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by

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# Approximating Random Variables by Stochastic Integrals

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**Abstract:** Let $X$ be a semimartingale and $\Theta$ the space of all predictable $X$-integrable processes $\vartheta$ such that $\int \vartheta \, dX$ is in the space $\mathcal{S}^2$ of semimartingales. We consider the problem of approximating a given random variable $H \in \mathcal{L}^2$ by a stochastic integral $\int_0^T \vartheta_s \, dX_s$, with respect to the $\mathcal{L}^2$-norm. If $X$ is special and has the form $X = X_0 + M + \int \alpha \, d\langle M \rangle$, we construct a solution in feedback form under the assumptions that $\int \alpha^2 \, d\langle M \rangle$ is deterministic and that $H$ admits a strong F-S decomposition into a constant, a stochastic integral of $X$ and a martingale part orthogonal to $M$. We provide sufficient conditions for the existence of such a decomposition, and we give several applications to quadratic optimization problems arising in financial mathematics.

**Key words:** semimartingales, stochastic integrals, strong F-S decomposition, mean-variance tradeoff, option pricing, financial mathematics

**1991 Mathematics Subject Classification:** 60G48, 60H05, 90A09

**Running head:** $\mathcal{L}^2$-approximation by stochastic integrals

## 0. Introduction

In this paper, we study an approximation problem arising naturally in financial mathematics. Let $X$ be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ and denote by $\Theta$ the space of all predictable $X$-integrable processes $\vartheta$ such that $\int \vartheta \, dX$ is in the space $\mathcal{S}^2$ of semimartingales. Given an $\mathcal{F}_T$-measurable random variable $H \in \mathcal{L}^2$ and a constant $c \in I\!R$, we then consider the optimization problem

$$\text{(0.1)} \qquad \text{Minimize } E\left[\left(H - c - \int_0^T \vartheta_s \, dX_s\right)^2\right] \text{ over all } \vartheta \in \Theta.$$

If we also vary $c$, we thus want to approximate a random variable by the sum of a constant and a stochastic integral of $X$.

This problem has a very natural interpretation in financial mathematics, in particular in the theory of option pricing and option hedging. Think of $X_t$ as the discounted price at time $t$ of some risky asset (e.g., a stock) and of $\vartheta$ as a dynamic portfolio strategy, where $\vartheta_t$ describes the number of shares of $X$ to be held at time $t$. If we assume that there also exists some riskless asset (e.g., a bank account) with discounted price 1 at all times, then every $\vartheta \in \Theta$ determines a self-financing trading strategy whose value process is given by $c + \int \vartheta \, dX$, where $c \in I\!R$ denotes the initial capital at time 0. For a more detailed exposition, we refer to Harrison/Pliska (1981). In this context, the random variable $H$ is then interpreted as a contingent claim or random loss to be suffered at time $T$, and so (0.1) corresponds to minimizing the expected square of the net loss, $H - c - \int_0^T \vartheta_s \, dX_s$, at time $T$. This problem was previously studied in various forms of generality in Duffie/Richardson (1991), Schäl (1994), Schweizer (1992), Hipp (1993) and Schweizer (1993a, 1993b). Here we extend their results to the case of a general semimartingale in continuous time.

Once the basic problem (0.1) has been solved and if there is a nice dependence of the solution $\xi^{(c)}$ on $c$, one can readily give solutions to various optimization problems with quadratic criteria. These applications are discussed in section 4; they contain in particular the optimal choice of initial capital and strategy, the strategies minimizing the variance of $H - c - \int_0^T \vartheta_s \, dX_s$ either with or without the constraint of a fixed mean, and the approximation of a riskless asset.

Throughout the paper, $X$ will be an $I\!R^d$-valued semimartingale in $\mathcal{S}^2_{\mathrm{loc}}$. For ease of exposition, however, we formulate the results in this introduction only for $d = 1$. We assume that $X$ has a canonical decomposition of the form

$$X = X_0 + M + \int \alpha \, d\langle M \rangle$$

and call

$$\widetilde{K}_t := \int_0^t \frac{\alpha_s^2}{1 + \alpha_s^2 \Delta \langle M \rangle_s} \, d\langle M \rangle_s \qquad , \qquad 0 \le t \le T$$

the extended mean-variance tradeoff process of $X$. Our main result in section 2 then states that (0.1) has a solution $\xi^{(c)}$ for every $c \in I\!R$ if $\widetilde{K}$ is deterministic and if $H$ admits a decomposition of the form

$$H = H_0 + \int_0^T \xi_s^H \, dX_s + L_T^H \qquad P\text{-a.s.} \tag{0.2}$$

with $H_0 \in I\!R$, $\xi^H \in \Theta$ and $L^H$ a square-integrable martingale orthogonal to $\int \vartheta \, dM$ for every $\vartheta$. Moreover, $\xi^{(c)}$ is explicitly given in feedback form as the solution of

$$\xi_t^{(c)} = \xi_t^H + \frac{\alpha_t}{1 + \alpha_t^2 \Delta \langle M \rangle_t} \left( V_{t-}^H - c - \int_0^{t-} \xi_s^{(c)} \, dX_s \right) \quad , \qquad 0 \leq t \leq T, \tag{0.3}$$

where

$$V_t^H := H_0 + \int_0^t \xi_s^H \, dX_s + L_t^H \quad , \qquad 0 \leq t \leq T$$

is the intrinsic value process of $H$. An outline of the proof is given in section 2 and full details are provided in section 3. The argument extends the technique introduced in Duffie/Richardson (1991) and Schweizer (1992) for a diffusion process to the case of a general semimartingale.

The assumption that $\widetilde{K}$ is a deterministic process is very strong, but unfortunately indispensable for both our proof and the validity of (0.3). On the other hand, a decomposition of the form (0.2) can be obtained in remarkable generality. By slightly adapting a result of Buckdahn (1993) on backward stochastic differential equations, we show in section 5 that every $\mathcal{F}_T$-measurable $H \in \mathcal{L}^2$ admits such a decomposition if $\widetilde{K}$ is bounded and has jumps bounded by a constant $b < \frac{1}{2}$. Section 6 concludes the paper with several examples. In the positive direction, we consider continuous processes admitting an equivalent martingale measure and a multidimensional jump-diffusion model. On the other hand, a counterexample shows that (0.3) in general no longer solves (0.1) if $\widetilde{K}$ is allowed to be random.

## 1. Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $I\!F = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness. $T > 0$ is a fixed finite time horizon, and we assume that $\mathcal{F} = \mathcal{F}_T$. For unexplained notation, terminology and results from martingale theory, we refer to Dellacherie/Meyer (1982) and Jacod (1979). Let $X = (X_t)_{0 \leq t \leq T}$ be an $I\!R^d$-valued semimartingale in $\mathcal{S}^2_{\mathrm{loc}}$; for the canonical decomposition

$$X = X_0 + M + A$$

of $X$, this means that $M \in \mathcal{M}^2_{0,\mathrm{loc}}$ and that the variation $|A^i|$ of the predictable finite variation part $A^i$ of $X^i$ is locally square-integrable for each $i$. We can and shall choose versions of $M$ and $A$ such that $M^i$ and $A^i$ are right-continuous with left limits (RCLL for short) for each $i$. We denote by $\langle M^i \rangle$ the sharp bracket process associated to $M^i$, and we shall assume that for each $i$,

$$A^i \ll \langle M^i \rangle \qquad \text{with predictable density } \alpha^i = (\alpha_t^i)_{0 \leq t \leq T}. \tag{1.1}$$

Throughout the sequel, we fix a predictable increasing RCLL process $B = (B_t)_{0 \le t \le T}$ null at 0 such that $\langle M^i \rangle \ll B$ for each $i$; for instance, we could choose $B = \sum\limits_{i=1}^{d} \langle M^i \rangle$. This implies $\langle M^i, M^j \rangle \ll B$ for all $i, j$, and we define the predictable matrix-valued process $\sigma = (\sigma_t)_{0 \le t \le T}$ by

$$\sigma_t^{ij} := \frac{d\langle M^i, M^j \rangle_t}{dB_t} \qquad \text{for } i, j = 1, \ldots, d, \tag{1.2}$$

so that each $\sigma_t^{ij}$ is a symmetric nonnegative definite $d \times d$ matrix. If we define the predictable $I\!R^d$-valued process $\gamma = (\gamma_t)_{0 \le t \le T}$ by

$$\gamma_t^i := \alpha_t^i \sigma_t^{ii} \qquad \text{for } i = 1, \ldots, d, \tag{1.3}$$

then (1.1) and (1.2) imply that for each $i$,

$$A_t^i = \int_0^t \gamma_s^i \, dB_s \qquad , \qquad 0 \le t \le T. \tag{1.4}$$

**Definition.** The space $L^2_{(\mathrm{loc})}(M)$ consists of all predictable $I\!R^d$-valued processes $\vartheta = (\vartheta_t)_{0 \le t \le T}$ such that the process

$$\left( \int_0^t \vartheta_s^* \sigma_s \vartheta_s \, dB_s \right)_{0 \le t \le T} \qquad \text{is (locally) integrable,}$$

where $^*$ denotes transposition. The space $L^2_{(\mathrm{loc})}(A)$ consists of all predictable $I\!R^d$-valued processes $\vartheta = (\vartheta_t)_{0 \le t \le T}$ such that the process

$$\left( \int_0^t \left| \vartheta_s^* \gamma_s \right| dB_s \right)_{0 \le t \le T} \qquad \text{is (locally) square-integrable.}$$

Finally, we set $\Theta := L^2(M) \cap L^2(A)$.

If $\vartheta \in L^2_{(\mathrm{loc})}(M)$, the stochastic integral $\int \vartheta \, dM$ is well-defined, in $\mathcal{M}^2_{0(,\mathrm{loc})}$, and

$$\left\langle \int \vartheta \, dM, \int \psi \, dM \right\rangle_t = \int_0^t \vartheta_s^* \sigma_s \psi_s \, dB_s \qquad , \qquad 0 \le t \le T \tag{1.5}$$

for $\vartheta, \psi \in L^2_{\mathrm{loc}}(M)$. If $\vartheta \in L^2_{(\mathrm{loc})}(A)$, the process

$$\int_0^t \vartheta_s^* \, dA_s := \sum_{i=1}^{d} \int_0^t \vartheta_s^i \, dA_s^i = \int_0^t \vartheta_s^* \gamma_s \, dB_s \qquad , \qquad 0 \le t \le T \tag{1.6}$$

is well-defined as a Riemann-Stieltjes integral and has (locally) square-integrable variation $\left|\int \vartheta^* dA\right| = \int |\vartheta^*\gamma|\,dB$. For any $\vartheta \in \Theta$, the stochastic integral process

$$G_t(\vartheta) := \int_0^t \vartheta_s\,dX_s \qquad , \qquad 0 \le t \le T$$

is therefore well-defined and a semimartingale in $\mathcal{S}^2$ with canonical decomposition

$$G(\vartheta) = \int \vartheta\,dM + \int \vartheta^* dA. \tag{1.7}$$

We remark that the stochastic integral $\int \vartheta\,dM$ cannot be defined as the sum $\sum_{i=1}^d \int \vartheta^i dM^i$ in general; this is why we refrain from using the notation $\int \vartheta^* dM$. On the other hand, the notation $\int \vartheta^* dA$ makes sense due to (1.6).

Having set up the model, the *basic problem* we now want to study is

$$\text{Given } H \in \mathcal{L}^2 \text{ and } c \in I\!R, \text{ minimize } E\left[\big(H - c - G_T(\vartheta)\big)^2\right] \text{ over all } \vartheta \in \Theta. \tag{1.8}$$

In order to solve (1.8), we shall have to impose additional assumptions on $X$ and $H$. We first introduce the predictable matrix-valued process $J = (J_t)_{0\le t\le T}$ by setting

$$J_t^{ij} := \sum_{0<s\le t} \Delta A_s^i \Delta A_s^j \qquad \text{for } i,j = 1,\dots,d, \tag{1.9}$$

where $\Delta U_t := U_t - U_{t-}$ denotes the jump of $U$ at time $t$ for any RCLL process $U$. By (1.4), $J$ can be written as

$$J_t^{ij} = \int_0^t \kappa_s^{ij}\,dB_s \qquad , \qquad 0 \le t \le T, \tag{1.10}$$

where the predictable matrix-valued process $\kappa = (\kappa_t)_{0\le t\le T}$ is given by

$$\kappa_t^{ij} := \gamma_t^i \gamma_t^j \Delta B_t \qquad , \qquad 0 \le t \le T, \text{ for } i,j = 1,\dots,d.$$

Since $B$ is increasing, each $\kappa_t^{ij}$ is a symmetric nonnegative definite $d\times d$ matrix. The following terminology was partly introduced in Schweizer (1993c):

**Definition.** We say that $X$ *satisfies the structure condition (SC)* if there exists a predictable $I\!R^d$-valued process $\widehat{\lambda} = (\widehat{\lambda}_t)_{0\le t\le T}$ such that

$$\sigma_t \widehat{\lambda}_t = \gamma_t \qquad P\text{-a.s. for all } t \in [0,T] \tag{1.11}$$

and

$$\widehat{K}_t := \int_0^t \widehat{\lambda}_s^* \gamma_s\,dB_s < \infty \qquad P\text{-a.s. for all } t \in [0,T]. \tag{1.12}$$

We then choose an RCLL version of $\widehat{K}$ and call it the *mean-variance tradeoff (MVT) process* of $X$.

**Definition.** We say that $X$ *satisfies the extended structure condition (ESC)* if there exists a predictable $I\!R^d$-valued process $\widetilde{\lambda} = (\widetilde{\lambda}_t)_{0 \le t \le T}$ such that

$$(\sigma_t + \kappa_t)\widetilde{\lambda}_t = \gamma_t \qquad P\text{-a.s. for all } t \in [0,T] \tag{1.13}$$

and

$$\widetilde{K}_t := \int_0^t \widetilde{\lambda}_s^* \gamma_s \, dB_s < \infty \qquad P\text{-a.s. for all } t \in [0,T]. \tag{1.14}$$

We then choose an RCLL version of $\widetilde{K}$ and call it the *extended mean-variance tradeoff (EMVT) process* of $X$.

**Remarks. 1)** If $A$ is continuous, then $\kappa \equiv 0$ by (1.9) and (1.10); hence conditions (SC) and (ESC) are equivalent in that case. The exact relation between (SC) and (ESC) is shown in Lemma 1, and sufficient conditions for (SC) are provided in Schweizer (1993c). For instance, every continuous adapted process admitting an equivalent local martingale measure satisfies (SC).

**2)** For $d = 1$, the name "mean-variance tradeoff" can be heuristically explained in the following way: since $\sigma, \widehat{\lambda}, \alpha, \gamma$ are all scalars, equation (1.11) reduces to

$$\sigma_t \widehat{\lambda}_t = \sigma_t \alpha_t$$

by (1.3). Thus we can choose

$$\widehat{\lambda}_t = \alpha_t = \frac{dA_t}{d\langle M \rangle_t} = \frac{\text{“}E[dX_t|\mathcal{F}_{t-}]\text{”}}{\text{“Var}[dX_t|\mathcal{F}_{t-}]\text{”}};$$

of course, the last term is not rigorously defined.

