Local risk-minimization for
multidimensional assets and payment streams

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Abstract: One of the earliest concepts for hedging and pricing in incomplete financial markets has been the quadratic criterion of local risk-minimization. However, definitions and theory have so far been established only for the case of a single (one-dimensional) risky asset. We extend the approach to a general multidimensional setting and prove that the basic martingale characterization result for locally risk-minimizing strategies still holds true. In comparison with existing literature, the self-contained presentation is more streamlined, and a number of earlier imposed technical conditions are no longer needed. As a minor extension, we show how payment streams (instead of final payoffs only) can be handled as well.

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0. Introduction

Since its inception 20 years ago in [9], local risk-minimization has become a popular criterion for hedging and pricing in incomplete financial markets. It has been and is still being used in many different areas, including transaction costs, American options, insider models, credit and default risk, life insurance liabilities, etc. Indeed, a Google Scholar search in early August 2007 with the key phrase "local risk-minimization" (enclosed in quotation marks) returned well over 100 genuine hits. One reason for this popularity lies in the fact that although the definition of locally risk-minimizing strategies is rather technical, they can be computed very easily and fairly explicitly in quite general semimartingale models. For some background reading, we refer to the survey article [13].

A closer look at the existing literature reveals, somewhat surprisingly, that local risk-minimization has up to now been defined and studied only for the case where the financial market contains one single risky asset. In this paper, we generalize the approach to $I\!\!R^d$-valued asset price processes X and show that one obtains the same martingale characterization of locally risk-minimizing strategies as for $d = 1$. We also remove several technical restrictions imposed in the original formulation, and extend the results from European contingent claims to payment streams. The overall presentation is deliberately kept self-contained.

The paper is structured as follows. Section 1 introduces the setup, defines local risk-minimization and formulates its equivalent characterization as the main result in Theorem 1.6. Section 2 contains two auxiliary results, and Section 3 presents a general convergence result for certain quantities appearing in our analysis. In Section 4, we prove the main result and comment on its relation to the existing literature, and Section 5 briefly presents the link to the Föllmer-Schweizer decomposition.

1. Setting, problem formulation and main result

This section explains the basic problem, introduces required terminology and concepts, and formulates our main result.

We start with a filtered probability space $(\Omega, \mathcal{F}, I\!\!F, P)$, where $T > 0$ is a finite time horizon and the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ satisfies the usual conditions. All processes are indexed by time t with $0 \le t \le T$. Discounted asset prices are given by an $I\!\!R^d$-valued RCLL semimartingale $X = (X_t)_{0 \le t \le T}$, and we assume that X satisfies the *structure condition (SC)*. This means that X is special with canonical decomposition

$$X = X_0 + M + A = X_0 + M + \int d\langle M \rangle \lambda,$$

where M is in $\mathcal{M}^2_{0,\mathrm{loc}}$ and λ is $I\!\!R^d$-valued, predictable and in $L^2_{\mathrm{loc}}(M)$, so that the *mean-variance tradeoff* process $K := \int \lambda \, dA = \int \lambda^{\mathrm{tr}} d\langle M \rangle \lambda$ satisfies $K_T < \infty$ P-a.s. It is well

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known that (SC) is related to an absence-of-arbitrage condition; see [2] and [12].

Let B be a bounded, strictly increasing, predictable (real-valued) process null at 0 such that $\langle M^i, M^j \rangle \ll B$ for all i, j. One example is $B_t := \tanh\left(t + \sum_{i=1}^{d} \langle M^i \rangle_t\right)$, and we shall see later that the choice of B does not affect our main result. We denote by $P_B := P \otimes B$ the finite measure on $\left(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T])\right)$ given by $P_B[D] := E\left[\int_0^T I_D(\omega, s) \, dB_s(\omega)\right]$ and define the matrix-valued predictable process σ by $d\langle M \rangle = \sigma \, dB$. Each $\sigma_t(\omega)$ is a nonnegative definite symmetric $d \times d$-matrix, and we note for future use that

$$\left\langle \int \vartheta \, dM, \int \zeta \, dM \right\rangle = \int \vartheta^{\mathrm{tr}} d\langle M \rangle \zeta = \int \vartheta^{\mathrm{tr}} \sigma \zeta \, dB \qquad \text{for } \vartheta, \zeta \in L^2_{\mathrm{loc}}(M)$$

and that hence $\int \delta^{\mathrm{tr}} \sigma \delta \, dB$ is bounded iff $\left\langle \int \delta \, dM \right\rangle$ is bounded. Moreover, we point out that $\int \delta \, dX$ is in $\mathcal{S}^2(P)$ (see below) for every $\mathbb{R}^d$-valued predictable process δ such that $\left\langle \int \delta \, dM \right\rangle$ and $\int |\delta \, dA|$ are both bounded. This will be important later when we define small perturbations.

Definition 1.1. The space Θ_S consists of all $\mathbb{R}^d$-valued predictable processes ϑ such that the stochastic integral process $\int \vartheta \, dX$ is well-defined and in the space $\mathcal{S}^2(P)$ of semimartingales. This means that

$$E\left[\int_0^T \vartheta_s^{\mathrm{tr}} d\langle M \rangle_s \, \vartheta_s + \left(\int_0^T |\vartheta_s \, dA_s|\right)^2\right] < \infty$$

or equivalently that $\int_0^T \vartheta_s^{\mathrm{tr}} \sigma_s \vartheta_s \, dB_s + \left(\int_0^T |\vartheta_s^{\mathrm{tr}} \sigma_s \lambda_s| \, dB_s\right)^2 \in L^1(P)$. An L^2-*strategy* is a pair $\varphi = (\vartheta, \eta)$, where $\vartheta \in \Theta_S$ and η is a real-valued adapted process such that the *value process* $V(\varphi) := \vartheta^{\mathrm{tr}} X + \eta$ is right-continuous and square-integrable, i.e., $V_t(\varphi) \in L^2(P)$ for all $t \in [0, T]$. φ is called *0-achieving* if $V_T(\varphi) = 0$ P-a.s.

As usual, a strategy $\varphi = (\vartheta, \eta)$ describes how we trade in the financial market given by X. At time t, we hold ϑ_t^i shares of asset i for $i = 1, \ldots, d$ and have the amount η_t in a riskless bank account with zero interest rate and hence constant value 1. We next consider a *payment stream* $H = (H_t)_{0 \le t \le T}$ kept fixed throughout the sequel. Mathematically, H is right-continuous, adapted, real-valued and square-integrable; the interpretation is that $H_t \in L^2(P)$ represents the total payments on $[0, t]$ arising due to some financial contract. A *European contingent claim* with maturity T would have $H_t = 0$ for all $t < T$ and just an $\mathcal{F}_T$-measurable payoff $H_T \in L^2(P)$ due at time T; in general, the process H involves both cash inflows and outlays, and can but need not be of finite variation. We want to hedge H in a quadratic sense, and so first assign to each L^2-strategy a cost and a quadratic risk process.

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Definition 1.2. Fix a payment stream H. The *(cumulative) cost process* of an L^2-strategy $\varphi = (\vartheta, \eta)$ is

$$C_t^H(\varphi) := H_t + V_t(\varphi) - \int_0^t \vartheta_s \, dX_s, \qquad 0 \le t \le T.$$

φ is called *self-financing* (for H) if $C^H(\varphi)$ is constant, and *mean-self-financing* if $C^H(\varphi)$ is a martingale (which is then square-integrable). The *risk process* of φ is

$$R_t^H(\varphi) := E\left[\left(C_T^H(\varphi) - C_t^H(\varphi)\right)^2 \Big| \mathcal{F}_t\right], \qquad 0 \le t \le T.$$

As usual, $C_t^H(\varphi)$ describes the cumulative costs on $[0, t]$ from paying according to H and trading according to φ; see [6] and [5]. One difference to the well-known situation in dimension $d = 1$ and with a European contingent claim H_T is that we use here a different attribution of value and costs; this was suggested in [7] and also used in [8] and [1]. Indeed, our total cost C_T^H is the same as in the approach in [11], but we use strategies with $V_T = 0$ instead of $V_T = H_T$. This is (with hindsight) better suited for an extension to payment streams.

