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t-Motives & Galois Representations

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For a given problem, you don't need to know that much, usually —
and, besides, very simple ideas will often work.

Jean-Pierre Serre [**CL**]

A casual preface

The preface tries to give a non-specialist taste of what Galois theory and this thesis are about.

§1. Galois groups. Suppose one is interested in solving polynomial equations. Such an equation

$$a_r X^r + a_{r-1} X^{r-1} + \cdots + a_0 = 0, \tag{1}$$

where the coefficients a_i are rational numbers (fractions), is called **solvable by radicals**, if the solutions α can be obtained from the coefficients a_i in a finite sequence of steps, each of which may involve addition, subtraction, multiplication, division, or taking n-th roots. For example, if the degree $r = 2$, then

$$\alpha = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}.$$

Actually, every polynomial equation of degree at most 4 is solvable in radicals, as there exist similar universal formulas.

A question which puzzled 18th century arithmeticians is[1]

> *Can polynomial equations of degree at least 5 be solved by radicals?*

Around 1830, Evariste Galois came up with a theory of polynomial equations which not only answered this question but in fact introduced new structures that would revolutionize algebra. Beyond that, he provided mathematics with one of its most fascinating biographies, by leading a very short but agitated life.

> All night long he had spent the fleeting hours feverishly dashing off his scientific last will and testament, writing against time to glean a few of the great things in his teeming mind before the dead which he saw could overtake him. Time after time he broke off to scribble in the margin "I have not time; I have not time," and passed on to the next frantically scrawled outline. What he wrote in those last desperate hours before the dawn will keep generations of mathematician busy for hundreds of years. He had found, once and for all, the true solution of a riddle which had tormented mathematicians for centuries: under what conditions can an equation be solved? [**Bel**], p. 375

[1] For an account on the history of this topic and of algebra in general, there is e.g. [**Wae**].

Thus, at least according to E.T. Bell and mathematical folklore[2], Galois scribbled down the elements of his theory on the eve of his fatal duel.

Originally, the equation $X^2 = -1$ *had no solution.*
Then the two solutions i *and* $-i$ *were created.*
But there is absolutely no way to tell who is i *and who is* $-i$.
That is Galois theory.
S.S. Abhyankar [**Abh**]

If we take a polynomial equation (1) with rational coefficients, some solutions α may not be defined over the field $\mathbb{Q}$ of rational numbers itself, but then we can consider the smallest field K which contains $\mathbb{Q}$ and all the solutions of (1). Just like we could replace i by $-i$ without making any difference, we obtain a finite set of symmetries of the field K which leave $\mathbb{Q}$ fixed; these symmetries form a group, called the **Galois group** $\mathrm{Gal}(K/\mathbb{Q})$. Galois observed that if $r \geq 5$, then this group in general does not have a 'solvable' structure, which implied that polynomial equations of degree at least 5 cannot be solved by radicals.

The field $\bar{\mathbb{Q}}$ which contains $\mathbb{Q}$ and the solutions of all polynomial equations over $\mathbb{Q}$ is called the **algebraic closure** of $\mathbb{Q}$, and to this extension we can again associate an (infinite) group, the absolute Galois group $\Gamma_{\mathbb{Q}}$. This single group now encodes all the information on algebraic extensions of $\mathbb{Q}$. Unfortunately, its structure is tremendously complicated and a great deal of modern number theory is directly related to trying to understand it. One idea is to study it by its action on vector spaces, i.e. by its linear representations.

§2. Galois representations of $\Gamma_{\mathbb{Q}}$**.** Suppose one is interested in the question:[3]

(Fermat's last theorem) *Does the equation*

$$X^n + Y^n = 1 \tag{2}$$

have any rational solutions for X *and* Y*, both different from zero, if the integer* n *is at least* 3*?*

Around 1637, Pierre de Fermat wrote down this problem in his copy of Diophantos' Arithmeticae, and went on to say that he could show by a very elegant argument that no such solutions exist, but that the margin was too narrow to give it. Thus he would haunt generations and generations of mathematicians, as they would not be able to find any proof for his so-called 'Last Theorem', but beyond their stubbornness, they loved the problem dearly because all attempts to solve it generated good theories anyway.

Before Wiles finally did prove, in 1994, that there are no such solutions, it was already known, by another theory, that there could be at most a finite number of solutions. If one considers the equation (2) over the complex numbers, then the solutions form a real surface. In 1984, Faltings proved:

[2] See [**Rot**] for a demystified account.

[3] Those also interested in the story surrounding the question should read [**Sin**].

> (Mordell conjecture) *The number of rational solutions of any system of polynomial equations whose complex solutions form a surface with at least 2 handles, is finite.*

If one considers the equation (2) over the complex numbers, then the solutions form a surface indeed, and the number of handles equals

$$(n-1)(n-2)/2,$$

so this theorem applies if $n \geq 4$ (For $n = 3$, Fermat's last theorem can be proved by an straightforward number theory argument).

Not only are Faltings's and Wiles's theorems two landmarks of 20th century mathematics, they also illustrate perfectly the prominence of Galois theory. They study linear representations of the absolute Galois group, i.e. the action of $\Gamma_{\mathbb{Q}}$ on certain vector spaces (over the field of ℓ-adic numbers). In other words, they consider systems of polynomial equations which arise from algebro-geometric objects (elliptic curves, abelian varieties) and which carry a linear structure. The above two problems can be reformulated into equivalent statements on the representation-theoretical properties of these **Galois representations** (the Tate conjecture [**CS**], resp. the Taniyama-Weil conjecture [**CSS**]).

