathbb{F}^S \subseteq \mathbb{G}$ which means that S is $\mathbb{G}$-adapted, then $c + G_T(\vartheta)$ is $\mathcal{G}_T$-measurable and setting $\widetilde{H} := E[H \,|\, \mathcal{G}_T]$, we can write the objective in (1.1) as

$$E\big[\big(H - c - G_T(\vartheta)\big)^2\big] = \|H - \widetilde{H}\|_{L^2}^2 + \|\widetilde{H} - c - G_T(\vartheta)\|_{L^2}^2.$$

So we only need to minimise the second summand over (c, ϑ), and this is the classic MVH problem in the filtration $\mathbb{G}$ for the $\mathcal{G}_T$-measurable payoff $\widetilde{H}$. For different models and with different techniques, this has been studied by Pham (2001), Kohlmann et al. (2007), Makogin et al. (2017), among others. An analogous reduction for (1.2) when $\mathbb{F}^S = \mathbb{G}$ is for instance given in Xiong & Zhou (2007), and related work for the different criterion of local risk-minimisation, but still with $\mathbb{F}^S \subseteq \mathbb{G}$, can be found in Ceci et al. (2014b, 2017).

Once we abandon the assumption $\mathbb{F}^S \subseteq \mathbb{G}$ so that S is not $\mathbb{G}$-adapted in general, the literature becomes much more sparse. Nevertheless, this situation occurs very naturally, for instance if we have delayed or time-discrete information. Probably the first paper in this direction is due to Di Masi et al. (1995) who studied (1.1) in a specific model where S is in addition a martingale. More precisely, they were actually looking for a risk-minimising strategy, in the sense of Föllmer & Sondermann (1986), with $\mathbb{G}$-predictable strategies; but the resulting optimal integrand is in the martingale case the same as for (1.1). The case where S is a general locally square-integrable local martingale was subsequently solved by Schweizer (1994), and alternative presentations with extra applications appeared in Ceci et al. (2014c,a), again in the martingale case. The only work on (1.1) for an $\mathbb{F}$-semimartingale S not adapted to $\mathbb{G}$ seems due to Mania et al. (2008, 2009). They were able to obtain results on (1.1) via the martingale optimality principle and general BSDEs; but their assumptions are rather restrictive and for instance already exclude the classic Black–Scholes model of geometric Brownian motion. For (1.2) with S not $\mathbb{G}$-adapted, the PhD thesis of Šikić (2015) studies the special case where $\mathbb{G}$ models delayed information and S evolves as an additive or multiplicative random walk in discrete time. Finally, Christiansen & Steffensen (2013) consider (1.2) with geometric Brownian motion for S and with deterministic information and strategies parametrised by proportions of wealth. They give a verification theorem for the corresponding HJB equation, but do not prove the existence of a solution.

In this paper, we give explicit solutions to (1.1) and (1.2) under two assumptions:

(1.3) $\mathbb{G} = \mathbb{F}^{\mathrm{det}}$ is the zero-information filtration, meaning that all strategies must be deterministic functions.

This can be viewed as a worst case scenario because $\mathbb{F}^{\mathrm{det}}$ is the smallest possible filtration we can think of. Accordingly, the solutions to (1.1) and (1.2) for $\mathbb{F}^{\mathrm{det}}$ yield upper respectively lower bounds on the hedging error respectively mean-variance performance achievable with strategies from any filtration $\mathbb{G}$. Note in particular that S is not adapted to $\mathbb{F}^{\mathrm{det}}$ as soon as it contains some randomness; so then $\mathbb{F}^S \not\subseteq \mathbb{F}^{\mathrm{det}}$. The corresponding space $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ of strategies is defined later in Section 2.2.

(1.4) S is a time-dependent affine function of a square-integrable martingale, meaning that $S_t = S_0 + f(t) + g(t)Y_t$, $t \in [0,T]$, for functions f, g with $f(0) = 0$, $g(0) = 1$ and $Y \in \mathcal{M}_0^2$. We call S a type (A) semimartingale.

It turns out that the interplay between $\mathbb{F}^{\mathrm{det}}$ and S of type (A) is just right for allowing us to study (1.1) and (1.2) for $\mathbb{F}^{\mathrm{det}}$. Interestingly, (1.4) also follows almost from (1.3) if we add one of the key conditions in Mania et al. (2008, 2009), namely that S should have the form $S = S_0 + M + \int \lambda \,\mathrm{d}\langle M\rangle$ with $\langle M\rangle$ and λ both adapted to $\mathbb{G} = \mathbb{F}^{\mathrm{det}}$. However, our techniques are quite different from those in Mania et al. (2008, 2009) and strongly exploit the type (A) structure of S. Under (1.3) and (1.4), we obtain the solution of (1.1) for $\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ as an explicit transformation of the integrand Π^H in the Galtchouk–Kunita–Watanabe decomposition of H with respect to the martingale part M of S. The solution of (1.2) for $\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ is given explicitly in terms of quantities one can compute from S in $\mathbb{G} = \mathbb{F}^{\mathrm{det}}$.

The rest of the paper is structured as follows. After we fix some notation in the next subsection, Section 2 studies type (A) semimartingales, introduces the relevant space $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ of strategies and shows in Theorem 2.11 the key result that any stochastic integral $\delta \bullet S_T$ with $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ can be written as the sum of a constant and a stochastic integral $\vartheta \bullet M_T$ with respect to M, where the constant and the integrand $\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ are explicitly given in terms of δ. Moreover, the corresponding linear operator $\delta \mapsto \mathcal{A}[\delta] = \vartheta$ is a continuous and open bijection from $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ to itself. Section 3 first gives sufficient conditions on S for the linear subspace $G_T(\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})) \subseteq L^2$ to be closed in L^2, which guarantees the existence of solutions to (1.1) and (1.2) for $\Theta = \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$. Combining this with the results on the operator $\mathcal{A}$ yields the solutions to (1.1) and (1.2) in explicit form. Finally, Section 4 shows that under suitable integrability, both arithmetic and exponential Lévy models are type (A) semimartingales, works out the explicit solutions from Section 3 in terms of the Lévy triplet, and illustrates the hedging results for the case of a European call option in the Bachelier and Black–Scholes models.

1.1. *Notation*

We work with a time horizon $T \in (0, \infty)$ and on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and completeness. We also assume that $\mathcal{F}_0$ is trivial and for simplicity that $\mathcal{F} = \mathcal{F}_T$. Stochastic processes $X = (X_t)_{t\in[0,T]}$ are denoted by Greek or by capital letters, and their time indices are written as subscripts. In contrast, functions $a : [0, T] \to \mathbb{R}$ are denoted by small letters, with their time arguments in brackets, like $t \mapsto a(t)$. We can, and often do, identify a function a on $[0, T]$ with a process A via $A_t(\omega) := a(t)$ for $(\omega, t) \in \Omega \times [0, T]$. Purely formally, however, functions and processes are different objects because their domains of definition are not the same. Finally, we denote by $X^*_t := \sup_{0\le s\le t} |X_s|$, $t \in [0, T]$, the supremum process of X.

For a finite variation (FV) function a on $[0, T]$, we denote by $|\mathrm{d}a|$ the variation measure of the signed Lebesgue–Stieltjes (LS) measure associated to a, and by $L^p(\mathrm{d}a) := L^p(|\mathrm{d}a|)$ for $p \in [1, \infty)$ the Banach space of $|\mathrm{d}a|$-equivalence classes of Borel-measurable functions h on $[0, T]$ with $\int_0^T |h(t)|^p |\mathrm{d}a(t)| < \infty$. For an FV process A, we write $\mathrm{d}A$ and $|\mathrm{d}A|$ for the ω-wise LS measures on $[0, T]$ of A and of the variation of A, respectively. All integrals $\int_a^b$ are over $(a, b]$.

All our semimartingales X are with respect to P and $\mathbb{F}$, real-valued and have RCLL trajectories $t \mapsto X_t(\omega)$ for P-a.a. ω. In particular, we view FV functions as nonrandom semimartingales and choose them to be RCLL. We write $[S, X]$ for the quadratic covariation of two semimartingales S, X, and $\langle M, N\rangle$ for the predictable quadratic covariation of two locally square-integrable local martingales M, N. We set $[X] := [X, X]$ and $\langle M\rangle := \langle M, M\rangle$. If S is a special semimartingale, we write $S = S_0 + M + A$ for its canonical decomposition into $S_0 \in \mathbb{R}$, local martingale part M and predictable FV part A, both latter null at zero. We denote by $\mathcal{M}^2_0$ the set of all square-integrable martingales null at zero. A semimartingale S is in $\mathcal{S}^2$ if it is special with $\|M^*_T\|_{L^2} + \| \int_0^T |\mathrm{d}A_t| \|_{L^2} < \infty$, and $\mathcal{S}^2_0 := \{S \in \mathcal{S}^2 : S_0 = 0\}$. In particular, $\mathcal{M}^2_0 \subseteq \mathcal{S}^2_0$. Finally, $\bullet$ denotes stochastic integration; so $\vartheta \bullet S = \int \vartheta \,\mathrm{d}S$.

2. Type (A) Semimartingales and Deterministic Integrands

In this section, we introduce a particular class of semimartingales and study their integrals of deterministic functions.

2.1. *Basics*

Definition 2.1. Let $f, g : [0, T] \to \mathbb{R}$ be FV (and RCLL) functions with $f(0) = 0$ and $g(0) = 1$. Take $Y \in \mathcal{M}_0^2$ and $S_0 \in \mathbb{R}$. We call a stochastic process $S = (S_t)_{t\in[0,T]}$ of the form

$$S_t = S_0 + f(t) + g(t)Y_t, \quad t \in [0, T], \tag{2.1}$$

a *semimartingale of type (A)* or *type (A) semimartingale*. We sometimes write (2.1) as $S = S_0 + f + gY$, and we use the shorthand notation $S \mathrel{\widehat{=}} (S_0, f, g, Y)$.

Remark 2.2. 1) The capital letter A stands for "affine function of a martingale".

2) Section 4 shows that (suitably integrable) arithmetic and exponential Lévy processes are type (A) semimartingales.

Our first simple result shows that type (A) semimartingales are square-integrable and determines their canonical decomposition.

Lemma 2.3. *Let $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ be a type (A) semimartingale. Then:*

1) The product gY is in $\mathcal{S}_0^2$ with canonical decomposition

$$gY = g \bullet Y + Y_- \bullet g. \tag{2.2}$$

2) S is in $\mathcal{S}^2$, and its canonical decomposition $S = S_0 + M + A$ is given by

$$M = g \bullet Y, \tag{2.3}$$

$$A = f + Y_- \bullet g. \tag{2.4}$$

Proof. 1) The Borel function g can be identified with an $\mathbb{F}$-predictable process, and so we obtain (2.2) directly from Proposition I.4.49 b) in Jacod & Shiryaev (2003). Any FV function is (chosen) RCLL and hence uniformly in t bounded on compact intervals. Using $Y \in \mathcal{M}_0^2$ therefore gives

$$[g \bullet Y]_T \le [Y]_T \sup_{t\in[0,T]} |g(t)|^2 \in L^1,$$

$$\int_0^T |Y_{t-}||\mathrm{d}g(t)| \le \sup_{t\in[0,T]} |Y_t| \int_0^T |\mathrm{d}g(t)| \in L^2.$$

In view of (2.2), this shows that $gY \in \mathcal{S}_0^2$.

2) Because $S = S_0 + f + gY$ is the sum of $S_0 + gY \in \mathcal{S}^2$ and the FV function f, it is in $\mathcal{S}^2$. Moreover, part 1) gives $S = S_0 + f + gY = S_0 + g \bullet Y + f + Y_- \bullet g$ which yields (2.3) and (2.4) . □

Definition 2.4. The *deterministic filtration* $\mathbb{F}^{\mathrm{det}} = (\mathcal{F}_t^{\mathrm{det}})_{t\in[0,T]}$ is defined by $\mathcal{F}_t^{\mathrm{det}} := \sigma(\mathcal{N})$, $t \in [0, T]$, where $\mathcal{N}$ denotes the collection of P-nullsets in $\mathcal{F}_T$.

It is easy to verify that each $\mathcal{F}_t^{\mathrm{det}}$ is P-trivial so that any $\mathcal{F}_t^{\mathrm{det}}$-measurable random variable is P-a.s. nonrandom. By approximating nonnegative $\mathbb{F}^{\mathrm{det}}$-predictable processes pointwise by adapted left-continuous ones, and arguing for the latter via a monotone class argument and dominated convergence, one can also verify the (unsurprising) fact that *any* $\mathbb{F}^{\mathrm{det}}$*-predictable process on* $\Omega\times[0,T]$ *is indistinguishable from a Borel function on* $[0,T]$. To be more precise: We can identify any Borel function $h:[0,T]\to\mathbb{R}$ up to indistinguishability with the $\mathbb{F}^{\mathrm{det}}$-predictable process $\vartheta:\Omega\times[0,T]\to\mathbb{R}$ given by $\vartheta_t(\omega)=h(t)$ for all $\omega\in\Omega$ and $t\in[0,T]$. We omit the details and refer to Lemma 10.6 in Zivoi (2017).

The next result shows that for $N\in\mathcal{M}_0^2$, the $\mathbb{F}^{\mathrm{det}}$-compensator $\langle N\rangle^{\mathrm{p},\mathbb{F}^{\mathrm{det}}}$ of $\langle N\rangle$ can be identified with the Borel function $t\mapsto E[\langle N\rangle_t]$.

Lemma 2.5. *1) Fix* $Y\in\mathcal{M}_0^2$ *and define* $\mathfrak{y}^{\mathrm{det}}(t):=E[\langle Y\rangle_t]$ *for* $t\in[0,T]$*. For every nonnegative Borel function* δ *on* $[0,T]$*, we then have*

$$E\bigg[\int_0^T\delta(t)\,\mathrm{d}\langle Y\rangle_t\bigg]=\int_0^T\delta(t)\,\mathrm{d}\mathfrak{y}^{\mathrm{det}}(t).\tag{2.5}$$

2) For $S\mathrel{\widehat{=}}(S_0,f,g,Y)$ *with canonical decomposition* $S=S_0+M+A$*, the function* $\mathfrak{m}^{\mathrm{det}}(t)=E[\langle M\rangle_t]$*,* $t\in[0,T]$*, is given by*

$$\mathrm{d}\mathfrak{m}^{\mathrm{det}}(t)=g^2(t)\,\mathrm{d}\mathfrak{y}^{\mathrm{det}}(t),\tag{2.6}$$

and for any Borel function $\delta\in L^1(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$*, we have*

$$E\bigg[\int_0^T\delta(t)\,\mathrm{d}\langle M\rangle_t\bigg]=\int_0^T\delta(t)\,\mathrm{d}\mathfrak{m}^{\mathrm{det}}(t).\tag{2.7}$$

Proof. 1) Like $\langle Y\rangle$, $\mathfrak{y}^{\mathrm{det}}$ is increasing and null at zero, hence of FV and RCLL. Next, (2.5) holds by linearity for $\mathbb{R}$-linear combinations δ of indicators $\mathbf{1}_{(a,b]}$ with $0\le a<b\le T$, and it extends to nonnegative Borel functions by standard measure-theoretic induction and monotone integration.

2) Because $M=g\bullet Y$ by (2.3), we have $\langle M\rangle=g^2\bullet\langle Y\rangle$. As an FV function, g is Borel-measurable, and so both (2.6) and (2.7) follow from part 1). $\square$

Definition 2.6. For $M\in\mathcal{M}_0^2$, we set $P_M:=P\otimes\langle M\rangle$ and denote by $L^2(M)$ the Hilbert space of P_M-equivalence classes of $\mathbb{F}$-predictable processes $\Pi=(\Pi_t)_{t\in[0,T]}$ with

$$\|\Pi\|_{L^2(M)}:=(E_M[\Pi^2])^{1/2}=\bigg(E\bigg[\int_0^T\Pi_t^2\,\mathrm{d}\langle M\rangle_t\bigg]\bigg)^{1/2}<\infty.$$

The associated scalar product is denoted by $(\,\cdot\,,\,\cdot\,)_{L^2(M)}$. Similarly, $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ is the Hilbert space of $\mathrm{d}\mathfrak{m}^{\mathrm{det}}$-equivalence classes of Borel functions π on $[0,T]$ with

$$\|\pi\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}:=\bigg(\int_0^T|\pi(t)|^2\,\mathrm{d}\mathfrak{m}^{\mathrm{det}}(t)\bigg)^{1/2}<\infty.$$

The corresponding scalar product is denoted by $(\,\cdot\,,\,\cdot\,)_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}$.