Remark 1.3. For later use, we observe that if $\varphi = (\vartheta, \eta)$ is a 0-achieving and mean-self-financing L^2-strategy for H, then φ is uniquely determined from ϑ (and of course H). To see this, use the definition and martingale property of $C^H(\varphi)$ and $V_T(\varphi) = 0$ to obtain

(1.1)
$$\eta_t = V_t(\varphi) - \vartheta_t^{\mathrm{tr}} X_t$$
$$= E\left[H_T - \int_0^T \vartheta_s \, dX_s \,\Big|\, \mathcal{F}_t\right] - H_t + \int_0^t \vartheta_s \, dX_s - \vartheta_t^{\mathrm{tr}} X_t$$
$$= E\left[H_T - H_t - \int_t^T \vartheta_s \, dX_s \,\Big|\, \mathcal{F}_t\right] - \vartheta_t^{\mathrm{tr}} X_t.$$

Clearly, $R^H(\varphi)$ is then determined by ϑ as well. ◇

As in [11], we want to minimize $R^H(\varphi)$ with respect to small perturbations of φ, to be introduced next. A *partition* of $[0, T]$ is a set $\tau = \{t_0, t_1, \dots, t_k\} \subseteq [0, T]$ with $0 = t_0 < t_1 < \cdots < t_k = T$, and its *mesh size* is $|\tau| := \max\{t_{i+1} - t_i \,|\, t_i, t_{i+1} \in \tau\}$. Note that k may vary with τ. A sequence $(\tau_n)_{n \in \mathbb{N}}$ of partitions is *increasing* if $\tau_n \subseteq \tau_{n+1}$ for all n; it *tends to the identity* if $\lim_{n \to \infty} |\tau_n| = 0$. To each partition τ, we associate on $\Omega \times [0, T]$ the σ-fields

$$\mathcal{P}^\tau := \sigma\left(\left\{D_0 \times \{0\}, D_i \times (t_i, t_{i+1}] \,\big|\, t_i, t_{i+1} \in \tau, D_0 \in \mathcal{F}_0, D_i \in \mathcal{F}_{t_i}\right\}\right),$$
$$\mathcal{O}^\tau := \sigma\left(\left\{D_0 \times \{0\}, D_{i+1} \times (t_i, t_{i+1}] \,\big|\, t_i, t_{i+1} \in \tau, D_0 \in \mathcal{F}_0, D_{i+1} \in \mathcal{F}_{t_{i+1}}\right\}\right).$$

One easily sees that $\sigma\left(\bigcup_{n\in I\!N}\mathcal{P}^{\tau_n}\right)$ equals the predictable σ-field $\mathcal{P}$ for any sequence of partitions tending to the identity; hence $\mathcal{P}^{\tau_n}$ increases to $\mathcal{P}$ if (τ_n) is in addition increasing.

Definition 1.4. A pair $\Delta = (\delta, \varepsilon)$ consisting of an $I\!R^d$-valued predictable process δ and an adapted real-valued process ε is called a *small perturbation* if $\left\langle \int \delta\, dM \right\rangle = \int \delta^{\mathrm{tr}}\sigma\delta\, dB$ and $|\delta^{\mathrm{tr}}\sigma\lambda|$ $\left(\text{and hence also } \int |\delta\, dA| = \int |\delta^{\mathrm{tr}}\sigma\lambda|\, dB\right)$ are bounded (uniformly in t, ω), the process $V(\Delta) := \delta^{\mathrm{tr}}X + \varepsilon$ is square-integrable, and $V_T(\Delta) = 0$ P-a.s. (Note that Δ need not be an L^2-strategy, although $\delta \in \Theta_S$ and Δ is 0-achieving: We do not require that $V(\Delta)$ is right-continuous.) For each subinterval $(s, t]$ of $[0, T]$, we then define the small perturbation

$$(1.2) \qquad \Delta|_{(s,t]} := \begin{cases} \left(\delta I_{]\!]s,t]\!]}, \varepsilon I_{[\![s,t[\![}\right) & \text{if } t < T, \\[2mm] \left(\delta I_{]\!]s,T]\!]}, \varepsilon I_{[\![s,T]\!]}\right) & \text{if } t = T. \end{cases}$$

For an L^2-strategy φ, a small perturbation Δ and a partition τ of $[0, T]$, we set

$$(1.3) \qquad r^{\tau}[\varphi, \Delta; H] := \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}^H\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) - R_{t_i}^H(\varphi)}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]}.$$

Recall that the payment stream H is fixed throughout. Note that $V(\psi)$, $C^H(\psi)$, $R^H(\psi)$ and hence $r^{\tau}[\varphi, \Delta; H]$ are well-defined without any requirement of right-continuity for $V(\psi)$.

As in [11], $r^{\tau}[\varphi, \Delta; H]$ is a measure for the increase of quadratic risk when φ is perturbed locally by Δ. Note that we have slightly modified the original definition of $\Delta|_{(s,t]}$ in [11] to account for the special case $t = T$. This allows us to drop the assumption, imposed in [11], that M is P-a.s. continuous at T, since we no longer require for a small perturbation that $\delta_T = 0$. In fact, every $I\!R^d$-valued predictable process δ with $\left\langle \int \delta\, dM \right\rangle$ and $|\delta^{\mathrm{tr}}\sigma\lambda|$ both bounded can be extended to a small perturbation $\Delta = (\delta, \varepsilon)$, given by the 0-achieving mean-self-financing L^2-strategy associated to δ by Remark 1.3.

The second (and important) difference to [11] is the denominator of $r^{\tau}[\varphi, \Delta; H]$ in (1.3). For a vector-valued M, we cannot divide by the (perhaps non-invertible) matrix $E\left[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i}\right]$; the appropriate "time scale" is now instead given by B. Our choice of B as strictly increasing will also simplify some arguments later on.

Definition 1.5. Fix a payment stream H. An L^2-strategy φ is called *locally risk-minimizing* for H if for every small perturbation Δ and every increasing sequence $(\tau_n)_{n\in I\!N}$ of partitions tending to the identity, we have

$$\liminf_{n\to\infty} r^{\tau_n}[\varphi, \Delta; H] \geq 0 \qquad P_B\text{-a.e.}$$

Our main result is now the following extension of Proposition 2.3 in [11], or Theorem 3.3 in [13].

Theorem 1.6. *Suppose the $\mathbb{R}^d$-valued semimartingale X satisfies the structure condition (SC) and let H be a payment stream. If the mean-variance tradeoff process $K = \int \lambda^{\mathrm{tr}} d\langle M \rangle \lambda$ (or, equivalently, A) is continuous, the following are equivalent for an L^2-strategy φ:*

1) *φ is locally risk-minimizing for H.*

2) *φ is 0-achieving and mean-self-financing, and the cost process $C^H(\varphi)$ is strongly orthogonal to M.*

In particular, the concept "locally risk-minimizing" does not depend on the choice of B.

We prove Theorem 1.6 in Section 4 where we also provide additional comments.

In our proof, we shall need small perturbations which satisfy the additional requirement that $\varepsilon_t = 0$ for all $t < T$. The next auxiliary result shows how this can be achieved.

Lemma 1.7. *Suppose X satisfies the structure condition (SC). For any $\mathbb{R}^d$-valued predictable process δ, there is a sequence $\Delta^m = (\delta^m, \varepsilon^m)$, $m \in \mathbb{N}$, of small perturbations such that $\varepsilon_t^m = 0$ for $t < T$ and each m, and $\lim_{m \to \infty} \delta^m = \delta$ P_B-a.e. More precisely, δ^m has the form*

$$(1.4) \qquad \delta^m = \delta I_{[\![0, \varrho_m]\!]} I_{\{|\delta| \leq m\}} I_{\{\delta^{\mathrm{tr}} \sigma \delta \leq m\}} I_{\{|\delta^{\mathrm{tr}} \sigma \lambda| \leq m\}}$$

for an increasing sequence of stopping times $\varrho_m \nearrow T$ P-a.s.