Anyway, the excitement about Wiles's proof that was still in the air certainly boosted my motivation when I started my Ph.D. research on Galois representations. A second good excuse for bringing up Faltings' theorem here, is that some of the essential ideas of its proof had been developed in Zarhin's work on the (ℓ-adic) Tate conjectures over function fields of finite characteristic, i.e. finite extensions of the field $\mathbb{F}_p(t)$ of rational functions in one variable over the finite field $\mathbb{F}_p$ of p elements. This and many other examples of 'transplantation' of pieces of theory motivate why one would want to do number theory without dealing with numbers: the function field case often serves as a terrific analogue for the number field $\mathbb{Q}$. This thesis will be a study of Galois representations over function fields.

§3. Galois representations associated to τ-sheaves. Let p be a prime number and K a field of characteristic p, i.e. where p times 1 equals 0. For such a field, we have the identity

$$(X+Y)^p = X^p + Y^p. \tag{3}$$

The field K^{sep} which contains K and the roots of all polynomials over K whose derivative is nonzero is called the separable closure of K. The absolute Galois group Γ_K is defined as the group of symmetries of K^{sep} which leave K invariant.

Take an invertible $r \times r$ matrix A with coefficients a_{ij} lying in K, and look at the following system of r algebraic equations in r variables $X_1, \ldots, X_r$:

$$\begin{cases} X_1^p = a_{11}X_1 + a_{21}X_2 + \cdots + a_{r1}X_r \\ X_2^p = a_{12}X_1 + a_{22}X_2 + \cdots + a_{r2}X_r \\ \qquad\vdots \\ X_r^p = a_{1r}X_1 + a_{2r}X_2 + \cdots + a_{rr}X_r \end{cases}$$

or, for short:

$$(X_1^p, \dots, X_r^p) = (X_1, \dots, X_r) \cdot A. \tag{4}$$

By the identity (3), the set $W(A)$ of solutions $(X_1, \dots, X_r) \in (K^{\mathrm{sep}})^r$ for (4) is a vector space over the finite field $\mathbb{F}_p$ with p elements, and one proves that it has dimension r. The absolute Galois group Γ_K permutes these solutions, so we obtain an action of Γ_K on W. Thus, in finite characteristic, Galois representations can be obtained from such a matrix A; the converse is true as well.

Consider the power series ring $K^{\mathrm{sep}}[[t]]$, consisting of infinite power series

$$S := \sum_{i=0}^{\infty} s_i t^i = s_0 + s_1 t + s_2 t^2 + \dots$$

with coefficients s_i lying in K^{sep}. We define an operation σ on it as follows:

$${}^{\sigma}S := \sum_{i=0}^{\infty} s_i^p t^i.$$

Choose an invertible matrix A with entries in the power series ring $K[[t]]$. For power series $S_j = \sum_{i=0}^{\infty} s_{ij} t^i \in K^{\mathrm{sep}}[[t]]$, we look at the equation

$$({}^{\sigma}S_1, \dots, {}^{\sigma}S_r) = (S_1, \dots, S_r) \cdot A. \tag{5}$$

This equation actually involves an infinite number of polynomial equations in the infinite number of variables s_{ij}. The set $T_t(A)$ of solutions

$$(S_1, \dots, S_r) \in K^{\mathrm{sep}}[[t]]^r$$

is again endowed with a linear structure and an action of the Galois group Γ_K.

Finally, suppose we have a matrix A defined over $K[t]$. We can then, analogously to $T_\ell(A)$ in the above, construct a 'Galois module' $T_\ell(A)$, for all irreducible polynomials ℓ in $\mathbb{F}_p[t]$, and thus obtain a **system** of Galois representations. We can rephrase this in a more intrinsic way by introducing Drinfeld's τ-modules[4]. A τ-module M over $K[t]$ is a free module over the ring $K[t]$, together with a σ-semilinear map τ (σ acts trivially on t and by raising-to-the-p-th-power on K). If we denote by A the matrix representing τ with respect to some basis for M, then we can associate to M a system of Galois representations $T_\ell(M)$ as before.

The first part of this thesis deals with properties of these systems associated to a τ-module. Consider the set of equations (5) in an infinite number of variables $s_{ij}^{(t)}$ which define $T_t(M)$, and, at the same time, consider analogous sets of equations in variables $s_{ij}^{(\ell)}$ for the other $T_\ell(M)$. A question one would like to answer is to give a qualitative description of the (infinite) Galois group corresponding to the field extension defined by the solutions of these equations. A naive formulation of the so-called adelic Mumford-Tate conjecture is that, under certain conditions on the τ-module M (more precisely, no nontrivial endomorphisms plus a condition on the determinant module),

[4]Using more efficient language, we will actually, instead of τ-modules, consider τ-sheaves.

Every permutation of the solutions of these equations that

- *respects all the linear relations and*
- *leaves a particular finite set of variables* $s_{ij}^{(\ell)}$ *invariant*

is a Galois symmetry.[5]

On the other hand, and on a deeper level, τ-sheaves are also closely related to abelian t-modules and Anderson's t-motives, basic structures in the arithmetic of function fields. These offer a striking, even if poorly understood, counterpart for motives in algebraic geometry. In the second part of this thesis, we will explain how general results on τ-sheaves and Galois representations shed new light on the structure of t-motives.

Und so hat es auch schon damals, als Ulrich Mathematiker wurde,
Leute gegeben, die den Zusammenbruch der Europäischen Kultur voraussagten,
weil kein Glaube, keine Liebe, keine Einfalt, keine Güte mehr
im Menschen wohne, und bezeichnenderweise sind sie alle
in ihrer Jugend- und Schulzeit schlechte Mathematiker gewesen.
Robert Musil [**Mus**] I §11 p. 40

[5] In more cryptic terms: the image of the Galois representation is almost as large as 'possible'. See the artist's impression on page v!

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Zürich, September 30, 2001

Contents

Introduction

First, we state definitions and fix notations for the central objects considered: Drinfeld's τ-sheaves and Anderson's t-motives, with their associated systems of Galois representations. In section **II**, we give a concise overview of the present work, its evolution and its main results. Section **III** contains a survey in Dutch.