Because $\mathbb{F}^{\mathrm{det}}$-predictable processes can be identified with Borel functions and due to (2.7), the space $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ can be identified with $L^2(M) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$, the space of equivalence classes of $\mathbb{F}^{\mathrm{det}}$-predictable $\pi \in L^2(M)$. With a slight abuse of notation, we sometimes write $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) = L^2(M) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$. Together with the usual Itô isometry in $L^2(M)$, we thus obtain for π and ψ in $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ the $\mathbb{F}^{\mathrm{det}}$-*Itô isometry*

$$\begin{aligned}(\pi \bullet M_T, \psi \bullet M_T)_{L^2} &= E\bigg[\int_0^T \pi(t)\psi(t)\,\mathrm{d}\langle M\rangle_t\bigg] \\ &= \int_0^T \pi(t)\psi(t)\,\mathrm{d}\mathfrak{m}^{\mathrm{det}}(t) = (\pi,\psi)_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}. \qquad (2.8)\end{aligned}$$

2.2. *The space* $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$

For $S \in \mathcal{S}^2$ with canonical decomposition $S = S_0 + M + A$, we denote by $\mathbf{\Theta}(S)$ the Banach space of equivalence classes of $\mathbb{F}$-predictable processes $\Pi = (\Pi_t)_{t\in[0,T]}$ with

$$\|\Pi\|_{\mathbf{\Theta}(S)} := \|(\Pi \bullet M)^*_T\|_{L^2} + \bigg\|\int_0^T |\Pi_t||\mathrm{d}A_t|\bigg\|_{L^2} < \infty.$$

This implies that $\Pi \bullet S \in \mathcal{S}^2_0$. We then use the notation $\mathbf{\Theta}(S) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$ for the $\mathbb{F}^{\mathrm{det}}$-predictable members of $\mathbf{\Theta}(S)$. While $L^2(M) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}}) = L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$, the semimartingale case needs a slightly different class of integrands than $\mathbf{\Theta}(S) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$.

Definition 2.7. For $S \,\widehat{=}\, (S_0, f, g, Y)$, set $\mathrm{d}\mathfrak{s}^{\mathrm{det}} := |\mathrm{d}f| + |\mathrm{d}g| + \mathrm{d}\mathfrak{m}^{\mathrm{det}}$ and define by

$$\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) := L^1(\mathrm{d}f) \cap L^1(\mathrm{d}g) \cap L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$$

the Banach space of $\mathrm{d}\mathfrak{s}^{\mathrm{det}}$-equivalence classes of Borel functions ϑ on $[0,T]$ such that

$$\|\vartheta\|_{\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})} := \|\vartheta\|_{L^1(\mathrm{d}f)} + \|\vartheta\|_{L^1(\mathrm{d}g)} + \|\vartheta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} < \infty.$$

Our next result compares the norms $\|\cdot\|_{\mathbf{\Theta}(S)}$ and $\|\cdot\|_{\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})}$ for Borel functions and shows in particular that with the usual identification of functions as nonrandom processes, we can write $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) \subseteq \mathbf{\Theta}(S) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$.

Remark 2.8. To be precise, both $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ and $\mathbf{\Theta}(S) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$ are spaces not of stochastic processes ϑ, but of equivalence classes $[\vartheta]$. The above inclusion statement then means that for any equivalence class $[\vartheta] \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, there is an equivalence class $[\vartheta'] \in \mathbf{\Theta}(S) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$ such that $[\vartheta] \subseteq [\vartheta']$. An analogous comment applies in the sequel to all statements of the form $L^p(\mu) \subseteq L^q(\nu)$.

Lemma 2.9. *Fix* $S \,\widehat{=}\, (S_0, f, g, Y)$. *There exists a constant* $K \in (0,\infty)$ *such that*

$$\|(\vartheta \bullet S)^*_T\|_{L^2} \le \|\vartheta\|_{\mathbf{\Theta}(S)} \le K\|\vartheta\|_{\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})}, \quad \forall \vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}). \qquad (2.9)$$

(For Borel functions $\vartheta \notin \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$*, the right inequality holds trivially.)*

Proof. The left inequality is immediate from the definition of the norm $\|\cdot\|_{\mathbf{\Theta}(S)}$. For the right one, we set $|||\vartheta|||_{\mathbf{\Theta}(S)} := \| \int_0^T |\vartheta(s)||\mathrm{d}A_s|\|_{L^2} + \|\vartheta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}$ and first note that for any $\vartheta \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) = L^2(M) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}})$, the BDG inequality and (2.8) yield the estimate $\|(\vartheta \bullet M)^*_T\|_{L^2} \leq K_1 \|\vartheta\|_{L^2(M)} = K_1\|\vartheta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}$. We therefore obtain

$$\|\vartheta\|_{\mathbf{\Theta}(S)} \leq \left\| \int_0^T |\vartheta(s)||\mathrm{d}A_s| \right\|_{L^2} + K_1\|\vartheta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} \leq \max(1, K_1)|||\vartheta|||_{\mathbf{\Theta}(S)}.$$

On the other hand, using from Lemma 2.3 that $S = S_0 + M + A$ with $M = g \bullet Y$ and $A = f + Y_- \bullet g$ gives for $\vartheta^n := \vartheta \mathbf{1}_{\{|\vartheta| \leq n\}}$ that

$$\begin{aligned} \left\| \int_0^T |\vartheta^n(t)||\mathrm{d}A_t| \right\|_{L^2} &\leq \int_0^T |\vartheta^n(t)||\mathrm{d}f(t)| + \|Y^*_T\|_{L^2} \int_0^T |\vartheta^n(t)||\mathrm{d}g(t)| \\ &\leq K_2\big(\|\vartheta^n\|_{L^1(\mathrm{d}f)} + \|\vartheta^n\|_{L^1(\mathrm{d}g)}\big) \end{aligned}$$

with $K_2 = \max(1, \|Y^*_T\|_{L^2})$. This implies $|||\vartheta^n|||_{\mathbf{\Theta}(S)} \leq \max(K_2, 1)\|\vartheta^n\|_{\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})}$, and letting $n \to \infty$ yields $|||\vartheta|||_{\mathbf{\Theta}(S)} \leq \max(K_2, 1)\|\vartheta\|_{\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})}$, by monotone integration on the LHS and due to $\vartheta^n \overset{n\to\infty}{\longrightarrow} \vartheta$ in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ on the RHS. Putting everything together gives (2.9). $\square$

2.3. *The key results*

This section contains the heart of all our subsequent results, which are all based on the integration by parts formula: For two RCLL FV functions $F, G : [0, T] \to \mathbb{R}$,

$$F(T)G(T) - F(t)G(t) = \int_t^T F(u)\,\mathrm{d}G(u) + \int_t^T G(u-)\,\mathrm{d}F(u), \quad t \in [0, T]. \tag{2.10}$$

Proposition 2.10. *Fix $S \mathrel{\widehat{=}} (S_0, f, g, Y)$. For any $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, we have*

$$\int_0^T \delta(t)\,\mathrm{d}S_t = \int_0^T \delta(t)\,\mathrm{d}f(t) + \int_0^T \left(g(t)\delta(t) + \int_t^T \delta(u)\,\mathrm{d}g(u)\right)\mathrm{d}Y_t \quad P\text{-a.s.} \tag{2.11}$$

Proof. Fix $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) = L^1(\mathrm{d}f) \cap L^1(\mathrm{d}g) \cap L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$. By Lemma 2.9, the LHS in (2.11) is well defined. Because δ belongs to $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) = L^1(\mathrm{d}f) \cap L^1(\mathrm{d}g) \cap L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$, the function $t \mapsto \int_t^T \delta(u)\,\mathrm{d}g(u) = \int_{(t,T]} \delta(u)\,\mathrm{d}g(u)$ is of FV and RCLL, hence bounded and Y-integrable. Finally, by the associativity of stochastic integrals and the formula $M = g \bullet Y$ from Lemma 2.3, $g\delta$ is Y-integrable if and only if δ is M-integrable. So the RHS in (2.11) is also well defined.

Because Lemma 2.3 gives $\mathrm{d}S = \mathrm{d}f + Y_-\,\mathrm{d}g + g\,\mathrm{d}Y$, we now obtain

$$\int_0^T \delta(t)\,\mathrm{d}S_t = \int_0^T \delta(t)\,\mathrm{d}f(t) + \int_0^T Y_{t-}\delta(t)\,\mathrm{d}g(t) + \int_0^T g(t)\delta(t)\,\mathrm{d}Y_t \quad P\text{-a.s.} \tag{2.12}$$

Again Lemma 2.3 gives for any G of FV that $\mathrm{d}(GY) = G\,\mathrm{d}Y + Y_-\,\mathrm{d}G$, and so we obtain $G(T)Y_T = \int_0^T G(t)\,\mathrm{d}Y_t + \int_0^T Y_{t-}\,\mathrm{d}G(t)$ because $Y_0 = 0$. Choosing $G = \int \delta\,\mathrm{d}g$

yields

$$\int_0^T Y_{t-}\delta(t)\,\mathrm{d}g(t) = \int_0^T \big(G(T) - G(t)\big)\,\mathrm{d}Y_t = \int_0^T \left(\int_t^T \delta(u)\,\mathrm{d}g(u)\right)\mathrm{d}Y_t,$$

and plugging this back into (2.12) directly gives (2.11). □

The crucial result in Proposition 2.10 is that any stochastic integral $\delta \bullet S_T$ of S with a deterministic integrand δ can be written as the sum of a constant and a stochastic integral $\psi \bullet Y_T$ of Y with another deterministic integrand ψ. Moreover, the constant $\int_0^T \delta(t)\,\mathrm{d}f(t)$ and the integrand $\psi(t) = g(t)\delta(t) + \int_t^T \delta(u)\,\mathrm{d}g(u)$ are even given explicitly. However, analysing the properties of ψ as a function of δ turns out to be rather difficult, and for the question whether the space of all (final values of) stochastic integrals $\int_0^T \delta(t)\,\mathrm{d}S_t$ is closed in L^2, it is much better to work with the martingale part M of S instead of with Y. Because $M = g \bullet Y$ by Lemma 2.3, we can pass from the Y-integrand ψ to an M-integrand simply by dividing by g, provided that $g \neq 0$. Doing that transformation automatically brings up the linear operator $\mathcal{A}$ appearing in the next result.

Theorem 2.11. *Fix $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ and assume that g satisfies*

$$\inf_{t\in[0,T]} |g(t)| > 0. \tag{2.13}$$

For any $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, we define on $[0,T]$ the Borel functions

$$t \mapsto \mathcal{A}[\delta](t) := \delta(t) + \frac{1}{g(t)} \int_t^T \delta(u)\,\mathrm{d}g(u), \tag{2.14}$$

$$t \mapsto \mathcal{A}^{\leftarrow}[\delta](t) := \delta(t) - \int_t^T \frac{\delta(u)}{g(u-)}\,\mathrm{d}g(u). \tag{2.15}$$

(Both integrals are over $(t,T]$.) Then the following statements hold true:

1) $\mathcal{A}, \mathcal{A}^{\leftarrow} : \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) \to \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ are well defined.

2) $\mathcal{A} \circ \mathcal{A}^{\leftarrow} = \mathrm{Id}$, i.e., $\mathcal{A}^{\leftarrow}$ is a right inverse of $\mathcal{A}$ on $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$.

3) $\mathcal{A}^{\leftarrow} \circ \mathcal{A} = \mathrm{Id}$, i.e., $\mathcal{A}^{\leftarrow}$ is also a left inverse of $\mathcal{A}$ on $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$. Together with 2), this means that $\mathcal{A}^{\leftarrow}$ is the (unique) inverse $\mathcal{A}^{-1}$ of $\mathcal{A}$.

4) For any $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, we have

$$\int_0^T \mathcal{A}^{-1}[\delta](t)\,\mathrm{d}f(t) = \int_0^T \delta(t)\,\mathrm{d}\mathfrak{a}(t), \tag{2.16}$$

where the FV (and RCLL) function $\mathfrak{a} : [0,T] \to \mathbb{R}$ is given by

$$\mathrm{d}\mathfrak{a}(t) := \mathrm{d}f(t) - \frac{f(t-)}{g(t-)}\,\mathrm{d}g(t). \tag{2.17}$$

5) For any $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, we have

$$\int_0^T \delta(t)\,\mathrm{d}S_t = \int_0^T \delta(t)\,\mathrm{d}f(t) + \int_0^T \mathcal{A}[\delta](t)\,\mathrm{d}M_t \quad P\text{-a.s.} \tag{2.18}$$

Proof. Fix $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) = L^1(\mathrm{d}f) \cap L^1(\mathrm{d}g) \cap L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$.

1) From (2.13), we get $\sup_{t\in[0,T]} |1/g(t)| < \infty$, and $\mathcal{A}[\delta] = \delta + (1/g)\int^T \delta\,\mathrm{d}g$ is the sum of $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ and $(1/g)\int^T \delta\,\mathrm{d}g$. In the latter product, the first factor $1/g$ is uniformly bounded, and because δ is in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, the second factor $\int^T \delta\,\mathrm{d}g$ is of FV and RCLL and hence bounded on $[0,T]$. But all bounded Borel functions belong to $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, and so we get $(1/g)\int^T \delta\,\mathrm{d}g \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, and hence $\mathcal{A}[\delta] \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, whenever $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$. An analogous argument shows that $\mathcal{A}^{\leftarrow}[\delta] \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ whenever $\delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$; this also uses (2.13), to deduce that δ/g_- is in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ like δ.

2) Inserting $\mathcal{A}[\delta] = \delta + (1/g)\int^T \delta\,\mathrm{d}g$ and $\mathcal{A}^{\leftarrow}[\delta] = \delta - \int^T \delta/g_-\,\mathrm{d}g$ yields

$$
\begin{aligned}
&(\mathcal{A}\circ\mathcal{A}^{\leftarrow})[\delta](t)\\
&= \mathcal{A}^{\leftarrow}[\delta](t) + \frac{1}{g(t)}\int_t^T \mathcal{A}^{\leftarrow}[\delta](u)\,\mathrm{d}g(u)\\
&= \delta(t) - \int_t^T \frac{\delta(u)}{g(u-)}\,\mathrm{d}g(u) + \frac{1}{g(t)}\int_t^T \bigg(\delta(u) - \int_u^T \frac{\delta(z)}{g(z-)}\,\mathrm{d}g(z)\bigg)\mathrm{d}g(u). \qquad (2.19)
\end{aligned}
$$

Applying the integration by parts formula (2.10) to $F(t) = \int_t^T \delta(u)/g(u-)\,\mathrm{d}g(u)$ and $G = g$ yields, after noting that $F(T) = 0$,

$$
\begin{aligned}
-g(t)\int_t^T \frac{\delta(u)}{g(u-)}\,\mathrm{d}g(u) &= F(T)G(T) - F(t)G(t)\\
&= \int_t^T \bigg(\int_u^T \frac{\delta(z)}{g(z-)}\,\mathrm{d}g(z)\bigg)\mathrm{d}g(u) - \int_t^T \delta(u)\,\mathrm{d}g(u).
\end{aligned}
$$

Dividing by $g(t)$ and plugging the result back into (2.19) yields $(\mathcal{A}\circ\mathcal{A}^{\leftarrow})[\delta] = \delta$.

3) Inserting $\mathcal{A}[\delta] = \delta + (1/g)\int^T \delta\,\mathrm{d}g$ and $\mathcal{A}^{\leftarrow}[\delta] = \delta - \int^T \delta(u)/g(u-)\,\mathrm{d}g(u)$ yields

$$
\begin{aligned}
(\mathcal{A}^{\leftarrow}\circ\mathcal{A})[\delta](t) &= \mathcal{A}[\delta](t) - \int_t^T \frac{\mathcal{A}[\delta](u)}{g(u-)}\,\mathrm{d}g(u)\\
&= \delta(t) + \frac{1}{g(t)}\int_t^T \delta(u)\,\mathrm{d}g(u)\\
&\quad - \int_t^T \frac{1}{g(u-)}\bigg(\delta(u) + \frac{1}{g(u)}\int_u^T \delta(z)\,\mathrm{d}g(z)\bigg)\mathrm{d}g(u). \qquad (2.20)
\end{aligned}
$$

Applying the integration by parts formula (2.10) to $F(t) = \int_t^T \delta(u)\,\mathrm{d}g(u)$ and the FV function $G = 1/g$ shows, with $F(T) = 0$,

$$
\begin{aligned}
-\frac{1}{g(t)}\int_t^T \delta(u)\,\mathrm{d}g(u) &= F(T)G(T) - F(t)G(t)\\
&= \int_t^T \bigg(\int_u^T \delta(z)\,\mathrm{d}g(z)\bigg)\mathrm{d}\bigg(\frac{1}{g(u)}\bigg) - \int_t^T \frac{\delta(u)}{g(u-)}\,\mathrm{d}g(u).
\end{aligned}
$$

Inserting this back into (2.20) yields

$$(\mathcal{A}^{\leftarrow} \circ \mathcal{A})[\delta](t) = \delta(t) - \int_t^T \left(\int_u^T \delta(z) \, \mathrm{d}g(z) \right) \left(\mathrm{d}\Big(\frac{1}{g(u)}\Big) + \frac{1}{g(u)g(u-)} \, \mathrm{d}g(u) \right).$$

But now a careful application of the chain rule, including the jumps of g, shows that $d(1/g) = -1/(gg_-) \, \mathrm{d}g$. So the last term vanishes and we obtain 3).