Proof. By the structure condition (SC), X is in $\mathcal{S}_{\mathrm{loc}}^2(P)$, and so there are stopping times $\varrho_m \nearrow T$ P-a.s. with $X_{\varrho_m}^* := \sup_{0 \leq t \leq \varrho_m} |X_t| \in L^2(P)$ for each m. Now define the predictable process δ^m by (1.4) and $\Delta^m := (\delta^m, \varepsilon^m)$ by

$$\varepsilon_t^m := -I_{\{T\}}(t)(\delta_t^m)^{\mathrm{tr}} X_t.$$

Then ε^m is adapted with $\varepsilon_t^m = 0$ for all $t < T$, and $|(\delta^m)^{\mathrm{tr}} \sigma \lambda|$ and $\langle \int \delta^m dM \rangle$ are both bounded, since B is bounded. Moreover, $V_T(\Delta^m) = 0$ and for $t < T$, (1.4) gives $|V_t(\Delta^m)| = |(\delta_t^m)^{\mathrm{tr}} X_t| \leq m X_{\varrho_m}^* \in L^2(P)$ so that Δ^m is a small perturbation. Finally, (1.4) clearly implies $\delta^m \longrightarrow \delta$ P_B-a.e. as $m \to \infty$. **q.e.d.**

2. Preliminary results

This section prepares the ground by proving two auxiliary results. Although these are analogous to earlier work in [11], we provide full details since our definitions here are slightly

different and the results are a bit more general. We first show that when searching for locally risk-minimizing L^2-strategies, we can restrict our attention to 0-achieving and mean-self-financing ones. This is a more general version of Lemma 2.1 in [11]; thanks to our choice of B, we do not need the assumption that $\langle M \rangle$ (for $d = 1$) is P-a.s. strictly increasing on $[0, T]$.

Proposition 2.1. *Suppose X satisfies the structure condition (SC) and fix a payoff process H. Then any locally risk-minimizing L^2-strategy is 0-achieving and mean-self-financing.*

Proof. Let $\varphi = (\vartheta, \eta)$ be locally risk-minimizing; then φ is 0-achieving by definition. Define $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ by $\hat{\vartheta} \equiv \vartheta$ and like in (1.1)

$$\hat{\eta}_t := E\left[H_T - \int_0^T \vartheta_s \, dX_s \,\middle|\, \mathcal{F}_t\right] - H_t + \int_0^t \vartheta_s \, dX_s - \vartheta_t^{\mathrm{tr}} X_t, \qquad 0 \le t \le T,$$

choosing a right-continuous version of the martingale given by the conditional expectations. Then $\hat{\varphi}$ is an L^2-strategy, because $\hat{\vartheta}$ is in Θ_S, $\hat{\eta}$ is adapted, and $V(\hat{\varphi})$ is like H and $\int \vartheta \, dX$ right-continuous and square-integrable. Moreover, $\hat{\varphi}$ is clearly 0-achieving and by definition of $\hat{\eta}$ also mean-self-financing. Because $C_T^H(\hat{\varphi}) = H_T - \int_0^T \vartheta_s \, dX_s = C_T^H(\varphi)$, we thus have

(2.1) $$C_t^H(\hat{\varphi}) = E[C_T^H(\hat{\varphi}) \,|\, \mathcal{F}_t] = E[C_T^H(\varphi) \,|\, \mathcal{F}_t] \qquad \text{for any } t \in [0, T].$$

Moreover, $\Delta := \hat{\varphi} - \varphi = (0, \hat{\eta} - \eta) =: (\delta, \varepsilon)$ is like $\hat{\varphi}$ and φ an L^2-strategy and 0-achieving, and since $\delta \equiv 0$, Δ is even a small perturbation.

Now take any partition τ of $[0, T]$ and $t_i, t_{i+1} \in \tau$. Since $\Delta|_{(t_i, t_{i+1}]}$ is like Δ a small perturbation and hence 0-achieving, we get $V_T\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) = 0 = V_T(\varphi)$ and thus

$$C_T^H\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) = C_T^H(\varphi),$$

since $\delta \equiv 0$. For $t_i < T$, we have by (1.2) and (2.1)

$$C_{t_i}^H\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) = H_{t_i} + V_{t_i}\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) - \int_0^{t_i} \vartheta_s \, dX_s$$

$$= H_{t_i} + \vartheta_{t_i}^{\mathrm{tr}} X_{t_i} + \hat{\eta}_{t_i} - \int_0^{t_i} \vartheta_s \, dX_s$$

$$= C_{t_i}^H(\hat{\varphi})$$

$$= E[C_T^H(\varphi) \,|\, \mathcal{F}_{t_i}].$$

Therefore we obtain by using (2.1)

$$R_{t_i}^H\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) = E\left[\left(C_T^H(\varphi) - E[C_T^H(\varphi) \,|\, \mathcal{F}_{t_i}]\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right]$$

$$= E\left[\left(C_T^H(\varphi) - C_{t_i}^H(\varphi)\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right] - \left(C_{t_i}^H(\varphi) - E[C_T^H(\varphi) \,|\, \mathcal{F}_{t_i}]\right)^2$$

$$= R_{t_i}^H(\varphi) - \left(C_{t_i}^H(\varphi) - E[C_T^H(\varphi) \,|\, \mathcal{F}_{t_i}]\right)^2$$

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and hence

$$(2.2) \qquad r^\tau[\varphi, \Delta; H] = \sum_{t_i, t_{i+1} \in \tau} \frac{R_{t_i}^H\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) - R_{t_i}^H(\varphi)}{E[B_{t_{i+1}} - B_{t_i} \mid \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]}$$

$$= -\sum_{t_i, t_{i+1} \in \tau} \frac{\left(C_{t_i}^H(\varphi) - E[C_T^H(\varphi) \mid \mathcal{F}_{t_i}]\right)^2}{E[B_{t_{i+1}} - B_{t_i} \mid \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]}.$$

This already indicates that for a non-mean-self-financing L^2-strategy, $r^\tau[\varphi, \Delta; H]$ will probably become negative somewhere, and we now show that this persists asymptotically.

Since φ is locally risk-minimizing, $\liminf_{n\to\infty} r^{\tau_n}[\varphi, \Delta; H] \geq 0$ P_B-a.e. for every increasing sequence (τ_n) of partitions tending to the identity. Now take such a sequence and assume that for some $n_0 \in I\!N$ and some $t_0 \in \tau_{n_0}$, we have

$$C_{t_0}^H(\varphi) \neq E[C_T^H(\varphi) \mid \mathcal{F}_{t_0}] \qquad \text{on a set } \Gamma \in \mathcal{F} \text{ with } P[\Gamma] > 0.$$

Since we can and do choose both $C^H(\varphi)$ and $E[C_T^H(\varphi) \mid I\!F]$ right-continuous, there are for every $\omega \in \Gamma$ positive $\beta(\omega), \gamma(\omega)$ such that

$$\left|\left(C_t^H(\varphi) - E[C_T^H(\varphi) \mid \mathcal{F}_t]\right)(\omega)\right| \geq \gamma(\omega) > 0 \qquad \text{for } \omega \in \Gamma \text{ and every } t \in [t_0, t_0 + 2\beta(\omega)].$$

Moreover, B is bounded by some constant k, say; hence we can also choose versions of the conditional expectations such that for all $n \in I\!N$ and all $t_i, t_{i+1} \in \tau_n$,

$$E[B_{t_{i+1}} - B_{t_i} \mid \mathcal{F}_{t_i}](\omega) \leq k \qquad \text{for all } \omega \in \Omega.$$