I. τ-Sheaves and t-motives

§1. Algebraic τ-sheaves.

1. *τ-Modules.* Let R be a commutative ring and σ an endomorphism of R. For an R-module M, we define the R-module

$$\sigma^* M := R_\sigma \otimes_R M,$$

where R_σ is the ring R, viewed as an R-algebra via σ. Any R-linear homomorphism

$$\tau : \sigma^* M \to M$$

can be regarded as a map $M \to M$ which is σ-semi-linear, i.e.

$$\tau(r \cdot m) = {}^{\sigma}r \cdot \tau(m)$$

for $r \in R$ and $m \in M$.

DEFINITION 0.1.

i) A **τ-module** (M, τ) (for short: M) **over** R is a finitely generated projective R-module endowed with an injective morphism

$$\tau : \sigma^* M \to M.$$

ii) A **morphism of τ-modules** is an R-linear morphism respecting the action of τ. An **isogeny** between τ-modules is an injective morphism of τ-modules whose cokernel is a torsion R-module.

iii) The **tensor product** $M_1 \otimes M_2$ of two τ-modules has $M_1 \otimes_R M_2$ as the underlying R-module and a τ-action defined by

$$\tau(m_1 \otimes m_2) = \tau m_1 \otimes \tau m_2$$

for $m_i \in M_i$.

iv) A τ-module is called **smooth** if τ is an isomorphism. It is called **trivial** if it is isomorphic to a direct sum of copies of the τ-module whose underlying R-module is R itself and where

$$\tau : R_\sigma \to R : 1 \mapsto 1.$$

v) If all nontrivial sub-τ-modules of a given τ-module M are isogenous to M, then M is called **simple**.

2. *τ-Sheaves.* Let $\mathbb{F}_q$ be a finite field with q elements, and let p denote its characteristic. We fix an absolutely irreducible affine smooth curve $\mathcal{C}$ with constant field $\mathbb{F}_q$, called the base curve. We denote its function field by F and put

$$\mathbf{A} := H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}).$$

For any $\mathbb{F}_q$-scheme X, the coefficient scheme, we consider the product

$$\mathcal{C}_X := \mathcal{C} \times_{\mathbb{F}_q} X.$$

If X is an affine scheme Spec B, we also write $\mathcal{C}_B := \mathcal{C}_X$.

Denote by $\varphi : X \to X$ the Frobenius morphism defined by the map

$$x \mapsto {}^{\varphi}x := x^q$$

on $\mathcal{O}_X$. We then endow the scheme $\mathcal{C}_X$ with the endomorphism $\sigma := id \times \varphi$. The following object, closely related to Drinfeld's shtukas and F-crystals, often appears under the name 'φ-sheaf' as well (cf. [**TW**]):

DEFINITION 0.2. (Drinfeld) A **τ-sheaf** (M, τ) (for short: M) **of rank** $r \geq 1$**, defined on** $\mathcal{C}_X$**,** is a locally free $\mathcal{O}_{\mathcal{C}_X}$-module of finite rank r, endowed with an injective morphism

$$\tau : \sigma^* M \to M.$$

A **morphism of τ-sheaves** is a $\mathcal{O}_{\mathcal{C}_X}$-linear morphism which respects the action of τ.

EXAMPLE 0.3. The most elementary example is given by putting $\mathcal{C} := \mathbb{A}^1$, and, for some field K containing $\mathbb{F}_q$, setting $X :=$ Spec K. We will identify the τ-sheaf M with its module of global sections, a free $K[t]$-module of finite rank endowed with a σ-semi-linear injective morphism, where the endomorphism σ acts as Frobenius on K and trivially on t.

Let us fix a basis for M. We express τ with respect to this basis by means of a matrix Δ in $\mathrm{Mat}_{r \times r}(K[t])$, the ring of r by r matrices over $K[t]$. If we write $\mathbf{m} := (m_1, \ldots, m_r)$, then Δ is determined by

$$\tau(\mathbf{m}) = \mathbf{m} \cdot \Delta.$$

If we replace $\mathbf{m}$ by another basis $\mathbf{m}' = \mathbf{m} \cdot U$, with $U \in \mathrm{GL}_r(K[t])$, then τ is represented by

$$U^{-1} \cdot \Delta \cdot {}^{\sigma}U,$$

where the matrix ${}^{\sigma}U$ is obtained by applying σ to the entries of U.

REMARK 0.4. As is explained in [**TW**], if $\mathcal{C}_X$ is affine, every locally free sheaf M on $\mathcal{C}_X$ injects into a free sheaf M_e of finite rank as a direct factor:

$$M_e = M \oplus M_0.$$

We then define a (noninjective!) $\mathcal{O}_{\mathcal{C}_X}$-linear homomorphism $\tau_e : \sigma^* M_e \to M_e$ by

$$\tau_e := \tau \oplus 0.$$

The pair (M_e, τ) (which is not a τ-sheaf in our terminology) is called an **free extension by zero** of M.

3. *Characteristic.* For a τ-sheaf M on $\mathcal{C}_X$, the cokernel coker τ of τ is supported on a closed subscheme of codimension at least 1 of $\mathcal{C}_X$, as τ is an injective morphism.