4) Choose $G = f$ and $F(t) = \int_t^T \delta(u)/g(u-) \, \mathrm{d}g(u)$, apply the integration by parts formula (2.10) for $t = 0$ and use $F(T) = 0$, $G(0) = f(0) = 0$ to obtain

$$0 = \int_0^T \left(\int_t^T \frac{\delta(u)}{g(u-)} \, \mathrm{d}g(u) \right) \mathrm{d}f(t) - \int_0^T f(t-) \frac{\delta(t)}{g(t-)} \, \mathrm{d}g(t).$$

This gives in view of 3) that

$$\begin{aligned} \int_0^T \mathcal{A}^{\leftarrow}[\delta](t) \, \mathrm{d}f(t) &= \int_0^T \left(\delta(t) - \int_t^T \frac{\delta(u)}{g(u-)} \, \mathrm{d}g(u) \right) \mathrm{d}f(t) \\ &= \int_0^T \delta(t) \, \mathrm{d}f(t) - \int_0^T \delta(t) \frac{f(t-)}{g(t-)} \, \mathrm{d}g(t) \\ &= \int_0^T \delta(t) \, \mathrm{d}\mathfrak{a}(t), \end{aligned}$$

by the definition of $\mathfrak{a}$.

5) Because $\mathrm{d}M_t = g(t) \, \mathrm{d}Y_t$ by Lemma 2.3, (2.18) follows directly from (2.11) and the definition (2.14) of $\mathcal{A}[\delta]$. □

Remark 2.12. 1) Using the product rule and again $d(1/g) = -1/(gg_-) \, \mathrm{d}g$, we can rewrite $\mathrm{d}\mathfrak{a}$ from (2.17) as

$$\mathrm{d}\mathfrak{a}(t) = g(t) \, \mathrm{d}\left(\frac{f}{g} \right)(t). \tag{2.21}$$

2) Condition (2.13) clearly implies that the filtrations $\mathbb{F}^Y$ and $\mathbb{F}^M$ generated by Y and M, respectively, coincide. However, we do not know if the condition $\mathbb{F}^Y = \mathbb{F}^M$ alone is sufficient to let us obtain our results.

Theorem 2.11 shows that under the small extra condition (2.13) on g, the transformation from the S-integrand δ to the M-integrand $\mathcal{A}[\delta]$ in the representation (2.18) is given by an invertible linear operator on the space $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, and provides an explicit formula for the operator. This is very useful in the subsequent analysis. In the sequel, whenever we assume (2.13), we drop the notation $\mathcal{A}^{\leftarrow}$ and simply write $\mathcal{A}^{-1}$.

3. Quadratic Problems with Deterministic Integrands

This section has three parts. We always work with a type (A) semimartingale S and first provide sufficient conditions on S for the space

$$G_T\big(\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})\big) := \{\vartheta \bullet S_T : \vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})\}$$

of stochastic integrals to be closed in L^2. Combining these results with the representation from Theorem 2.11, we can then solve a quadratic hedging problem for general payoffs and a mean-variance portfolio selection problem, both for zero-information (deterministic) strategies.

3.1. *Closedness and weighted norm inequalities*

We begin with an auxiliary result which does not need any extra condition on g.

Lemma 3.1. *For $S \mathrel{\widehat{=}} (S_0, f, g, Y)$, the following are equivalent:*

a) $\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) = L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$.

b) There exists a constant $K \in (0, \infty)$ such that

$$\|\delta\|_{L^1(\mathrm{d}f)} + \|\delta\|_{L^1(\mathrm{d}g)} \leq K \|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}, \quad \forall \delta \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}). \tag{3.1}$$

c) $|\mathrm{d}f| + |\mathrm{d}g| \ll \mathrm{d}\mathfrak{m}^{\mathrm{det}}$ *with* $\gamma := (|\mathrm{d}f| + |\mathrm{d}g|)/\mathrm{d}\mathfrak{m}^{\mathrm{det}} \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$.

d) $\mathrm{d}\mathfrak{s}^{\mathrm{det}} \ll \mathrm{d}\mathfrak{m}^{\mathrm{det}}$ *with* $\mathrm{d}\mathfrak{s}^{\mathrm{det}}/\mathrm{d}\mathfrak{m}^{\mathrm{det}} \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$.

Proof. b) $\Rightarrow$ a): The definition of $\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) = L^1(\mathrm{d}f) \cap L^1(\mathrm{d}g) \cap L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ directly gives the inclusion "$\subseteq$", and "$\supseteq$" follows from (3.1). See also Remark 2.8.

c) $\Rightarrow$ b): The Cauchy–Schwarz inequality gives for $\delta \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ that

$$\|\delta\|_{L^1(\mathrm{d}f)} + \|\delta\|_{L^1(\mathrm{d}g)} = \int_0^T |\delta(t)| \gamma(t) \, \mathrm{d}\mathfrak{m}^{\mathrm{det}}(t) \leq \|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} \|\gamma\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}.$$

This is (3.1) with $K = \|\gamma\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}$.

a) $\Rightarrow$ c): It is well known that for any finite measures μ, ν and any $p, q \in [1, \infty)$, the inclusion $L^p(\nu) \subseteq L^q(\mu)$ implies $\nu \ll \mu$. So with the definition of $\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, a) yields $|\mathrm{d}f| \ll \mathrm{d}\mathfrak{m}^{\mathrm{det}}$ and $|\mathrm{d}g| \ll \mathrm{d}\mathfrak{m}^{\mathrm{det}}$ so that γ is well defined and in $L^1(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$. If $\gamma \notin L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$, then also $\gamma + 1 = \frac{\mathrm{d}\mathfrak{s}^{\mathrm{det}}}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}} \notin L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$, and by Cauchy–Schwarz, there must then exist some $\beta \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ with $(\gamma+1)\beta \notin L^1(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$. But now we can use the definitions of $\gamma + 1$, $\mathrm{d}\mathfrak{s}^{\mathrm{det}}$ and $\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ together with Cauchy–Schwarz to compute

$$\begin{aligned}
\|(\gamma+1)\beta\|_{L^1(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} &= \|\beta\|_{L^1(\mathrm{d}\mathfrak{s}^{\mathrm{det}})} \\
&\leq \|\beta\|_{L^1(\mathrm{d}f)} + \|\beta\|_{L^1(\mathrm{d}g)} + \|\beta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} \|1\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} \\
&\leq \max(1, \|1\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}) \|\beta\|_{\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})} < \infty
\end{aligned}$$

because β is in $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) = \boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ by a). This contradiction shows that γ is in $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$.

c) $\Leftrightarrow$ d): This is clear from the definition of $\mathrm{d}\mathfrak{s}^{\mathrm{det}}$ in Definition 2.7. $\square$

Definition 3.2. We say that $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ satisfies $D_2(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ if there exists a constant $K \in (0, \infty)$ such that we have (3.1), i.e.,

$$\|\delta\|_{L^1(\mathrm{d}f)} + \|\delta\|_{L^1(\mathrm{d}g)} \leq K \|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}, \quad \forall \delta \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}).$$

Because the assumptions (2.13), i.e., $\inf_{t\in[0,T]}|g(t)|>0$, and $D_2(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ together frequently occur in later results, we introduce the following definition.

Definition 3.3. We call $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ *standard* if both (2.13) and $D_2(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ hold.

Corollary 3.4. *If $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ is standard, then $\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\mathrm{det}}$ exists and is in $\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$.*

Proof. According to Lemma 3.1, $\gamma := (|\mathrm{d}f| + |\mathrm{d}g|)/\mathrm{d}\mathfrak{m}^{\mathrm{det}}$ is in $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ because S satisfies $D_2(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$. We can then rewrite $\mathrm{d}\mathfrak{a}(t)$ from (2.17) as

$$\mathrm{d}\mathfrak{a}(t) = \left(\frac{\mathrm{d}f}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}(t) - \frac{f(t-)}{g(t-)}\frac{\mathrm{d}g}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}(t)\right)\mathrm{d}\mathfrak{m}^{\mathrm{det}}(t)$$

to see that $\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\mathrm{det}}$ exists $\mathrm{d}\mathfrak{m}^{\mathrm{det}}$-a.e. Moreover, thanks to (2.13), we have that $K = \sup_{t\in[0,T]}|f(t-)/g(t-)| < \infty$, and so the triangle inequality implies

$$\frac{|\mathrm{d}\mathfrak{a}|}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}(t) \le \frac{|\mathrm{d}f|}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}(t) + \left|\frac{f(t-)}{g(t-)}\right|\frac{|\mathrm{d}g|}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}(t) \le \max(1,K)\gamma(t) \quad \mathrm{d}\mathfrak{m}^{\mathrm{det}}\text{-a.e.}$$

So $\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\mathrm{det}}$ is in $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) = \boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ by Lemma 3.1 again. □

Theorem 3.5. *Let $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ be standard. Then the linear operator $\mathcal{A}$ from (2.14) is a continuous bijection with continuous inverse $\mathcal{A}^{-1}$ given by $\mathcal{A}^{\leftarrow}$ from (2.15), and there exists a constant $K \in (0,\infty)$ such that*

$$\frac{1}{K}\|\vartheta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} \le \|\vartheta \bullet S_T\|_{L^2} \le K\|\vartheta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}, \quad \forall \vartheta \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}). \tag{3.2}$$

As a consequence, $G_T(\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})) = \{\vartheta \bullet S_T : \vartheta \in \boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})\}$ is closed in L^2.

Proof. First of all, $D_2(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ implies by Lemma 3.1 that $\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) = L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$. Next, (2.14), (2.13), $\mathfrak{m}^{\mathrm{det}}(T) < \infty$ and $D_2(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ yield

$$\begin{aligned}
\|\mathcal{A}[\delta]\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} &\le \|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} + \left\|\frac{1}{g}\int_{\cdot}^{T}\delta\,\mathrm{d}g\right\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} \\
&\le \|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} + \left(\sup_{t\in[0,T]}\frac{1}{|g(t)|}\right)\|\delta\|_{L^1(\mathrm{d}g)}\mathfrak{m}^{\mathrm{det}}(T) \\
&\le \left(1 + K\mathfrak{m}^{\mathrm{det}}(T)\sup_{t\in[0,T]}\frac{1}{|g(t)|}\right)\|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}.
\end{aligned}$$

This shows that $\mathcal{A} : L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) \to L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ is continuous. But by Theorem 2.11, $\mathcal{A}$ is invertible, hence surjective, and so the open mapping theorem implies that it is open and its inverse $\mathcal{A}^{-1}$ is continuous as well.

For (3.2), the right inequality follows directly from Lemma 2.9. For the left one, we write $\vartheta \bullet S_T = \int_0^T \vartheta(t)\,\mathrm{d}f(t) + \mathcal{A}[\vartheta] \bullet M_T$ as in (2.11) and use the martingale property of $\mathcal{A}[\vartheta] \bullet M$, the $\mathbb{F}^{\mathrm{det}}$-Itô isometry (2.8) and the continuity of $\mathcal{A}^{-1}$ to obtain

$$\|\vartheta \bullet S_T\|_{L^2}^2 = \left| \int_0^T \vartheta(t)\,\mathrm{d}f(t) \right|^2 + \|\mathcal{A}[\vartheta] \bullet M_T\|_{L^2}^2 \geq \|\mathcal{A}[\vartheta]\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}^2 \geq k\|\vartheta\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}^2.$$

Finally, (3.2) shows that the linear subspace $G_T(\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})) \subseteq L^2$ is norm-equivalent to the Hilbert space $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$, and therefore it is closed in L^2. □

With the above results, we can now solve our two quadratic optimisation problems.

3.2. *Mean-variance hedging*

In this section, we solve the *mean-variance hedging* (MVH) problem

$$\text{minimise } \|H - c - \vartheta \bullet S_T\|_{L^2} \text{ over } (c, \vartheta) \in \mathbb{R} \times \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}). \tag{3.3}$$

In other words, we want to find a zero-information (because ϑ must be deterministic) self-financing strategy (c, ϑ) with initial capital c which minimises the mean squared error between the final wealth $c + \vartheta \bullet S_T$ and a given time-T financial payoff $H \in \mathcal{L}^2$. We recall from Section 2.1 that $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) \subseteq L^2(M)$ and $\|\cdot\|_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})} = \|\cdot\|_{L^2(M)}$ on $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$. We also recall that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$.

To prepare for the main result, fix $H \in \mathcal{L}^2$ and denote by

$$H = E[H] + \Pi^H \bullet M_T + L_T^H \quad P\text{-a.s.} \tag{3.4}$$

its Galtchouk–Kunita–Watanabe (GKW) decomposition with respect to M, where Π^H is in $L^2(M)$ and $L^H \in \mathcal{M}_0^2$ is strongly orthogonal to M. Recall that $\langle N \rangle^{\mathrm{p},\mathbb{F}^{\mathrm{det}}}$ is the $\mathbb{F}^{\mathrm{det}}$-predictable dual projection of the quadratic variation process of $N \in \mathcal{M}_0^2$ and define

$$\pi^H := E_M[\Pi^H \,|\, \mathcal{P}(\mathbb{F}^{\mathrm{det}})] = \frac{\mathrm{d}(\int \Pi^H \,\mathrm{d}\langle M \rangle)^{\mathrm{p},\mathbb{F}^{\mathrm{det}}}}{\mathrm{d}\langle M \rangle^{\mathrm{p},\mathbb{F}^{\mathrm{det}}}} \quad \mathrm{d}\mathfrak{m}^{\mathrm{det}}\text{-a.e.}; \tag{3.5}$$

the representation in terms of a Radon–Nikodým derivative follows from Section 4.3 in Schweizer (1994). We identify π^H with a Borel function on $[0, T]$ and recall from Lemma 2.5 that the $\mathbb{F}^{\mathrm{det}}$-predictable projections in (3.5) can be identified with expectation functions. As a conditional expectation, π^H is the unique element in $L^2(M) \cap \mathcal{P}(\mathbb{F}^{\mathrm{det}}) = L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$ such that

$$(\Pi^H - \pi^H, \delta)_{L^2(M)} = 0, \quad \forall \delta \in L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}). \tag{3.6}$$

We also recall from (2.17) and (2.15) the formulas for $\mathrm{d}\mathfrak{a}$ and $\mathcal{A}^{\leftarrow}$, respectively.

Note that π^H is by construction always in $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})$, but could fail to lie in the smaller space $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$. The first main result of this section is the following theorem. We postpone its proof until the end of the proof of Theorem 3.8 below.

Theorem 3.6. *Suppose $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ satisfies* (2.13)*. If $\pi^H = E_M[\Pi^H \mid \mathcal{P}(\mathbb{F}^{\mathrm{det}})]$ is in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, then the solution (c^H, ϑ^H) to the MVH problem for $H \in \mathcal{L}^2$ exists and is given by*

$$c^H = E[H] - \int_0^T \pi^H(t)\,\mathrm{d}\mathfrak{a}(t), \tag{3.7}$$

$$\vartheta^H = \mathcal{A}^{-1}[\pi^H] \quad \mathrm{d}\mathfrak{s}^{\mathrm{det}}\text{-}a.e. \tag{3.8}$$

Corollary 3.7. *If $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ is standard, then the MVH problem admits a solution for every $H \in \mathcal{L}^2$, and the solution is then given by* (3.7) *and* (3.8)*.*

Proof. If S is standard, it satisfies (2.13) and $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) = \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ by Lemma 3.1. Thus $\pi^H \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ and Theorem 3.6 is directly applicable. □

If $\pi^H = E_M[\Pi^H \mid \mathcal{P}(\mathbb{F}^{\mathrm{det}})]$ does not belong to $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$, we can still construct ε-optimal solutions of the MVH problem. For that purpose, we introduce

$$\mathbf{dist}_S(H) := \inf_{(c,\vartheta)\in\mathbb{R}\times\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})} \|H - c - \vartheta \bullet S_T\|^2_{L^2}.$$

Theorem 3.8. *Suppose $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ satisfies* (2.13) *and fix $H \in \mathcal{L}^2$. Then we have*

$$\mathbf{dist}_S(H) = \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2}, \tag{3.9}$$

and for any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $(c^\varepsilon, \vartheta^\varepsilon)$ defined by

$$c^\varepsilon := E[H] - \int_0^T \pi^H(t)\mathbf{1}_{\{|\pi^H(t)|\le N(\varepsilon)\}}\,\mathrm{d}\mathfrak{a}(t),$$
$$\vartheta^\varepsilon := \mathcal{A}^{-1}[\pi^H\mathbf{1}_{\{|\pi^H|\le N(\varepsilon)\}}]$$

is in $\mathbb{R}\times\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ with $\|H - c^\varepsilon - \vartheta^\varepsilon \bullet S_T\|^2_{L^2} \le \mathbf{dist}_S(H) + \varepsilon$.