Now fix $\omega \in \Gamma$. Since (τ_n) is increasing and tends to the identity, we can find for every $t \in (t_0, t_0 + \beta(\omega)]$ and all sufficiently large n time points $t_j, t_{j+1} \in \tau_n$ with $t \in (t_j, t_{j+1}] \subseteq (t_0, t_0 + \beta(\omega)]$. This implies by (2.2) that for such t,

$$r^{\tau_n}[\varphi, \Delta; H](t, \omega) = -\frac{\left(C_{t_j}^H(\varphi) - E[C_T^H(\varphi) \mid \mathcal{F}_{t_j}]\right)^2(\omega)}{E[B_{t_{j+1}} - B_{t_j} \mid \mathcal{F}_{t_j}](\omega)} \leq -\frac{\gamma^2(\omega)}{k} < 0$$

for all large enough n and therefore

$$(2.3) \quad \liminf_{n\to\infty} r^{\tau_n}[\varphi, \Delta; H](t, \omega) \leq -\frac{\gamma^2(\omega)}{k} < 0 \qquad \text{for all } \omega \in \Gamma \text{ and } t \in (t_0, t_0 + \beta(\omega)].$$

Because B is strictly increasing and $P[\Gamma] > 0$, (2.3) yields

$$P_B\left[\liminf_{n\to\infty} r^{\tau_n}[\varphi, \Delta; H] < 0\right] \geq E\left[I_\Gamma(\omega)\left(B_{t_0+\beta(\omega)} - B_{t_0}\right)(\omega)\right] > 0$$

which contradicts the assumption that φ is locally risk-minimizing.

The above argument shows that if φ is locally risk-minimizing, we have with probability 1 that $C_t^H(\varphi) = E[C_T^H(\varphi) \,|\, \mathcal{F}_t]$ simultaneously for all $t \in \bigcup_{n \in I\!N} \tau_n =: \mathcal{D}$. Since (τ_n) tends to the identity, $\mathcal{D}$ is dense in $[0, T]$, and so right-continuity of $C^H(\varphi)$ and $E[C_T^H(\varphi) \,|\, I\!F]$ yields that with probability 1, $C_t^H(\varphi) = E[C_T^H(\varphi) \,|\, \mathcal{F}_t]$ for all $t \in [0, T]$. So φ is indeed mean-self-financing. **q.e.d.**

Our second auxiliary result shows that we can decompose $r^\tau[\varphi, \Delta; H]$ into a sum of four quantities. This will be useful later to analyze the asymptotic behaviour of $(r^{\tau_n}[\varphi, \Delta; H])_{n \in I\!N}$. Analogous results can be found in Section 2 of [11] and Section 3 of [10].

Proposition 2.2. *Assume that X satisfies the structure condition (SC) and fix a payment stream H. For every 0-achieving mean-self-financing L^2-strategy $\varphi = (\vartheta, \eta)$, every small perturbation $\Delta = (\delta, \varepsilon)$ and every partition τ of $[0, T]$, we then have*

$$r^\tau[\varphi, \Delta; H] = A_1^\tau + A_2^\tau + A_3^\tau + A_4^\tau,$$

where

$$A_1^\tau = E_B\left[(\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta \,\big|\, \mathcal{P}^\tau\right],$$

$$A_2^\tau = \sum_{t_i, t_{i+1} \in \tau} \frac{\mathrm{Var}\left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\bigg|\, \mathcal{F}_{t_i}\right]}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]},$$

$$A_3^\tau = \sum_{t_i, t_{i+1} \in \tau} \frac{\mathrm{Cov}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dM_s - \left(C_{t_{i+1}}^H(\varphi) - C_{t_i}^H(\varphi)\right), \int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\bigg|\, \mathcal{F}_{t_i}\right)}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]},$$

$$A_4^\tau = \sum_{t_i, t_{i+1} \in \tau} \frac{\left(E\left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\bigg|\, \mathcal{F}_{t_i}\right] + \varepsilon_{t_i}\right)^2}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]},$$

and μ^H is the integrand from the Galtchouk-Kunita-Watanabe decomposition of the martingale $C^H(\varphi)$ with respect to M.

Proof. For any small perturbation $\Delta' = (\delta', \varepsilon')$, we denote by $C^H(\vartheta + \delta')$ the cost process of the unique 0-achieving mean-self-financing L^2-strategy determined by $\vartheta + \delta'$; see Remark 1.3 or the construction as in (1.1) of $\hat\varphi$ in the proof of Proposition 2.1. Since φ itself is mean-self-financing, $C^H(\varphi) = C^H(\vartheta)$. Just plugging in the definitions gives

$$(2.4) \qquad C_T^H\left(\varphi + \Delta|_{(t_i, t_{i+1}]}\right) = C_T^H(\varphi) - \int_{t_i}^{t_{i+1}} \delta_s \, dX_s = C_T^H\left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]}\right)$$

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and for $t_i < T$, by using the definitions and (2.4),

$$C_{t_i}^H \left(\varphi + \Delta|_{(t_i, t_{i+1}]} \right) = H_{t_i} + \vartheta_{t_i}^{\mathrm{tr}} X_{t_i} + \eta_{t_i} + \varepsilon_{t_i} - \int_0^{t_i} \vartheta_s \, dX_s = C_{t_i}^H(\varphi) + \varepsilon_{t_i},$$

$$C_{t_i}^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right) = E \left[C_T^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right) \mid \mathcal{F}_{t_i} \right] = C_{t_i}^H(\varphi) - E \left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right].$$

Combining the above with the martingale property of $C^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right)$ yields

(2.5) $R_{t_i}^H \left(\varphi + \Delta|_{(t_i, t_{i+1}]} \right)$

$$= E \left[\left(C_T^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right) - C_{t_i}^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right) - \varepsilon_{t_i} - E \left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right] \right)^2 \middle| \mathcal{F}_{t_i} \right]$$

$$= R_{t_i}^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right) + \left(\varepsilon_{t_i} + E \left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right] \right)^2$$

$$= \mathrm{Var} \left[C_T^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right) \mid \mathcal{F}_{t_i} \right] + \left(\varepsilon_{t_i} + E \left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right] \right)^2.$$

Using (2.4) and $X = X_0 + M + A$ now gives

(2.6) $\mathrm{Var} \left[C_T^H \left(\vartheta + \delta I_{]\!]t_i, t_{i+1}]\!]} \right) \mid \mathcal{F}_{t_i} \right] = \mathrm{Var} \left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right] + \mathrm{Var} \left[C_T^H(\vartheta) - \int_{t_i}^{t_{i+1}} \delta_s \, dM_s \,\middle|\, \mathcal{F}_{t_i} \right]$

$$- 2 \mathrm{Cov} \left(C_T^H(\vartheta) - \int_{t_i}^{t_{i+1}} \delta_s \, dM_s, \int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right).$$

Since $C^H(\vartheta) = C^H(\varphi)$ and $\int \delta \, dM$ are both martingales and $\int_{t_i}^{t_{i+1}} \delta_s \, dA_s$ is $\mathcal{F}_{t_{i+1}}$-measurable, the last term in (2.6) equals

$$+ 2 \mathrm{Cov} \left(\int_{t_i}^{t_{i+1}} \delta_s \, dM_s - \left(C_{t_{i+1}}^H(\varphi) - C_{t_i}^H(\varphi) \right), \int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right).$$

Moreover, $C^H(\varphi) = C_0^H(\varphi) + \int \mu^H dM + L^H$ is the Galtchouk-Kunita-Watanabe decomposition of $C^H(\varphi)$ and so

(2.7) $\mathrm{Var} \left[C_T^H(\vartheta) - \int_{t_i}^{t_{i+1}} \delta_s \, dM_s \,\middle|\, \mathcal{F}_{t_i} \right]$

$$= \mathrm{Var} \left[C_{t_{i+1}}^H(\varphi) - C_{t_i}^H(\varphi) - \int_{t_i}^{t_{i+1}} \delta_s \, dM_s \,\middle|\, \mathcal{F}_{t_i} \right]$$

$$= R_{t_i}^H(\varphi) + \mathrm{Cov} \left(\int_{t_i}^{t_{i+1}} \delta_s \, dM_s, \int_{t_i}^{t_{i+1}} \delta_s \, dM_s - 2 \left(C_{t_{i+1}}^H(\varphi) - C_{t_i}^H(\varphi) \right) \,\middle|\, \mathcal{F}_{t_i} \right)$$

$$= R_{t_i}^H(\varphi) + E \left[\int_{t_i}^{t_{i+1}} d \left\langle \int \delta \, dM, \int \delta \, dM - 2 C^H(\varphi) \right\rangle_s \,\middle|\, \mathcal{F}_{t_i} \right]$$

$$= R_{t_i}^H(\varphi) + E \left[\int_{t_i}^{t_{i+1}} \delta_s^{\mathrm{tr}} \, d\langle M \rangle_s (\delta_s - 2\mu_s^H) \,\middle|\, \mathcal{F}_{t_i} \right].$$