DEFINITION 0.5. Consider a morphism $\iota : X \to \mathcal{C}$ and its graph $\Gamma(\iota)$ in $\mathcal{C}_X$.

i) We say that a τ-sheaf M on $\mathcal{C}_X$ has **characteristic** ι, if coker τ is supported on the graph $\Gamma(\iota)$ and if the restriction coker $\tau|_{\Gamma(\iota)}$ is a locally free $\mathcal{O}_{\Gamma(\iota)}$-module of a constant rank d, which is called the **dimension** of M.
ii) For a field K containing $\mathbb{F}_q$, consider a morphism $\iota : \operatorname{Spec} K \to \mathcal{C}$. If the image of the generic point η of Spec K via ι is the generic point of $\mathcal{C}$, then we say that the characteristic is **generic**, and it is called **special** otherwise.
iii) Finally, consider a morphism $\iota : \operatorname{Spec} K \to \mathcal{C}$ and let X be a connected $\mathbb{F}_q$-scheme with function field K. We then call a closed point x of X (or its associated valuation v_x on K) **finite** if ι extends to a morphism $\iota : X \to \mathcal{C}$, and **infinite** if not. If x is finite, then we call $\iota(x)$ the **residual characteristic point at** x.

EXAMPLE 0.6. In the context of example 0.3, the determinant of a matrix Δ representing τ is independent of the choice of a basis $\mathbf{m}$ up to a unit in K. Given a morphism $\iota : \operatorname{Spec} K \to \mathbb{A}^1$ induced by a map

$$\iota^* : \mathbb{F}_q[t] \to K : t \mapsto \theta,$$

the τ-module M has characteristic ι if the determinant of Δ equals

$$h \cdot (t - \theta)^d,$$

where h is a unit in K and d the dimension of M. The characteristic is generic if and only if θ is transcendental over $\mathbb{F}_q$. A valuation v is finite with respect to ι if and only if $v(\theta) \geq 0$.

§2. Galois representations.

1. *ℓ-Adic τ-sheaves.* Let ℓ be a closed point of $\mathcal{C}$ and X an $\mathbb{F}_q$-scheme. We denote by $\mathbf{A}_\ell$ the completion with respect to the ℓ-adic topology of the local ring at ℓ of regular functions on $\mathcal{C}$, and by F_ℓ its field of fractions. Also, let κ_ℓ be the residue field of $\mathbf{A}_\ell$.

For a closed point ℓ of $\mathcal{C}$ and an $\mathbb{F}_q$-scheme X, we let $\hat{\mathcal{C}}_{X,\ell}$ be the formal completion of $\mathcal{C} \times_{\mathbb{F}_q} X$ along $\{\ell\} \times X$ and $\mathcal{O}_{\hat{\mathcal{C}}_{X,\ell}}$ its structure ring. The Frobenius morphism φ on X induces the endomorphism $\sigma := id \times \varphi$ on $\hat{\mathcal{C}}_{X,\ell}$.

Let $\pi_1(X)$ denote the arithmetic fundamental group of X (cf. [**SGA1**] Exposé No. 5). We have the following fundamental correspondence:

PROPOSITION 0.7 (Drinfeld, [**TW**] Prop. 6.2). *The category of smooth τ-sheaves of rank r on $\hat{\mathcal{C}}_{X,\ell}$ is antiequivalent to the category of X-schemes of free $\mathbf{A}_\ell$-modules of rank r with continuous $\pi_1(X)$-action.*

We now recall the definition of the functor T establishing this antiequivalence (cf. [**TW**] §6). Let N be a locally free $\mathcal{O}_X$-module endowed with a morphism

$$\tau : \varphi^* N \to N.$$

For every X-scheme X', we define the X'-valued points of the scheme of $\mathbb{F}_q$-modules $T(N)$ by

$$T(N)(X') := \left\{ f \in \mathrm{Hom}_{\mathcal{O}_X}(N, \mathcal{O}_{X'});\ f \circ \tau = \varphi \circ f \right\}. \tag{6}$$

For every $n \geq 1$, let $\mathcal{C}^n_{X,\ell}$ denote the n-th formal neighborhood of ℓ. We denote the closed embedding

$$\mathcal{C}^n_{X,\ell} \hookrightarrow \hat{\mathcal{C}}_{X,\ell}$$

by i^n_ℓ and the morphism $\mathcal{C}^n_{X,\ell} \to X$ by j_n. For an ℓ-adic τ-sheaf $\hat{M}_\ell$, we set

$$M^n_\ell := (j_n)_*(i^n_\ell)^*\hat{M}_\ell, \tag{7}$$

a locally free $\mathcal{O}_X$-module endowed with a morphism

$$\tau : \varphi^* M^n_\ell \to M^n_\ell.$$

We thus obtain an injective system of τ-modules M^n_ℓ on X, which yields a projective system $T(M^n_\ell)$ of schemes of $\mathbb{F}_q$-modules. We put

$$T(\hat{M}_\ell) := \varprojlim T(M^n_\ell).$$

This module $T(\hat{M}_\ell)$ naturally carries the structure of a scheme of $\mathbf{A}_\ell$-modules of some rank $r' \leq r$, with equality holding if and only if $\hat{M}_\ell$ is smooth.

2. *Tate modules.* For a field K containing $\mathbb{F}_q$, let K^{sep} be the separable closure of K and Γ_K the absolute Galois group

$$\mathrm{Gal}(K^{\mathrm{sep}}/K) \cong \pi_1(\mathrm{Spec}\ K)$$

of K. The functor T associates to each ℓ-adic τ-sheaf $\hat{M}_\ell$ over $\hat{\mathcal{C}}_{K,\ell}$ the scheme $T(\hat{M}_\ell)$ of free $\mathbf{A}_\ell$-modules of finite rank with continuous Γ_K-action. Let M be a τ-sheaf on $\mathcal{C}_X$. We can associate to M an ℓ-adic τ-sheaf on $\hat{\mathcal{C}}_{X,\ell}$ via

$$\hat{M}_\ell := \mathcal{O}_{\hat{\mathcal{C}}_{X,\ell}} \otimes_{\mathcal{O}_{\mathcal{C}_X}} M. \tag{8}$$