Proof. Fix $(c, \vartheta) \in \mathbb{R}\times\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$. Using $H = E[H] + \Pi^H \bullet M_T + L^H_T$ from (3.4) together with $\vartheta \bullet S_T = \int_0^T \vartheta(t)\,\mathrm{d}f(t) + \mathcal{A}[\vartheta]\bullet M_T$ from (2.11), we obtain that P-a.s.,

$$H - c - \vartheta\bullet S_T = \left(E[H] - c - \int_0^T \vartheta(t)\,\mathrm{d}f(t)\right) + (\pi^H - \mathcal{A}[\vartheta])\bullet M_T + (\Pi^H - \pi^H)\bullet M_T + L^H_T.$$

By Theorem 2.11, $\pi^H - \mathcal{A}[\vartheta]$ is in $L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}}) \subseteq L^2(M)$. Using (3.6), the strong orthogonality of L^H and M and the Itô isometry implies

$$\begin{aligned}\|H - c - \vartheta\bullet S_T\|^2_{L^2} &= \left|E[H] - c - \int_0^T \vartheta(t)\,\mathrm{d}f(t)\right|^2 + \|\pi^H - \mathcal{A}[\vartheta]\|^2_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}\\ &\quad + \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2}\\ &\ge \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2}.\end{aligned} \tag{3.10}$$

Because (c,ϑ) was arbitrary, this shows $\mathbf{dist}_S(H) \geq \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2}$.

To prove the converse inequality and show the existence of ε-optimal pairs, we construct $(c^n,\vartheta^n)_{n\in\mathbb{N}} \subseteq \mathbb{R}\times\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$ with

$$\|H - c^n - \vartheta^n \bullet S_T\|^2_{L^2} \overset{n\to\infty}{\longrightarrow} \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2}.$$

To that end, we set

$$\pi^H_n := \pi^H \mathbf{1}_{\{|\pi^H|\leq n\}}, \quad c^H_n := E[H] - \int_0^T \pi^H_n(t)\,\mathrm{d}\mathfrak{a}(t), \quad \vartheta^n := \mathcal{A}^{-1}[\pi^H_n]. \tag{3.11}$$

Then π^H_n is bounded, hence in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$, and $(c^n,\vartheta^n) \in \mathbb{R}\times\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$. Theorem 2.11 thus implies that $\vartheta^H_n = \mathcal{A}^{-1}[\pi^H_n] \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$ and $\int_0^T \vartheta^H_n(t)\,\mathrm{d}f(t) = \int_0^T \pi^H_n(t)\,\mathrm{d}\mathfrak{a}(t)$ by (2.16). So we obtain

$$c^H_n = E[H] - \int_0^T \vartheta^H_n(t)\,\mathrm{d}f(t), \tag{3.12}$$

and we also have $\vartheta^H_n \bullet S_T = \int_0^T \vartheta^H_n(t)\,\mathrm{d}f(t) + \mathcal{A}[\vartheta^H_n]\bullet M_T$ P-a.s. from (2.11) in Theorem 2.11. Combining this with (3.4), (3.12), $\mathcal{A}[\vartheta^H_n] = \pi^H_n$ and the definition of π^H_n thus yields

$$\begin{aligned} H - c^H_n - \vartheta^H_n \bullet S_T &= \left(E[H] - c^H_n - \int_0^T \vartheta^H_n(t)\,\mathrm{d}f(t)\right) + (\Pi^H - \mathcal{A}[\vartheta^H_n])\bullet M_T + L^H_T \\ &= (\Pi^H - \pi^H)\bullet M_T + L^H_T + (\pi^H \mathbf{1}_{\{|\pi^H|>n\}})\bullet M_T \quad P\text{-a.s.} \end{aligned}$$

Because $(\Pi^H - \pi^H, \pi^H\mathbf{1}_{\{|\pi^H|>n\}})_{L^2(M)} = 0$ by (3.6) and L^H and M are strongly orthogonal, the Itô isometry then yields

$$\|H - c^n - \vartheta^n \bullet S_T\|^2_{L^2} = \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2} + \|\pi^H \mathbf{1}_{\{|\pi^H|>n\}}\|^2_{L^2(\mathrm{d}\mathfrak{m}^{\det})}.$$

But $\pi^H \in L^2(\mathrm{d}\mathfrak{m}^{\det})$ implies that $\pi^H\mathbf{1}_{\{|\pi^H|>n\}} \overset{n\to\infty}{\longrightarrow} 0$ in $L^2(\mathrm{d}\mathfrak{m}^{\det})$ and therefore

$$\|H - c^n - \vartheta^n \bullet S_T\|^2_{L^2} \overset{n\to\infty}{\longrightarrow} \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2}.$$

This shows that $\mathbf{dist}_S(H) \leq \|\Pi^H - \pi^H\|^2_{L^2(M)} + \|L^H_T\|^2_{L^2}$ and thus proves (3.9). Finally, choosing $(c^\varepsilon,\vartheta^\varepsilon)$ with $N = N(\varepsilon)$ such that $\|\pi^H\mathbf{1}_{\{|\pi^H|>N(\varepsilon)\}}\|^2_{L^2(\mathrm{d}\mathfrak{m}^{\det})} \leq \varepsilon$ gives via (3.11) an ε-optimal solution. □

We can now use part of the previous proof to argue Theorem 3.6.

Proof of Theorem 3.6. If π^H is in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$, then $\vartheta^H = \mathcal{A}^{-1}[\pi^H]$ is in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$ by Theorem 2.11. Thus we may choose c^H as in (3.7), and inserting $(c,\vartheta) = (c^H,\vartheta^H)$ in (3.10) yields $\|H - c^H - \vartheta^H \bullet S_T\|^2_{L^2} = \mathbf{dist}_S(H)$ by (3.9). This shows optimality of (c^H,ϑ^H). □

3.3. *Mean-variance portfolio selection*

In this section, we solve for $\alpha > 0$ the *mean-variance portfolio selection* (MVPS) problem

$$\text{maximise } E[\vartheta \bullet S_T] - \alpha \mathrm{Var}[\vartheta \bullet S_T] \text{ over } \vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) \tag{3.13}$$

with corresponding value function

$$MV_\alpha := \sup_{\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})} \big(E[\vartheta \bullet S_T] - \alpha \mathrm{Var}[\vartheta \bullet S_T]\big).$$

We write ϑ^{MV} for its solution if that exists.

It is well known that the MVPS problem is closely linked to the optimisation problem

$$\text{minimise } \|1 - \vartheta \bullet S_T\|_{L^2} \text{ over } \vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) \tag{3.14}$$

with solution ϑ° (if that exists). This is true quite generally, and one can in fact in (3.13) and (3.14) replace $G_T(\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}))$ and $\vartheta \bullet S_T$ with $\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ by an abstract linear subspace $\mathcal{G} \subseteq L^2$ and $g \in \mathcal{G}$; see Fontana & Schweizer (2012). In their framework, we take $\mathcal{G} = G_T(\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}))$, $Y \equiv 0$, $\gamma = 1/\alpha$ and note that $1 - \pi(1) = g^1 = \vartheta^\circ \bullet S_T$. If we define

$$\mathbf{dist}^\circ_S(1) := \inf_{\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})} \|1 - \vartheta \bullet S_T\|^2_{L^2},$$

then $E[\pi(1)] = E[(\pi(1))^2] = \|1 - g^1\|^2_{L^2} = \mathbf{dist}^\circ_S(1)$, and Remark 3.4 (4) in Fontana & Schweizer (2012) shows that

$$MV_\alpha < \infty \quad \Longleftrightarrow \quad \mathbf{dist}^\circ_S(1) > 0.$$

So (3.13) is only meaningful if $\mathbf{dist}^\circ_S(1) > 0$ or, equivalently, if 1 is not in the L^2-closure of $G_T(\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}))$. The link between ϑ^{MV} and ϑ° is then by Proposition 3.4 of Fontana & Schweizer (2012) as follows.

Lemma 3.9. *Suppose* $\mathbf{dist}^\circ_S(1) > 0$ *and* (3.14) *has a solution* $\vartheta^\circ \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$*. Then*

$$\begin{aligned} \vartheta^{MV} &= \frac{1}{2\alpha\, \mathbf{dist}^\circ_S(1)} \vartheta^\circ, \\ MV_\alpha &= \frac{1}{4\alpha}\left(\frac{1}{\mathbf{dist}^\circ_S(1)} - 1\right). \end{aligned}$$

To study ϑ° and $\mathbf{dist}^\circ_S(1)$, we begin with the following result.

Lemma 3.10. *Suppose that* $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ *satisfies* (2.13) *and denote by* $\mathrm{d}\mathfrak{a} = \mathrm{d}\mathfrak{a}^a + \mathrm{d}\mathfrak{a}^s$ *the Lebesgue decomposition of* $\mathrm{d}\mathfrak{a}$ *with respect to* $\mathrm{d}\mathfrak{m}^{\mathrm{det}}$ *into an absolutely continuous and a singular part. For any* ϑ *and* δ *in* $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$*, we then have*

$$\begin{aligned} (1 - \vartheta \bullet S_T, \delta \bullet S_T)_{L^2} = &\int_0^T \left(D_\vartheta \frac{\mathrm{d}\mathfrak{a}^a}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}(t) - \mathcal{A}[\vartheta](t)\right)\mathcal{A}[\delta](t)\,\mathrm{d}\mathfrak{m}^{\mathrm{det}}(t) \\ &+ D_\vartheta \int_0^T \mathcal{A}[\delta](t)\,\mathrm{d}\mathfrak{a}^s(t), \end{aligned}$$

where $D_\vartheta := 1 - \int_0^T \mathcal{A}[\vartheta](t)\,\mathrm{d}\mathfrak{a}(t)$.

Proof. Using (2.18) and (2.16) from Theorem 2.11, multiplying out and using (2.8) gives

$$\begin{aligned}(1-\vartheta\bullet S_T, \delta\bullet S_T)_{L^2} &= \left(1 - \int_0^T \vartheta(t)\,\mathrm{d}f(t)\right)\int_0^T \delta(t)\,\mathrm{d}f(t) \\ &\quad - (\mathcal{A}[\vartheta]\bullet M_T, \mathcal{A}[\delta]\bullet M_T)_{L^2} \\ &= \left(1 - \int_0^T \mathcal{A}[\vartheta](t)\,\mathrm{d}\mathfrak{a}(t)\right)\int_0^T \mathcal{A}[\delta](t)\,\mathrm{d}\mathfrak{a}(t) \\ &\quad - \int_0^T \mathcal{A}[\vartheta](t)\mathcal{A}[\delta](t)\,\mathrm{d}\mathfrak{m}^{\mathrm{det}}(t).\end{aligned}$$

Plugging in D_ϑ and the Lebesgue decomposition of $\mathrm{d}\mathfrak{a}$ then yields the result. □

To exploit Lemma 3.10, we recall that a strategy $\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ is a solution to (3.14) if and only if it satisfies the first order condition

$$(1-\vartheta\bullet S_T, \delta\bullet S_T)_{L^2} = 0, \quad \forall \delta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}). \tag{3.15}$$

Theorem 3.11. *Suppose* $S \,\widehat{=}\, (S_0, f, g, Y)$ *satisfies* (2.13) *and* $\mathrm{d}\mathfrak{s}^{\mathrm{det}} \ll \mathrm{d}\mathfrak{m}^{\mathrm{det}}$. *Then existence of a solution* $\vartheta^\circ \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ *to* (3.14) *plus* $\mathbf{dist}^\circ_S(1) > 0$ *is equivalent to*

$$\mathrm{d}\mathfrak{a} \ll \mathrm{d}\mathfrak{m}^{\mathrm{det}} \quad \textit{with} \quad \frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}} \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}).$$

In that case, we have the explicit formulas

$$\vartheta^\circ = D^\circ \mathcal{A}^{-1}\left[\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}\right] \quad \mathrm{d}\mathfrak{m}^{\mathrm{det}}\text{-}a.e., \tag{3.16}$$

$$D^\circ := \left(1 + \left\|\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}}\right\|^2_{L^2(\mathrm{d}\mathfrak{m}^{\mathrm{det}})}\right)^{-1} = \mathbf{dist}^\circ_S(1) \in (0,\infty). \tag{3.17}$$

In particular, if S *is standard, then* ϑ° *always exists and is given by* (3.16) *and* (3.17).

Proof. As in Lemma 3.10, $\mathrm{d}\mathfrak{a} = \mathrm{d}\mathfrak{a}^a + \mathrm{d}\mathfrak{a}^s$ is the Lebesgue decomposition of $\mathrm{d}\mathfrak{a}$ with respect to $\mathrm{d}\mathfrak{m}^{\mathrm{det}}$. Because $\mathcal{A} : \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}}) \to \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ is bijective by Theorem 2.11, combining (3.15) and Lemma 3.10 shows that a given $\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\mathrm{det}})$ solves (3.14) if and only if

$$D_\vartheta \frac{\mathrm{d}\mathfrak{a}^a}{\mathrm{d}\mathfrak{m}^{\mathrm{det}}} - \mathcal{A}[\vartheta] = 0 \ \mathrm{d}\mathfrak{m}^{\mathrm{det}}\text{-a.e.} \quad \text{and} \quad D_\vartheta\,\mathrm{d}\mathfrak{a}^s = 0, \tag{3.18}$$

where $D_\vartheta = 1 - \int_0^T \mathcal{A}[\vartheta](t)\,\mathrm{d}\mathfrak{a}(t)$.

Suppose first that there exists a strategy $\vartheta \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$ which solves (3.14). By combining $\mathbf{dist}_S^\circ(1) = E[\pi(1)] = E[1-g^1]$ with (2.18) and (2.16) from Theorem 2.11, we obtain

$$\mathbf{dist}_S^\circ(1) = E[1-\vartheta \bullet S_T] = 1 - \int_0^T \vartheta(t)\,\mathrm{d}f(t) = 1 - \int_0^T \mathcal{A}[\vartheta](t)\,\mathrm{d}\mathfrak{a}(t) = D_\vartheta.$$

Because $\mathbf{dist}_S^\circ(1) > 0$ by assumption, (3.18) implies $\mathrm{d}\mathfrak{a}^s = 0$, hence $\mathrm{d}\mathfrak{a} \ll \mathrm{d}\mathfrak{m}^{\det}$, and

$$D_\vartheta \frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}} - \mathcal{A}[\vartheta] = 0 \quad \mathrm{d}\mathfrak{m}^{\det}\text{-a.e.} \tag{3.19}$$

But $\mathrm{d}\mathfrak{m}^{\det} \ll \mathrm{d}\mathfrak{s}^{\det} = |\mathrm{d}f| + |\mathrm{d}g| + \mathrm{d}\mathfrak{m}^{\det}$, and so the assumption $\mathrm{d}\mathfrak{s}^{\det} \ll \mathrm{d}\mathfrak{m}^{\det}$ implies that $\mathrm{d}\mathfrak{s}^{\det} \approx \mathrm{d}\mathfrak{m}^{\det}$. So (3.19) also holds $\mathrm{d}\mathfrak{s}^{\det}$-a.e. and implies, because $\mathcal{A}[\vartheta]$ is in $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$ like ϑ itself, that $\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\det}$ belongs to $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$ as well.