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Now we combine (2.5) – (2.7) to obtain

$$
R_{t_i}^H \left(\varphi + \Delta|_{(t_i, t_{i+1}]} \right) - R_{t_i}^H(\varphi)
$$

$$
= E\left[\int_{t_i}^{t_{i+1}} (\delta_s - 2\mu_s^H)^{\mathrm{tr}} \, d\langle M \rangle_s \delta_s \,\bigg|\, \mathcal{F}_{t_i} \right] + \mathrm{Var}\left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\bigg|\, \mathcal{F}_{t_i} \right]
$$

$$
+ 2\mathrm{Cov}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dM_s - \left(C_{t_{i+1}}^H(\varphi) - C_{t_i}^H(\varphi) \right), \int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\bigg|\, \mathcal{F}_{t_i} \right)
$$

$$
+ \left(E\left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\bigg|\, \mathcal{F}_{t_i} \right] + \varepsilon_{t_i} \right)^2.
$$

Dividing by $E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]$, multiplying by $I_{(t_i, t_{i+1}]}$ and summing over $t_i, t_{i+1} \in \tau$ gives $r^\tau[\varphi, \Delta; H]$ on the left-hand side and the terms $A_2^\tau, A_3^\tau, A_4^\tau$ on the right-hand side. Moreover, we also obtain on the right-hand side the term

$$
\sum_{t_i, t_{i+1} \in \tau} \frac{E\left[\int_{t_i}^{t_{i+1}} (\delta_s - 2\mu_s^H)^{\mathrm{tr}} \, d\langle M \rangle_s \delta_s \,\bigg|\, \mathcal{F}_{t_i} \right]}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]}
$$

$$
= \sum_{t_i, t_{i+1} \in \tau} \frac{E\left[\int_{t_i}^{t_{i+1}} (\delta_s - 2\mu_s^H)^{\mathrm{tr}} \sigma_s \delta_s \, dB_s \,\bigg|\, \mathcal{F}_{t_i} \right]}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]},
$$

and since this equals $E_B\left[(\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta \,\big|\, \mathcal{P}^\tau \right] = A_1^\tau$, the proof is complete. **q.e.d.**

3. Convergence results

We have seen in Proposition 2.2 that $r^\tau[\varphi, \Delta; H]$ can be split into terms A_i^τ with $i = 1, \ldots, 4$. This section studies the asymptotic behaviour of $(A_i^{\tau_n})_{n \in \mathbb{N}}$ along an increasing sequence of partitions (τ_n) tending to the identity. The results are very similar to those in [10].

For this section, we introduce for brevity the following *standing assumptions*:

(3.1) X satisfies the structure condition (SC);

   H is a payment stream;

   $\varphi = (\vartheta, \eta)$ is a 0-achieving and mean-self-financing L^2-strategy;

   $\Delta = (\delta, \varepsilon)$ is a small perturbation;

   $(\tau_n)_{n \in \mathbb{N}}$ is an increasing sequence of partitions of $[0, T]$ tending to the identity.

Additional conditions will be added where needed. As in Section 2, μ^H is the integrand in the Galtchouk-Kunita-Watanabe decomposition of $C^H(\varphi)$ with respect to M, and we write

$$
r^{\tau_n}[\varphi, \Delta; H] = A_1^n + A_2^n + A_3^n + A_4^n
$$

for each $n \in \mathbb{N}$ by Proposition 2.2, using A_i^n as shorthand for $A_i^{\tau_n}$.

Lemma 3.1. *Under the standing assumptions (3.1),*

$$(3.2) \qquad \lim_{n \to \infty} A_1^n = (\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta \qquad P_B\text{-a.e.}$$

Proof. By Proposition 2.2, $A_1^n = E_B\left[(\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta \,\middle|\, \mathcal{P}^{\tau_n}\right]$, and as observed in Section 1, $\mathcal{P}^{\tau_n}$ increases to the predictable σ-field $\mathcal{P}$ since (τ_n) is increasing and tends to the identity. Moreover, $(\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta$ is predictable and in $L^1(P_B)$ by the Cauchy-Schwarz inequality, since μ^H is in Θ_S and $\langle \int \delta \, dM \rangle$ is bounded. Hence (3.2) follows directly from the martingale convergence theorem. **q.e.d.**

It will later be important to know when the limit in (3.2) is nonnegative. The next result, a multidimensional extension of Proposition 1.1 in [10], settles this; note that its last condition on δ is satisfied whenever δ comes from a small perturbation $\Delta = (\delta, \varepsilon)$.

Proposition 3.2. *Assume (3.1) and fix $\mu \in \Theta_S$. If $\mu^{\mathrm{tr}} \sigma \mu = 0$ P_B-a.e., then for every $\delta \in \Theta_S$,*

$$(3.3) \qquad (\delta - 2\mu)^{\mathrm{tr}} \sigma \delta \geq 0 \qquad P_B\text{-a.e.}$$

Conversely, if (3.3) holds for every $\mathbb{R}^n$-valued predictable δ such that $\langle \int \delta \, dM \rangle$ and $|\delta^{\mathrm{tr}} \sigma \lambda|$ are both bounded, then $\mu^{\mathrm{tr}} \sigma \mu = 0$ P_B-a.e.

Proof. If $\mu^{\mathrm{tr}} \sigma \mu = 0$, also $\mu^{\mathrm{tr}} \sigma \delta = 0$ by the Cauchy-Schwarz inequality, and so (3.3) follows. Conversely, if (3.3) were valid for every $\delta \in \Theta_S$, we could choose $\delta := \mu$ and immediately obtain $\mu^{\mathrm{tr}} \sigma \mu \leq 0$ so that $\mu^{\mathrm{tr}} \sigma \mu = 0$ must hold since σ is nonnegative definite. In the general case where (3.3) holds only for the smaller class of δ as in the statement, we first define

$$\delta_* := \mu I_{\{\mu^{\mathrm{tr}} \sigma \mu > 0\}} \frac{\min\left(1, (\mu^{\mathrm{tr}} \sigma \mu)^{\frac{1}{2}}\right)}{(\mu^{\mathrm{tr}} \sigma \mu)^{\frac{1}{2}}}.$$

Then δ_* is clearly $\mathbb{R}^n$-valued and predictable, and $\langle \int \delta_* \, dM \rangle = \int \delta_*^{\mathrm{tr}} \sigma \delta_* \, dB$ is bounded since B is bounded and $0 \leq \delta_*^{\mathrm{tr}} \sigma \delta_* \leq 1$. Moreover, we have on the set $\{\mu^{\mathrm{tr}} \sigma \mu > 1\}$ that

$$(\delta_* - 2\mu)^{\mathrm{tr}} \sigma \delta_* = \left(\mu \frac{1}{(\mu^{\mathrm{tr}} \sigma \mu)^{\frac{1}{2}}} - 2\mu\right)^{\mathrm{tr}} \sigma \mu \frac{1}{(\mu^{\mathrm{tr}} \sigma \mu)^{\frac{1}{2}}} = 1 - 2(\mu^{\mathrm{tr}} \sigma \mu)^{\frac{1}{2}} < -1,$$

while on the set $\{0 < \mu^{\mathrm{tr}} \sigma \mu \leq 1\}$, we have

$$(\delta_* - 2\mu)^{\mathrm{tr}} \sigma \delta_* = (\mu - 2\mu)^{\mathrm{tr}} \sigma \mu = -\mu^{\mathrm{tr}} \sigma \mu < 0.$$