DEFINITION 0.8. Let K be a field K containing $\mathbb{F}_q$. Consider a τ-sheaf M over $\mathcal{C}_K$ and a closed point ℓ of $\mathcal{C}$.

i) For a closed point ℓ of $\mathcal{C}$, then the $\mathbf{A}_\ell[\Gamma_K]$-module

$$T_\ell(M) := T(\hat{M}_\ell)(\mathrm{Spec}\ K^{\mathrm{sep}})$$

is called the **Tate module of** M **at** ℓ. Its $\mathbf{A}_\ell$-rank equals the rank of $\hat{M}_\ell$ if and only if the latter is smooth. We also consider the dual $\mathbf{A}_\ell[\Gamma_K]$-module

$$H_\ell(M) := \mathrm{Hom}_{\mathbf{A}_\ell}(T_\ell(M), \mathbf{A}_\ell).$$

ii) Recall that κ_ℓ denotes the residue field at the closed point ℓ of $\mathcal{C}$. With the notations of (7), we define the ℓ**-torsion module** as the continuous $\kappa_\ell[\Gamma_K]$-module

$$W_\ell(M) := T(M^1_\ell)(\mathrm{Spec}\ K^{\mathrm{sep}}). \tag{9}$$

iii) We define an $F_\ell[\Gamma_K]$-module associated to M as follows

$$V_\ell(M) := F_\ell \otimes_{\mathbf{A}_\ell} T_\ell(M). \tag{10}$$

REMARK 0.9. By the injectivity of τ on M, the codimension of the support on $\mathcal{C}_X$ of its cokernel is at least 1. We remark that it follows from this that, if $X = \operatorname{Spec} K$, for a field K containing $\mathbb{F}_q$, then $\hat{M}_\ell$ is smooth for all but a finite number of closed points ℓ of $\mathcal{C}$. If X has generic characteristic ι, then $\hat{M}_\ell$ is smooth for all ℓ.

EXAMPLE 0.10. We take up Example 0.3. Let ℓ be the point of $\mathcal{C}$ corresponding to the ideal (t). The ring $\mathbf{A}_\ell$ is then isomorphic to the power series ring $\mathbb{F}_q[[t]]$. The Tate module $T_\ell(M)$ can be computed as follows: It is the $\mathbb{F}_q[[t]]$-module consisting of vectors $(X_1, \ldots, X_r) \in K^{\mathrm{sep}}[[t]]^{\oplus r}$ satisfying

$$({}^\sigma X_1, \ldots, {}^\sigma X_r) = (X_1, \ldots, X_r) \cdot \Delta. \tag{11}$$

The ℓ-adic τ-module $\hat{M}_\ell$ is smooth if $\det \Delta \in K[t]$ is not divisible by t, and then $T_\ell(M)$ has full rank r.

REMARK 0.11. The $F_\ell[\Gamma_K]$-modules $V_\ell(M)$ give rise to continuous representations

$$\rho_\ell : \Gamma_K \to \mathrm{Aut}_{F_\ell}(V_\ell(M)).$$

These representations form a strictly compatible system of Galois representations (cf. Thm. 3.3) in the sense of Serre ([**Se1**]) and, by work of Tamagawa, also the Tate and semisimplicity conjectures (cf. Thm. 3.7) are known to hold.

For any subset Λ of closed points of $\mathcal{C}$, we consider the rings

$$\begin{aligned} \kappa_\Lambda &:= \prod_{\ell \in \Lambda} \kappa_\ell \\ \text{and } F_\Lambda &:= \prod_{\ell \in \Lambda}{}' F_\ell, \end{aligned} \tag{12}$$

where the prime indicates that, if Λ is infinite, then we consider the restricted product, i.e. F_Λ is the subring of $\prod_{\ell \in \Lambda} F_\ell$ consisting of elements $(a_\ell)_\ell$ such that $a_\ell \in \mathbf{A}_\ell$ for almost all ℓ in Λ. We put:

$$\begin{aligned} W_\Lambda(M) &:= \prod_{\ell \in \Lambda} W_\ell(M) \\ \text{and } V_\Lambda(M) &:= \prod_{\ell \in \Lambda}{}' V_\ell(M). \end{aligned} \tag{13}$$

If Λ contains all closed points of $\mathcal{C}$, then we set $\kappa_{\mathrm{ad}} := \kappa_\Lambda$ and

$$W_{\mathrm{ad}}(M) := W_\Lambda(M); \tag{14}$$

idem for F_{ad} and $V_{\mathrm{ad}}(M)$.

§3. Anderson t-motives. We now review Anderson's definition of abelian t-modules and t-motives ([**An1**], §1). When speaking of t-motives, we will, as was done by Anderson himself, restrict ourselves to the case $\mathcal{C} = \mathbb{A}^1$, the affine line over $\mathbb{F}_q$. There should be no obstacles to generalize to the case where $\mathcal{C}$ is equal to $\bar{\mathcal{C}}\backslash\{\infty\}$, where $\bar{\mathcal{C}}$ is an absolutely irreducible smooth projective curve with constant field $\mathbb{F}_q$ and ∞ a closed point of $\bar{\mathcal{C}}$. We fix a ring isomorphism

$$\mathbf{A} = H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \cong \mathbb{F}_q[t].$$

1. *t-modules.* Let K be a field containing $\mathbb{F}_q$. The ring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K})$ of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_{a,K}$ is isomorphic to $K[\varphi]$, the skew polynomial ring generated by the Frobenius morphism $\varphi : \kappa \mapsto \kappa^q$ and with the commutation relation $\varphi \cdot \kappa = \kappa^q \cdot \varphi$, for all $\kappa \in K$. The ring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K}^{\oplus d})$ can then be identified with the matrix ring $\mathrm{Mat}_{d\times d}(K[\varphi])$.