Conversely, if $\mathrm{d}\mathfrak{a} \ll \mathrm{d}\mathfrak{m}^{\det}$ with $\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\det} \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$, Theorem 2.11 implies that $\mathcal{A}^{-1}[\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\det}] \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$, and $\|\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\det}\|_{L^2(\mathrm{d}\mathfrak{m}^{\det})} \le \|\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\det}\|_{\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})} < \infty$ shows that D° in (3.17) is well defined and in $(0,\infty)$. Because again $\mathrm{d}\mathfrak{s}^{\det} \approx \mathrm{d}\mathfrak{m}^{\det}$, we can also define ϑ° by (3.16) and obtain that $\vartheta^\circ \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$. Simply combining the definitions of D_{ϑ°, ϑ° and D° with (2.16) shows that

$$\begin{aligned} D_{\vartheta^\circ} &= 1 - D^\circ \int_0^T \mathcal{A}^{-1}\left[\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}}\right](t)\,\mathrm{d}f(t) \\ &= 1 - D^\circ \int_0^T \frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}}(t)\,\mathrm{d}\mathfrak{a}(t) \\ &= 1 - D^\circ \int_0^T \left(\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}}(t)\right)^2 \mathrm{d}\mathfrak{m}^{\det}(t) \\ &= \left(1 + \left\|\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}}\right\|^2_{L^2(\mathrm{d}\mathfrak{m}^{\det})}\right)^{-1} = D^\circ. \end{aligned}$$

But $\mathrm{d}\mathfrak{a} \ll \mathrm{d}\mathfrak{m}^{\det}$ implies $\mathrm{d}\mathfrak{a}^s = 0$ so that rewriting (3.16) with $D^\circ = D_{\vartheta^\circ}$ implies that ϑ° satisfies (3.18) and is therefore the solution to (3.14). Finally, the same computation as in the first step shows that $\mathbf{dist}_S^\circ(1) = D_{\vartheta^\circ} = D^\circ > 0$.

If S is standard, then $\mathrm{d}\mathfrak{s}^{\det} \ll \mathrm{d}\mathfrak{m}^{\det}$ by Lemma 3.1 and we have $\mathrm{d}\mathfrak{a} \ll \mathrm{d}\mathfrak{m}^{\det}$ with $\mathrm{d}\mathfrak{a}/\mathrm{d}\mathfrak{m}^{\det} \in \mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det})$ by Corollary 3.4. So the assertion follows from the first part of the present theorem. ☐

The solution to the MVPS problem (3.13) is now given as follows.

Theorem 3.12. *Suppose $S \mathrel{\widehat{=}} (S_0, f, g, Y)$ is standard. Then*

$$\begin{aligned} \vartheta^{MV} &= \frac{1}{2\alpha}\mathcal{A}^{-1}\left[\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}}\right], \\ MV_\alpha &= \frac{1}{4\alpha}\left\|\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}}\right\|^2_{L^2(\mathrm{d}\mathfrak{m}^{\det})}. \end{aligned}$$

Proof. This follows directly from combining Theorem 3.11 with Lemma 3.9. ☐

4. Examples

In this section, we work out the preceding theory in two classes of examples: arithmetic and exponential Lévy processes. Before starting, we need a small extra result for the MVH problem. Fix a payoff $H \in \mathcal{L}^2$ and denote by $H = E[H] + \Pi^H \bullet M_T + L_T^H$ P-a.s. its GKW decomposition with respect to a given $M \in \mathcal{M}_0^2$. In view of Theorem 3.6 and (3.5),

$$\pi^H = E_M[\Pi^H \mid \mathcal{P}(\mathbb{F}^{\mathrm{det}})] = \frac{\mathrm{d}(\int \Pi^H \,\mathrm{d}\langle M\rangle)^{\mathrm{p},\mathbb{F}^{\mathrm{det}}}}{\mathrm{d}\langle M\rangle^{\mathrm{p},\mathbb{F}^{\mathrm{det}}}} \tag{4.1}$$

is an important ingredient for the solution of the MVH problem (3.3).

Lemma 4.1. *Suppose that $M \in \mathcal{M}_0^2$ with $\mathrm{d}\langle M\rangle_t = \Psi_t^2 \,\mathrm{d}t$ for some $\mathbb{F}$-predictable process Ψ. Then for any $H \in \mathcal{L}^2$, the process π^H from* (4.1) *can be identified with the function*

$$\pi^H(t) = \frac{E[\Pi_t^H \Psi_t^2]}{E[\Psi_t^2]} \quad \mathrm{d}t\text{-}a.e. \tag{4.2}$$

Proof. Using $\mathrm{d}\langle M\rangle_t = \Psi_t^2 \,\mathrm{d}t$ and the Kunita–Watanabe inequality implies

$$E\bigg[\int_0^T |\Pi_t^H| \Psi_t^2 \,\mathrm{d}t\bigg] = E\bigg[\int_0^T |\Pi_t^H| \,\mathrm{d}\langle M\rangle_t\bigg] \le \|\langle M\rangle_T^{1/2}\|_{L^2} \|\Pi^H\|_{L^2(M)} < \infty.$$

By Fubini's theorem, $t \mapsto E[|\Pi_t^H|\Psi_t^2]$ is thus $\mathrm{d}t$-integrable on $[0,T]$ and so $E[|\Pi_t^H|\Psi_t^2] < \infty$ for $\mathrm{d}t$-a.a. $t \in [0,T]$. On the other hand, as $\langle M\rangle = \int \Psi_t^2 \,\mathrm{d}t$ is integrable, $t \mapsto E[\Psi_t^2]$ is $\mathrm{d}t$-integrable and $E[\Psi_t^2] < \infty$ for $\mathrm{d}t$-a.a. $t \in [0,T]$. If we set $0/0 := 1$, the quotient $E[\Pi_t^H \Psi_t^2]/E[\Psi_t^2]$ is therefore well defined for $\mathrm{d}t$-a.a. $t \in [0,T]$. Using dominated convergence and Fubini's theorem gives for all bounded Borel functions δ that

$$\begin{aligned} E\bigg[\int_0^T \delta(t)\Pi_t^H \,\mathrm{d}\langle M\rangle_t\bigg] &= \lim_{n\to\infty} E\bigg[\int_0^T \delta(t)\Pi_t^H \Psi_t^2 \mathbf{1}_{\{|\Pi_t^H|\Psi_t^2 \le n\}} \,\mathrm{d}t\bigg] \\ &= \lim_{n\to\infty} \int_0^T E[\delta(t)\Pi_t^H \Psi_t^2 \mathbf{1}_{\{|\Pi_t^H|\Psi_t^2 \le n\}}] \,\mathrm{d}t \\ &= \int_0^T \delta(t) E[\Pi_t^H \Psi_t^2] \,\mathrm{d}t. \end{aligned}$$

Because δ was arbitrary, this yields $(\int \Pi^H \,\mathrm{d}\langle M\rangle)^{\mathrm{p},\mathbb{F}^{\mathrm{det}}} = \int E[\Pi_t^H \Psi_t^2] \,\mathrm{d}t$, and we find analogously that $\langle M\rangle^{\mathrm{p},\mathbb{F}^{\mathrm{det}}} = \int E[\Psi_t^2] \,\mathrm{d}t$. In view of (4.1), this implies (4.2). □

4.1. *Arithmetic Lévy models*

Both our example classes are built on Lévy processes. We recall (see for instance Theorem 3.1 in Cont & Tankov (2004)) that the Lévy triplet (b, Σ, ν) of a onedimensional Lévy process $L = (L_t)_{t\in[0,T]}$ is given by the Lévy–Khinchine representation $E[e^{izL_t}] = e^{t\psi(z)}$ for $z \in \mathbb{R}$, with characteristic exponent

$$\psi(z) := ibz - \frac{1}{2}\Sigma z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|\le 1\}})\,\nu(\mathrm{d}x). \tag{4.3}$$

We also need some integrability properties which are summarised in the next result. This is a combination of Propositions 3.13, 3.18 and 3.17 in Cont & Tankov (2004).

Proposition 4.2. *Let $L = (L_t)_{t\in[0,T]}$ be a Lévy process with Lévy triplet (b, Σ, ν) such that $\int_{\{|x|\ge 1\}} x^2\,\nu(\mathrm{d}x) < \infty$. Then the following statements hold:*

1) $E[L_t] = (b + \int_{\{|x|\ge 1\}} x\,\nu(\mathrm{d}x))t$, $t \in [0,T]$.

2) L is a martingale if and only if $b + \int_{\{|x|\ge 1\}} x\,\nu(\mathrm{d}x) = 0$.

3) If L is a martingale, then $(L_t^2 - E[L_t^2])_{t\in[0,T]}$ is a martingale as well, and we have $E[L_t^2] = (\Sigma + \int_{\mathbb{R}} x^2\,\nu(\mathrm{d}x))t$, $t \in [0,T]$.

In the rest of this subsection, we consider a Lévy process as in Proposition 4.2 and define $S := S_0 + L$ with $S_0 \in \mathbb{R}$. We also define the two constants

$$\mu_{\mathbf{a}} := b + \int_{\{|x|\ge 1\}} x\,\nu(\mathrm{d}x), \tag{4.4}$$

$$\sigma_{\mathbf{a}}^2 := \Sigma + \int_{\mathbb{R}} x^2\,\nu(\mathrm{d}x), \tag{4.5}$$

where the subscript $\mathbf{a}$ is mnemonic for "arithmetic Lévy".

Lemma 4.3. *Suppose that L is as in Proposition 4.2 and define the functions $f, g : [0,T] \to \mathbb{R}$ and the process $Y = (Y_t)_{t\in[0,T]}$ by*

$$f(t) := \mu_{\mathbf{a}} t, \quad g(t) := 1, \quad Y_t := L_t - \mu_{\mathbf{a}} t. \tag{4.6}$$

Then the following statements hold:

1) $Y \in \mathcal{M}_0^2$ with $\mathrm{d}\langle Y\rangle_t = \sigma_{\mathbf{a}}^2\,\mathrm{d}t$.

2) $S = S_0 + L$ is a type (A) semimartingale with quadruplet (S_0, f, g, Y) given by (4.6), and its canonical decomposition $S = S_0 + M + A$ is given by

$$M_t := Y_t, \quad A_t := \mu_{\mathbf{a}} t, \quad \textit{for } t \in [0,T].$$

In particular, we have

$$\mathrm{d}\langle M\rangle_t = \mathrm{d}\langle Y\rangle_t = \sigma_{\mathbf{a}}^2\,\mathrm{d}t. \tag{4.7}$$

3) We have $\mathrm{d}\mathfrak{a}(t) = \mu_{\mathbf{a}}\,\mathrm{d}t$ and $\mathrm{d}\mathfrak{m}^{\det}(t) = \sigma_{\mathbf{a}}^2\,\mathrm{d}t$, and if $\sigma_{\mathbf{a}}^2 \neq 0$, then

$$\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}} \equiv \frac{\mu_{\mathbf{a}}}{\sigma_{\mathbf{a}}^2}. \tag{4.8}$$

Proof. Clearly Y is a Lévy process with Lévy triplet $(b - \mu_{\mathbf{a}}, \Sigma, \nu)$ and hence a martingale by (4.4) and Proposition 4.2, 2). By Proposition 4.2, 3) and (4.5), $(Y_t^2 - \sigma_{\mathbf{a}} t)_{t\in[0,T]}$ is then also a martingale which proves 1). Writing S as

$$S_t = S_0 + \mu_{\mathbf{a}} t + (L_t - \mu_{\mathbf{a}} t) = S_0 + f(t) + g(t)Y_t, \quad t \in [0,T],$$

therefore immediately gives 2), and 3) follows from Lemma 2.5 and by inserting $f(t) = \mu_{\mathbf{a}} t$ and $g \equiv 1$ into the formula (2.21) for $\mathrm{d}\mathfrak{a}(t)$. $\square$

Lemma 4.4. *Suppose L is as in Proposition 4.2. If $\sigma_{\mathbf{a}}^2 \neq 0$, then $S = S_0 + L$ is standard with $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det}) = L^2(\mathrm{d}\mathfrak{m}^{\det}) = L^2(\mathrm{d}t)$, and for any $H \in \mathcal{L}^2$, we have $\pi^H(t) = E[\Pi_t^H]$ $\mathrm{d}t$-a.e.*

Proof. (4.6) gives $\|\cdot\|_{L^1(\mathrm{d}f)} = |\mu_{\mathbf{a}}|\|\cdot\|_{L^1(\mathrm{d}t)}$ and $\|\cdot\|_{L^1(\mathrm{d}g)} \equiv 0$. Moreover, (4.7) yields $\mathrm{d}\mathfrak{m}^{\det}(t) = \sigma_{\mathbf{a}}^2 \,\mathrm{d}t$ so that for δ bounded Borel, using $\sigma_{\mathbf{a}}^2 \neq 0$ gives

$$\|\delta\|_{L^1(\mathrm{d}f)} + \|\delta\|_{L^1(\mathrm{d}g)} = |\mu_{\mathbf{a}}|\|\delta\|_{L^1(\mathrm{d}t)} \leq \frac{|\mu_{\mathbf{a}}|\sqrt{T}}{|\sigma_{\mathbf{a}}|}\|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\det})}.$$

Hence $D_2(\mathrm{d}\mathfrak{s}^{\det})$ is satisfied and so $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det}) = L^2(\mathrm{d}\mathfrak{m}^{\det})$ by Lemma 3.1. Because $g \equiv 1$ satisfies (2.13), S is standard, and again using $\sigma_{\mathbf{a}}^2 \neq 0$ gives $L^2(\mathrm{d}\mathfrak{m}^{\det}) = L^2(\mathrm{d}t)$. Finally, because $\sigma_{\mathbf{a}}^2 \neq 0$, the formula for $\pi^H(t)$ follows directly from Lemma 4.1. $\square$

The solutions of our two quadratic optimisation problems in the arithmetic Lévy setting now look as follows.

Theorem 4.5. *Suppose L is as in Proposition 4.2, $\sigma_{\mathbf{a}}^2 \neq 0$ and $S = S_0 + L$. Then:*
1) For each $H \in \mathcal{L}^2$, the solution (c^H, ϑ^H) to the MVH problem (3.3) *exists and is given by*

$$c^H = E[H] - \mu_{\mathbf{a}} \int_0^T E[\Pi_t^H] \,\mathrm{d}t, \quad \vartheta^H(t) = E[\Pi_t^H] \quad \mathrm{d}t\text{-a.e.}$$

2) The solution to the MVPS problem (3.13) *exists and is given by*

$$\vartheta^{MV} \equiv \frac{1}{2\alpha}\frac{\mu_{\mathbf{a}}}{\sigma_{\mathbf{a}}^2}, \quad \text{with value } MV_\alpha = \frac{1}{4\alpha}\frac{\mu_{\mathbf{a}}^2}{\sigma_{\mathbf{a}}^2}T.$$

Proof. 1) Because S is standard by Lemma 4.4, (c^H, ϑ^H) exists for every $H \in \mathcal{L}^2$ by Corollary 3.7 and is given by (3.7), (3.8). Next, $\pi^H(t) = E[\Pi_t^H]$ $\mathrm{d}t$-a.e. by Lemma 4.4, and the formula for c^H follows by inserting $\mathrm{d}\mathfrak{a}(t) = \mu_{\mathbf{a}} \,\mathrm{d}t$ in (3.7). Finally, plugging f and g into the definition (2.14) shows that $\mathcal{A}[\delta] = \delta$, hence $\mathcal{A}^{-1} = \mathcal{A} = \mathrm{Id}$, and so (3.8) yields $\vartheta^H = \pi^H$.

2) Again using that S is standard, the formulas for ϑ^{MV} and MV_α follow directly from Theorem 3.12, $\mathcal{A}^{-1} = \mathrm{Id}$ and (4.8). $\square$

Remark 4.6. If $\nu \equiv 0$ is the zero measure, we recover for $S_t = S_0 + \mu_{\mathbf{a}} t + \sigma_{\mathbf{a}} W_t$, $t \in [0, T]$, the *Bachelier model* of arithmetic Brownian motion with drift $\mu_{\mathbf{a}} = b$ and volatility $\sigma_{\mathbf{a}} = \sqrt{\Sigma}$; see Section 4.3.1.

Remark 4.7. Let L be as in Proposition 4.2 and $\lambda > 0$. The Lévy Ornstein–Uhlenbeck process S (see Barndorff-Nielsen & Shephard (2001)) is then defined as

$$S_t = e^{-\lambda t}\left(S_0 + \int_0^t e^{\lambda s}\,\mathrm{d}L_s\right), \quad t \in [0,T], \tag{4.9}$$

and we claim that this is also a type (A) semimartingale. Indeed, using $\mu_{\mathbf{a}}$ from (4.4) to define $\widetilde{L}_t := L_t - \mu_{\mathbf{a}} t$ allows us to write the $\mathrm{d}L$-integral in (4.9) as

$$\int_0^t e^{\lambda s}\,\mathrm{d}L_s = \int_0^t e^{\lambda s}\,\mathrm{d}\widetilde{L}_s + \mu_{\mathbf{a}} \int_0^t e^{\lambda s}\,\mathrm{d}s, \quad t \in [0,T],$$

which is clearly the canonical decomposition of $\int e^{\lambda s}\,\mathrm{d}L_s$. Moreover, $\int e^{\lambda s}\,\mathrm{d}\widetilde{L}_s$ is in $\mathcal{M}_0^2$ because Lemma 4.3, 1) implies $\langle \int e^{\lambda s}\,\mathrm{d}\widetilde{L}_s \rangle_T = \sigma_{\mathbf{a}}^2 \int_0^T e^{2\lambda s}\,\mathrm{d}s$ P-a.s., which is nonrandom and hence integrable. Thus we can write S as

$$S_t = S_0 + S_0(e^{-\lambda t} - 1) + \mu_{\mathbf{a}} e^{-\lambda t} \int_0^t e^{\lambda s}\,\mathrm{d}s + e^{-\lambda t} \int_0^t e^{\lambda s}\,\mathrm{d}\widetilde{L}_s, \quad t \in [0,T],\ P\text{-a.s.},$$

and read off the quadruplet (S_0, f, g, Y) as

$$f(t) = S_0(e^{-\lambda t} - 1) + \mu_{\mathbf{a}} \frac{1 - e^{-\lambda t}}{\lambda}, \quad g(t) = e^{-\lambda t}, \quad Y_t = \int_0^t e^{\lambda s}\,\mathrm{d}\widetilde{L}_s.$$

This allows us to do more computations, but we do not give further details here.