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Since $\mu^{\mathrm{tr}}\sigma\mu \geq 0$ P_B-a.e., the above shows that

$$(3.4) \qquad (\delta_* - 2\mu)^{\mathrm{tr}}\sigma\delta_* < 0 \qquad P_B\text{-a.e. on } \{\mu^{\mathrm{tr}}\sigma\mu > 0\}.$$

Now let $\Delta^m = (\delta^m, \varepsilon^m)$, $m \in I\!N$, be the sequence of small perturbations constructed from δ_* via Lemma 1.7. Then $\delta^m = \delta_* I_{[\![0,\varrho_m]\!]} I_{\{|\delta_*| \leq m\}} I_{\{\delta_*^{\mathrm{tr}}\sigma\delta_* \leq m\}} I_{\{|\delta_*^{\mathrm{tr}}\sigma\lambda| \leq m\}}$ and therefore

$$(\delta^m - 2\mu)^{\mathrm{tr}}\sigma\delta^m = (\delta_* - 2\mu)^{\mathrm{tr}}\sigma\delta_* I_{[\![0,\varrho_m]\!]} I_{\{|\delta_*| \leq m\}} I_{\{\delta_*^{\mathrm{tr}}\sigma\delta_* \leq m\}} I_{\{|\delta_*^{\mathrm{tr}}\sigma\lambda| \leq m\}}$$
$$\longrightarrow (\delta_* - 2\mu)^{\mathrm{tr}}\sigma\delta_* \qquad P_B\text{-a.e. as } m \to \infty$$

since $\varrho_m \nearrow T$ P-a.s. By assumption, $(\delta^m - 2\mu)^{\mathrm{tr}}\sigma\delta^m \geq 0$ P_B-a.e. for every m, and therefore $(\delta_* - 2\mu)^{\mathrm{tr}}\sigma\delta_* \geq 0$ P_B-a.e. In view of (3.4), we must thus have $\mu^{\mathrm{tr}}\sigma\mu = 0$ P_B-a.e. **q.e.d.**

Remark 3.3. Combining Lemma 3.1 and Proposition 3.2 shows that $\liminf\limits_{n\to\infty} A_1^n \geq 0$ holds P_B-a.e. for every small perturbation if and only if $(\mu^H)^{\mathrm{tr}}\sigma\mu^H = 0$ P_B-a.e. Equivalently, this says that $\left\langle \int \mu^H dM \right\rangle \equiv 0$, or $\int \mu^H dM \equiv 0$, which means that $C^H(\varphi)$ is strongly orthogonal to M. This is one of the two key ingredients for the proof of Theorem 1.6. ◇

The second key ingredient in the proof of Theorem 1.6 is to show that the terms A_2^n, A_3^n, A_4^n are all asymptotically negligible. This will be achieved by combining an estimate with a general convergence result, and we now proceed to develop the latter. This is a slight generalization of Lemma 2.1 in [10].

As in (3.1), let $(\tau_n)_{n \in I\!N}$ be an increasing sequence of partitions of $[0, T]$ tending to the identity. Let $Y = (Y_t)_{0 \leq t \leq T}$ be an adapted real-valued process with $Y_0 = 0$. For $p > 0$ and $t \in [0, T]$, the *p-variation* of Y on $[0, t]$ along (τ_n) is

$$W_p(Y, t) := \sup_{n \in I\!N} \sum_{t_i, t_{i+1} \in \tau_n} \left| Y_{t_{i+1} \wedge t} - Y_{t_i \wedge t} \right|^p.$$

For any partition τ of $[0, T]$, we also define the processes

$$Q_p[Y, \tau](\omega, t) := \sum_{t_i, t_{i+1} \in \tau} \frac{|Y_{t_{i+1}} - Y_{t_i}|^p}{B_{t_{i+1}} - B_{t_i}}(\omega) I_{(t_i, t_{i+1}]}(t),$$

$$\tilde{Q}_p[Y, \tau](\omega, t) := \sum_{t_i, t_{i+1} \in \tau} \frac{E\left[|Y_{t_{i+1}} - Y_{t_i}|^p \,\middle|\, \mathcal{F}_{t_i}\right]}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]}(\omega) I_{(t_i, t_{i+1}]}(t).$$

Both are nonnegative and well-defined since B is strictly increasing; $Q_p[Y, \tau]$ is $\mathcal{O}^\tau$-measurable, and like at the end of Section 2, one readily verifies that

$$(3.5) \qquad \tilde{Q}_p[Y, \tau] = E_B\left[Q_p[Y, \tau] \,\middle|\, \mathcal{P}^\tau\right].$$

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If Y is increasing and Y_T is integrable, $P_Y = P \otimes Y$ denotes as in Section 1 the finite measure induced by P and Y on $\left(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T])\right)$.

Lemma 3.4. *If Y is adapted, null at 0, increasing and Y_T is integrable, then*

$$Q_1[Y, \tau] = \left.\frac{dP_Y}{dP_B}\right|_{\mathcal{O}^\tau} \qquad \text{and} \qquad \tilde{Q}_1[Y, \tau] = \left.\frac{dP_Y}{dP_B}\right|_{\mathcal{P}^\tau}$$

for every partition τ of $[0,T]$.

Proof. Since B is strictly increasing, $P_Y \ll P_B$. For any $t_i, t_{i+1} \in \tau$ and $D_{i+1} \in \mathcal{F}_{t_{i+1}}$,

$$E_B\left[Q_1[Y, \tau]I_{D_{i+1} \times (t_i, t_{i+1}]}\right] = E\left[\frac{Y_{t_{i+1}} - Y_{t_i}}{B_{t_{i+1}} - B_{t_i}}(B_{t_{i+1}} - B_{t_i})I_{D_{i+1}}\right] = E_Y\left[I_{D_{i+1} \times (t_i, t_{i+1}]}\right]$$

so that $\left.\frac{dP_Y}{dP_B}\right|_{\mathcal{O}^\tau} = Q_1[Y, \tau]$. The second assertion immediately follows from (3.5). **q.e.d.**

Proposition 3.5. *Assume (3.1) and let Y be an adapted process with $Y_0 = 0$. Let $1 \le r < p$ and suppose Y has integrable r-variation along (τ_n). If Y is continuous, then*

$$(3.6) \qquad \qquad \lim_{n \to \infty} Q_p[Y, \tau_n] = 0 \qquad P_B\text{-a.e.}$$

If in addition

$$(3.7) \qquad \qquad \sup_{n \in I\!N} Q_p[Y, \tau_n] \in L^1(P_B),$$

then we also have

$$(3.8) \qquad \qquad \lim_{n \to \infty} \tilde{Q}_p[Y, \tau_n] = 0 \qquad P_B\text{-a.e.}$$

Proof. We first note that since $p > r$,

$$Q_p[Y, \tau_n] \le Q_r[Y, \tau_n] \sup_{t_i, t_{i+1} \in \tau_n} |Y_{t_{i+1}} - Y_{t_i}|^{p-r},$$

and the second factor converges to 0 since Y is P-a.s. uniformly continuous on $[0,T]$. Hence it is enough for (3.6) to show that $\sup_{n \in I\!N} Q_r[Y, \tau_n] < \infty$ P_B-a.e. But if U denotes the r-variation of Y along (τ_n), we clearly have $|Y_{t_{i+1}} - Y_{t_i}|^r \le U_{t_{i+1}} - U_{t_i}$ and thus by Lemma 3.4

$$Q_r[Y, \tau_n] \le Q_1[U, \tau_n] = \left.\frac{dP_U}{dP_B}\right|_{\mathcal{O}^{\tau_n}}.$$

The last expression is a nonnegative P_B-supermartingale, hence P_B-a.e. convergent and therefore bounded in n P_B-a.e., giving (3.6). Due to (3.5) and (3.7), (3.8) then follows immediately from Hunt's lemma; see [3], V.45. **q.e.d.**

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4. The main result and its proof

We are now ready to prove our main result, recalled here for convenience.