DEFINITION 0.12. Let K be a field containing $\mathbb{F}_q$.

i) A **d-dimensional t-module** (E, ϕ_E) **(for short: E) defined over** K is an algebraic group E isomorphic to $\mathbb{G}_{a,K}^{\oplus d}$ endowed with an injective $\mathbb{F}_q$-algebra morphism
$$\phi_E : \mathbf{A} \to \mathrm{End}_{\mathbb{F}_q}(E).$$
ii) A **morphism of t-modules** is a morphism of the underlying algebraic groups which commutes with the action of $\mathbf{A}$.
iii) For a given ring morphism $\iota^* : \mathbf{A} \to K$, we say that an t-module E defined over K has **characteristic** ι^* if, for every $a \in \mathbf{A}$, the endomorphism on $\mathrm{Lie}(E)$ induced by $\phi_E(a)$ has single eigenvalue $\iota^*(a)$.

To a d-dimensional t-module E defined over K, we associate the K-vector space
$$M(E) := \mathrm{Hom}_{\mathbb{F}_q}(E, \mathbb{G}_{a,K})$$
of $\mathbb{F}_q$-linear algebraic homomorphisms $E \to \mathbb{G}_{a,K}$. The action of $\mathbf{A}$ on E induces an t-module structure on $M(E)$ via
$$a \cdot m := m \circ a,$$
for $m \in M(E)$ and $a \in \mathbf{A}$. This action commutes with the action of K, and therefore we can see $M(E)$ as a module over $K \otimes_{\mathbb{F}_q} \mathbf{A} = K[t]$.

If $M(E)$ is finitely generated over $K[t]$, it is automatically free of finite rank, by [**An1**], Lemma 1.4.5. The Frobenius endomorphism σ on $\mathbb{G}_{a,K}$ yields an injective $K[t]$-linear map
$$\sigma^* M(E) \to M(E),$$
which endows $M(E)$ with the structure of a τ-module over $K[t]$.

We fix a characteristic morphism $\iota : \mathrm{Spec}\, K \to \mathbb{A}^1$, defined by a map
$$\iota^* : \mathbf{A} \to K$$
(cf. Example 0.6). Remark that the τ-module $M(E)$ has characteristic ι if and only if E has characteristic ι^*. Finally, we define

DEFINITION 0.13. (Anderson)
Let E be a d-dimensional t-module over K with characteristic ι^*. If $M(E)$ is finitely generated over $K[t]$ (hence free, of some rank r), and $M(E)$ has characteristic ι, then E is called an **abelian t-module** and the τ-module $M(E)$ over $K[t]$ (or, equivalently, the associated τ-sheaf on $\mathbb{A}^1_K$) is called a **t-motive**, of dimension d and

rank r. A 1-dimensional abelian t-module is called a **Drinfeld module** (see [**Dr1**], [**AlB**], [**Go4**], etc.).

Any morphism $e : E \to E'$ of abelian t-modules E, E' induces a morphism

$$e^\star : M(E') \to M(E) : m \mapsto m \circ e$$

of the associated t-motives. We then have:

PROPOSITION 0.14 (Anderson ([**An1**] §1)). *The categories of abelian t-modules and t-motives are antiequivalent.*

2. *Purity.* Let ∞ be the point at infinity of the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$ such that $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. We consider the formal completion

$$\hat{\mathbb{P}}^1_{K,\infty}$$

of $\mathbb{P}^1_K$ along $\{\infty\} \times \operatorname{Spec} K$ and its structure ring $\mathcal{O}_{\hat{\mathbb{P}}^1_{K,\infty}}$. For a sheaf $\tilde{M}$ on $\mathbb{P}^1_K$, we set

$$\hat{M}_\infty := \mathcal{O}_{\hat{\mathbb{P}}^1_{K,\infty}} \otimes_{\mathcal{O}_{\mathbb{P}^1_K}} \tilde{M}.$$

DEFINITION 0.15 (Anderson ([**An1**] §1.9)). A τ-sheaf M on $\mathbb{A}^1_K$ is called **pure (of weight w)** if there exist $s \in \mathbb{N}$, and an extension $\tilde{M}$ of M to $\mathbb{P}^1_K$ such that

$$\tau^s((\sigma^s)^*\hat{M}_\infty) = t^{sw} \cdot \hat{M}_\infty.$$

PROPOSITION 0.16 (Anderson ([**An1**], Prop. 1.9.2)). *Let M be a τ-sheaf on $\mathbb{A}^1_K$ with characteristic ι and dimension d. If M is pure of weight w, then there exists an abelian t-module E of dimension d, defined over a finite inseparable extension K' of K, such that $M_{K'} \cong M(E)$. If r denotes the rank of M, then $rw = d$.*

3. *Drinfeld modules.*

DEFINITION 0.17 (Drinfeld ([**Dr1**], [**AlB**], [**Go4**], etc.)). Let $\mathcal{C} = \bar{\mathcal{C}} \backslash \{\infty\}$, where $\bar{\mathcal{C}}$ is an absolutely irreducible smooth projective curve with constant field $\mathbb{F}_q$ and ∞ a closed point of $\bar{\mathcal{C}}$. Put

$$\mathbf{A} := H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}),$$

and, for a field K containing $\mathbb{F}_q$, let $\iota^* : \mathbf{A} \to K$ be a ring morphism. A **Drinfeld A-module** ϕ defined over K is a 1-dimensional $\mathbf{A}$-module

$$\phi : \mathbf{A} \to \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K})$$

such that the induced action $\partial\phi : \mathbf{A} \to \operatorname{End}(\operatorname{Lie}(E))$ on the Lie algebra $\operatorname{Lie}(\mathbb{G}_{a,K})$ is given by ι^*, but $\phi \ncong \partial\phi$.

