SEMIMARTINGALES AND HEDGING IN INCOMPLETE MARKETS

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Introduction. Consider a stochastic process (X_t) which models the price evolution of a risk asset (e.g., a stock). Let the random variable H describe a payoff to be made at a fixed time $T > 0$ (e.g., a call option or stop-loss contract $H = (X_T - K)^+$). Two important questions in such a setting are:

1) What is a fair price (or insurance premium) for H at time 0?

2) What is a good hedging strategy (or insurance policy)?

We present here an approach to answering 2) in a very general framework with respect to a sort of mean-variance criterion. It may be worth mentioning that 1) is an open problem in this general case.

1. The hedging problem. We assume that the price process $X = X_0 + M + A$ of our stock is a special *semimartingale*. A *trading strategy* is a pair of processes (ξ_t), (η_t), where ξ is predictable and describes the number of shares we hold at time t and η is adapted and gives the amount held in some riskless asset whose value per unit is normalized to one. Such a strategy induces a wealth process $V_t = \xi_t X_t + \eta_t$ and a process of cumulative hedging costs $C_t = V_t - \int_0^t \xi_u \, dX_u$, wealth minus gains from trade. To measure the riskiness of a strategy, we introduce the process $R_t := \mathbf{E}[(C_T - C_t)^2 | \mathcal{F}_t]$. The hedging problem for a given contingent claim H can then be loosely formulated as

(HP) Find a strategy (ξ, η) with terminal wealth $V_T = H$ which has minimal local variances.

An exact statement is provided by the following

DEFINITION. A strategy $\varphi = (\xi, \eta)$ is called *optimal* if $V_T = H$ P-a.s. and if

$$\liminf_{n\to\infty} \sum_{t_i \in \tau_n} \left(R_{t_i}(\varphi + \Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\varphi)\right) I_{(t_i,t_{i+1}]} / \mathbf{E}\left[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} | \mathcal{F}_{t_i}\right] \geqq 0$$

holds $P \times \langle M\rangle$-a.e. on $\Omega \times [0, T]$, for all small perturbations Δ and for every increasing sequence (τ_n) of partitions of $[0, T]$ with $|\tau_n| \to 0$. For additional details and motivation, see [5].

2. Characterizing optimal strategies.

THEOREM. *Assume that M is square-integrable, A is continuous, $A \ll \langle M\rangle$ and that $dA/d\langle M\rangle \in \mathcal{L}\log^+ \mathcal{L}(P \times \langle M\rangle)$. Then the following statements are equivalent.*

1) *(ξ, η) is an optimal strategy for H.*

2) *The cost process $C(\xi, \eta)$ is a martingale and orthogonal to M.*

3) *$C(\xi, \eta)$ is a martingale, and ξ satisfies the optimality equation $\xi + \mu^{\xi,A} = \mu^H$. (Here, μ^H and $\mu^{\xi,A}$ denote respectively the integrands of the projections of H and $\int_0^T \xi_u \, dA_u$ on the stable subspace generated by M.)*

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Sketch of the proof (see [5] for details).

a) If C is not a martingale, one can construct a better strategy, using the fact that R is a conditional mean squared error. As a consequence, η is determined by ξ, and it will suffice to vary ξ.

b) If we compare ξ to some $\xi + \delta$, we obtain an expression of the form

$$\left(C - \int \delta \, dX\right)^2 - C^2 = -2C \int \delta \, dX + \left(\int \delta \, dX\right)^2.$$

Dividing by $\langle M \rangle$ and using a technical differentiation argument for semimartingales (see [4]) leads to the condition $C \int \delta \, dM = 0$, thus proving orthogonality.

For the purpose of actually finding optimal strategies (or equivalently for solving the optimality equation), the following formulation is more useful. See [1] for a proof.

LEMMA. *H admits an optimal strategy if and only if it has a decomposition*

$$H = H_0 + \int_0^T \xi_u^H \, dX_u + L_T^H, \tag{D}$$

where the martingale L^H is orthogonal to M. The optimal strategy is then determined by $\xi = \xi^H$ and $C = H_0 + L^H$.

Remark. There are two cases where a decomposition (D) is immediately available. The first is a so-called *complete* model, where every random variable has a representation as a stochastic integral of X; see [3]. The second is the case where $X \equiv M$ is a *martingale*, since (D) is then provided by the Kunita–Watanabe projection theorem; see [2]. The general incomplete semimartingale case is more difficult, and we shall sketch two possible lines of approach in the next sections.

3. The minimal martingale measure. The first idea is to try to get back to a martingale situation. Let $P^* \approx P$ be any martingale measure for X and consider the Kunita–Watanabe decomposition of H under P^*:

$$H = H_0^* + \int_0^T \xi_u^* \, dX_u + L_T^*,$$

where the P^*-martingales X and L^* are P^*-orthogonal. Then we can make two observations:

(i) If L^* is also a P-martingale and P-orthogonal to M, we obviously have the *existence* of a decomposition (D).

(ii) If every P-martingale L which is P-orthogonal to M is also a P^*-martingale and P^*-orthogonal to X, we obtain *uniqueness* for the decomposition (D), since the Kunita–Watanabe decomposition is unique.

Thus, the problem is to find a martingale measure $\widehat{P} \approx P$ satisfying (i) and (ii). Such a $\widehat{P}$ will be called a *minimal martingale measure.* Intuitively, $\widehat{P}$ is that martingale measure for X which is closest to the original measure P. The following result is proved in [1].

THEOREM 1. *If X is continuous, this approach works. Furthermore, $\widehat{P}$ is uniquely determined and can be found by minimizing a certain functional involving the relative entropy $H(\cdot \mid P)$.*

4. Incomplete information. A second idea is to examine claims H which are stochastic integrals with respect to some larger filtration $\widetilde{\mathbf{F}} \supseteq \mathbf{F}$. Let us assume that H has the form

$$H = H_0 + \int_0^T \tilde{\xi}_u \, dX_u$$

for some $\widetilde{\mathbf{F}}$-predictable integrand $\tilde{\xi}$. Since our strategies ξ have to be $\mathbf{F}$-predictable, $\tilde{\xi}$ cannot serve as an admissible strategy. However, one has the following result (see [1] and [6]).

THEOREM 2. *Assume that the decomposition $X = X_0 + M + A$ is the same under both filtrations $\mathbf{F}$ and $\widetilde{\mathbf{F}}$, and that $\langle M\rangle^{\widetilde{\mathbf{F}}} = \langle M\rangle^{\mathbf{F}}$. Then every H of the above form admits a decomposition* (D), *and the corresponding optimal strategy ξ^H can be obtained by projection:$\xi^H = \mathbf{E}_M\left[\tilde{\xi}\,|\,\mathcal{P}(\mathbf{F})\right]$, where $\mathbf{E}_M$ denotes expectation under the measure $P \times \langle M\rangle$, and $\mathcal{P}(\mathbf{F})$ is the predictable σ-field associated to $\mathbf{F}$.*

Intuitively, the conditions of the theorem say that the first and second-order structure of X is not changed by enlarging the filtration from $\mathbf{F}$ to $\widetilde{\mathbf{F}}$. This is sufficient since we essentially use a mean-variance criterion. The preceding method can for example be used to study a Black–Scholes type model with a random variance. See [1] for details of this application.

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