Theorem 1.6. *Suppose the $\mathbb{R}^d$-valued semimartingale X satisfies the structure condition (SC) and let H be a payment stream. If the mean-variance tradeoff process $K = \int \lambda^{\mathrm{tr}} d\langle M \rangle \lambda$ (or, equivalently, A) is continuous, the following are equivalent for an L^2-strategy φ:*

1) *φ is locally risk-minimizing for H.*

2) *φ is 0-achieving and mean-self-financing, and the cost process $C^H(\varphi)$ is strongly orthogonal to M.*

In particular, the concept "locally risk-minimizing" does not depend on the choice of B.

Proof. a) Thanks to Proposition 2.1, we can in both cases assume that φ is 0-achieving and mean-self-financing. Fix an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of partitions of $[0, T]$ tending to the identity and take a small perturbation $\Delta = (\delta, \varepsilon)$. By Lemma 1.7, we can (and later shall) choose Δ such that $\varepsilon_t = 0$ for all $t < T$. Now apply Proposition 2.2 to write for each $n \in \mathbb{N}$, using the shorthand A_i^n for $A_i^{\tau_n}$, that

$$(4.1) \qquad r^{\tau_n}[\varphi, \Delta; H] = A_1^n + A_2^n + A_3^n + A_4^n.$$

We want to argue next that A_2^n, A_3^n and A_4^n can all be neglected asymptotically.

b) Let $U := \int |\delta \, dA| = \int |\delta^{\mathrm{tr}} \sigma \lambda| \, dB$ denote the variation of $\int \delta \, dA$. For $t \leq t'$, we then have $\left| \int_t^{t'} \delta_s \, dA_s \right| \leq U_{t'} - U_t$, and the explicit expression for $A_2^{\tau_n}$ in Proposition 2.2 gives

$$
\begin{aligned}
A_2^n &= \sum_{t_i, t_{i+1} \in \tau_n} \frac{\mathrm{Var}\left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right]}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]} \\
&\leq \sum_{t_i, t_{i+1} \in \tau_n} \frac{E\left[(U_{t_{i+1}} - U_{t_i})^2 \,\middle|\, \mathcal{F}_{t_i} \right]}{E[B_{t_{i+1}} - B_{t_i} \,|\, \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]} \\
&= \tilde{Q}_2[U, \tau_n].
\end{aligned}
$$

Next use Cauchy-Schwarz and the notation $Y := \left\langle \int \delta \, dM \right\rangle + \left\langle C^H(\varphi) \right\rangle$ to get

$$
\begin{aligned}
&\left| \mathrm{Cov}\left(\int_{t_i}^{t_{i+1}} \delta_s \, dM_s - \left(C_{t_{i+1}}^H(\varphi) - C_{t_i}^H(\varphi) \right), \int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\middle|\, \mathcal{F}_{t_i} \right) \right|^2 \\
&\leq E\left[\left(\int_{t_i}^{t_{i+1}} \delta_s \, dM_s - \left(C_{t_{i+1}}^H(\varphi) - C_{t_i}^H(\varphi) \right) \right)^2 \,\middle|\, \mathcal{F}_{t_i} \right] E\left[\left(\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \right)^2 \,\middle|\, \mathcal{F}_{t_i} \right] \\
&\leq 2 E[Y_{t_{i+1}} - Y_{t_i} \,|\, \mathcal{F}_{t_i}] E\left[(U_{t_{i+1}} - U_{t_i})^2 \,\middle|\, \mathcal{F}_{t_i} \right].
\end{aligned}
$$

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Combining this with the expression for $A_3^{\tau_n}$ in Proposition 2.2 and Cauchy-Schwarz yields

$$|A_3^n| \le \sqrt{8} \left(\sum_{t_i, t_{i+1} \in \tau_n} \frac{E[Y_{t_{i+1}} - Y_{t_i} \mid \mathcal{F}_{t_i}]}{E[B_{t_{i+1}} - B_{t_i} \mid \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]} \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{t_i, t_{i+1} \in \tau_n} \frac{E\left[(U_{t_{i+1}} - U_{t_i})^2 \mid \mathcal{F}_{t_i} \right]}{E[B_{t_{i+1}} - B_{t_i} \mid \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]} \right)^{\frac{1}{2}}$$

$$= \sqrt{8} \left(\tilde{Q}_1[Y, \tau_n] \right)^{\frac{1}{2}} \left(\tilde{Q}_2[U, \tau_n] \right)^{\frac{1}{2}}.$$

Now $\langle \int \delta \, dM \rangle$ is bounded and $C^H(\varphi)$ is a square-integrable martingale; therefore Y is increasing and integrable so that $\tilde{Q}_1[Y, \tau_n] = \frac{dP_Y}{dP_B}\big|_{\mathcal{P}^{\tau_n}}$ by Lemma 3.4. So $(\tilde{Q}_1[Y, \tau_n])_{n \in \mathbb{N}}$ is a nonnegative P_B-supermartingale, thus convergent and hence bounded in n P_B-a.e.

Finally, the term A_4^n is by Proposition 2.2 always nonnegative. If the small perturbation $\Delta = (\delta, \varepsilon)$ has $\varepsilon_t = 0$ for all $t < T$, the explicit expression for $A_4^{\tau_n}$ gives

$$A_4^n = \sum_{t_i, t_{i+1} \in \tau_n} \frac{\left(E\left[\int_{t_i}^{t_{i+1}} \delta_s \, dA_s \,\Big|\, \mathcal{F}_{t_i} \right] \right)^2}{E[B_{t_{i+1}} - B_{t_i} \mid \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]}$$

$$\le \sum_{t_i, t_{i+1} \in \tau_n} \frac{E\left[(U_{t_{i+1}} - U_{t_i})^2 \mid \mathcal{F}_{t_i} \right]}{E[B_{t_{i+1}} - B_{t_i} \mid \mathcal{F}_{t_i}]} I_{(t_i, t_{i+1}]}$$

$$= \tilde{Q}_2[U, \tau_n].$$

Thus controlling $\tilde{Q}_2[U, \tau_n]$ is the key to understanding the asymptotics of $r^{\tau_n}[\varphi, \Delta; H]$.

c) We show below in step d) that

$$(4.2) \qquad \lim_{n \to \infty} \tilde{Q}_2[U, \tau_n] = 0 \qquad P_B\text{-a.e.}$$

Accepting this for the moment, let us prove the equivalence of 1) and 2). If φ is locally risk-minimizing, then $\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta; H] \ge 0$ P_B-a.e. for every small perturbation $\Delta = (\delta, \varepsilon)$. If we choose a Δ such that $\varepsilon_t = 0$ for all $t < T$, (4.2) and the estimates in step b) imply

$$\lim_{n \to \infty} A_i^n = 0 \qquad P_B\text{-a.e. for } i = 2, 3, 4,$$

and we know from Lemma 3.1 that

$$(4.3) \qquad \lim_{n \to \infty} A_1^n = (\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta \qquad P_B\text{-a.e.}$$

Hence we obtain from (4.1) that $(\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta \ge 0$ P_B-a.e. for every δ such that $\langle \int \delta \, dM \rangle$ and $|\delta^{\mathrm{tr}} \sigma \lambda|$ are bounded, and thus $(\mu^H)^{\mathrm{tr}} \sigma \mu^H = 0$ P_B-a.e. by Proposition 3.2. This shows that $C^H(\varphi)$ is strongly orthogonal to M.