REMARK 0.18. Prop. 0.16 implies, in particular, that any pure τ-sheaf M on $\mathbb{A}^1_K$ with dimension 1, corresponds, over a finite inseparable extension, to a Drinfeld module. Conversely, the t-motive $M(\phi)$ corresponding to a Drinfeld module ϕ, is always pure (cf. [**An1**] Prop. 4.1.1)

4. *Tate modules.* For a t-module E over K, one associates to every nonzero ideal ℓ of $\mathbf{A}$, with generator λ the **ℓ-torsion module**

$$E[\ell] = \ker(\phi_E(\lambda))(\bar{K}).$$

It is a finite $\mathbf{A}/\ell$-module endowed with a continuous action of Γ_K. For every nonzero maximal ideal ℓ, we consider the system of morphisms

$$\phi_E(\lambda^m) : E[\ell^{m+n}] \to E[\ell^n].$$

DEFINITION 0.19. The inverse limit

$$T_\ell(E) := \lim_{\leftarrow} E[\ell^n]$$

is the **ℓ-adic Tate module of** E.

If E is abelian with associated t-motive M and the characteristic is generic (to assure that $\hat{M}_\ell$ is smooth for every ℓ), then we have an isomorphism

$$T_\ell(E) \cong T_\ell(M) \tag{15}$$

of $\mathbf{A}_\ell[\Gamma_K]$-modules by [**An1**], prop. 1.8.3 (cf. [**Tag3**] as well).

5. *What's in a name?* The category $t-\mathrm{Mot}_K$ of t-motives over K is an additive tensor category. In Def. 0.8, we defined, for every closed point ℓ of $\mathbb{A}^1$, a contravariant functor V_ℓ from $t-\mathrm{Mot}_K$ to the category of finite dimensional F_ℓ-vector spaces with a continuous action of Γ_K. This can be considered as the ℓ-adic realization for t-motives.

Anderson, Gekeler etc. (see [**Go2**]) developed a 'de Rham' realization, a functor V_{dR} from $t-\mathrm{Mot}_K$ to the category of finite dimensional F_∞-vector spaces, where F_∞ is the completion of $\mathcal{O}_{\mathbb{P}^1}$ at ∞. For uniformizable t-motives (cf. 5.2), there is also the notion of a Betti realization, given by the corresponding lattice. Further, as we just saw, Anderson gave a definition of purity.

Thus, judging from its formal properties which compare very well with that of classical motives from algebraic geometry (as discussed in [**Se6**] and [**Se7**] e.g.), the category t-motives have a very 'motivic' nature indeed. However, nothing seems to be known yet about the relation with cohomology of algebraic varieties in characteristic p.

II. A bird's eyes' view

§1. The starting point[6] for this research project was given by Serre's famous theorem on the image of the absolute Galois group of a number field on the Tate modules of an elliptic curve:

THEOREM 0.20 (Serre ([**Se3**], 1972)). *Let E be an elliptic curve without potential*[7] *complex multiplication defined over a number field K with absolute Galois group Γ_K. For any prime number p, consider the Tate module $T_p(E)$ of E at p and the associated $\mathbb{Q}_p[\Gamma_K]$-module*

$$V_p(E) := \mathbb{Q}_p \otimes T_p(E).$$

The image of the 'adelic' representation of Γ_K on the restricted product

$$V_{\mathrm{ad}}(E) := \prod_p{}' V_p(E)$$

is open in $\mathrm{GL}_2(\prod'_p \mathbb{Q}_p)$, for the adelic topology.

Roughly speaking, the main ingredients of its proof include:

i) the fact that the Γ_K-modules $V_p(E)$ form a strictly compatible system of Galois representations and satisfy the Tate and semisimplicity conjecture;
ii) an application of the theory of p-adic Lie groups to show that, if E has no potential complex multiplication, the image of the representation

$$\rho_p : \Gamma_K \to \mathrm{GL}_2(\mathbb{Q}_p),$$

given by the continuous Γ_K-action on $V_p(E)$, is open for all p;
iii) the study of the action of tame inertia on $V_p(E)$;
iv) the construction of compatible systems of 1-dimensional Galois representations associated to Hecke characters (cf. [**Se1**]).

Combining results i), iii) and iv) with a classification of maximal subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$, Serre first proves: If E has no potential complex multiplication, then the Galois representation

$$\overline{\rho}_p : \Gamma_K \to \mathrm{GL}_2(\mathbb{F}_p)$$

given by the $\mathbb{F}_p[\Gamma_K]$-modules $E[p]$ of p-torsion points is surjective for almost all primes p. Thm. 0.20 follows from this by ii) and some group theory.

§2. Consider an affine, smooth, absolutely irreducible curve $\mathcal{C}$ with field of constants $\mathbb{F}_q$ en denote by $\mathbf{A}$ the ring of global functions. Analogously to Thm. 0.20, there is an 'adelic Mumford-Tate conjecture' on the image of the adelic representation associated to Drinfeld modules (cf. Conj. 3.17 as well):

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[6] I want to thank G. Cornelissen, J. Top, M. van der Put and J. Van Geel for suggesting this topic for my FWO research project.

[7] Let P be a property related to a field K. We say that the property holds **potentially**, if it holds for some finite extension K' of K.

CONJECTURE 0.21. *Let ϕ be a Drinfeld module (Def. 0.13) of rank r, defined over a global function field K, without potential complex multiplication and of generic characteristic. The image of the representation ρ_Λ of Γ_K on the module $V_\Lambda(\phi)$ (see (13)) is open, for any set Λ of closed points of $\mathcal{C}$.*

Using heavy machinery from the theory of algebraic groups and Serre's ideas on Frobenius tori, Pink proved this conjecture in [**Pi2**] (Thm. 0.1; 1997) in the case that Λ is a finite set.