4.2. *Exponential Lévy models*

For our second class of examples, we again first collect some integrability properties. These are from Propositions 3.18, 3.14 and 8.20 in Cont & Tankov (2004).

Proposition 4.8. *Let $L = (L_t)_{t \in [0,T]}$ be a Lévy process with Lévy triplet (b, Σ, ν) such that $\int_{\{|x| \geq 1\}} e^{2x}\,\nu(\mathrm{d}x) < \infty$. Then the following statements hold:*

1) e^L is a martingale if and only if $b + \frac{1}{2}\Sigma + \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{\{|x| \leq 1\}})\,\nu(\mathrm{d}x) = 0$.

2) We have $E[e^{2L_t}] < \infty$ and $E[e^{2L_t}] = e^{t\psi(-2i)}$, where ψ is from (4.3).

3) e^L is special with canonical decomposition $e^L = 1 + N + B$ given by

$$N_t := \sqrt{\Sigma} \int_0^t e^{L_{s-}}\,\mathrm{d}W_s + \int_{(0,t] \times \mathbb{R}} e^{L_{s-}} (e^x - 1)\,\widetilde{J}_L(\mathrm{d}s, \mathrm{d}x), \quad t \in [0,T],$$

where W is a Brownian motion, $\widetilde{J}_L(\mathrm{d}s, \mathrm{d}x)$ denotes the compensated Poisson random measure of L, and

$$B_t := \left(b + \frac{1}{2}\Sigma + \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{\{|x| \leq 1\}})\,\nu(\mathrm{d}x)\right) \int_0^t e^{L_{s-}}\,\mathrm{d}s, \quad t \in [0,T].$$

In the rest of this subsection, we consider a Lévy process as in Proposition 4.8

and define $S := S_0 e^L$, where $S_0 > 0$. We also define the three constants

$$\mu_{\mathbf{e}} := b + \frac{1}{2}\Sigma + \int_{\mathbb{R}} (e^x - 1 - x\mathbf{1}_{\{|x|\le 1\}})\,\nu(\mathrm{d}x), \tag{4.10}$$

$$\sigma_{\mathbf{e}}^2 := \Sigma + \int_{\mathbb{R}} (e^x - 1)^2\,\nu(\mathrm{d}x), \tag{4.11}$$

$$\lambda_{\mathbf{e}} := 2b + 2\Sigma + \int_{\mathbb{R}} (e^{2x} - 1 - 2x\mathbf{1}_{\{|x|\le 1\}})\,\nu(\mathrm{d}x), \tag{4.12}$$

where the subscript $\mathbf{e}$ is mnemonic for "exponential Lévy". We remark for later use that one can show that $\lambda_{\mathbf{e}} = \log E[e^{2L_1}]$ so that $E[e^{2L_t}] = e^{\lambda_{\mathbf{e}} t}$.

Lemma 4.9. *Suppose that L is as in Proposition 4.8 and define the functions $f, g : [0,T] \to \mathbb{R}$ and the process $Y = (Y_t)_{t\in[0,T]}$ by*

$$f(t) := S_0(e^{\mu_{\mathbf{e}} t} - 1), \quad g(t) := e^{\mu_{\mathbf{e}} t}, \quad Y_t := S_0(e^{L_t - \mu_{\mathbf{e}} t} - 1). \tag{4.13}$$

Then the following statements hold:

1) $Y \in \mathcal{M}_0^2$ with $\mathrm{d}\langle Y\rangle_t = S_0^2 \sigma_{\mathbf{e}}^2 e^{2(L_t - \mu_{\mathbf{e}} t)}\,\mathrm{d}t$.

2) $S = S_0 e^L$ with $S_0 > 0$ is a type (A) semimartingale with quadruplet (S_0, f, g, Y) given by (4.13), and its canonical decomposition $S = S_0 + M + A$ is given by

$$M_t = \int_0^t e^{\mu_{\mathbf{e}} s}\,\mathrm{d}Y_s, \quad A_t = \mu_{\mathbf{e}} \int_0^t S_s\,\mathrm{d}s, \quad \text{for } t \in [0,T]. \tag{4.14}$$

In particular, we have

$$\mathrm{d}\langle M\rangle_t = \sigma_{\mathbf{e}}^2 S_t^2\,\mathrm{d}t. \tag{4.15}$$

3) We have $\mathrm{d}\mathfrak{a}(t) = \mu_{\mathbf{e}} S_0\,\mathrm{d}t$ and $\mathrm{d}\mathfrak{m}^{\det}(t) = S_0^2 \sigma_{\mathbf{e}}^2 e^{\lambda_{\mathbf{e}} t}\,\mathrm{d}t$, and if $\sigma_{\mathbf{e}}^2 \neq 0$, then

$$\frac{\mathrm{d}\mathfrak{a}}{\mathrm{d}\mathfrak{m}^{\det}}(t) = \frac{1}{S_0}\frac{\mu_{\mathbf{e}}}{\sigma_{\mathbf{e}}^2} e^{-\lambda_{\mathbf{e}} t}, \quad t \in [0,T]. \tag{4.16}$$

Proof. 1) The process $\widetilde{L}_t = L_t - \mu_{\mathbf{e}} t$, $t \in [0,T]$, is a Lévy process with Lévy triplet $(b - \mu_{\mathbf{e}}, \Sigma, \nu)$. Hence $\widetilde{Y} := e^{\widetilde{L}}$ is an exponential Lévy process. Proposition 4.8, 1) and 2) and the definition of $\mu_{\mathbf{e}}$ in (4.10) then imply that $\widetilde{Y}$, and hence $Y = S_0(\widetilde{Y} - 1)$, is a martingale with $Y_T \in \mathcal{L}^2$ so that $Y \in \mathcal{M}_0^2$. According to Proposition 4.8, 3), applied to $\widetilde{L}$ instead of L, we can alternatively write $\widetilde{Y}$ as

$$\widetilde{Y}_t = e^{\widetilde{L}_t} = 1 + \sqrt{\Sigma}\int_0^t e^{\widetilde{L}_{s-}}\,\mathrm{d}W_s + \int_{(0,t]\times\mathbb{R}} e^{\widetilde{L}_{s-}}(e^x - 1)\,\widetilde{J}_L(\mathrm{d}s, \mathrm{d}x); \tag{4.17}$$

note that $\widetilde{J}_{\widetilde{L}} = \widetilde{J}_L$ and the FV part vanishes due to the definition of $\mu_{\mathbf{e}}$ in (4.10). But (4.17) is also the decomposition of $\widetilde{Y}$ into its continuous and purely discontinuous local martingale parts, and so the two processes on the RHS of (4.17) are strongly orthogonal. Using (4.11), $(1/S_0^2)\langle Y\rangle = \langle\widetilde{Y}\rangle$ is therefore given by

$$\frac{1}{S_0^2}\langle Y\rangle_t = \Sigma\int_0^t e^{2\widetilde{L}_{s-}}\,\mathrm{d}s + \int_{\mathbb{R}}(e^x-1)^2\,\nu(\mathrm{d}x)\int_0^t e^{2\widetilde{L}_{s-}}\,\mathrm{d}s$$
$$= \sigma_{\mathbf{e}}^2\int_0^t e^{2\widetilde{L}_s}\,\mathrm{d}s \quad P\text{-a.s.} \tag{4.18}$$

Note that we can replace $e^{\widetilde{L}_{s-}}$ by $e^{\widetilde{L}_s}$ in the ds-integral because $\widetilde{L}$ is RCLL so that P-a.s., we have $\widetilde{L}_{s-}\neq\widetilde{L}_s$ for at most countably many $s\in[0,T]$, which form a ds-nullset.

2) The identities $\widetilde{L}_t = L_t-\mu_{\mathbf{e}}t$ and

$$S_t = S_0e^{L_t} = S_0 + S_0(e^{\mu_{\mathbf{e}}t}-1) + e^{\mu_{\mathbf{e}}t}S_0(e^{\widetilde{L}_t}-1)$$
$$= S_0+f(t)+g(t)Y_t,\quad t\in[0,T],$$

show that S is a type (A) semimartingale. By Lemma 2.3, its canonical decomposition is given by $M=g\bullet Y$ and $A=f+Y_-\bullet g$, and plugging in f,g,Y from (4.13) yields (4.14); note that we can again can replace Y_- by Y, hence also S_- by S, in the ds-integral. Using $M=g\bullet Y$, (4.13), (4.18) and $S_0e^{\mu_{\mathbf{e}}t}e^{\widetilde{L}_t}=S_0e^{L_t}=S_t$ finally gives (4.15) via

$$\mathrm{d}\langle M\rangle_t = g^2(t)\,\mathrm{d}\langle Y\rangle_t = e^{2\mu_{\mathbf{e}}t}S_0^2\sigma_{\mathbf{e}}^2e^{2\widetilde{L}_t}\,\mathrm{d}t = \sigma_{\mathbf{e}}^2S_t^2\,\mathrm{d}t.$$

3) Inserting f and g from (4.13) into the defining formula (2.17) for $\mathrm{d}\mathfrak{a}(t)$ easily gives $\mathrm{d}\mathfrak{a}(t)=\mu_{\mathbf{e}}S_0\,\mathrm{d}t$. On the other hand, $\mathfrak{m}^{\det}(t)=E[\langle M\rangle_t]$ from Lemma 2.5 and (4.15) yield $\mathrm{d}\mathfrak{m}^{\det}(t)=\sigma_{\mathbf{e}}^2E[S_t^2]\,\mathrm{d}t$ via Fubini's theorem. To calculate $E[S_t^2]$, we use $S=S_0e^L$ and the definition (4.12) of $\lambda_{\mathbf{e}}$ to obtain $E[S_t^2]=S_0^2e^{\lambda_{\mathbf{e}}t}$. This gives the formula for $\mathrm{d}\mathfrak{m}^{\det}(t)$ and then also (4.16), proving 3). □

Lemma 4.10. *Suppose L is as in Proposition 4.8 and $S=S_0e^L$ with $S_0>0$. If $\sigma_{\mathbf{e}}^2\neq 0$, then S is standard with $\boldsymbol{\Theta}(\mathrm{d}\mathfrak{s}^{\det})=L^2(\mathrm{d}\mathfrak{m}^{\det})=L^2(\mathrm{d}t)$, and for every $H\in\mathcal{L}^2$, we have*

$$\pi^H(t) = \frac{E[\Pi_t^HS_t^2]}{E[S_t^2]} = E_R[\Pi_t^H] \quad \mathrm{d}t\text{-}a.e., \tag{4.19}$$

where $R\approx P$ is defined by $\mathrm{d}R/\mathrm{d}P := e^{\widehat{L}_T}$ with $\widehat{L}_t := 2L_t-\lambda_{\mathbf{e}}t$, $t\in[0,T]$.

Proof. For any bounded Borel function δ, (4.13) gives

$$\|\delta\|_{L^1(\mathrm{d}f)} = S_0\|\delta\|_{L^1(\mathrm{d}g)} = S_0|\mu_{\mathbf{e}}|\int_0^T|\delta(t)|e^{\mu_{\mathbf{e}}t}\,\mathrm{d}t.$$

On the other hand, using the expression for $\mathrm{d}\mathfrak{m}^{\det}(t)$ from Lemma 4.9, 3) to compute

$\|1\|^2_{L^2(\mathrm{d}\mathfrak{m}^{\det})} = S_0^2\sigma_{\mathbf{e}}^2(e^{\lambda_{\mathbf{e}}T}-1)/\lambda_{\mathbf{e}}$ gives via Cauchy–Schwarz that

$$\begin{aligned}\|\delta\|_{L^1(\mathrm{d}g)} &= \frac{|\mu_{\mathbf{e}}|}{S_0^2\sigma_{\mathbf{e}}^2}\int_0^T |\delta(t)|e^{(\mu_{\mathbf{e}}-\lambda_{\mathbf{e}})t}\,\mathrm{d}\mathfrak{m}^{\det}(t)\\ &\le \frac{|\mu_{\mathbf{e}}|\lambda_{\mathbf{e}}\max(1,e^{(\mu_{\mathbf{e}}-\lambda_{\mathbf{e}})T})}{S_0\sigma_{\mathbf{e}}(e^{\lambda_{\mathbf{e}}T}-1)}\|\delta\|_{L^2(\mathrm{d}\mathfrak{m}^{\det})}.\end{aligned}$$

This implies that $D_2(\mathrm{d}\mathfrak{s}^{\det})$ is satisfied and hence $\mathbf{\Theta}(\mathrm{d}\mathfrak{s}^{\det}) = L^2(\mathrm{d}\mathfrak{m}^{\det})$ by Lemma 3.1. Moreover, g from (4.13) clearly satisfies (2.13) so that S is standard. Finally, the density $t\mapsto \frac{\mathrm{d}\mathfrak{m}^{\det}}{\mathrm{d}t}(t) = S_0^2\sigma_{\mathbf{e}}^2e^{\lambda_{\mathbf{e}}t}$ is bounded away from 0 (because $\sigma_{\mathbf{e}}^2\neq 0$) and ∞ on $[0,T]$ so that we get $L^2(\mathrm{d}\mathfrak{m}^{\det}) = L^2(\mathrm{d}t)$.

For $H\in\mathcal{L}^2$, the first equality in (4.19) follows directly from (4.15) and Lemma 4.1. For the second, Step 3) in the proof of Lemma 4.9 gives $E[S_t^2] = S_0^2e^{\lambda_{\mathbf{e}}t}$ so that

$$\frac{S_t^2}{E[S_t^2]} = e^{2L_t-\lambda_{\mathbf{e}}t} =: e^{\widehat{L}_t},\quad t\in[0,T].$$

Clearly, $\widehat{L}$ is a Lévy process, and $e^{\widehat{L}}$ is integrable by Proposition 4.8, 2), with $E[e^{\widehat{L}_t}]\equiv 1$ by construction. Hence $e^{\widehat{L}}$ is a martingale, and π^H can be rewritten as

$$\pi^H(t) = \frac{E[\Pi_t^H S_t^2]}{E[S_t^2]} = E[\Pi_t^H e^{\widehat{L}_t}] = E[\Pi_t^H e^{\widehat{L}_T}] = E_R[\Pi_t^H]\quad \mathrm{d}t\text{-a.e.}$$

because Π_t^H is $\mathcal{F}_t$-measurable. □

After the preceding preparations, we can now present the solutions of our two quadratic optimisation problems in the exponential Lévy setting.