**3)** Intuitively, both $\widehat{K}$ and $\widetilde{K}$ measure the extent to which $X$ deviates from being a martingale. More precisely, a process $X$ satisfying (ESC) is a martingale if and only if $\widetilde{K}_T = 0$ $P$-a.s. In fact, the "only if" part is immediate if one notices that $\widetilde{K} = \int \widetilde{\lambda}^* dA$ by (1.14) and (1.6), and the "if" part can be proved by using the definitions of $\widetilde{K}$, $\widetilde{\lambda}$ and $\kappa$. In the same way, a process $X$ satisfying (SC) is a martingale if and only if $\widehat{K}_T = 0$ $P$-a.s.

The next result summarizes some elementary properties of $\widetilde{\lambda}$ and $\widehat{\lambda}$; as they are straightforward to verify from the definitions, we omit the proof.

**Lemma 1. 1)** *$X$ satisfies (SC) if and only if $X$ satisfies (ESC) and*

$$\int_0^t \frac{1}{1 - \Delta \widetilde{K}_s} \, d\widetilde{K}_s < \infty \qquad P\text{-a.s. for all } t \in [0,T];$$

*in particular, we then have $\Delta\widetilde{K}_t < 1$ $P$-a.s. for all $t \in [0,T]$. If $X$ satisfies (SC), $\widehat{\lambda}$ and $\widetilde{\lambda}$ can be constructed from each other by*

$$\widehat{\lambda}_t = \frac{\widetilde{\lambda}_t}{1-\Delta\widetilde{K}_t} \qquad , \qquad \widetilde{\lambda}_t = \frac{\widehat{\lambda}_t}{1+\Delta\widehat{K}_t},$$

*and $\widehat{K}$, $\widetilde{K}$ are then related by*

$$\widehat{K}_t = \int_0^t \frac{1}{1-\Delta\widetilde{K}_s}\, d\widetilde{K}_s \qquad , \qquad \widetilde{K}_t = \int_0^t \frac{1}{1+\Delta\widehat{K}_s}\, d\widehat{K}_s.$$

**2)** *Suppose that $X$ satisfies (SC). Then the process $\widehat{K}$ does not depend on the choice of $\widehat{\lambda}$ and is locally bounded. Any $\widehat{\lambda}$ satisfying (1.11) and (1.12) is in $L^2_{\mathrm{loc}}(M)$, and the stochastic integral $\int \widehat{\lambda}\, dM$ is well-defined, in $\mathcal{M}^2_{0,\mathrm{loc}}$ and does not depend on the choice of $\widehat{\lambda}$. Finally, we then have $\widehat{K} = \left\langle \int \widehat{\lambda}\, dM \right\rangle$.*

**3)** *Suppose that $X$ satisfies (ESC). Then the process $\widetilde{K}$ does not depend on the choice of $\widetilde{\lambda}$ and is locally bounded. Any $\widetilde{\lambda}$ satisfying (1.13) and (1.14) is in $L^2_{\mathrm{loc}}(M)$, and the stochastic integral $\int \widetilde{\lambda}\, dM$ is well-defined, in $\mathcal{M}^2_{0,\mathrm{loc}}$ and does not depend on the choice of $\widetilde{\lambda}$. Finally, we then have $\widetilde{K} = \left\langle \int \widetilde{\lambda}\, dM \right\rangle + \left[ \int \widetilde{\lambda}^* dA \right]$.*

For some purposes, it is useful to have an alternative description of the space $\Theta$. Recall that $L(X)$ denotes the set of all $I\!R^d$-valued $X$-integrable predictable processes.

**Lemma 2.** *If $X$ satisfies (1.1), then*

$$\Theta = \left\{ \vartheta \in L(X) \,\middle|\, \int \vartheta\, dX \in \mathcal{S}^2 \right\} =: \Theta'.$$

*If in addition $X$ satisfies (SC) and $\widehat{K}_T$ is bounded, then $\Theta = L^2(M)$.*

**Proof.** Since the variation of $\int \vartheta^* dA$ is given by $\int |\vartheta^*\gamma|\, dB$, it is clear that $\Theta'$ contains $L^2(M) \cap L^2(A)$. Conversely, $X$ is special and $\int \vartheta\, dX$ is special for any $\vartheta \in \Theta'$; hence $\int \vartheta\, dM$ and $\int \vartheta^* dA$ both exist in the usual sense by Théorème 2 of Chou/Meyer/Stricker (1980), and $\int \vartheta\, dX \in \mathcal{S}^2$ thus implies that $\vartheta \in L^2(M) \cap L^2(A)$. Finally,

$$\begin{aligned}
\int_0^T |\vartheta_s^*\gamma_s|\, dB_s &= \int_0^T \left|\vartheta_s^*\sigma_s\widehat{\lambda}_s\right| dB_s \\
&\le \int_0^T (\vartheta_s^*\sigma_s\vartheta_s)^{\frac{1}{2}} \left(\widehat{\lambda}_s^*\sigma_s\widehat{\lambda}_s\right)^{\frac{1}{2}} dB_s \\
&\le \left(\widehat{K}_T\right)^{\frac{1}{2}} \left( \int_0^T \vartheta_s^*\sigma_s\vartheta_s\, dB_s \right)^{\frac{1}{2}}
\end{aligned}$$

shows that $L^2(M) \subseteq L^2(A)$ if $\widehat{K}_T$ is bounded.

**q.e.d.**

## 2. The main theorem

Throughout this section, we shall assume that $X$ is given as in section 1. In order to formulate our central result on the solution of (1.8), we introduce the following

**Definition.** We say that a random variable $H \in \mathcal{L}^2$ admits a *strong F-S decomposition* if $H$ can be written as

$$H = H_0 + \int_0^T \xi_s^H \, dX_s + L_T^H \qquad P\text{-a.s.,} \tag{2.1}$$

where $H_0 \in I\!R$ is a constant, $\xi^H \in \Theta$ is a strategy and $L^H = (L_t^H)_{0 \le t \le T}$ is in $\mathcal{M}^2$ with $E\left[L_0^H\right] = 0$ and strongly orthogonal to $\int \vartheta \, dM$ for every $\vartheta \in L^2(M)$.

**Remarks. 1)** If $X$ is a locally square-integrable martingale, then such a decomposition always exists. In fact, (2.1) is then the well-known *Galtchouk-Kunita-Watanabe decomposition* obtained by projecting $H$ on the space $G_T\big(L^2(X)\big)$ which is closed in $\mathcal{L}^2$ since the stochastic integral is an isometry by the local martingale property of $X$. For more details, see Kunita-Watanabe (1967), Galtchouk (1975) and Meyer (1977).

**2)** Under some additional assumptions on $X$, it was shown by Föllmer/Schweizer (1991) and Schweizer (1991) that $H$ admits a decomposition (2.1) if and only if there exists a *locally risk-minimizing* trading strategy for $H$. A more general decomposition of the type (2.1) was then studied by Ansel/Stricker (1992) whose terminology we adopt (and adapt) here. In particular, these authors prove the uniqueness of such a generalized decomposition and give sufficient conditions for its existence in the case $d = 1$. For the case where $X$ is continuous, their results were extended to the multidimensional case $d > 1$ in Schweizer (1993c). Using a recent result of Buckdahn (1993) on backward stochastic differential equations, we shall provide sufficient conditions for a strong F-S decomposition in section 5.

**3)** In a discrete-time framework, a strong F-S decomposition exists for any $H \in \mathcal{L}^2$ if $X$ has a bounded MVT process; see Proposition 2.6 of Schweizer (1993b). In that case, Theorem 2.1 of Schweizer (1993b) even shows that $G_T(\Theta)$ is *closed* in $\mathcal{L}^2$ although the stochastic integral is not an isometry in general. Both these results are proved by backward induction in discrete time and thus suggest an approach using backward stochastic differential equations. We shall provide an analogue of the first result in section 5 under an additional condition on the jumps of $\widetilde{K}$; the question of closedness of $G_T(\Theta)$ in $\mathcal{L}^2$ remains open so far.

**Theorem 3.** *Suppose that $X$ satisfies (ESC) and that the EMVT process $\widetilde{K}$ of $X$ is deterministic. If $H \in \mathcal{L}^2$ admits a strong F-S decomposition, then (1.8) has a solution $\xi^{(c)} \in \Theta$ for any $c \in I\!R$. It is given as the solution of the equation*

$$\xi_t^{(c)} = \xi_t^H + \widetilde{\lambda}_t \left( V_{t-}^H - c - G_{t-}(\xi^{(c)}) \right) \qquad , \qquad 0 \le t \le T, \tag{2.2}$$

*where the process $V^H = (V_t^H)_{0 \le t \le T}$ is defined by*

$$V_t^H := H_0 + \int_0^t \xi_s^H \, dX_s + L_t^H \qquad , \qquad 0 \le t \le T. \tag{2.3}$$

**Sketch of proof.** Since the actual argument is rather lengthy, we give here only the idea of the proof and provide full details in the next section. The first step is to show by standard arguments and estimates for stochastic differential equations that (2.2) has indeed a solution $\xi^{(c)}$ and that $\xi^{(c)} \in \Theta$. Since $G_T(\Theta)$ is a linear subspace of the Hilbert space $\mathcal{L}^2$, the projection theorem implies that a strategy $\xi \in \Theta$ solves (1.8) if and only if

$$E\left[\big(H - c - G_T(\xi)\big)G_T(\vartheta)\right] = 0 \qquad \text{for all } \vartheta \in \Theta. \tag{2.4}$$

By (2.3) and (2.1), $H = V_T^H$ $P$-a.s.; to prove (2.4), we thus fix $\xi, \vartheta \in \Theta$ and define the function $f : [0,T] \to I\!R$ by

$$f(t) := E\left[\big(V_t^H - c - G_t(\xi)\big)\, G_t(\vartheta)\right] \qquad , \qquad 0 \le t \le T.$$

Then the theorem will be proved if we show that $f(T) = 0$ for $\xi = \xi^{(c)}$ and arbitrary $\vartheta$. Now the product rule and some computations give

$$\begin{aligned} f(t) = E&\left[\int_0^t \vartheta_s^* \Big((\sigma_s + \kappa_s)(\xi_s^H - \xi_s) + \gamma_s\big(V_{s-}^H - c - G_{s-}(\xi)\big)\Big)\, dB_s\right] \\ &+ E\left[\int_0^t \gamma_s^*(\xi_s^H - \xi_s) G_{s-}(\vartheta)\, dB_s\right] ; \end{aligned}$$

inserting $\xi = \xi^{(c)}$ hence yields by (2.2), (1.13) and (1.14)

$$f(t) = -E\left[\int_0^t \Big(V_{s-}^H - c - G_{s-}(\xi^{(c)})\Big)\, G_{s-}(\vartheta)\, d\widetilde{K}_s\right] = -\int_0^t f(s-)\, d\widetilde{K}_s,$$

since $\widetilde{K}$ is deterministic. Thus $f \equiv 0$ for any $\vartheta \in \Theta$ if $\xi = \xi^{(c)}$, so $\xi^{(c)}$ solves (1.8).

**Remarks. 1)** The above scheme of proof is essentially due to Duffie/Richardson (1991). In a model where $X$ is geometric Brownian motion, they considered the random variable $H = kX_T^1$ and introduced the function $f$ with $V^H$ replaced by a *tracking process* $Z$, i.e., a process with $Z_T = H$ $P$-a.s. For their special choice of $H$, $Z$ is easy to guess directly. In the same framework for $X$, their approach was extended to general random variables $H$ by Schweizer (1992) who pointed out the possibility of systematically choosing $V^H$ as the tracking process. The present work now considers the case where $X$ is a general semimartingale in $\mathcal{S}^2_{\mathrm{loc}}$ and provides a large class of examples where the conditions of Theorem 3 are satisfied.

**2)** In a discrete-time framework, problem (1.8) was also considered by Schäl (1994) and Schweizer (1993a, 1993b). Whereas Schäl (1994) worked under the assumption that the MVT process is deterministic, the results of Schweizer (1993b) show that (1.8) can be solved in discrete time under the sole assumption that the EMVT process is *bounded*. It is at present an open question whether this result can be extended to the continuous-time case in full generality.

## 3. Proof of the main theorem

In this section, we give a detailed proof of Theorem 3. We shall assume throughout the section that $X$ is given as in section 1. More specific assumptions about $X$ and $H$ will be stated when they are necessary.

### 3.1. Construction of the strategy $\xi^{(c)}$

The first step of the proof consists in showing that $\xi^{(c)}$ is well-defined by (2.2) and in $\Theta$.

**Proposition 4.** *Suppose that $X$ satisfies (ESC) and that the EMVT process $\widetilde{K}$ of $X$ is deterministic. If $H$ admits a strong F-S decomposition, then for every $c \in I\!R$, there exists a strategy $\xi^{(c)} \in \Theta$ with*

$$\xi^{(c)} = \xi^H + \widetilde{\lambda}\left(V_-^H - c - G_-(\xi^{(c)})\right) \qquad \big(\textit{with equality in } L^2(M)\big), \tag{3.1}$$

*where $V^H$ is given by (2.3).*

**Proof.** 1) By (1.13) and (1.14),

$$\int_0^T \widetilde{\lambda}_s^* \sigma_s \widetilde{\lambda}_s \, dB_s \le \int_0^T \left|\widetilde{\lambda}_s^* \gamma_s\right| \, dB_s = \int_0^T \widetilde{\lambda}_s^* \gamma_s \, dB_s = \widetilde{K}_T,$$

and since $\widetilde{K}_T$ is deterministic, hence bounded, we conclude that $\widetilde{\lambda}$ is in $\Theta$. Thus the processes

$$Z_t := -\int_0^t \widetilde{\lambda}_u \, dX_u \qquad , \qquad 0 \le t \le T$$

$$Y_t := \int_0^t \left(\xi_u^H + \widetilde{\lambda}_u (V_{u-}^H - c)\right) dX_u \qquad , \qquad 0 \le t \le T$$

are well-defined and semimartingales. By Theorem V.7 of Protter (1990), the equation

$$U_t = Y_t + \int_0^t U_{s-} \, dZ_s \qquad , \qquad 0 \le t \le T \tag{3.2}$$

has therefore a unique strong solution $U$ which is also a semimartingale.

2) Since $\xi^H \in \Theta$ and $L^H \in \mathcal{M}^2$ by the strong F-S decomposition of $H$, it is clear from (2.3) that $\sup\limits_{0 \le u \le T} \left|V_u^H - c\right| \in \mathcal{L}^2$. Since $\widetilde{K}$ is deterministic, hence bounded, this implies that

$$\sup_{0 \le t \le T} E\left[Y_{t-}^2\right] < \infty. \tag{3.3}$$

In fact, the definition of $Y$ yields

$$Y_t^2 \le 2\left(\int_0^t \xi_u^H \, dX_u\right)^2 + 4\left(\int_0^t (V_{u-}^H - c)\widetilde{\lambda}_u \, dM_u\right)^2 + 4\left(\int_0^t (V_{u-}^H - c)\widetilde{\lambda}_u^* \, dA_u\right)^2$$

and therefore

$$\sup_{0\le t\le T} E\left[Y_{t-}^2\right] \le 2E\left[\sup_{0\le t\le T}\left(\int_0^t \xi_u^H\,dX_u\right)^2\right] + 4\sup_{0\le t\le T} E\left[\int_0^t (V_{u-}^H - c)^2\widetilde{\lambda}_u^*\sigma_u\widetilde{\lambda}_u\,dB_u\right]$$
$$+ 4\sup_{0\le t\le T} E\left[\left(\int_0^t (V_{u-}^H - c)\,d\widetilde{K}_u\right)^2\right]$$

by (1.5), (1.6) and (1.14). But the first term on the right-hand side is finite since $\xi^H \in \Theta$, and the third is dominated by

$$4E\left[\widetilde{K}_T\int_0^T \sup_{0\le u\le T}(V_u^H - c)^2\,d\widetilde{K}_u\right] \le 4\big\|\widetilde{K}_T\big\|_\infty^2 E\left[\sup_{0\le u\le T}(V_u^H - c)^2\right] < \infty.$$

Finally, the second term is majorized by

$$4E\left[\int_0^T\left(\sup_{0\le u\le T}(V_u^H - c)^2\right)\widetilde{\lambda}_u^*(\sigma_u + \kappa_u)\widetilde{\lambda}_u\,dB_u\right] \le 4\big\|\widetilde{K}_T\big\|_\infty E\left[\sup_{0\le u\le T}(V_u^H - c)^2\right] < \infty,$$

because $\kappa$ is nonnegative definite. This proves (3.3).