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Conversely, suppose that $C^H(\varphi)$ is strongly orthogonal to M so that $(\mu^H)^{\mathrm{tr}} \sigma \mu^H = 0$ P_B-a.e. and thus, again by Proposition 3.2, $(\delta - 2\mu^H)^{\mathrm{tr}} \sigma \delta \geq 0$ P_B-a.e. for every δ such that (δ, ε) is a small perturbation for some ε. By (4.2) and the estimates in step b),

$$\lim_{n \to \infty} A_i^n = 0 \qquad P_B\text{-a.e. for } i = 2, 3,$$

and $\liminf_{n \to \infty} A_4^n \geq 0$ P_B-a.e. since $A_4^n \geq 0$. Combining this with (4.1) and (4.3) yields for every small perturbation Δ that $\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta; H] \geq 0$ P_B-a.e., and so φ is locally risk-minimizing.

d) It remains to prove (4.2). Since δ comes from a small perturbation and B is bounded, both $|\delta^{\mathrm{tr}} \sigma \lambda|$ and $U = \int |\delta^{\mathrm{tr}} \sigma \lambda| \, dB = \int |\delta \, dA|$ are bounded as well. Moreover, U is continuous (because $K = \int \lambda^{\mathrm{tr}} \sigma \lambda \, dB$ is so, by assumption) and null at 0 with bounded 1-variation so that Proposition 3.5 with $r = 1$ and $p = 2$ yields $\lim_{n \to \infty} Q_2[U, \tau_n] = 0$ P_B-a.e. Hence (4.2) will follow from Proposition 3.5 once we prove that

(4.4)
$$\sup_{n \in \mathbb{N}} Q_2[U, \tau_n] \in L^1(P_B).$$

But since U is increasing and $\frac{dP_U}{dP_B} = |\delta^{\mathrm{tr}} \sigma \lambda|$, we get from Lemma 3.4

$$Q_2[U, \tau_n] = \sum_{t_i, t_{i+1} \in \tau_n} \frac{\left(U_{t_{i+1}} - U_{t_i}\right)^2}{B_{t_{i+1}} - B_{t_i}} I_{(t_i, t_{i+1}]}$$
$$\leq U_T \, Q_1[U, \tau_n]$$
$$= U_T \left. \frac{dP_U}{dP_B} \right|_{\mathcal{O}^{\tau_n}}$$
$$= U_T E_B \left[|\delta^{\mathrm{tr}} \sigma \lambda| \,\big|\, \mathcal{O}^{\tau_n} \right],$$

and because U_T and $|\delta^{\mathrm{tr}} \sigma \lambda|$ are both bounded by some constant, so is $Q_2[U, \tau_n]$, uniformly in n. This gives (4.4) and thus completes the proof.   **q.e.d.**

Apart from providing a streamlined exposition, the results in this paper extend earlier work on local risk-minimization in three directions:

1) We treat payment streams $(H_t)_{0 \leq t \leq T}$ instead of European contingent claims H_T due at time T, thus extending to the general semimartingale setting work done by [7] for the case where X is a martingale. This has independently also been done in [1]. However, we point out that passing from H_T to (H_t) is quite simple and constitutes no major contribution, as will again become apparent in the next section.

2) We treat a multidimensional setting with $d \geq 1$ risky assets by allowing X to be $\mathbb{R}^d$-valued. The feasibility of this extension was announced in [13], but the work has not been done in the literature so far. While technically not very difficult, it needs a careful formulation and treatment, and we view this as one of our two main contributions.

3) We remove several technical conditions on the underlying price process X; only the structure condition (SC) and continuity of A are required, thanks to the improved formulation of the basic criterion. This is our second main contribution.

To emphasize the improvements made here, we briefly look at the classical case from [11] where X is one-dimensional and only a European contingent claim $H_T \in L^2(P, \mathcal{F}_T)$ is considered. In comparison with [11], we no longer need the assumption (X2) that $\langle M \rangle$ is P-a.s. strictly increasing on $[0, T]$, nor (X5) that X is P-a.s. continuous at T, nor any global integrability on A or λ as in (X4). We work with a smaller (more restrictive) class of small perturbations than in [11]; but the equivalent characterization of local risk-minimality via part 2) of Theorem 1.6 is the same as in Proposition 2.3 of [11], and so our approach here is equivalent to the one in [11]. In particular, this also shows that

(4.5)   the notion of *pseudo-optimality* introduced in [13] for an L^2-strategy

              coincides with local risk-minimality.

5. A simple application

To round off the paper, we present in this section the link between local risk-minimization and the Föllmer-Schweizer decomposition. This is quite simple and well known and only done for completeness. We use the same setup as in Section 1.

Definition 5.1. An $\mathcal{F}_T$-measurable random variable $Y \in L^2(P)$ admits a *Föllmer-Schweizer decomposition* if it can be written as

(5.1) $$Y = Y^{(0)} + \int_0^T \vartheta_s^Y \, dX_s + L_T^Y \qquad P\text{-a.s.,}$$

where $Y^{(0)} \in L^2(P)$ is $\mathcal{F}_0$-measurable, ϑ^Y is in Θ_S, and the process $L^Y = (L_t^Y)$ is a (right-continuous) square-integrable martingale null at 0 and strongly orthogonal to M.

Thanks to Theorem 1.6, we obtain

Proposition 5.2. *Suppose the $\mathbb{R}^d$-valued semimartingale X satisfies the structure condition (SC) and the mean-variance tradeoff process $K = \int \lambda^{\mathrm{tr}} d\langle M \rangle \lambda$ (or, equivalently, A) is continuous. Then a payment stream H admits a locally risk-minimizing L^2-strategy φ if and only if H_T admits a Föllmer-Schweizer decomposition. In that case, $\varphi = (\vartheta, \eta)$ is given by*

(5.2) $$\vartheta = \vartheta^{H_T}, \qquad \eta = V^{H_T} - (\vartheta^{H_T})^{\mathrm{tr}} X$$

with

$$(5.3) \qquad V_t^{H_T} := H_T^{(0)} + \int_0^t \vartheta_s^{H_T} dX_s + L_t^{H_T} - H_t, \qquad 0 \le t \le T,$$

and then

$$(5.4) \qquad C_t^H(\varphi) = H_T^{(0)} + L_t^{H_T}, \qquad 0 \le t \le T.$$

Proof. If H_T has a Föllmer-Schweizer decomposition (5.1), then (5.2) and (5.3) define an L^2-strategy φ whose cost process is given by (5.4). Hence φ is mean-self-financing and also 0-achieving by (5.1) and thus locally risk-minimizing for H by Theorem 1.6. Conversely, if $\varphi = (\vartheta, \eta)$ is locally risk-minimizing for H, we can write the condition $V_T(\varphi) = 0$ as

$$H_T = C_T^H(\varphi) + \int_0^T \vartheta_s \, dX_s = C_0^H(\varphi) + \int_0^T \vartheta_s \, dX_s + \left(C_T^H(\varphi) - C_0^H(\varphi) \right),$$

and so we have (5.1) for H_T with

$$H_T^{(0)} := C_0^H(\varphi), \qquad \vartheta^{H_T} := \vartheta, \qquad L^{H_T} := C^H(\varphi) - C_0^H(\varphi);$$

note that L^{H_T} is a martingale and strongly orthogonal to M by Theorem 1.6. **q.e.d.**

Proposition 5.2 is a slight generalization (to payment streams) of Proposition (2.24) in [4]; see also Proposition 3.4 in [13]. Apart from subtracting the process H in (5.3), the proof remains unchanged. It is interesting to note that the extension from European contingent claims to payment streams involves no difficulties at all and that the key quantity to examine is only the total payment H_T. This is due to the fact that our strategies need not be self-financing, so that any intermediate payments can simply be added to the costs.

Remark 5.3. Proposition 5.2 gives a link between local risk-minimization and the Föllmer-Schweizer decomposition which holds true in full generality. Finding the Föllmer-Schweizer decomposition, however, is not always easy. It has been shown in [4] and [13] that (up to some mild integrability conditions) this decomposition can be obtained as the Galtchouk-Kunita-Watanabe (GKW) decomposition under the so-called minimal martingale measure $\widehat{P}$, *if X is continuous*. For discontinuous X, the result in Proposition 5.2 is still true, but using the GKW decomposition under $\widehat{P}$ may fail to give the correct decomposition. A more detailed discussion of this issue is given in [14], whose authors in particular point out and improve upon some incorrect work in the literature. $\diamond$

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