Theorem 4.11. *If L is as in Proposition 4.8, $\sigma_{\mathbf{e}}^2\neq 0$ and $S=S_0e^L$ with $S_0>0$, then:*

1) For each $H\in\mathcal{L}^2$, the solution (c^H,ϑ^H) to the MVH problem (3.3) *exists and is given by*

$$c^H = E[H]-\mu_{\mathbf{e}}S_0\int_0^T\pi^H(t)\,\mathrm{d}t,\quad \vartheta^H(t) = \pi^H(t)-\mu_{\mathbf{e}}\int_t^T\pi^H(u)\,\mathrm{d}u\quad \mathrm{d}t\text{-}a.e.$$

2) The solution to the MVPS problem (3.13) *exists and is given by*

$$\begin{aligned}\vartheta^{MV}(t) &= \frac{1}{2\alpha}\frac{\mu_{\mathbf{e}}e^{-\lambda_{\mathbf{e}}T}}{S_0\lambda_{\mathbf{e}}\sigma_{\mathbf{e}}^2}\big(\mu_{\mathbf{e}}+(\lambda_{\mathbf{e}}-\mu_{\mathbf{e}})e^{\lambda_{\mathbf{e}}(T-t)}\big)\quad \mathrm{d}t\text{-}a.e.,\\ MV_\alpha &= \frac{1}{4\alpha}\frac{\mu_{\mathbf{e}}^2}{\sigma_{\mathbf{e}}^2}\frac{1-e^{-\lambda_{\mathbf{e}}T}}{\lambda_{\mathbf{e}}}.\end{aligned}$$

Proof. This argument parallels the proof of Theorem 4.5, and so we only point out the differences. In view of Lemma 4.10, computing $\pi^H(t) = E_R[\Pi_t^H]$ from Π_t^H depends via R also on the model for L or S. The formula for c^H follows from (3.7) via $\mathrm{d}\mathfrak{a}(t) = \mu_{\mathbf{e}}S_0\,\mathrm{d}t$. Plugging f and g into the definition (2.15) of $\mathcal{A}^{-1} = \mathcal{A}^{\leftarrow}$ yields

$$\mathcal{A}^{-1}[\delta](t) = \delta(t)-\mu_{\mathbf{e}}\int_t^T\delta(u)\,\mathrm{d}u,\quad t\in[0,T],$$

so that the formula for ϑ^H follows from (3.8). The formulas for ϑ^{MV} and MV_α use Theorem 3.12, the expression for $\mathcal{A}^{-1}$, (4.16) and $\mathrm{d}\mathfrak{m}^{\det}(t) = S_0^2\sigma_{\mathbf{e}}^2 e^{\lambda_{\mathbf{e}}t}\,\mathrm{d}t$, together with some straightforward computations. $\square$

Remark 4.12. In the special case where $\nu \equiv 0$ is the zero measure, we recover for $S_t = S_0 e^{L_t} = S_0 e^{bt+\sqrt{\Sigma}W_t}$ by Proposition 4.8, 3) via $\mathrm{d}S_t = \sqrt{\Sigma}S_t\,\mathrm{d}W_t + \mu_{\mathbf{e}}S_t\,\mathrm{d}t$ the *Black–Scholes model* of geometric Brownian motion with volatility $\sigma_{\mathbf{e}} = \sqrt{\Sigma}$ and drift $\mu_{\mathbf{e}} = b + \frac{1}{2}\sigma_{\mathbf{e}}^2$; see Section 4.3.2.

4.3. *Explicit hedging results for calls and puts*

To illustrate our results more concretely, we present in this section the optimal deterministic hedging strategies for European call options in the Bachelier and Black–Scholes models. So the payoff in the MVH problem is $H = (S_T - K)^+$, and we show how to obtain (c^H, ϑ^H) from Theorem 3.6 in two specific models for S.

4.3.1. *The Bachelier model*

Suppose that $S_t = S_0 + \sigma W_t + \mu t$ with a Brownian motion W and constants $S_0 \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$. The filtration $\mathbb{F}$ is generated by W (and augmented by the P-null-sets from $\mathcal{F}_T^W$) so that W has the martingale representation property in $\mathbb{F}$ and S admits a unique equivalent martingale measure Q^* under which $S = S_0 + \sigma W^*$ for a Q^*-Brownian motion W^*. The canonical decomposition $S = S_0 + M + A$ is given by $M_t = \sigma W_t$, $A_t = \mu t$, and since the Lévy triplet of the underlying L is $(b, \Sigma, \nu) = (\mu, \sigma^2, 0)$, we get in (4.4) that $\mu_{\mathbf{a}} = \mu$.

To find the GKW decomposition of H with respect to M, we first compute

$$
\begin{aligned}
V_t^H &:= E[H \,|\, \mathcal{F}_t] = E[(S_T - K)^+ \,|\, \mathcal{F}_t] \\
&= E\big[\big(S_0 + M_t + \sigma(W_T - W_t) + A_T - K\big)^+ \,\big|\, \mathcal{F}_t\big] \\
&= E\bigg[\bigg(\sigma\sqrt{T-t}\,\frac{W_T - W_t}{\sqrt{T-t}} - \widetilde{K}_t\bigg)^+ \,\bigg|\, \mathcal{F}_t\bigg],
\end{aligned}
$$

where we set $\widetilde{K}_t := K - S_0 - \mu T - M_t$. This is $\mathcal{F}_t$-measurable, and $(W_T - W_t)/\sqrt{T-t}$ is independent of $\mathcal{F}_t$ with a standard normal distribution. A straightforward computation therefore yields

$$
\begin{aligned}
V_t^H &= \sigma\sqrt{T-t}\,\phi\bigg(\frac{S_0 + M_t + \mu T - K}{\sigma\sqrt{T-t}}\bigg) \\
&\quad + (S_0 + M_t + \mu T - K)\Phi\bigg(\frac{S_0 + M_t + \mu T - K}{\sigma\sqrt{T-t}}\bigg) \\
&=: v(t, M_t),
\end{aligned}
$$

where $\phi = \Phi'$ and Φ denote the density and the cumulative distribution function of the standard normal distribution. By applying Itô's formula and exploiting the fact that V^H is a P-martingale, we obtain the GKW decomposition

$$H = V_T^H = v(0, M_0) + \int_0^T \frac{\partial v}{\partial x}(t, M_t)\, \mathrm{d}M_t = E[H] + \int_0^T \Pi_t^H \, \mathrm{d}M_t \quad P\text{-a.s.}$$

with

$$\Pi_t^H = \frac{\partial v}{\partial x}(t, M_t) = \Phi\left(\frac{S_t + \mu(T-t) - K}{\sigma\sqrt{T-t}}\right), \quad t \in [0, T],$$

and

$$E[H] = V_0^H = \sigma\sqrt{T}\,\phi\left(\frac{S_0 + \mu T - K}{\sigma\sqrt{T}}\right) + (S_0 + \mu T - K)\Phi\left(\frac{S_0 + \mu T - K}{\sigma\sqrt{T}}\right).$$

By Theorem 4.5, the solution (c^H, ϑ^H) to the MVH problem is therefore given by

$$\begin{aligned} c^H &= E[H] - \mu \int_0^T \vartheta^H(t)\, \mathrm{d}t, \\ \vartheta^H(t) &= E[\Pi_t^H] = E\left[\Phi\left(\frac{S_t + \mu(T-t) - K}{\sigma\sqrt{T-t}}\right)\right], \quad t \in [0, T]. \end{aligned}$$

Remark 4.13. For a European put option whose payoff is

$$H' = (K - S_T)^+ = H - (S_T - K),$$

we obtain $V_t^{H'} = V_t^H - (S_0 + M_t + \mu T - K)$ and therefore $\Pi_t^{H'} = \Pi_t^H - 1$ and hence $\pi^{H'}(t) = \pi^H(t) - 1$. Because π^H has values in $(0, 1)$, hedging a put with a deterministic strategy thus always involves a short position in S, exactly like the full information strategy.

While we cannot compute c^H and ϑ^H in closed form, we can illustrate our results numerically. We choose the parameters $T = 1$, $S_0 = 100$, $\mu = 5\%$, $\sigma = 20\%$ and consider the three strikes $K = 95, 100, 105$ so that the option starts out in the money, at the money or out of the money, respectively. Table 1 gives for each case the full information price $E_{Q^*}[H]$ and the zero-information price c^H.

Table 1. Call prices in the Bachelier model.

	$K = 95$	$K = 100$	$K = 105$
full information price	10.73	7.98	5.73
zero-information price	10.50	7.73	5.48

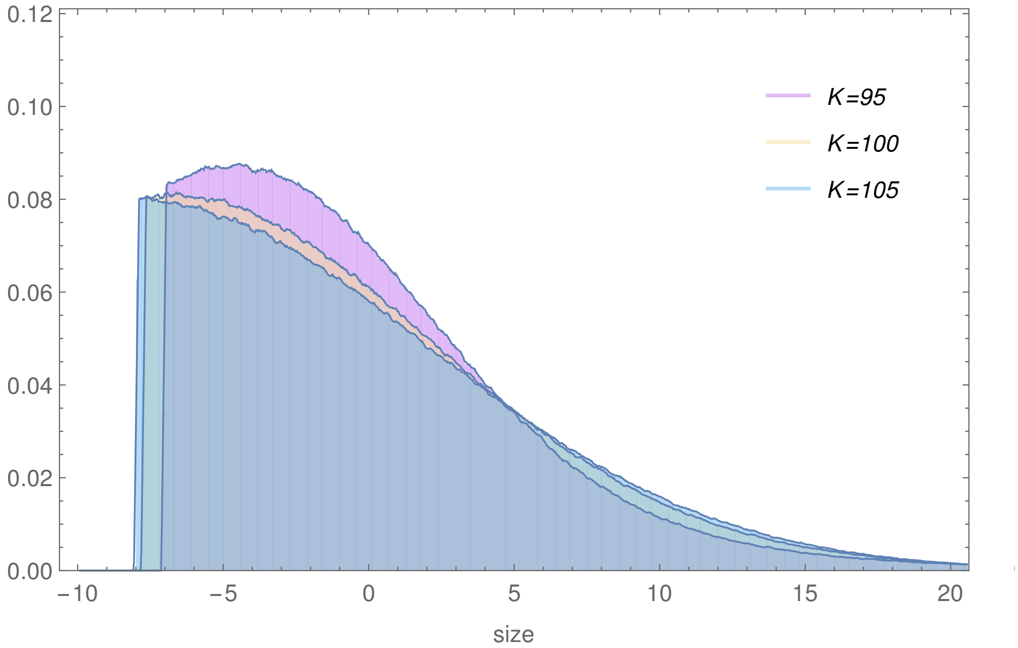

Figure 1. P&L histograms in the Bachelier model.

Figure 1 provides for each of the three above cases the histogram of the hedging error (P&L) $H - c^H - \int_0^T \vartheta^H(t)\,\mathrm{d}S_t$ resulting from the optimal strategy. The latter was calculated by discretising the time interval $[0,T]$ into $N = 100$ steps and using numerical integration. The histograms are then based on a sample of 5×10^6 sample paths, where the stochastic integral was calculated by the Euler–Maruyama method (with also $N = 100$ points) and using the strategy obtained above. A summary of the corresponding statistical quantities is given in Table 2, and Figure 2 presents for each case the optimal deterministic strategy ϑ^H.

Table 2. P&L statistics in the Bachelier model.

	$K = 95$	$K = 100$	$K = 105$
mean	0.00	0.00	0.00
variance	30.14	34.66	36.37
10% quantile	-5.85	-6.50	-6.72
25% quantile	-4.13	-4.62	-4.79
50% quantile	-1.12	-1.21	-1.23
75% quantile	2.80	3.32	3.53
90% quantile	7.13	8.11	8.47

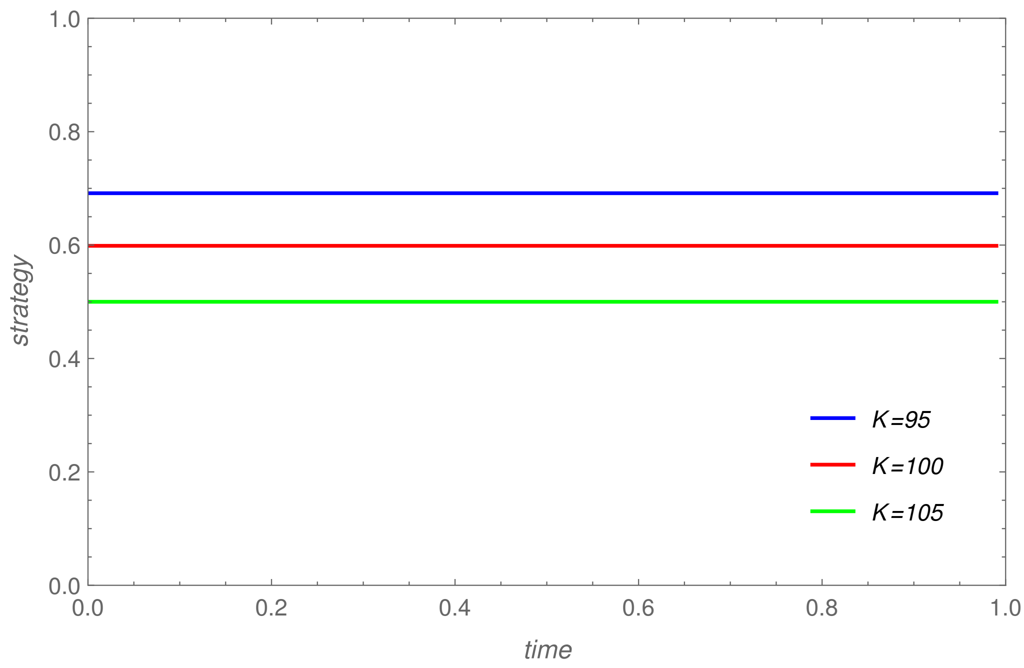

Figure 2. Optimal strategy in the Bachelier model for various strike prices.

Finally, we compare for the at-the-money case $K = 100$ the optimal deterministic strategy ϑ^H and the full information perfect hedging strategy ϑ^*. The latter is obtained by computing $V_t^* = E_{Q^*}[H \,|\, \mathcal{F}_t]$, $t \in [0,T]$, and representing H as

$$H = V_T^* = E_{Q^*}[H] + \int_0^T \vartheta_t^* \,\mathrm{d}S_t \quad P\text{-a.s.}$$

with

$$\vartheta_t^* = \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right), \quad t \in [0,T]. \tag{4.20}$$

The computations are analogous to those for V^H and Π^H, except that we can formally set $\mu = 0$ because we work under the martingale measure Q^*. We first present in Figure 3 a few realisations of ϑ^* together with ϑ^H. Figure 4 then plots ϑ^H against the quantiles of ϑ^* at 10%, 20%, ..., 90%, using the explicit expression in (4.20).

Remark 4.14. Our numerical computations seem to indicate that the optimal deterministic strategy $t \mapsto \vartheta^H(t)$ is constant in t. It is not difficult to show that $\vartheta^H(0) = \vartheta^H(T)$, but we have so far not been able to prove the full constancy.

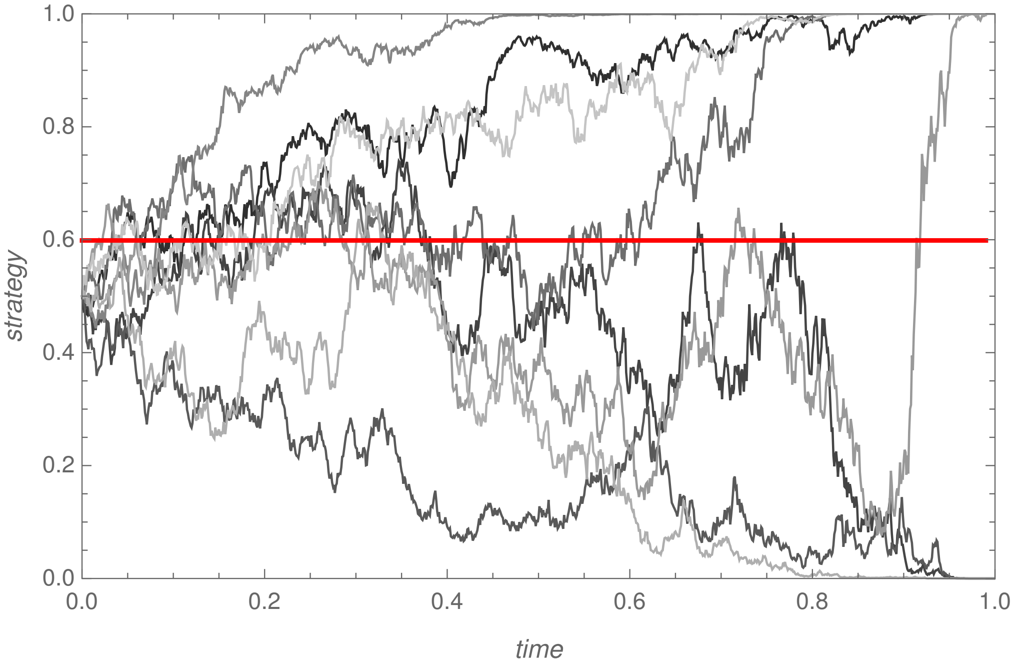

Figure 3. Bachelier model, $K = 100$ (at the money): ϑ^H (red) versus some realisations of ϑ^*.