3) From (3.3) and the fact that $\widetilde{K}$ is deterministic, we obtain

$$\sup_{0\le t\le T} E\left[U_{t-}^2\right] < \infty. \tag{3.4}$$

To see this, define the function $h$ on $[0,T]$ by $h(t) := E\left[U_{t-}^2\right]$. Then (3.2) and the definitions of $Y$ and $Z$ imply as in step 2)

$$\begin{aligned} h(t) &\le 2E\left[Y_{t-}^2\right] + 4E\left[\left(\int_0^{t-} U_{s-}\widetilde{\lambda}_s\,dM_s\right)^2\right] + 4E\left[\left(\int_0^{t-} U_{s-}\widetilde{\lambda}_s^*\,dA_s\right)^2\right] \\ &\le 2E\left[Y_{t-}^2\right] + 4\int_0^{t-} E\left[U_{s-}^2\right]\,d\widetilde{K}_s + 4\big\|\widetilde{K}_T\big\|_\infty\int_0^{t-} E\left[U_{s-}^2\right]\,d\widetilde{K}_s \\ &\le 2E\left[Y_{t-}^2\right] + 4\left(1 + \big\|\widetilde{K}_T\big\|_\infty\right)\int_0^t h(s)\,d\widetilde{K}_s, \end{aligned}$$

where the second inequality uses Fubini's theorem and the fact that $\widetilde{K}$ is deterministic. From Gronwall's inequality, we conclude that

$$h(t) \le 2\exp\left(4\left(1 + \big\|\widetilde{K}_T\big\|_\infty\right)\widetilde{K}_T\right)\sup_{0\le s\le t} E\left[Y_{s-}^2\right],$$

and so (3.4) follows from (3.3).

4) Again since $\widetilde{K}$ is deterministic, (3.4) implies that

$$\vartheta := \widetilde{\lambda}\left(V^H_- - c - U_-\right) \in \Theta. \tag{3.5}$$

In fact, (1.14) yields

$$\left(\int_0^T |\vartheta^*_s \gamma_s|\, dB_s\right)^2 = \left(\int_0^T \left|V^H_{s-} - c - U_{s-}\right|\, d\widetilde{K}_s\right)^2 \le \left\|\widetilde{K}_T\right\|_\infty \int_0^T (V^H_{s-} - c - U_{s-})^2\, d\widetilde{K}_s$$

and therefore $\vartheta \in L^2(A)$ by (3.4), since $\widetilde{K}$ is deterministic. Furthermore,

$$\int_0^T \vartheta^*_s \sigma_s \vartheta_s\, dB_s = \int_0^T \left(V^H_{s-} - c - U_{s-}\right)^2 \widetilde{\lambda}^*_s \sigma_s \widetilde{\lambda}_s\, dB_s \le \int_0^T \left(V^H_{s-} - c - U_{s-}\right)^2\, d\widetilde{K}_s$$

by (1.14) and (1.13), since $\kappa$ is nonnegative definite. Because $\widetilde{K}$ is deterministic, (3.4) implies that $\vartheta \in L^2(M)$, hence $\vartheta \in \Theta$.

5) Due to (3.5), we can now define a strategy $\xi^{(c)} \in \Theta$ by setting

$$\xi^{(c)} := \xi^H + \widetilde{\lambda}\left(V^H_- - c - U_-\right).$$

Then the definitions of $Y$ and $Z$ imply that

$$G_t(\xi^{(c)}) = \int_0^t \xi^{(c)}_s\, dX_s = Y_t + \int_0^t U_{s-}\, dZ_s = U_t \qquad P\text{-a.s. for all } t \in [0,T]$$

by (3.2) so that $G(\xi^{(c)})$ satisfies the stochastic differential equation

$$G_t(\xi^{(c)}) = Y_t + \int_0^t G_{s-}(\xi^{(c)})\, dZ_s = G_t(\xi^H) + \int_0^t \widetilde{\lambda}_s \left(V^H_{s-} - c - G_{s-}(\xi^{(c)})\right) dX_s$$

for $t \in [0,T]$. Hence the special semimartingale

$$\begin{aligned} &G(\xi^{(c)}) - G(\xi^H) - \int \widetilde{\lambda}\left(V^H_- - c - G_-(\xi^{(c)})\right) dX \\ &= \int \left(\xi^{(c)} - \xi^H - \widetilde{\lambda}\left(V^H_- - c - G_-(\xi^{(c)})\right)\right) dX \end{aligned}$$

is indistinguishable from 0, and this implies in particular that its integrand must be 0 in $L^2(M)$, thus proving (3.1).

**q.e.d.**

**Remark.** A closer look at the preceding proof reveals that we do not really need the full strength of the assumption that $\widetilde{K}$ is deterministic. The same argument still works if there exists a deterministic function $\widetilde{k} : [0,T] \to I\!R$ such that $\widetilde{k} - \widetilde{K}$ is $P$-a.s. increasing. However, this condition is not sufficient to prove Theorem 3 by our methods, and so we have refrained from stating Proposition 4 in this slightly more general form.

### 3.2. An auxiliary technical result

The following lemma is a technical tool which is crucial in the proof of Theorem 3. It allows us to restrict attention to bounded strategies $\vartheta$ in the definition of the function $f$, and it also lets us exploit stopping techniques in the subsequent arguments. We denote by $P_B$ the Doléans measure of the process $B$ on the product space $\Omega \times [0,T]$, and we recall that an increasing sequence $(T_m)_{m\in I\!\!N}$ of stopping times is called stationary if $P$-a.s. the sequence $\big(T_m(\omega)\big)_{m\in I\!\!N}$ is constant from some $m_0(\omega)$ on.

**Lemma 5.** *For fixed $H \in \mathcal{L}^2$, $c \in I\!\!R$ and $\xi \in \Theta$, the following statements are equivalent:*

**a)** *$\xi$ solves (1.8).*
**b)** $E\big[\big(H - c - G_T(\xi)\big)G_T(\vartheta)\big] = 0$ *for all $\vartheta \in \Theta$.*
**c)** $E\big[\big(H - c - G_T(\xi)\big)G_T(\vartheta)\big] = 0$ *for all bounded $\vartheta \in \Theta$.*
**d)** *For every bounded $\vartheta \in \Theta$, there exists a stationary sequence $(T_m)_{m\in I\!\!N}$ of stopping times such that $T_m \nearrow T$ $P$-a.s. and*

$$E\left[\big(H - c - G_T(\xi)\big)G_T\left(\vartheta I_{]\!]0,T_m]\!]}\right)\right] = 0 \qquad \textit{for all } m \in I\!\!N.$$

**Proof.** 1) Since $\xi$ is in $\Theta$ and $G_T(\Theta)$ is a linear subspace of the Hilbert space $\mathcal{L}^2$, the equivalence of a) and b) follows directly from the projection theorem, and it is clear that b) implies c) and c) implies d).

2) Consider now any sequence $(\vartheta^m)_{m\in I\!\!N}$ of $I\!\!R^d$-valued predictable processes with the following properties:

$$\vartheta^m \longrightarrow \vartheta \qquad P_B\text{-a.e. for some } \vartheta \in \Theta, \tag{3.6}$$

$$\int_0^T \sup_{m\in I\!\!N} |(\vartheta^m_s - \vartheta_s)^*\gamma_s| \, dB_s \in \mathcal{L}^2 \tag{3.7}$$

and

$$\int_0^T \sup_{m\in I\!\!N} \big((\vartheta^m_s - \vartheta_s)^*\sigma_s(\vartheta^m_s - \vartheta_s)\big) \, dB_s \in \mathcal{L}^1. \tag{3.8}$$

Then $G_T(\vartheta^m)$ tends to $G_T(\vartheta)$ in $\mathcal{L}^2$. In fact, (3.6) implies that both $(\vartheta^m - \vartheta)^*\gamma$ and $(\vartheta^m - \vartheta)^*\sigma(\vartheta^m - \vartheta)$ converge to 0 $P_B$-a.e. Then (3.7) yields by dominated convergence first

$$\int_0^T (\vartheta^m_s - \vartheta_s)^*\gamma_s \, dB_s \longrightarrow 0 \qquad P\text{-a.s.},$$

hence also in $\mathcal{L}^2$ again by (3.7), so that

$$\int_0^T (\vartheta_s^m)^* dA_s \longrightarrow \int_0^T \vartheta_s^*\, dA_s \qquad \text{in } \mathcal{L}^2$$

by (1.6). In the same way, (3.8) yields

$$\int_0^T (\vartheta_s^m - \vartheta_s)^* \sigma_s (\vartheta_s^m - \vartheta_s)\, dB_s \longrightarrow 0 \qquad \text{in } \mathcal{L}^1$$

by twice using the dominated convergence theorem. But the last convergence means that $\vartheta^m$ tends to $\vartheta$ in $L^2(M)$, and this implies

$$\int_0^T \vartheta_s^m\, dM_s \longrightarrow \int_0^T \vartheta_s\, dM_s \qquad \text{in } \mathcal{L}^2$$

by the isometry property of the stochastic integral, hence the assertion by (1.7).

3) To show that c) implies b), we now fix $\vartheta \in \Theta$ and define a sequence of bounded predictable processes $\vartheta^m$ by setting $\psi^m := -m \vee (\vartheta \wedge m)$ and

$$\vartheta^m := \psi^m I_{\{|(\psi^m)^*\gamma| \le |\vartheta^*\gamma|\}} I_{\{(\psi^m - \vartheta)^*\sigma(\psi^m - \vartheta) \le \vartheta^*\sigma\vartheta\}} I_{\{(\psi^m)^*\sigma\psi^m \le \vartheta^*\sigma\vartheta\}}.$$

Then $(\vartheta^m)_{m\in\mathbb{N}}$ satisfies (3.6) – (3.8), for by the definition of $\vartheta^m$ we have

$$\int_0^T \sup_{m\in\mathbb{N}} |(\vartheta_s^m - \vartheta_s)^*\gamma_s|\, dB_s \le 2 \int_0^T |\vartheta_s^*\gamma_s|\, dB_s \in \mathcal{L}^2,$$

since $\vartheta \in L^2(A)$, and

$$\int_0^T \sup_{m\in\mathbb{N}} \big((\vartheta_s^m - \vartheta_s)^*\sigma_s(\vartheta_s^m - \vartheta_s)\big)\, dB_s \le \int_0^T \vartheta_s^*\sigma_s\vartheta_s\, dB_s \in \mathcal{L}^1,$$

because $\vartheta \in L^2(M)$. Hence 2) implies that $G_T(\vartheta^m)$ tends to $G_T(\vartheta)$ in $\mathcal{L}^2$, and since each $\vartheta^m$ is in $\Theta$, b) follows from c).

4) Finally we show that d) implies c). To that end, fix a bounded $\vartheta \in \Theta$ and a sequence $(T_m)_{m\in\mathbb{N}}$ of stopping times as in d). If we define predictable processes $\vartheta^m$ by

$$\vartheta^m := \vartheta I_{]\!]0,T_m]\!]},$$

then $(\vartheta^m)_{m\in\mathbb{N}}$ again satisfies (3.6) – (3.8). In fact, stationarity and $T_m \nearrow T$ $P$-a.s. imply that

$$\vartheta_t^m \longrightarrow \vartheta_t \qquad P\text{-a.s. for all } t \in [0,T],$$

hence (3.6). Furthermore, the definition of $\vartheta^m$ implies that

$$\int_0^T \sup_{m\in\mathbb{N}} |(\vartheta_s^m - \vartheta_s)^*\gamma_s|\, dB_s \le 2 \int_0^T |\vartheta_s^*\gamma_s|\, dB_s \in \mathcal{L}^2,$$

since $\vartheta \in L^2(A)$, and by the nonnegative definiteness of $\sigma$, we have

$$\int_0^T \sup_{m \in I\!N} \big( (\vartheta^m_s - \vartheta_s)^* \sigma_s (\vartheta^m_s - \vartheta_s) \big) \, dB_s \leq \sup_{m \in I\!N} \int_{T_m}^T \vartheta^*_s \sigma_s \vartheta_s \, dB_s \leq \int_0^T \vartheta^*_s \sigma_s \vartheta_s \, dB_s \in \mathcal{L}^1,$$

since $\vartheta \in L^2(M)$. Thus 2) implies that $G_T(\vartheta^m)$ tends to $G_T(\vartheta)$ in $\mathcal{L}^2$, and so c) follows from d).

**q.e.d.**

**Remark.** It is important for later applications that the sequence $(T_m)_{m \in I\!N}$ of stopping times can depend on $\vartheta$; this is clearly allowed by the formulation in d).

### 3.3. Proof that $\xi^{(c)}$ is optimal

We begin with a preliminary technical result:

**Lemma 6.** *Suppose that $L \in \mathcal{M}^2$ is strongly orthogonal to $\int \vartheta \, dM$ for every $\vartheta \in L^2(M)$. For all strategies $\psi, \vartheta \in \Theta$, we then have*

$$E\Big[ [G(\psi) + L, G(\vartheta)]_t \Big] = E\left[ \int_0^t \vartheta^*_s (\sigma_s + \kappa_s) \psi_s \, dB_s \right] \qquad , \qquad 0 \leq t \leq T.$$

**Proof.** 1) By the bilinearity of the square bracket, we have

$$\begin{aligned} (3.9) \quad [G(\psi) + L, G(\vartheta)] = {} & \left[ \int \psi \, dM, \int \vartheta \, dM \right] + \left[ \int \psi^* dA, \int \vartheta^* dA \right] + \left[ L, \int \vartheta \, dM \right] \\ & + \left[ \int \psi \, dM + L, \int \vartheta^* dA \right] + \left[ \int \psi^* dA, \int \vartheta \, dM \right]. \end{aligned}$$

Since $\int \psi \, dM$ and $\int \vartheta \, dM$ are both in $\mathcal{M}^2_0$, $[\int \psi \, dM, \int \vartheta \, dM] - \langle \int \psi \, dM, \int \vartheta \, dM \rangle$ is a martingale null at 0 and therefore

$$E\left[ \left[ \int \psi \, dM, \int \vartheta \, dM \right]_t \right] = E\left[ \left\langle \int \psi \, dM, \int \vartheta \, dM \right\rangle_t \right] = E\left[ \int_0^t \vartheta^*_s \sigma_s \psi_s \, dB_s \right]$$

by (1.5). Furthermore, $\int \psi^* dA$ and $\int \vartheta^* dA$ are both of finite variation; this implies that

$$\begin{aligned} \left[ \int \psi^* dA, \int \vartheta^* dA \right]_t &= \sum_{0 < s \leq t} \Delta \left( \int \psi^* dA \right)_s \Delta \left( \int \vartheta^* dA \right)_s \\ &= \sum_{0 < s \leq t} \sum_{i,j=1}^d \psi^i_s \Delta A^i_s \Delta A^j_s \vartheta^j_s \\ &= \int_0^t \vartheta^*_s \kappa_s \psi_s \, dB_s \end{aligned}$$

by (1.9) and (1.10). Since $L \in \mathcal{M}^2$ is strongly orthogonal to $\int \vartheta \, dM$ for every $\vartheta \in L^2(M)$, $\left[L, \int \vartheta \, dM\right]$ is a martingale null at 0 for every $\vartheta \in L^2(M)$. Thus it is enough to show that the fourth and fifth term on the right-hand side of (3.9) are both martingales null at 0.