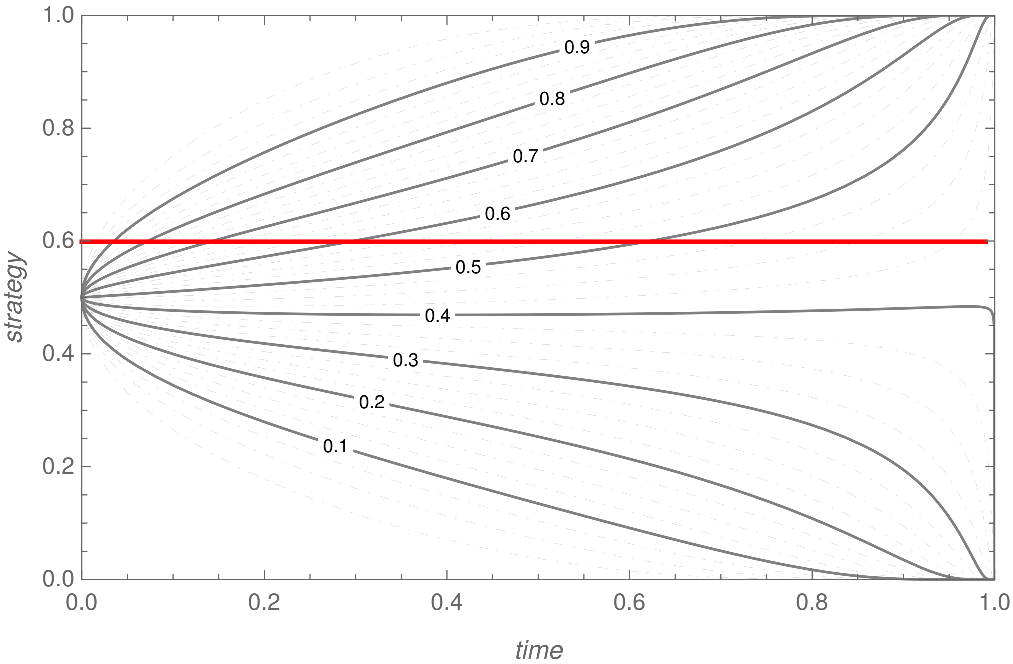

Figure 4. Bachelier model, $K = 100$ (at the money): ϑ^H (red) versus quantiles of ϑ^*.

4.3.2. *The Black–Scholes model*

Now consider the case where $S_t = S_0 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$ with a Brownian motion W and constants $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$. The filtration $\mathbb{F}$ is again generated by W (and augmented by the P-nullsets from $\mathcal{F}^W_T$). The canonical decomposition $S = S_0 + M + A$ is here given by $\mathrm{d}M_t = \sigma S_t \,\mathrm{d}W_t$, $\mathrm{d}A_t = \mu S_t \,\mathrm{d}t$, and because now $(b, \Sigma, \nu) = (\mu - \frac{1}{2}\sigma^2, \sigma^2, 0)$, we get in (4.10) that $\mu_{\mathbf{e}} = \mu$.

As for the Bachelier model, we start by computing the GKW decomposition of H with respect to M. To that end, we introduce the martingale $\widehat{S}_t := e^{-\mu t} S_t$, $t \in [0, T]$, and note that $\mathrm{d}\widehat{S}_t = \sigma \widehat{S}_t \,\mathrm{d}W_t = e^{-\mu t}\,\mathrm{d}M_t$. We then consider again

$$\begin{aligned}
V^H_t &:= E[H \,|\, \mathcal{F}_t] = E[(S_T - K)^+ \,|\, \mathcal{F}_t] \\
&= E\big[\big(\widehat{S}_t e^{\mu T} e^{\sigma(W_T - W_t) - \frac{1}{2}\sigma^2(T-t)} - K\big)^+ \,\big|\, \mathcal{F}_t\big] \\
&= e^{\mu T}\widehat{S}_t E\Big[\Big(e^{\sigma\sqrt{T-t}\frac{W_T - W_t}{\sqrt{T-t}} - \frac{1}{2}\sigma^2(T-t)} - \widehat{K}_t\Big)^+ \,\Big|\, \mathcal{F}_t\Big],
\end{aligned}$$

where $\widehat{K}_t := \frac{K}{e^{\mu T}\widehat{S}_t}$ is $\mathcal{F}_t$-measurable and $(W_T - W_t)/\sqrt{T-t}$ is independent of $\mathcal{F}_t$ with a standard normal distribution. A routine computation gives

$$\begin{aligned}
V^H_t &= e^{\mu T}\widehat{S}_t \Phi\bigg(\frac{\log(\widehat{S}_t/K) + \mu T + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\bigg) \\
&\quad - K\Phi\bigg(\frac{\log(\widehat{S}_t/K) + \mu T - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\bigg) \\
&=: v(t, \widehat{S}_t),
\end{aligned}$$

and the same argument as in the Bachelier case then yields the GKW decomposition

$$H = V^H_T = v(0, S_0) + \int_0^T \frac{\partial v}{\partial x}(t, \widehat{S}_t)\,\mathrm{d}\widehat{S}_t = E[H] + \int_0^T \Pi^H_t \,\mathrm{d}M_t \quad P\text{-a.s.}$$

with

$$\begin{aligned}
\Pi^H_t &= e^{-\mu t}\frac{\partial v}{\partial x}(t, \widehat{S}_t) = e^{\mu(T-t)}\Phi\bigg(\frac{\log(S_t/K) + (\mu + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\bigg), \quad t \in [0, T], \\
E[H] &= e^{\mu T} S_0 \Phi\bigg(\frac{\log(S_0/K) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\bigg) - K\Phi\bigg(\frac{\log(S_0/K) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\bigg),
\end{aligned}$$

by straightforward but slightly tedious calculations. By Theorem 4.11, the solution (c^H, ϑ^H) to the MVH problem is then given by

$$\begin{aligned}
c^H &= E[H] - \mu S_0 \int_0^T \pi^H(t)\,\mathrm{d}t, \\
\vartheta^H(t) &= \pi^H(t) - \mu \int_t^T \pi^H(u)\,\mathrm{d}u, \quad t \in [0, T],
\end{aligned}$$

with π^H due to Lemma 4.10 given by

$$\pi^H(t) = \frac{E[\Pi^H_t S_t^2]}{E[S_t^2]}, \quad t \in [0, T].$$

By using the formulas for Π_t^H and S_t and simplifying, this can be rewritten as

$$\pi^H(t) = e^{\mu(T-t)} E\left[e^{2(\sigma W_t - \sigma^2 t)} \Phi\left(\frac{\log(S_0/K) + \sigma W_t - \sigma^2 t + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T-t}}\right)\right]$$

which can readily be computed numerically.

As in the Bachelier case, we illustrate our results numerically, with the same parameters $T = 1$, $S_0 = 100$, $\mu = 5\%$, $\sigma = 20\%$ and strikes $K = 95, 100, 105$, and using the same numerical methods. For each case, Table 3 gives the full information price $E_{Q^*}[H]$ and the zero-information price c^H, and Figure 5 provides the histogram from 5×10^6 simulations of the hedging error (P&L) $H - c^H - \int_0^T \vartheta^H(t)\,\mathrm{d}S_t$ resulting from the optimal strategy. A summary of the corresponding statistical quantities is given in Table 4.

Table 3. Call prices in the Black–Scholes model.

	$K = 95$	$K = 100$	$K = 105$
full information price	10.51	7.97	5.90
zero-information price	10.13	7.54	5.45

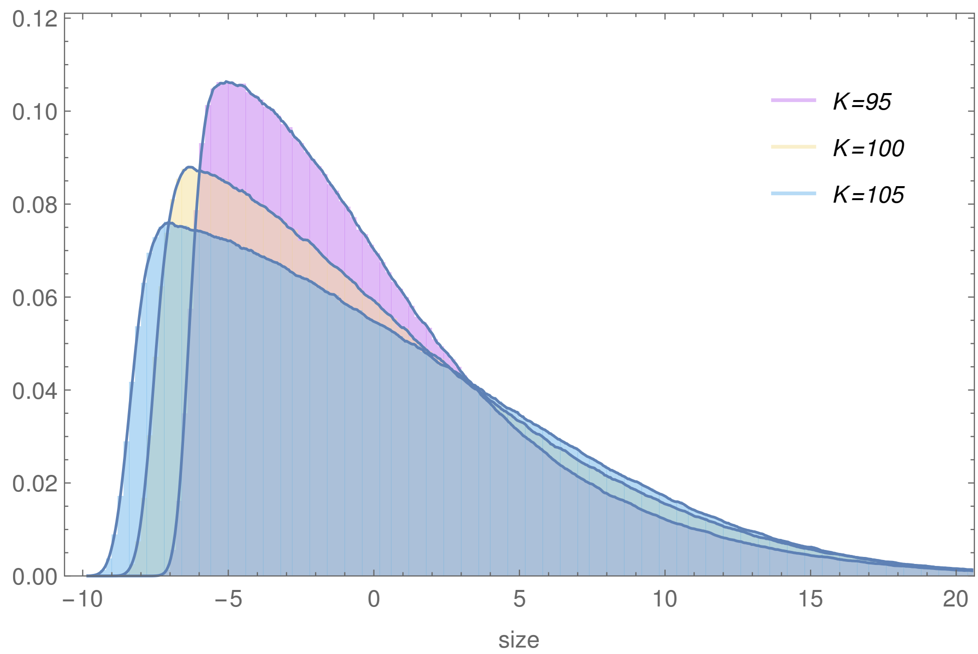

Figure 5. P&L histograms in the Black–Scholes model.

Table 4. P&L statistics in the Black–Scholes model.

	$K = 95$	$K = 100$	$K = 105$
mean	0.00	0.00	0.00
variance	27.43	34.59	39.47
10% quantile	-5.39	-6.39	-7.04
25% quantile	-3.96	-4.64	-5.02
50% quantile	-1.29	-1.32	-1.25
75% quantile	2.60	3.36	3.77
90% quantile	7.27	8.37	8.79

Figure 6 presents for each case the optimal deterministic strategy ϑ^H and a few realisations of the full information perfect hedging strategy ϑ^*, which is of course given by the familiar Black–Scholes delta hedge

$$\vartheta_t^* = \Phi\left(\frac{\log(S_t/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right), \quad t \in [0,T].$$

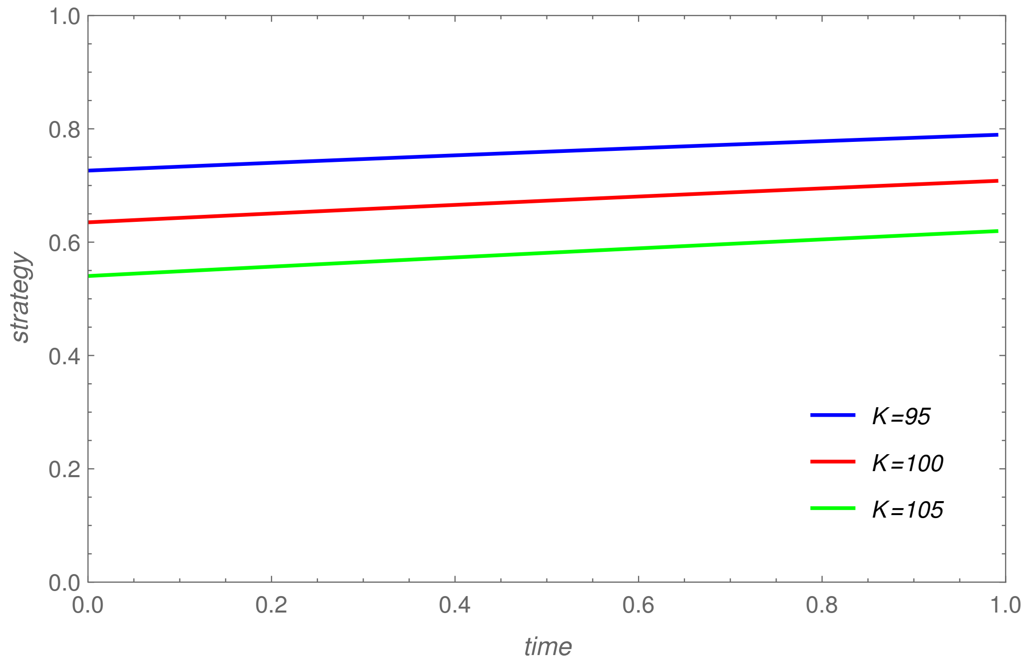

Figure 6. Optimal strategy in the Black–Scholes model for various strike prices.

Finally, we again compare for the at-the-money case $K = 100$ the optimal deterministic strategy ϑ^H and the full information perfect hedging strategy ϑ^*. As in Section 4.3.1, we show in Figure 7 a few realisations of ϑ^* together with ϑ^H, and in Figure 8 a plot of ϑ^H against the quantiles of ϑ^* at 10%, 20%, …, 90%.

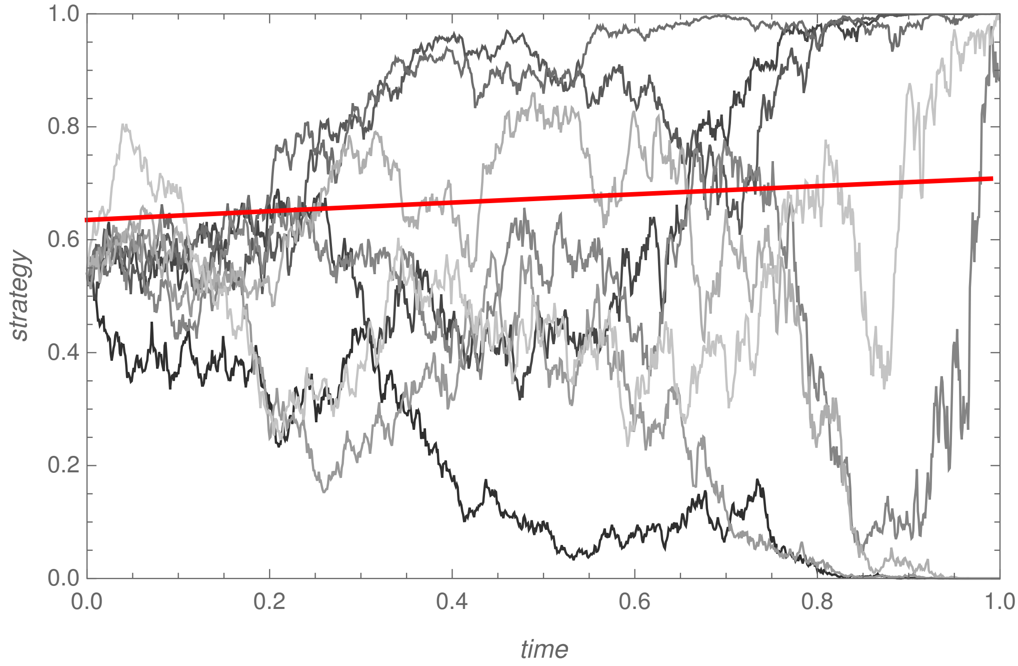

Figure 7. Black–Scholes model, $K = 100$ (at the money): ϑ^H (red) versus some realisations of ϑ^*.

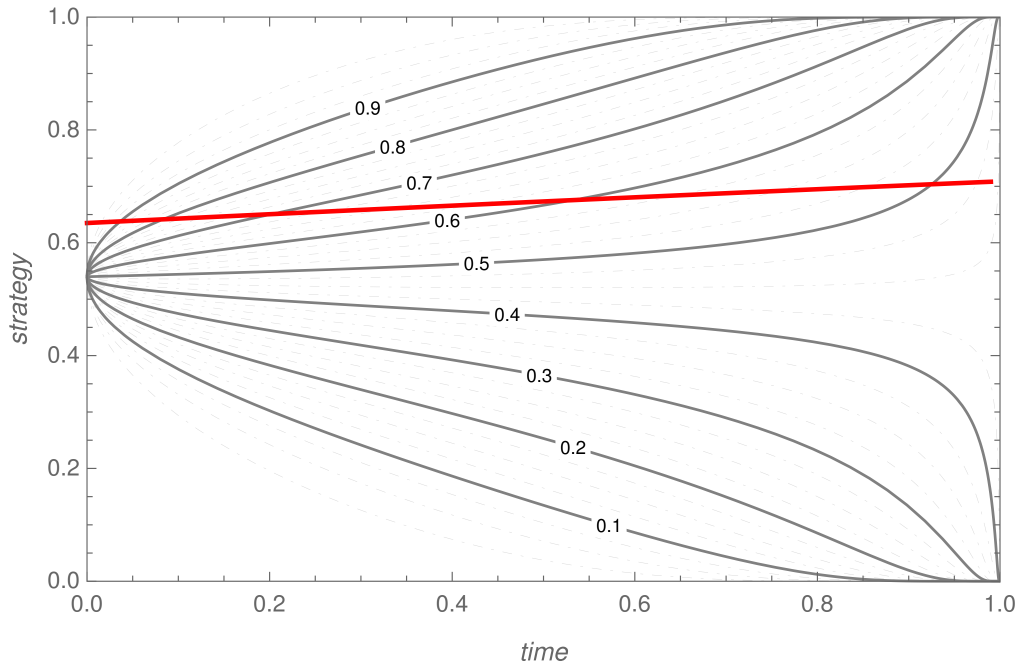

Figure 8. Black–Scholes model, $K = 100$ (at the money): ϑ^H (red) versus quantiles of ϑ^*.

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