2) Now take any $Y \in \mathcal{M}^2$ and any predictable finite variation process $C$ null at 0 with $|C|_T \in \mathcal{L}^2$. Then we claim that $[Y, C]$ is a martingale null at 0. In fact,

$$[Y, C]_t = \sum_{0<s\leq t} \Delta Y_s \Delta C_s \qquad , \qquad 0 \leq t \leq T$$

is a local martingale null at 0 by Yoeurp's lemma, and

$$\begin{aligned}
\sup_{0\leq t\leq T} \left|[Y, C]_t\right| &\leq \left(\sum_{0<s\leq T} (\Delta Y_s)^2\right)^{\frac{1}{2}} \left(\sum_{0<s\leq T} (\Delta C_s)^2\right)^{\frac{1}{2}} \\
&\leq ([Y]_T)^{\frac{1}{2}} \sum_{0<s\leq T} |\Delta C_s| \\
&\leq ([Y]_T)^{\frac{1}{2}} \, |C|_T \in \mathcal{L}^1
\end{aligned}$$

shows that this local martingale is actually a true martingale. Applying this result once with $Y := \int \psi \, dM + L$, $C := \int \vartheta^* dA$ and once with $Y := \int \vartheta \, dM$, $C := \int \psi^* dA$ completes the proof.

**q.e.d.**

**Proof of Theorem 3.** Now we can assemble the previous results to prove the main theorem. So fix $H \in \mathcal{L}^2$ and $c \in I\!R$ and assume that the conditions of Theorem 3 are satisfied. Then the strategy $\xi^{(c)} \in \Theta$ is well-defined by (3.1) due to Proposition 4. Fix any bounded $\vartheta \in \Theta$ and define a sequence of stopping times by

$$T_m := T \wedge \inf \left\{ t \geq 0 \middle| \left|V_t^H\right| + \left|G_t(\xi^{(c)})\right| + |G_t(\vartheta)| \geq m \right\}.$$

Then $(T_m)_{m\in I\!N}$ is stationary, increases to $T$ $P$-a.s., and $V_-^H$, $G_-(\xi^{(c)})$ and $G_-(\vartheta)$ are all bounded on $[\![0, T_m]\!]$ for each $m$. Define the function $f : [0, T] \to I\!R$ by

$$f(t) := E\left[\left(V_t^H - c - G_t(\xi^{(c)})\right) G_t\left(\vartheta I_{]\!]0,T_m]\!]}\right)\right] \qquad , \qquad 0 \leq t \leq T.$$

If we can show that $f(T) = 0$ for each $m$, then Lemma 5 will imply that $\xi^{(c)}$ solves (1.8), since $V_T^H = H$ $P$-a.s. by (2.3) and (2.1), and $\vartheta$ was arbitrary. Fix $m \in I\!N$. Since

$$V^H - c - G(\xi^{(c)}) = H_0 - c + G(\xi^H - \xi^{(c)}) + L^H$$

by (2.3), the product rule implies that

$$\begin{aligned}
(3.10) \quad \left(V_t^H - c - G_t(\xi^{(c)})\right) G_t\left(\vartheta I_{]\!]0,T_m]\!]}\right) &= \int_0^t \left(V_{s-}^H - c - G_{s-}(\xi^{(c)})\right) I_{]\!]0,T_m]\!]}(s) \vartheta_s \, dX_s \\
&\quad + \int_0^t G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right) (\xi_s^H - \xi_s^{(c)}) \, dX_s \\
&\quad + \int_0^t G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right) \, dL_s^H \\
&\quad + \left[G(\xi^H - \xi^{(c)}) + L^H, G\left(\vartheta I_{]\!]0,T_m]\!]}\right)\right]_t .
\end{aligned}$$

But $V^H_-$ and $G_-(\xi^{(c)})$ are bounded on $[\![0,T_m]\!]$ and $\vartheta$ is in $\Theta$; thus the process $\int \left(V^H_- - c - G_-(\xi^{(c)})\right) I_{]\!]0,T_m]\!]}\vartheta\, dM$ is a martingale null at 0. Moreover, $G_-\left(\vartheta I_{]\!]0,T_m]\!]}\right)$ is bounded due to our choice of $T_m$, and so the processes $\int G_-\left(\vartheta I_{]\!]0,T_m]\!]}\right)(\xi^H - \xi^{(c)})\, dM$ and $\int G_-\left(\vartheta I_{]\!]0,T_m]\!]}\right) dL^H$ are also martingales null at 0. Taking expectations in (3.10) and using Lemma 6 therefore yields

$$
\begin{aligned}
f(t) &= E\left[\int_0^t \left(V^H_{s-} - c - G_{s-}(\xi^{(c)})\right) I_{]\!]0,T_m]\!]}(s)\vartheta^*_s\, dA_s + \int_0^t G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right)(\xi^H_s - \xi^{(c)}_s)^* dA_s \right.\\
&\qquad \left. + \int_0^t I_{]\!]0,T_m]\!]}(s)\vartheta^*_s(\sigma_s + \kappa_s)(\xi^H_s - \xi^{(c)}_s)\, dB_s\right]\\
&= E\left[\int_0^t I_{]\!]0,T_m]\!]}(s)\vartheta^*_s\left(\gamma_s\left(V^H_{s-} - c - G_{s-}(\xi^{(c)})\right) + (\sigma_s + \kappa_s)(\xi^H_s - \xi^{(c)}_s)\right) dB_s\right]\\
&\quad + E\left[\int_0^t G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right)(\xi^H_s - \xi^{(c)}_s)^*\gamma_s\, dB_s\right]
\end{aligned}
$$

by (1.4). But now (3.1) and (1.13) show that the first term vanishes by our choice of $\xi^{(c)}$, and again using (3.1) to rewrite the second one, we obtain

$$
\begin{aligned}
f(t) &= -E\left[\int_0^t \left(V^H_{s-} - c - G_{s-}(\xi^{(c)})\right) G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right)\widetilde{\lambda}^*_s\gamma_s\, dB_s\right]\\
&= -\int_0^t E\left[\left(V^H_{s-} - c - G_{s-}(\xi^{(c)})\right) G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right)\right] d\widetilde{K}_s
\end{aligned}
$$

by (1.14) and Fubini's theorem, since $\widetilde{K}$ is deterministic. It is now not difficult to show that

$$
E\left[\left(V^H_{s-} - c - G_{s-}(\xi^{(c)})\right) G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right)\right] = f(s-) \tag{3.11}
$$

for each $s \in (0,T]$. In fact, $V^H_u$, $G_u(\xi^{(c)})$ and $G_u\left(\vartheta I_{]\!]0,T_m]\!]}\right)$ converge to $V^H_{s-}$, $G_{s-}(\xi^{(c)})$ and $G_{s-}\left(\vartheta I_{]\!]0,T_m]\!]}\right)$, respectively, as $u$ increases to $s$, and as

$$
\sup_{0\le u\le T}\left|V^H_u\right| \quad , \quad \sup_{0\le u\le T}\left|G_u(\xi^{(c)})\right| \quad , \quad \sup_{0\le u\le T}\left|G_u\left(\vartheta I_{]\!]0,T_m]\!]}\right)\right|
$$

are all in $\mathcal{L}^2$, (3.11) follows from the dominated convergence theorem. Thus $f$ satisfies the integral equation

$$
f(t) = -\int_0^t f(s-)\, d\widetilde{K}_s \qquad , \qquad 0 \le t \le T;
$$

since this has a unique solution by Theorem V.7 of Protter (1990) (recall that $\widetilde{K}$ is RCLL, hence a semimartingale), we conclude that $f \equiv 0$, and so the proof of Theorem 3 is complete.

**q.e.d.**

## 4. Applications

In this section, we use Theorem 3 to solve several optimization problems with quadratic criteria. Unless explicitly stated otherwise, we always assume that $X$ is given as in section 1 and satisfies the assumptions of Theorem 3. We also fix a random variable $H$ in $\mathcal{L}^2$ admitting a strong F-S decomposition.

### 4.1. Explicit computations and auxiliary results

**Lemma 7.** *For any $c \in I\!R$,*

$$E\left[V_t^H - c - G_t(\xi^{(c)})\right] = (H_0 - c)\mathcal{E}(-\widetilde{K})_t \qquad , \qquad 0 \le t \le T. \tag{4.1}$$

**Proof.** Since $V^H - c - G(\xi^{(c)}) = H_0 - c + G(\xi^H - \xi^{(c)}) + L^H$ by (2.3) and since $\int(\xi^H - \xi^{(c)})\,dM$, $L^H$ are martingales, we have

$$\begin{aligned} h(t) &:= E\left[V_t^H - c - G_t(\xi^{(c)})\right] \\ &= H_0 - c + E\left[\int_0^t (\xi_s^H - \xi_s^{(c)})^* dA_s\right] \\ &= H_0 - c - \int_0^t E\left[V_{s-}^H - c - G_{s-}(\xi^{(c)})\right] d\widetilde{K}_s \end{aligned}$$

by (3.1), (1.14) and Fubini's theorem, since $\widetilde{K}$ is deterministic. A similar argument as for (3.11) shows that

$$E\left[V_{s-}^H - c - G_{s-}(\xi^{(c)})\right] = h(s-);$$

hence $h$ satisfies the integral equation

$$h(t) = H_0 - c - \int_0^t h(s-)\,d\widetilde{K}_s \qquad , \qquad 0 \le t \le T,$$

and so (4.1) follows from Theorem II.36 of Protter (1990).

**q.e.d.**

**Lemma 8.** *For any $c \in I\!R$,*

$$E\left[\left(V_t^H - c - G_t(\xi^{(c)})\right)^2\right] = (H_0 - c)^2\mathcal{E}(-\widetilde{K})_t + g(t), \qquad , \qquad 0 \le t \le T, \tag{4.2}$$

*where $g : [0,T] \to I\!R$ is the unique RCLL solution of the equation*

$$g(t) = E\left[(L_0^H)^2\right] + E\left[\langle L^H\rangle_t\right] - \int_0^t g(s-)\,d\widetilde{K}_s \qquad , \qquad 0 \le t \le T. \tag{4.3}$$

**Proof.** By Theorem V.7 of Protter (1990), (4.3) has indeed a unique solution. Now define $h : [0,T] \to I\!R$ by

$$h(t) := E\left[\left(V_t^H - c - G_t(\xi^{(c)})\right) L_t^H\right].$$

Since $L^H$ and $\int (\xi^H - \xi^{(c)})\, dM$ are strongly orthogonal, we obtain

$$E\left[L_t^H G_t(\xi^H - \xi^{(c)})\right] = E\left[L_t^H \int_0^t (\xi_s^H - \xi_s^{(c)})^* dA_s\right] = E\left[\int_0^t L_{s-}^H (\xi_s^H - \xi_s^{(c)})^* dA_s\right]$$

by Theorem VI.61 of Dellacherie/Meyer (1982) and an approximation argument to account for the fact that $L^H$ is not bounded, but only in $\mathcal{M}^2$. Thus (2.3) implies that

$$\begin{aligned}
h(t) &= E\left[L_t^H G_t(\xi^H - \xi^{(c)})\right] + E\left[(L_t^H)^2\right] \\
&= E\left[\int_0^t L_{s-}^H (\xi_s^H - \xi_s^{(c)})^* dA_s\right] + E\left[(L_0^H)^2\right] + E\left[\langle L^H\rangle_t\right] \\
&= E\left[(L_0^H)^2\right] + E\left[\langle L^H\rangle_t\right] - \int_0^t E\left[\left(V_{s-}^H - c - G_{s-}(\xi^{(c)})\right) L_{s-}^H\right] d\widetilde{K}_s,
\end{aligned}$$

where the last equality uses (3.1), (1.14) and the fact that $\widetilde{K}$ is deterministic. A similar argument as for (3.11) shows that

$$E\left[\left(V_{s-}^H - c - G_{s-}(\xi^{(c)})\right) L_{s-}^H\right] = h(s-);$$

hence $h$ satisfies the integral equation

$$h(t) = E\left[(L_0^H)^2\right] + E\left[\langle L^H\rangle_t\right] - \int_0^t h(s-)\, d\widetilde{K}_s \qquad , \qquad 0 \le t \le T$$

and therefore by uniqueness coincides with $g$. Now the same arguments as in the proof of Theorem 3 yield for arbitrary $\vartheta \in \Theta$

$$E\left[\left(V_t^H - c - G_t(\xi^{(c)})\right) G_t(\vartheta)\right] = 0 \qquad , \qquad 0 \le t \le T,$$

and so we deduce from (2.3) and (4.1) that

$$\begin{aligned}
E\left[\left(V_t^H - c - G_t(\xi^{(c)})\right)^2\right] &= E\left[\left(V_t^H - c - G_t(\xi^{(c)})\right)\left(H_0 - c + G_t(\xi^H - \xi^{(c)}) + L_t^H\right)\right] \\
&= (H_0 - c)^2 \mathcal{E}(-\widetilde{K})_t + h(t),
\end{aligned}$$

hence (4.2).

**q.e.d.**

Equation (4.3) for the function $g$ not only has a unique solution; there also exists an explicit expression for $g$ which can for instance be found in Théorème (6.8) of Jacod (1979).

This allows us to give an explicit formula for the minimal risk $E\left[\left(H-c-G_T(\xi^{(c)})\right)^2\right]$ as a function of the initial capital $c$. The result generalizes a previous computation of Duffie/Richardson (1991) and provides the continuous-time analogue of the results of Schäl (1994) and Schweizer (1993b). For ease of exposition, we only treat here the case where $\Delta\widetilde{K} < 1$. This is no severe restriction since we have $0 \le \Delta\widetilde{K} \le 1$ in any case. In fact, (1.14), (1.13), (1.5), (1.10) and (1.6) imply that

$$\Delta\widetilde{K}_t = \widetilde{\lambda}_t^*(\sigma_t+\kappa_t)\widetilde{\lambda}_t = \widetilde{\lambda}_t^*\sigma_t\widetilde{\lambda}_t\Delta B_t + \sum_{i,j=1}^d \widetilde{\lambda}_t^i\kappa_t^{ij}\widetilde{\lambda}_t^j\Delta B_t = \Delta\left\langle \int \widetilde{\lambda}\,dM\right\rangle_t + \left(\Delta\widetilde{K}_t\right)^2$$

is a real solution of the equation $x = c + x^2$ with $c \ge 0$. Since the solutions of this equation are $\frac{1}{2} \pm \sqrt{\frac{1}{4}-c}$ and since there exists a real solution, we conclude that $c \le \frac{1}{4}$ and $0 \le x \le 1$.

**Corollary 9.** *Suppose that*

$$\Delta\widetilde{K}_t = \widetilde{\lambda}_t^*\Delta A_t < 1 \qquad P\text{-a.s. for } t \in [0,T].$$

*Then we have for any* $c \in \mathbb{R}$

$$\min_{\vartheta\in\Theta} E\left[\left(H-c-G_T(\vartheta)\right)^2\right] = E\left[\left(H-c-G_T(\xi^{(c)})\right)^2\right] \tag{4.4}$$

$$= \mathcal{E}(-\widetilde{K})_T\left((H_0-c)^2 + E\left[(L_0^H)^2\right] + \int_0^T \frac{1}{\mathcal{E}(-\widetilde{K})_s}\,d\left(E\left[\langle L^H\rangle_s\right]\right)\right).$$

*If* $\widetilde{K}$ *is continuous, (4.4) simplifies to*

$$E\left[\left(H-c-G_T(\xi^{(c)})\right)^2\right] = e^{-\widetilde{K}_T}\left((H_0-c)^2 + E\left[(L_0^H)^2\right]\right) + E\left[\int_0^T e^{-(\widetilde{K}_T-\widetilde{K}_s)}\,d\langle L^H\rangle_s\right]. \tag{4.5}$$

**Proof.** By Theorem 3 and Lemma 8, it is clearly enough to compute the value $g(T)$. Since $\Delta\widetilde{K} < 1$, Théorème (6.8) of Jacod (1979) implies that $g(t)$ is given by

$$\mathcal{E}(-\widetilde{K})_t\left(E\left[(L_0^H)^2\right] + \int_0^t \frac{1}{\mathcal{E}(-\widetilde{K})_{s-}}\,d\left(E\left[\langle L^H\rangle_s\right]\right) - \int_0^t \frac{1}{\mathcal{E}(-\widetilde{K})_s}\,d\left[E\left[\langle L^H\rangle\right], -\widetilde{K}\right]_s\right)$$

for every $t \in [0,T]$. Because $E\left[\langle L^H\rangle\right]$ and $\widetilde{K}$ are both RCLL and of finite variation,

$$\left[E\left[\langle L^H\rangle\right], \widetilde{K}\right]_t = \sum_{0<s\le t}\Delta\left(E\left[\langle L^H\rangle_s\right]\right)\Delta\widetilde{K}_s = \int_0^t \Delta\widetilde{K}_s\,d\left(E\left[\langle L^H\rangle_s\right]\right)$$

by Theorem VIII.19 of Dellacherie/Meyer (1982). Furthermore,

$$\frac{1}{\mathcal{E}(-\widetilde{K})_{s-}} = \frac{1}{\mathcal{E}(-\widetilde{K})_s} \left( 1 + \frac{\Delta\left(\mathcal{E}(-\widetilde{K})\right)_s}{\mathcal{E}(-\widetilde{K})_{s-}} \right) = \frac{1}{\mathcal{E}(-\widetilde{K})_s}(1 - \Delta\widetilde{K}_s)$$

by the definition of the stochastic exponential, and thus we obtain (4.4). If $\widetilde{K}$ is continuous, then $\mathcal{E}(-\widetilde{K}) = \exp(-\widetilde{K})$ and (4.4) simplifies to

$$g(T) = e^{-\widetilde{K}_T} E\left[(L_0^H)^2\right] + \int_0^T e^{-(\widetilde{K}_T - \widetilde{K}_s)}\, d\left(E\left[\langle L^H\rangle_s\right]\right). \tag{4.6}$$

Now take any sequence $(\tau_n)_{n\in\mathbb{N}}$ of partitions of the interval $[0,T]$ whose mesh size $|\tau_n| := \max\limits_{t_i,t_{i+1}\in\tau_n} |t_{i+1} - t_i|$ tends to 0. Due to the continuity of $\widetilde{K}$, Theorem I.49 of Protter (1990) implies that

$$\int_0^T e^{-(\widetilde{K}_T - \widetilde{K}_s)}\, d\left(E\left[\langle L^H\rangle_s\right]\right) = \lim_{n\to\infty} \sum_{t_i\in\tau_n} e^{-(\widetilde{K}_T - \widetilde{K}_{t_i})}\Big(E\left[\langle L^H\rangle_{t_{i+1}}\right] - E\left[\langle L^H\rangle_{t_i}\right]\Big)$$

and

$$\int_0^T e^{-(\widetilde{K}_T - \widetilde{K}_s)}\, d\langle L^H\rangle_s = \lim_{n\to\infty} \sum_{t_i\in\tau_n} e^{-(\widetilde{K}_T - \widetilde{K}_{t_i})}\Big(\langle L^H\rangle_{t_{i+1}} - \langle L^H\rangle_{t_i}\Big) \qquad P\text{-a.s.}$$

Since $\widetilde{K}$ is increasing and $L^H \in \mathcal{M}^2$, the sums on the right-hand side of the last equation are bounded by $\langle L^H\rangle_T \in \mathcal{L}^1$. Hence we obtain

$$\int_0^T e^{-(\widetilde{K}_T - \widetilde{K}_s)}\, d\left(E\left[\langle L^H\rangle_s\right]\right) = E\left[\int_0^T e^{-(\widetilde{K}_T - \widetilde{K}_s)}\, d\langle L^H\rangle_s\right]$$

by the dominated convergence theorem, and combining this with (4.6) yields (4.5).

**q.e.d.**

### 4.2. The optimal choice of initial capital and strategy

As a first application, consider now the problem

$$\text{Minimize } E\left[\big(H - V_0 - G_T(\vartheta)\big)^2\right] \text{ over all pairs } (V_0, \vartheta) \in \mathbb{R} \times \Theta. \tag{4.7}$$

This can be interpreted as choosing an initial capital $V_0$ and a self-financing trading strategy $\vartheta$ so as to minimize the expected net quadratic loss at time $T$. In particular, $V_0$ is then the $\Theta$-approximation price of $H$ as defined in Schweizer (1993d).

**Corollary 10.** *Under the assumptions of Theorem 3, the solution of (4.7) is given by the pair* $\big(H_0, \xi^{(H_0)}\big)$.

**Proof.** Since the function $g$ defined by (4.3) does not depend on $c$, it is clear from Lemma 8 that the mapping $c \mapsto E\left[\big(H - c - G_T(\xi^{(c)})\big)^2\right]$ is minimized by $c^* = H_0$. For any pair $(c, \vartheta)$, the definitions of $\xi^{(c)}$ and $c^*$ therefore imply that

$$E\left[\big(H - c - G_T(\vartheta)\big)^2\right] \geq E\left[\Big(H - c - G_T(\xi^{(c)})\Big)^2\right] \geq E\left[\Big(H - c^* - G_T(\xi^{(c^*)})\Big)^2\right].$$

**q.e.d.**

### 4.3. The variance-minimizing strategy

Consider next the problem

$$\text{(4.8)} \qquad \text{Minimize } \operatorname{Var}[H - G_T(\vartheta)] \text{ over all } \vartheta \in \Theta.$$

In a very special case for both $X$ and $H$, this was solved by Richardson (1989) and Duffie/ Richardson (1991); the next result gives the solution in our general framework. Note that in contrast to Duffie/Richardson (1991), our argument remains the same whether $X$ is a martingale or not.

**Corollary 11.** *Under the assumptions of Theorem 3, the solution of (4.8) is given by the strategy* $\xi^{(H_0)}$.

**Proof.** With the same notations as in the proof of Corollary 10, we have for every $\vartheta \in \Theta$

$$\begin{aligned}
\operatorname{Var}[H - G_T(\vartheta)] &= E\left[\big(H - E[H - G_T(\vartheta)] - G_T(\vartheta)\big)^2\right] \\
&\geq E\left[\Big(H - E[H - G_T(\vartheta)] - G_T(\xi^{(E[H - G_T(\vartheta)])})\Big)^2\right] \\
&\geq E\left[\Big(H - c^* - G_T(\xi^{(c^*)})\Big)^2\right] \\
&\geq \operatorname{Var}\left[H - c^* - G_T(\xi^{(c^*)})\right] \\
&= \operatorname{Var}\left[H - G_T(\xi^{(c^*)})\right],
\end{aligned}$$

where the first inequality uses the definition of $\xi^{(c)}$ with $c := E[H - G_T(\vartheta)]$ and the second the definition of $c^*$.

**q.e.d.**

### 4.4. The mean-variance frontier

The third problem we address is

$$\text{(4.9)} \qquad \begin{aligned} &\text{Given } m \in I\!R, \text{ minimize } \operatorname{Var}[H - G_T(\vartheta)] \text{ over all } \vartheta \in \Theta \\ &\text{satisfying the constraint } E[H - G_T(\vartheta)] = m. \end{aligned}$$

We first show that for every $c \in I\!R$, $\xi^{(c)}$ is *$H$-mean-variance efficient* in the sense that

$$\mathrm{Var}\left[H - G_T(\xi^{(c)})\right] \le \mathrm{Var}[H - G_T(\vartheta)]$$

for every $\vartheta \in \Theta$ such that

$$E[H - G_T(\vartheta)] = E\left[H - G_T(\xi^{(c)})\right].$$

To see this, let $m = E\left[H - G_T(\xi^{(c)})\right]$, take any $\vartheta \in \Theta$ with $E[H - G_T(\vartheta)] = m$ and use the definition of $\xi^{(c)}$ to obtain

$$\begin{aligned}
\mathrm{Var}[H - G_T(\vartheta)] &= \mathrm{Var}[H - c - G_T(\vartheta)] \\
&= E\left[\big(H - c - G_T(\vartheta)\big)^2\right] - (m-c)^2 \\
&\ge E\left[\Big(H - c - G_T(\xi^{(c)})\Big)^2\right] - \Big(E\left[H - c - G_T(\xi^{(c)})\right]\Big)^2 \\
&= \mathrm{Var}\left[H - c - G_T(\xi^{(c)})\right] \\
&= \mathrm{Var}\left[H - G_T(\xi^{(c)})\right].
\end{aligned}$$

Like (4.8), also (4.9) was solved by Richardson (1989) and Duffie/Richardson (1991) in a very special case, and we now generalize their result to our situation. Note that the assumption $\widetilde{K}_T \neq 0$ below is equivalent to assuming that $X$ is not a martingale; see section 1.

**Corollary 12.** *Assume the conditions of Theorem 3 and suppose that $\widetilde{K}_T \neq 0$. For every $m \in I\!R$, the solution of (4.9) is then given by $\xi^{(c_m)}$ with*

$$c_m = \frac{m - H_0 \mathcal{E}(-\widetilde{K})_T}{1 - \mathcal{E}(-\widetilde{K})_T}. \tag{4.10}$$

**Proof.** Fix $m \in I\!R$. By the $H$-mean-variance efficiency of $\xi^{(c)}$, it is enough to show that there exists $c \in I\!R$ with $E\left[H - G_T(\xi^{(c)})\right] = m$, since the corresponding strategy $\xi^{(c)}$ will then solve (4.9). But Lemma 7 implies that for every $c \in I\!R$

$$E\left[H - G_T(\xi^{(c)})\right] = H_0 \mathcal{E}(-\widetilde{K})_T + c\left(1 - \mathcal{E}(-\widetilde{K})_T\right),$$

and this equals $m$ if $c$ is given by $c_m$ in (4.10); note that $c_m$ is well-defined since $\mathcal{E}(-\widetilde{K})_T \neq 1$ by the assumption that $\widetilde{K}_T \neq 0$.

**q.e.d.**

### 4.5. Approximation of a riskless asset

As a last application, consider now the problem (1.8) in the special case where $H \equiv 1$ and $c = 0$. The strategy $\xi^{(c)} = \xi^{(0)}$ by definition then solves the problem

$$\text{Minimize } E\left[\big(1 - G_T(\vartheta)\big)^2\right] \text{ over all } \vartheta \in \Theta. \tag{4.11}$$

This can be interpreted as approximating in $\mathcal{L}^2$ the riskless payoff 1 by the terminal wealth achievable by a self-financing trading strategy $\vartheta$. Such a question is of some interest in practice since it may happen that we have several risky assets $X^1, \dots, X^d$, but no riskless asset at our disposal. The assumption $c = 0$ is then quite natural, since the absence of a riskless asset makes it impossible to transfer an initial capital from time 0 to time $T$.

**Proposition 13.** *Under the assumptions of Theorem 3, the solution of (4.11) is given by the strategy*

$$\xi_t^{(0)} = \widetilde{\lambda}_t \mathcal{E}\left(-\int \widetilde{\lambda}\, dX\right)_{t-} \quad , \qquad 0 \le t \le T. \tag{4.12}$$

*The corresponding gains process $G(\xi^{(0)})$ is*

$$G_t(\xi^{(0)}) = 1 - \mathcal{E}\left(-\int \widetilde{\lambda}\, dX\right)_t \quad , \qquad 0 \le t \le T. \tag{4.13}$$

*For every $t \in [0,T]$, $\xi^{(0)}$ also solves the problem*

$$\text{Minimize } E\left[\left(1 - G_t(\vartheta)\right)^2\right] \text{ over all } \vartheta \in \Theta, \tag{4.14}$$

*and we have*

$$\begin{aligned} E\left[G_t(\xi^{(0)})\right] &= 1 - \mathcal{E}(-\widetilde{K})_t, \\ \operatorname{Var}\left[G_t(\xi^{(0)})\right] &= \mathcal{E}(-\widetilde{K})_t\left(1 - \mathcal{E}(-\widetilde{K})_t\right). \end{aligned} \tag{4.15}$$

**Proof.** It is obvious that the strong F-S decomposition of $H \equiv 1$ is given by $H_0 = 1$, $\xi^H \equiv 0$ and $L^H \equiv 0$. Since $V^H \equiv 1$, (2.2) therefore implies that $1 - G(\xi^{(0)})$ satisfies the equation

$$1 - G_t(\xi^{(0)}) = 1 - \int_0^t \left(1 - G_{s-}(\xi^{(0)})\right) \widetilde{\lambda}_s\, dX_s \quad , \qquad 0 \le t \le T,$$

hence

$$1 - G_t(\xi^{(0)}) = \mathcal{E}\left(-\int \widetilde{\lambda}\, dX\right)_t \quad , \qquad 0 \le t \le T,$$

and this proves (4.13) and (4.12). The same argument as in the proof of Theorem 3 shows that $\xi^{(0)}$ solves (4.14). Finally, $L^H \equiv 0$ implies that $g \equiv 0$ by (4.3), so Lemma 7 and Lemma 8 yield

$$E\left[1 - G_t(\xi^{(0)})\right] = \mathcal{E}(-\widetilde{K})_t = E\left[\left(1 - G_t(\xi^{(0)})\right)^2\right]$$

and therefore (4.15).

**q.e.d.**

### 4.6. The martingale case

In this subsection, we take a brief look at the simplifications of the preceding results in the case where $X$ is a local martingale, i.e., $A \equiv 0$. First of all, $\Theta$ then coincides with $L^2(M)$ and $G(\Theta)$ is just the stable subspace of $\mathcal{M}_0^2$ generated by $M - M_0 = X - X_0$. Since $G_T(\Theta)$ is therefore a *closed* subspace of $\mathcal{L}^2$, it is clear that (1.8) has a unique solution for every $H \in \mathcal{L}^2$, and every $H \in \mathcal{L}^2$ admits a strong F-S decomposition which is given by the well-known Galtchouk-Kunita-Watanabe decomposition of $H$ with respect to the local martingale $X$. The process $\widetilde{\lambda}$ is identically 0, and therefore

$$\xi^{(c)} = \xi^H = \xi^{(H_0)}$$

for every $c \in I\!R$ by (3.1). Finally $G(\vartheta)$ is a martingale for every $\vartheta \in \Theta$, so

$$E\left[H - G_T(\vartheta)\right] = E[H] = H_0 \qquad \text{for every } \vartheta \in \Theta$$

and thus it is clear that (4.9) can only have a solution for $m = H_0$.

## 5. Existence of a strong F-S decomposition

In this section, we give a sufficient condition on $X$ to ensure that every $H \in \mathcal{L}^2$ admits a strong F-S decomposition. Basically, this is a consequence of a recent result by Buckdahn (1993) on backward stochastic differential equations. To keep the paper self-contained and since our case is not exactly covered by Buckdahn's results, we nevertheless provide complete proofs here. Unless stated differently, we shall assume that $X$ is given as in section 1 and satisfies (SC). First of all, we need some notation:

**Definition.** $\mathcal{R}^2$ denotes the space of all real-valued adapted RCLL processes $U = (U_t)_{0 \le t \le T}$ such that

$$\|U\|_{\mathcal{R}^2} := \left\| \sup_{0 \le t \le T} |U_t| \right\|_{\mathcal{L}^2} < \infty.$$

By $\mathcal{I}^2(M)^\perp$, we denote the space of all martingales $L \in \mathcal{M}^2$ such that $E[L_0] = 0$ and $L$ is strongly orthogonal to $\int \vartheta \, dM$ for every $\vartheta \in L^2(M)$. In other words, $\mathcal{I}^2(M)^\perp$ is the orthogonal complement in $\mathcal{M}^2$ of the stable subspace generated by $M$. Finally, $\mathcal{B}^2$ denotes the Banach space $\mathcal{R}^2 \times L^2(M) \times \mathcal{I}^2(M)^\perp$ with any of the equivalent norms

$$\|(U,\vartheta,L)\|_a := a\|U\|_{\mathcal{R}^2} + \left\| \left( \int_0^T \vartheta_s^* \sigma_s \vartheta_s \, dB_s + \langle L \rangle_T \right)^{\frac{1}{2}} \right\|_{\mathcal{L}^2}$$

for $a > 0$. Note that this definition coincides with the one by Buckdahn (1993) if the components of $M$ are pairwise orthogonal.

**Definition.** Fix a random variable $H \in \mathcal{L}^2$, a process $\varrho \in L^2(M)$ and an $I\!R^d$-valued predictable RCLL process $C = (C_t)_{0 \le t \le T}$ of finite variation null at 0 such that $\int \vartheta^* dC$ is in $\mathcal{R}^2$ for every $\vartheta \in L^2(M)$. The mapping $\psi^C_{H,\varrho} : \mathcal{B}^2 \to \mathcal{B}^2$ is then defined by

$$\psi^C_{H,\varrho}(U,\vartheta,L) := (\widetilde{U}, \widetilde{\vartheta}, \widetilde{L}),$$

where $\widetilde{U}$ is an RCLL version of

$$\widetilde{U}_t := E\left[ H - \int_t^T (\varrho_s + \vartheta_s)^* dC_s \middle| \mathcal{F}_t \right] \quad , \qquad 0 \le t \le T, \tag{5.1}$$

and $\widetilde{\vartheta}$ and $\widetilde{L}$ are given by the Galtchouk-Kunita-Watanabe decomposition

$$H - \int_0^T (\varrho_s + \vartheta_s)^* dC_s = E\left[ H - \int_0^T (\varrho_s + \vartheta_s)^* dC_s \right] + \int_0^T \widetilde{\vartheta}_s \, dM_s + \widetilde{L}_T;$$

see Jacod (1979), Théorème (4.35) and Proposition (4.26).

From the definition of $\psi^C_{H,\varrho}$, it is clear that $(\widetilde{U}, \widetilde{\vartheta}, \widetilde{L})$ satisfies the equation

$$\widetilde{U}_t = H - \int_t^T (\varrho_s + \vartheta_s)^* dC_s - \int_t^T \widetilde{\vartheta}_s \, dM_s - (\widetilde{L}_T - \widetilde{L}_t) \quad , \qquad 0 \le t \le T. \tag{5.2}$$

To find a strong F-S decomposition of a given $H \in \mathcal{L}^2$, we shall therefore look for a fixed point $(V^H, \xi^H, L)$ of the mapping $\psi^A_{H,0}$, since we then obtain from (5.2) that

$$H = H_0 + \int_0^T \xi^H_s \, dX_s + L^H_T \qquad P\text{-a.s.} \tag{5.3}$$

with $H_0 := E\left[V^H_0\right]$ and $L^H := L + V^H_0 - E\left[V^H_0\right]$.

**Proposition 14.** *Suppose that $C$ has the form $C = \int \sigma\nu \, dB$ for some predictable $\mathbb{R}^d$-valued process $\nu$. If $C$ satisfies*

$$\widehat{K}^C_T := \int_0^T \nu^*_s \sigma_s \nu_s \, dB_s \le \delta < 1 \qquad P\text{-a.s. for some constant } \delta, \tag{5.4}$$

*then $\psi^C_{H,\varrho}$ has a unique fixed point in $\mathcal{B}^2$ for every pair $(H, \varrho) \in \mathcal{L}^2 \times L^2(M)$.*

**Proof.** Note first that (5.4) ensures that $\psi^C_{H,\varrho}$ is well-defined since by the Cauchy-Schwarz inequality,

$$\left( \left| \int \vartheta^* dC \right|_T \right)^2 = \left( \int_0^T |\vartheta^*_s \sigma_s \nu_s| \, dB_s \right)^2 \le \widehat{K}^C_T \int_0^T \vartheta^*_s \sigma_s \vartheta_s \, dB_s \in \mathcal{L}^1.$$

Following Buckdahn (1993), we now show that $\psi^C_{H,\varrho}$ is a contraction on $(\mathcal{B}^2, \|\cdot\|_a)$ for suitable $a$. First of all, (5.1) implies that

$$\left|\widetilde{U}_t - \widetilde{U'_t}\right| = \left| E\left[ \int_t^T (\vartheta'_s - \vartheta_s)^* dC_s \middle| \mathcal{F}_t \right] \right| \le E\left[ \int_0^T |(\vartheta'_s - \vartheta_s)^* \sigma_s \nu_s| \, dB_s \middle| \mathcal{F}_t \right]$$

and therefore

$$\left\|\widetilde{U}-\widetilde{U'}\right\|_{\mathcal{R}^2} \leq 2\left\|\int_0^T |(\vartheta'_s-\vartheta_s)^*\sigma_s\nu_s|\,dB_s\right\|_{\mathcal{L}^2} \leq 2\big\|\widehat{K}^C_T\big\|_\infty^{\frac{1}{2}}\,\|\vartheta'-\vartheta\|_{L^2(M)}$$

by the Doob and Cauchy-Schwarz inequalities. Moreover, (5.2) shows that

$$\begin{aligned}\int_0^T(\widetilde{\vartheta}_s-\widetilde{\vartheta'_s})\,dM_s+\widetilde{L}_T-\widetilde{L}_0-\widetilde{L'_T}+\widetilde{L'_0} &= \int_0^T(\vartheta'_s-\vartheta_s)^*dC_s-\widetilde{U}_0+\widetilde{U'_0}\\ &= \int_0^T(\vartheta'_s-\vartheta_s)^*dC_s-E\left[\int_0^T(\vartheta'_s-\vartheta_s)^*dC_s\middle|\mathcal{F}_0\right]\end{aligned}$$

and so we obtain

$$\begin{aligned}&\left\|\left(\int_0^T(\widetilde{\vartheta}_s-\widetilde{\vartheta'_s})^*\sigma_s(\widetilde{\vartheta}_s-\widetilde{\vartheta'_s})\,dB_s+\left\langle\widetilde{L}-\widetilde{L}'\right\rangle_T\right)^{\frac{1}{2}}\right\|_{\mathcal{L}^2}\\ &=\left(E\left[\left(\int_0^T(\widetilde{\vartheta}_s-\widetilde{\vartheta'_s})\,dM_s+\widetilde{L}_T-\widetilde{L}_0-\widetilde{L'_T}+\widetilde{L'_0}\right)^2\right]\right)^{\frac{1}{2}}\\ &\leq\left\|\int_0^T(\vartheta'_s-\vartheta_s)^*\sigma_s\nu_s\,dB_s\right\|_{\mathcal{L}^2}\\ &\leq\big\|\widehat{K}^C_T\big\|_\infty^{\frac{1}{2}}\,\|\vartheta'-\vartheta\|_{L^2(M)}\,.\end{aligned}$$

Putting these estimates together, we obtain

$$\begin{aligned}\big\|\psi^C_{H,\varrho}(U,\vartheta,L)-\psi^C_{H,\varrho}(U',\vartheta',L')\big\|_a &= \left\|(\widetilde{U}-\widetilde{U'},\widetilde{\vartheta}-\widetilde{\vartheta}',\widetilde{L}-\widetilde{L}')\right\|_a\\ &\leq(2a+1)\big\|\widehat{K}^C_T\big\|_\infty^{\frac{1}{2}}\,\|\vartheta'-\vartheta\|_{L^2(M)}\\ &\leq(2a+1)\sqrt{\delta}\,\|(U,\vartheta,L)-(U',\vartheta',L')\|_a\,,\end{aligned}$$

and so (5.4) implies that $\psi^C_{H,\varrho}$ is indeed a contraction on $(\mathcal{B}^2,\|\cdot\|_a)$ for $0<a<\frac{1-\sqrt{\delta}}{2\sqrt{\delta}}$. This completes the proof.

**q.e.d.**

**Theorem 15.** *Suppose that $X$ satisfies (SC) and that the MVT process $\widehat{K}$ of $X$ is bounded and satisfies*

$$\sup\left\{\Delta\widehat{K}_\tau\,\middle|\,\tau\text{ stopping time}\right\}<1. \tag{5.5}$$

*Then every $H \in \mathcal{L}^2$ admits a strong F-S decomposition.*

**Proof.** As in Buckdahn (1993), we show by a backward induction argument that $\psi^A_{H,0}$ has a fixed point in $\mathcal{B}^2$ for every $H \in \mathcal{L}^2$. Since $\widehat{K}$ is bounded, (5.5) implies the existence of stopping times $0 = \tau_0 < \tau_1 < \ldots < \tau_n = T$ such that

$$\widehat{K}_{\tau_j} - \widehat{K}_{\tau_{j-1}} \le \delta < 1 \qquad P\text{-a.s. for } j = 1, \ldots, n \text{ and some constant } \delta. \tag{5.6}$$

Define the processes $C^j$ and $D^j$ by setting

$$C^j_t := \int_0^t I_{]\!]\tau_{j-1},T]\!]}(s)\, dA_s \qquad , \qquad 0 \le t \le T, \text{ for } j = 1, \ldots, n+1,$$

$$D^j_t := C^j_t - C^{j+1}_t = \int_0^t I_{]\!]\tau_{j-1},\tau_j]\!]}(s)\, dA_s \qquad , \qquad 0 \le t \le T, \text{ for } j = 1, \ldots, n.$$

Due to (5.6),

$$\widehat{K}^{D^j}_T = \widehat{K}_{\tau_j} - \widehat{K}_{\tau_{j-1}} \le \delta < 1 \qquad P\text{-a.s. for } j = 1, \ldots, n$$

and so each $\psi^{D^j}_{0,\varrho}$ has a unique fixed point $(U, \vartheta, L) \in \mathcal{B}^2$ for every $\varrho \in L^2(M)$ by Proposition 14. Moreover, the definition of $\psi^{D^j}_{0,\varrho}$ shows that $\vartheta$ is given by the integrand in the Galtchouk-Kunita-Watanabe decomposition of

$$-\int_0^T (\varrho_s + \vartheta_s)^* dD^j_s = -\int_0^T I_{]\!]\tau_{j-1},\tau_j]\!]}(s)(\varrho_s + \vartheta_s)^* dA_s,$$

and since this random variable is $\mathcal{F}_{\tau_j}$-measurable, we conclude that $\vartheta = 0$ on $]\!]\tau_j, T]\!]$.

Now fix $H \in \mathcal{L}^2$. Due to (5.6),

$$\widehat{K}^{C^n}_T = \widehat{K}_{\tau_n} - \widehat{K}_{\tau_{n-1}} \le \delta < 1 \qquad P\text{-a.s.},$$

and so Proposition 14 implies that $\psi^{C^n}_{H,0}$ has a unique fixed point $(V^n, \xi^n, L^n)$ in $\mathcal{B}^2$. Assuming that $\psi^{C^j}_{H,0}$ has a fixed point $(V^j, \xi^j, L^j)$ in $\mathcal{B}^2$, we denote by $(U^{j-1}, \vartheta^{j-1}, R^{j-1})$ the unique fixed point of $\psi^{D^{j-1}}_{0,\xi^j}$. Since $\vartheta^{j-1} = 0$ on $]\!]\tau_{j-1}, T]\!]$, we obtain

$$\begin{aligned}
\int (\xi^j)^* dC^j + \int (\xi^j + \vartheta^{j-1})^* dD^{j-1} &= \int \left(\xi^j I_{]\!]\tau_{j-1},T]\!]} + (\xi^j + \vartheta^{j-1}) I_{]\!]\tau_{j-2},\tau_{j-1}]\!]}\right)^* dA \\
&= \int I_{]\!]\tau_{j-2},T]\!]} (\xi^j + \vartheta^{j-1})^* dA \\
&= \int (\xi^j + \vartheta^{j-1})^* dC^{j-1},
\end{aligned}$$

and (5.2) therefore yields

$$V_t^j + U_t^{j-1} = H - \int_t^T (\xi_s^j)^* dC_s^j - \int_t^T \xi_s^j \, dM_s - (L_T^j - L_t^j)$$

$$- \int_t^T (\xi_s^j + \vartheta_s^{j-1})^* dD_s^{j-1} - \int_t^T \vartheta_s^{j-1} \, dM_s - (R_T^{j-1} - R_t^{j-1})$$

$$= H - \int_t^T (\xi_s^j + \vartheta_s^{j-1})^* dC_s^{j-1} - \int_t^T (\xi_s^j + \vartheta_s^{j-1}) \, dM_s - (L_T^j + R_T^{j-1} - L_t^j - R_t^{j-1}).$$

By (5.2), this shows that $(V^j + U^{j-1}, \xi^j + \vartheta^{j-1}, L^j + R^{j-1})$ is a fixed point of $\psi_{H,0}^{C^{j-1}}$. By induction, $\psi_{H,0}^{A} = \psi_{H,0}^{C^1}$ therefore has a fixed point $(V^H, \xi^H, L)$ in $\mathcal{B}^2$, and since $\Theta = L^2(M)$ by Lemma 2, we obtain the strong F-S decomposition of $H$ as in (5.3).

**q.e.d.**

As an immediate consequence, we deduce

**Corollary 16.** *Suppose that $X$ satisfies (ESC) and the EMVT process $\widetilde{K}$ is deterministic and satisfies*

$$\sup\left\{\Delta \widetilde{K}_\tau \,\middle|\, \tau \text{ stopping time}\right\} < \frac{1}{2}. \tag{5.7}$$

*Then (1.8) admits a solution $\xi^{(c)} \in \Theta$ for every $H \in \mathcal{L}^2$ and every $c \in I\!R$.*

**Proof.** By Lemma 1 and (5.7), $X$ satisfies (SC) and $\widehat{K}_T$ is bounded (even deterministic) and satisfies (5.5). By Lemma 2, $\Theta = L^2(M)$ and so we can apply Theorem 15 and Theorem 3.

**q.e.d.**

We conclude this section by relating the strong F-S decomposition to the *minimal signed local martingale measure* $\widehat{P}$ for $X$. To that end, we recall that $X$ satisfies (SC) and define the minimal martingale density $\widehat{Z} \in \mathcal{M}^2_{\mathrm{loc}}$ by $\widehat{Z} := \mathcal{E}\left(-\int \widehat{\lambda}\, dM\right)$. Then $\widehat{Z}$ satisfies

$$d\widehat{Z}_t = -\widehat{Z}_{t-} \widehat{\lambda}_t \, dM_t,$$

and this implies that $\widehat{Z}L$ is in $\mathcal{M}_{\mathrm{loc}}$ for every $L \in \mathcal{I}^2(M)^\perp$. Moreover, one can show by using the product rule, Yoeurp's lemma and (SC) that $\widehat{Z}X$ is in $\mathcal{M}_{\mathrm{loc}}$ and $\widehat{Z}G(\vartheta)$ is in $\mathcal{M}_{0,\mathrm{loc}}$ for every $\vartheta \in \Theta$.

Now assume that $\widehat{K}_T = \left\langle \int \widehat{\lambda}\, dM \right\rangle_T$ is bounded. Then Théorème II.2 of Lepingle/Mémin (1978) implies that $\widehat{Z}$ is in $\mathcal{M}^2$, and this allows us to define a signed measure $\widehat{P} \ll P$ on $\mathcal{F}$ with $\widehat{P}[\Omega] = 1$ by setting

$$\frac{d\widehat{P}}{dP} := \widehat{Z}_T \in \mathcal{L}^2(P).$$

The preceding arguments show that $\widehat{Z}G(\vartheta)$ is in $\mathcal{M}^1_0(P)$ for every $\vartheta \in \Theta$, hence

$$\widehat{E}[G_T(\vartheta)] = 0 \qquad \text{for every } \vartheta \in \Theta,$$

and so $\widehat{P}$ is a *signed* $\Theta$*-martingale measure* in the sense of Schweizer (1993d). Moreover, the facts that $\widehat{Z}X \in \mathcal{M}_{\rm loc}(P)$ and $\widehat{Z}L \in \mathcal{M}^1(P)$ for every $L \in \mathcal{I}^2(M)^\perp$ justify calling $\widehat{P}$ the minimal signed local martingale measure for $X$; see Föllmer/Schweizer (1991), Ansel/Stricker (1992) and Schweizer (1993c). If $\widehat{Z}$ is strictly positive, we can even replace "signed" by "equivalent" throughout.

**Lemma 17.** *Suppose that $X$ satisfies (SC), the MVT process $\widehat{K}$ of $X$ is bounded and $H \in \mathcal{L}^2(P)$ admits a strong F-S decomposition. Then the process $\widehat{Z}V^H$ is in $\mathcal{M}^1(P)$, where $V^H$ is given by (2.3). In particular, we have*

$$H_0 = \widehat{E}[H].$$

*If $\widehat{Z}$ is strictly positive, then we also have*

$$V_t^H = \widehat{E}[H|\mathcal{F}_t] \quad , \quad 0 \le t \le T. \tag{5.8}$$

**Proof.** By definition, $\xi^H \in \Theta$ and $L^H \in \mathcal{I}^2(M)^\perp$; hence the preceding arguments yield

$$\widehat{Z}V^H = \widehat{Z}\left(H_0 + G(\xi^H) + L^H\right) \in \mathcal{M}^1(P).$$

Since $V_T^H = H$ $P$-a.s., $\widehat{Z}_0 = 1$ and $E\left[L_0^H\right] = 0$, we deduce

$$\widehat{E}[H] = E\big[\widehat{Z}_0 V_0^H\big] = H_0.$$

Finally, the last assertion follows from the Bayes rule.

**q.e.d.**

## 6. Examples

In this section, we illustrate the preceding results by means of several examples.

### 6.1. Continuous processes admitting an equivalent martingale measure

Consider first any continuous adapted $I\!R^d$-valued process $X$. If we assume that $X$ admits an *equivalent local martingale measure*, i.e., there exists a probability measure $P^* \approx P$ such that $X$ is a local $(P^*, I\!F)$-martingale, then $X$ is in $\mathcal{S}^2_{\rm loc}(P)$ and satisfies (1.1) and (SC); see Ansel/Stricker (1992) or Theorem 1 of Schweizer (1993c). Moreover, $\widehat{K}$ is continuous and so (5.5) is trivially satisfied; thus Theorem 15 implies that every $H \in \mathcal{L}^2(P)$ admits a strong F-S decomposition if $\widehat{K}_T$ is bounded. If $\widehat{K}$ is even deterministic, then the optimization problem (1.8) admits a solution $\xi^{(c)}$ for every pair $(c, H) \in I\!R \times \mathcal{L}^2(P)$.

This example generalizes previous results of Schweizer (1993a, 1993c) who obtained a strong F-S decomposition under the slightly more restrictive assumption that $\widehat{K}_T$ is bounded and $H$ is in $\mathcal{L}^{2+\varepsilon}(P)$ for some $\varepsilon > 0$. On the other hand, the method used there allows to give an explicit description not only of $V^H$, but also of the processes $\xi^H$ and $L^H$. To see this, we note that continuity of $X$ and boundedness of $\widehat{K}_T$ imply that the minimal martingale density $\widehat{Z}$ is strictly positive and in $\mathcal{M}^r(P)$ for every $r < \infty$, so $\widehat{P}$ is a probability measure equivalent to $P$, and $X$ is a continuous local $(\widehat{P}, I\!\!F)$-martingale. The strong F-S decomposition of $H \in \mathcal{L}^{2+\varepsilon}(P)$ can then be obtained by setting

$$V_t^H := \widehat{E}[H|\mathcal{F}_t] \qquad , \qquad 0 \le t \le T$$

as in (5.8) and

$$L_t^H := V_t^H - E\left[V_0^H\right] - \int_0^t \xi_s^H \, dX_s \qquad , \qquad 0 \le t \le T,$$

where $\xi^H$ denotes the integrand with respect to $X$ in the Galtchouk-Kunita-Watanabe decomposition of $H$ under $\widehat{P}$. Using the Burkholder-Davis-Gundy inequalities, one can moreover deduce additional integrability properties of $\xi^H$ and $L^H$ from information about the integrability of $H$. For more details, see Schweizer (1993a, 1993c).

### 6.2. A multidimensional jump-diffusion model

As a second class of examples, we consider a fairly general jump-diffusion model where $X$ is given as the solution of the stochastic differential equation

$$\text{(6.1)} \qquad dX_t^i = X_{t-}^i \left( \mu_t^i \, dt + \sum_{j=1}^n v_t^{ij} \, dW_t^j + \sum_{k=1}^m \varphi_t^{ik} \, dN_t^k \right) \qquad , \qquad 0 \le t \le T$$

for $i = 1, \ldots, d$, with all $X_0^i > 0$. Without special mention, all processes will be defined for $t \in [0, T]$. In (6.1), $W = (W^1, \ldots, W^n)^*$ is an $n$-dimensional Brownian motion and $N = (N^1, \ldots, N^m)^*$ is an $m$-variate point process with deterministic intensity $\nu = (\nu^1, \ldots, \nu^m)^*$; this is equivalent to saying that $N^1, \ldots, N^m$ are independent Poisson processes with intensities $\nu^1, \ldots, \nu^m$, respectively. $W$ and $N$ are then automatically independent. We shall take $d \le n+m$ so that in financial terms, there are more sources of uncertainty in the market than assets available for trade. $I\!\!F = (\mathcal{F}_t)_{0 \le t \le T}$ denotes the $P$-augmentation of the filtration generated by $W$ and $N$, and $\mathcal{F} = \mathcal{F}_T$. The coefficients $\mu = (\mu^1, \ldots, \mu^d)^*$, $v = (v^{ij})_{i=1,\ldots,d; j=1,\ldots,n}$ and $\varphi = (\varphi^{ik})_{i=1,\ldots,d; k=1,\ldots,m}$ are assumed to be predictable processes and (for simplicity) $P$-a.s. bounded, uniformly in $t$ and $\omega$. We also assume that $\nu$ is bounded uniformly in $t$,

$$\text{(6.2)} \qquad \nu^k(t) > 0 \qquad , \qquad 0 \le t \le T, \text{ for } k = 1, \ldots, m$$

and

$$\text{(6.3)} \qquad \varphi_t^{ik} > -1 \qquad P\text{-a.s. for } t \in [0, T],\ i = 1, \ldots, d \text{ and } k = 1, \ldots, m.$$

We define the $d \times m$ matrix-valued process $\psi$ by $\psi_t^{ik} := \varphi_t^{ik} \sqrt{\nu^k(t)}$ for $t \in [0, T]$ and impose the additional condition that

(6.4) the matrix $\Sigma_t := v_t v_t^* + \psi_t \psi_t^*$ is $P$-a.s. strongly nondegenerate, uniformly in $t$ and $\omega$, i.e., there exists a constant $\varepsilon > 0$ such that for all $t \in [0,T]$,

$$x^* \Sigma_t \, x \geq \varepsilon \|x\|^2 \qquad P\text{-a.s. for all } x \in I\!R^d.$$

This implies that $\Sigma_t$ is $P$-a.s. invertible for each $t$ with $\left\|\Sigma_t^{-1}\right\| \leq \frac{1}{\varepsilon}$ and that the process $\varrho = (\varrho^1, \dots, \varrho^d)^*$ defined by

$$\varrho_t := \Sigma_t^{-1}\big(\mu_t + \varphi_t \, \nu(t)\big) = (v_t v_t^* + \psi_t \psi_t^*)^{-1} \left(\mu_t + \varphi_t \, \nu(t)\right) \qquad , \qquad 0 \leq t \leq T$$

is $P$-a.s. bounded, uniformly in $t$ and $\omega$. Finally, we assume that

$$(6.5) \qquad (\varphi_t^* \varrho_t)^k \leq 1 - \delta \qquad P\text{-a.s. for } t \in [0,T],\ k = 1, \dots, m \text{ and some constant } \delta > 0.$$

For future reference, we introduce the notation $x \,\square\, y$ for the coordinatewise product of two vectors $x, y \in I\!R^m$:

$$(x \,\square\, y)^k := x^k y^k \qquad \text{for } k = 1, \dots, m.$$

**Remark.** Since jump-diffusion models for stock prices have recently been used by several authors, we provide here a brief comparison of our assumptions to those made in other papers and point out the relevant differences. We should like to emphasize, though, that all these papers are concerned with optimization problems different from (1.8); the overlap only concerns the basic model used for $X$.

**1)** The paper by Jeanblanc-Picqué/Pontier (1990) considers the case where $d = 2$ and $n = m = 1$ so that there are only one Brownian motion and one independent Poisson process. The matrix $\Sigma_t$ is then given by

$$\begin{pmatrix} |v_t^1|^2 + |\varphi_t^1|^2 \nu(t) & v_t^1 v_t^2 + \varphi_t^1 \varphi_t^2 \nu(t) \\ \\ v_t^1 v_t^2 + \varphi_t^1 \varphi_t^2 \nu(t) & |v_t^2|^2 + |\varphi_t^2|^2 \nu(t) \end{pmatrix},$$

its determinant is

$$\left| v_t^1 \varphi_t^2 - v_t^2 \varphi_t^1 \right|^2 \nu(t),$$

and so (6.4) is by (6.2) equivalent to the condition (1.5) of Jeanblanc-Picqué/Pontier (1990) that

$$\left| v_t^1 \varphi_t^2 - v_t^2 \varphi_t^1 \right| \geq \alpha > 0 \qquad P\text{-a.s. for } t \in [0,T] \text{ and some constant } \alpha.$$

A similar computation yields

$$\varphi_t^* \varrho_t = 1 - \frac{\mu_t^2 v_t^1 - \mu_t^1 v_t^2}{v_t^2 \varphi_t^1 - v_t^1 \varphi_t^2} \frac{1}{\nu(t)}$$

so that our condition (6.5) is by (6.2) a uniform version of their condition (1.6) which is necessary for absence of arbitrage. The crucial difference to our situation is that they assume $d = 2 = m + n$. This implies that not only the driving process $(W, N)$ (as explained in the remark below) but also $X$ itself has the martingale representation property. Hence every random variable $H \in \mathcal{L}^2$ is the sum of a constant and a stochastic integral with respect to $X$, without an additional term $L_T^H$ as in (2.1). In the language of financial mathematics, this means that $X$ yields a *complete market*; see Harrison/Pliska (1983). The importance of the assumption $d = m + n$ is therefore explained by the well-known fact that most optimization problems are substantially easier to solve in a complete than in an incomplete situation.

**2)** Shirakawa (1990a) considers essentially the same basic model as we do and studies the problem of finding sufficient conditions for the existence of an equivalent martingale measure for $X$. He shows in his Theorem 4.1 that *absence of arbitrage* in a first sense implies the existence of predictable processes $\pi = (\pi^1, \dots, \pi^n)^*$ and $\chi = (\chi^1, \dots, \chi^m)^*$ such that $\chi^k > 0$ for each $k$ and

$$\mu + \varphi\nu = v\pi + \varphi(\nu - \chi).$$

$\pi$ and $\chi$ are interpreted as *risk premium* processes associated to $W$ and $N$, respectively. Theorem 4.4 of Shirakawa (1990a) then shows that absence of arbitrage in a (stronger) second sense even implies the existence of an equivalent martingale measure for $X$. Our assumptions (6.4) and (6.5) imply the same conclusions; in fact, we can take $\pi := v^*\varrho$ and $\chi := \nu - (\varphi^*\varrho) \,\square\, \nu$, the interpretation of $\pi$ and $\chi$ as risk premia is provided by (6.7) and (6.8) below, and an equivalent martingale measure will be exhibited below. Thus we see again that our assumptions are closely related to a no-arbitrage condition on $X$. However, we have not pursued any further the issue of explicitly constructing an arbitrage opportunity from a violation of (6.5); for an approach in that direction, see Jeanblanc-Picqué/Pontier (1990).

**3)** The problem addressed in Shirakawa (1990b) is essentially the same as in Jeanblanc-Picqué/Pontier (1990), but for the case where both $W$ and $N$ are multidimensional. He also assumes that $d = n+m$ and this implies that his assumptions are practically the same as ours; (6.4) and (6.5) correspond to his Assumption 2.4. The clue to seeing this is the observation that for $d = m + n$, a slight modification of his Lemma 2.3 shows that

$$\Sigma_t^{-1} = D_t^* D_t \qquad , \qquad 0 \le t \le T,$$

where the matrix-valued process $D$ is defined by

$$D_t := \begin{pmatrix} (v_t^* v_t)^{-1} v_t^* \left(\mathrm{Id}_{d\times d} - \varphi_t F_t^{-1} \varphi_t^* E_t\right) \\ \frac{1}{\sqrt{\nu(t)}} \,\square\, \left(F_t^{-1}\varphi_t^* E_t\right) \end{pmatrix} \qquad , \qquad 0 \le t \le T,$$

with $\frac{1}{\sqrt{\nu}} := \left(\frac{1}{\sqrt{\nu^1}}, \dots, \frac{1}{\sqrt{\nu^m}}\right)^*$,

$$E_t := \mathrm{Id}_{d\times d} - v_t (v_t^* v_t)^{-1} v_t^* \qquad , \qquad 0 \le t \le T$$

and

$$F_t := \varphi_t^* E_t \varphi_t \qquad , \qquad 0 \le t \le T.$$

Establishing the correspondences between his conditions and ours is then a matter of straightforward but tedious computations.

**4)** The same model as in Shirakawa (1990b) is also studied in Xue (1992). His main contribution is to provide a rigorous proof of the martingale representation result used without proof in Jeanblanc-Picqué/Pontier (1990) and Shirakawa (1990b); see also Galtchouk (1976). In contrast to our situation, Xue (1992) also considers the complete case $d = m + n$. Apart from that, his conditions are almost identical to ours; he also assumes (6.4), and (6.5) is (although without the bound being uniform) implicitly used in his construction of the equivalent martingale measure by the appeal to his Theorem I.6.1.

Using (6.3), (6.4) and the boundedness of $\mu, v, \varphi, \nu$, one can show by a similar argument as in Xue (1992) that $X$ belongs to the space $\mathcal{S}^p$ of semimartingales for every $p < \infty$. The canonical decomposition $X = X_0 + M + A$ is given by

$$M_t^i = \sum_{j=1}^n \int_0^t X_{s-}^i v_s^{ij}\, dW_s^j + \sum_{k=1}^m \int_0^t X_{s-}^i \varphi_s^{ik} \left(dN_s^k - \nu^k(s)\, ds\right) \qquad , \qquad 0 \le t \le T$$

and

$$A_t^i = \int_0^t X_{s-}^i \left(\mu_s^i + \left(\varphi_s\, \nu(s)\right)^i\right) ds \qquad , \qquad 0 \le t \le T$$

for $i = 1, \ldots, d$. It is easy to see that $X$ satisfies (1.1) and (SC), and if we choose $B_t := t$ for all $t \in [0, T]$, the processes $\widehat{\lambda}$ and $\widehat{K}$ are given by

$$\widehat{\lambda}_t^i = \frac{1}{X_{t-}^i} \varrho_t^i \qquad , \qquad 0 \le t \le T, \text{ for } i = 1, \ldots, d$$

and

$$\widehat{K}_t = \int_0^t \left(\mu_s + \varphi_s\, \nu(s)\right)^* (v_s v_s^* + \psi_s \psi_s^*)^{-1} \left(\mu_s + \varphi_s\, \nu(s)\right) ds \qquad , \qquad 0 \le t \le T.$$

For details of these computations, we refer to Schweizer (1993a). Due to the boundedness of $\mu, \varphi, \nu$ and the nondegeneracy of $\Sigma$, $\widehat{K}$ is continuous and bounded, and Theorem 15 therefore implies that every $H \in \mathcal{L}^2$ admits a strong F-S decomposition. If we assume in addition that

(6.6) the process $\left(\left(\mu_t + \varphi_t\, \nu(t)\right)^* \left(v_t v_t^* + \psi_t \psi_t^*\right)^{-1} \left(\mu_t + \varphi_t\, \nu(t)\right)\right)_{0 \le t \le T}$ is deterministic,

then (1.8) can be solved for every pair $(c, H) \in I\!R \times \mathcal{L}^2$. This generalizes Corollary II.8.5 of Schweizer (1993a).

**Remarks. 1)** As equivalent martingale measure for $X$, we can choose the minimal signed local martingale measure $\widehat{P}$. Using (6.3), (6.4), (6.5) and the boundedness of $\mu, v, \varphi, \nu$, one can in fact show that $\widehat{Z}$ is strictly positive and in $\mathcal{M}^r(P)$ for every $r < \infty$; hence $\widehat{P} \approx P$, and $X$ is in $\mathcal{M}^p(\widehat{P})$ for every $p < \infty$. Moreover, Girsanov's theorem implies that

$$\widehat{W}_t := W_t + \int_0^t v_s^* \varrho_s\, ds \qquad , \qquad 0 \le t \le T \tag{6.7}$$

is an $n$-dimensional Brownian motion with respect to $\widehat{P}$ and $I\!F$, and that $N$ is an $m$-variate point process with $(\widehat{P}, I\!F)$-intensity

$$\widehat{\nu}_t := \nu(t) - (\varphi_t^* \varrho_t) \,\square\, \nu(t) \qquad , \qquad 0 \le t \le T. \tag{6.8}$$

For details, see Schweizer (1993a).

**2)** For random variables $H \in \mathcal{L}^{2+\varepsilon}(P)$ with some $\varepsilon > 0$, the existence of a strong F-S decomposition was also established in Schweizer (1993a) by a different method. The argument there used the fact that with respect to its own filtration $I\!\!F$, the process $(W, N)$ has the *martingale representation property:* every $F \in \mathcal{L}^2(P)$ can be written as

$$F = E[F] + \sum_{j=1}^{n} \int_0^T f_s^j \, dW_s^j + \sum_{k=1}^{m} \int_0^T g_s^k \left( dN_s^k - \nu^k(s) \, ds \right) \qquad P\text{-a.s.}$$

for predictable processes $f = (f^1, \ldots, f^n)^*$ and $g = (g^1, \ldots, g^m)^*$ satisfying

$$\sum_{j=1}^{n} E\left[ \int_0^T \left| f_s^j \right|^2 \, ds \right] + \sum_{k=1}^{m} E\left[ \int_0^T \left| g_s^k \right|^2 \nu^k(s) \, ds \right] < \infty.$$

Applying this result to $F := H\widehat{Z}_T$ allows to give a fairly explicit construction of the processes $V^H$, $\xi^H$ and $L^H$ in terms of $f, g$ and $H$. The (somewhat lengthy) details can be found in Schweizer (1993a).

**3)** In contrast to the case where $X$ is continuous, the strong F-S decomposition can here not be obtained as the Galtchouk-Kunita-Watanabe decomposition under $\widehat{P}$, since the corresponding $\widehat{P}$-martingale $\widehat{L}$ will typically not be a $P$-martingale.

Consider now the special case $m = 0$ so that (6.1) is the standard multidimensional diffusion model introduced by Bensoussan (1984) and generalized by Karatzas/Lehoczky/Shreve/Xu (1991). Conditions (6.2), (6.3) and (6.5) then disappear, and (6.4) can be relaxed to the assumption that

(6.9) $\qquad$ the matrix $v_t v_t^*$ is $P$-a.s. invertible for every $t \in [0, T]$,

if we impose in addition the condition

$$\int_0^T \left\| v_s^* \varrho_s \right\|^2 \, ds \le C < \infty \qquad P\text{-a.s. for some constant } C; \tag{6.10}$$

this guarantees that $\widehat{K}_T$ is bounded. Condition (6.9) follows immediately from the standard assumption in Karatzas/Lehoczky/Shreve/Xu (1991) that the matrix $v_t$ has full rank $d \le n$ $P$-a.s. for every $t \in [0, T]$. Condition (6.10) is also quite usual; it is for instance satisfied if $v^* \varrho$ is $P$-a.s. bounded, uniformly in $t$ and $\omega$. Finally, (6.6) reduces to the assumption that

$$\left( \mu_t^* (v_t v_t^*)^{-1} \mu_t \right)_{0 \le t \le T} \quad \text{is deterministic.}$$

In particular, if we choose $d = 1$ (one asset available for trade), $m = 0$ (no Poisson component), $n = 2$ (two driving Wiener processes) and

$$v_t^1 = v_t r_t \qquad , \qquad v_t^2 = v_t \sqrt{1 - (r_t)^2} \qquad , \qquad \mu_t = m_t$$

with $|r_t| \le 1$, then (6.4) is equivalent to assuming that $(v_t)$ is bounded away from 0, uniformly in $t$ and $\omega$, and (6.6) translates into the assumption that

$$\left( \frac{m_t}{v_t} \right)_{0 \le t \le T} \quad \text{is deterministic.}$$

Thus we recover the results of Schweizer (1992) as a special case.

### 6.3. A counterexample

Our third and final example is a counterexample which shows that Theorem 3 is in general no longer true if we remove the assumption that the EMVT process $\widetilde{K}$ is deterministic. More precisely, we shall prove that the strategy $\xi^{(c)}$ defined by (2.2) need not be optimal in that case. For that purpose, suppose that $X$ is given by

$$X_t = W_t + \int_0^t \mu_s \, ds \qquad , \qquad 0 \le t \le T,$$

where $W$ is a Brownian motion with respect to $P$ and $I\!F$, $\mu$ is an $I\!F$-adapted process bounded uniformly in $t$ and $\omega$, and $I\!F = I\!F^X$ is the $P$-augmentation of the filtration generated by $X$. Such a model can easily be constructed using an argument from Karatzas/Xue (1991). In fact, one can start from any sufficiently large filtration $\mathbb{G}$, a $(P,\mathbb{G})$-Brownian motion $B$ and a bounded $\mathbb{G}$-adapted process $m$, set

$$X_t := B_t + \int_0^t m_s \, ds \qquad , \qquad 0 \le t \le T$$

and then choose $\mu$ as the $I\!F^X$-optional projection of $m$ and $W$ as

$$W_t := B_t + \int_0^t (m_s - \mu_s) \, ds = X_t - \int_0^t \mu_s \, ds \qquad , \qquad 0 \le t \le T.$$

Since $\mu$ is bounded, the minimal martingale density $\widehat{Z}$ is strictly positive and in $\mathcal{M}^r(P)$ for every $r < \infty$; hence $\widehat{P} \approx P$ on $\mathcal{F}_T$. By Girsanov's theorem, $X$ is a Brownian motion under $\widehat{P}$ and therefore has the representation property with respect to its own filtration $I\!F$. Moreover, $\frac{1}{\widehat{Z}} = \mathcal{E}\left(\int \mu \, dX\right)$ is in $\mathcal{M}^r(\widehat{P})$ for every $r < \infty$, and this allows us to conclude that every $H \in \mathcal{L}^{2+\varepsilon}(P, \mathcal{F}_T)$ for some $\varepsilon > 0$ can be written as

$$H = \widehat{E}[H] + \int_0^T \xi_s^H \, dX_s \qquad P\text{-a.s.} \tag{6.11}$$

for some $I\!F$-predictable process $\xi^H$ satisfying

$$E\left[\int_0^T (\xi_s^H)^2 \, ds\right] < \infty;$$

the last assertion follows from the Burkholder-Davis-Gundy inequality. Since

$$\Theta = L^2(M) = \left\{ \text{all } I\!F\text{-predictable } \vartheta \text{ such that } E\left[\int_0^T \vartheta_s^2 \, ds\right] < \infty \right\},$$

(6.11) implies that $G_T(\Theta)$ contains $\bigcup\limits_{\varepsilon>0} \mathcal{L}^{2+\varepsilon}(P,\mathcal{F}_T)$.

**Proposition 18.** *Denote by $\zeta \in \Theta$ the integrand in the representation*

$$\widehat{Z}_T = E\big[\widehat{Z}_T^2\big] + \int_0^T \zeta_s \, dX_s \qquad P\text{-a.s.} \tag{6.12}$$

*For every $H \in \mathcal{L}^{2+\varepsilon}(P,\mathcal{F}_T)$ with $\varepsilon > 0$, the solution of (1.8) is then given by*

$$\psi^{(c)} := \xi^H + \frac{c - \widehat{E}[H]}{E\big[\widehat{Z}_T^2\big]}\, \zeta.$$

**Proof.** First of all, $\psi^{(c)}$ is in $\Theta$ since both $\xi^H$ and $\zeta$ are. Furthermore, (6.11) and (6.12) imply that

$$H - c - G_T(\psi^{(c)}) = \widehat{E}[H] - c - \int_0^T \frac{c - \widehat{E}[H]}{E\big[\widehat{Z}_T^2\big]}\, \zeta_s \, dX_s = \frac{\widehat{E}[H] - c}{E\big[\widehat{Z}_T^2\big]}\, \widehat{Z}_T$$

and therefore

$$E\Big[\Big(H - c - G_T(\psi^{(c)})\Big)\, G_T(\vartheta)\Big] = \frac{\widehat{E}[H] - c}{E\big[\widehat{Z}_T^2\big]}\, \widehat{E}\left[\int_0^T \vartheta_s \, dX_s\right] = 0$$

for every bounded $I\!\!F$-predictable process $\vartheta$, since $X$ is a $(\widehat{P}, I\!\!F)$-Brownian motion. Thus $\psi^{(c)}$ solves (1.8) by Lemma 5.

**q.e.d.**

Now consider the strategy $\xi^{(c)}$ defined by (2.2). Since

$$\xi_t^{(c)} = \xi_t^H + \mu_t \left(V_{t-}^H - c - G_{t-}(\xi^{(c)})\right)$$

and

$$V_t^H = \widehat{E}[H] + \int_0^t \xi_s^H \, dX_s = \widehat{E}[H] + G_t(\xi^H)$$

by (6.11), the process $U := \widehat{E}[H] - c + G(\xi^H - \xi^{(c)})$ satisfies the stochastic differential equation

$$U_t = \widehat{E}[H] - c - \int_0^t U_{s-}\mu_s \, dX_s \qquad , \qquad 0 \le t \le T.$$

Hence we deduce from (6.11) that

$$H - c - G_T(\xi^{(c)}) = U_T = \left(\widehat{E}[H] - c\right) \mathcal{E}\left(-\int \mu \, dX\right)_T = \left(\widehat{E}[H] - c\right) \widehat{Z}_T \exp\left(-\int_0^T \mu_s^2 \, ds\right).$$

If we now suppose that $\xi^{(c)}$ solves (1.8), then Lemma 5 implies that the probability measure $Q$ with density

$$\frac{dQ}{dP} := \text{const.}\left(\widehat{E}[H] - c\right)\widehat{Z}_T \exp\left(-\int_0^T \mu_s^2\,ds\right)$$

on $\mathcal{F}_T$ is an equivalent martingale measure for $X$. But since $X$ has the representation property under $\widehat{P}$, Théorème (11.3) and Corollaire (11.4) of Jacod (1979) imply that $Q$ must coincide with $\widehat{P}$ so that

$$\int_0^T \mu_s^2\,ds \quad \text{must be deterministic.}$$

Thus we see that $\xi^{(c)}$ will in general not solve (1.8). To make the counterexample more precise, we could start by defining $W$, $\mu$ and $X$ on $[0,\infty)$ and then apply the preceding arguments to some $T > 0$ such that $\int_0^T \mu_s^2\,ds$ is not deterministic; this will always exist unless $\mu$ itself is deterministic.

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