It was not so hard to realize that Serre's ideas, which deal with 2-dimensional representations, together with Pink's result, already allow to prove Conj. 0.21 for Drinfeld modules of rank at most 2. This line of thought was worked out in [**Ga1**]. To deal with the case of general rank, more ideas are needed, in particular on subgroups of finite algebraic groups (work of Larsen-Pink [**LP**]) and on the absolute irreducibility of the residual representations $\overline{\rho}_\ell$, which will be explained in upcoming work by Pink and Traulsen.

§3. The underlying motive in this research project was to generalize the techniques that are used in the above result from the case Drinfeld modules to arbitrary t-modules. In the first place, we wanted to study the action of inertia on the Tate modules associated to E. If we let K is a valued field containing $\mathbb{F}_q$, then this requires that we find a suitable model[8] for the t-module E with coefficients in the valuation ring R of K whose reduction modulo the maximal ideal of R yields some useful information.

For Drinfeld modules, there is a satisfying theory of models (see [**Tag2**], §1). We know, for example, that, for every Drinfeld module ϕ, there potentially exists a 'stable' model, i.e. a Drinfeld module with coefficients in R, isomorphic to ϕ and whose reduction is a Drinfeld module over the residue field of R, whose rank r' is possibly smaller than the rank r of ϕ. The model is called good if $r = r'$.

Assume that K is complete. Drinfeld's proposition on Tate uniformization (Prop. 2.10) then says: There exists a good Drinfeld module ϕ' over R and an $\mathbf{A}$-lattice H in K^{sep} (cf. Def. 6.15) such that we have an exact sequence

$$0 \to H \to \phi' \to \phi \to 0 \tag{16}$$

of *rigid analytic* spaces endowed with an $\mathbf{A}$-action.

As a consequence of this, the inertia group I_K acts potentially unipotently on all of its Tate modules $T_\ell(\phi)$, except at the residual characteristic point ℓ' (Def. 0.5.iv), if it exists. Compare this to the classical monodromy theorem on p-adic representations, stating that the inertia group of a local field of residual characteristic $p' \neq p$ acts potentially unipotently.

Unfortunately, extrapolating this satisfactory situation to general (abelian) t-modules is impossible. This is shown by the existence of a 'nonsemistable' abelian t-module E (see Prop. 2.11), where the action of inertia on the Tate modules $T_\ell(E)$ is not always unipotent for $\ell \neq \ell'$.

[8] This line of thought was worked out in the project proposal 'Bad reduction of t-modules' submitted to the Swiss Science Foundation.

But, as it turned out (Thm. 6.22), one can reinterpret the analytic structure of a Drinfeld module ϕ (cf. equation (16)) into an exact sequence

$$0 \to \hat{N} \to \hat{M}(\phi) \to \hat{M}(\phi') \to 0 \tag{17}$$

of τ-sheaves on $\mathbb{A}^1_K$, seen as a rigid analytic space. Here $\hat{M}(\phi)$ (resp. $\hat{M}(\phi')$) is the analytic τ-sheaf associated to the t-motive of ϕ (resp. ϕ'). This triggered the idea to forget about the t-modules (for a while) and concentrate on developing a theory of models for τ-sheaves instead. This strategy proved quite fruitful end allowed us to get some new insights into the theory of Galois representations associated with τ-sheaves on the one hand, and the arithmetic structure of t-motives on the other hand.

§4. Overview. Let us now make a tour through some of the main results in this thesis. Let X be an irreducible Dedekind $\mathbb{F}_q$-scheme with function field K and M a τ-sheaf on $\mathcal{C}_K$.

Chapter 1. If $X = \operatorname{Spec} R$, for a discrete valuation $\mathbb{F}_q$-algebra R, then it suffices, in order to define a model $\mathcal{M}$ for M on $\mathcal{C}_X$ (Def. 1.1), to give its stalk at the generic point of the special fibre. This observation by L. Lafforgue allows us to show for a given τ-sheaf M the existence

- of nondegenerate models $\mathcal{M}$ for M on $\mathcal{C}_{R'}$ (Def. 1.6), for a finite separable extension R' of R, en
- of a maximal model $\mathcal{M}^{\max}$ for M (Def. 1.12), which satisfies a Néron-type mapping property.

Suppose that $X = \operatorname{Spec} R$, where R is now a complete discrete valuation $\mathbb{F}_q$-algebra whose residue field is algebraic over $\mathbb{F}_q$. If we have a τ-sheaf $\mathcal{M}$ with nondegenerate reduction, then we prove that it is possible to lift this reduction *analytically* to an analytic sub-τ-sheaf $\mathcal{N}$ of $\mathcal{M}$ with good reduction, at least upon replacing $\mathcal{C}$ by an open subscheme. As a consequence, we obtain:

THEOREM 1.26. *For every analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathcal{C}}_K$, there exists*

- *a nonempty open subscheme $\mathcal{C}' \subset \mathcal{C}$,*
- *a finite extension R' of R, with fraction field K', and*
- *a filtration*

$$0 = \widetilde{N}_0 \subset \widetilde{N}_1 \subset \cdots \subset \widetilde{N}_n = \widetilde{M}|_{\widetilde{\mathcal{C}}'_{K'}} \tag{18}$$

by saturated analytic sub-τ-sheaves on $\widetilde{\mathcal{C}}'_{K'}$

such that the subquotients $\widetilde{M}_i := \widetilde{N}_i / \widetilde{N}_{i-1}$ have good models over $\widetilde{\mathcal{C}}'_{R'}$.

This generalizes the result (17) on Drinfeld modules.

Chapter 2. By the correspondence between ℓ-adic τ-sheaves and Galois representations, one immediately deduces from Thm. 1.26 that, for a τ-sheaf M on $\mathcal{C}_K$, the action of inertia acts potentially unipotently on the Tate modules $T_\ell(M)$, for all but a finite number of closed points ℓ of $\mathcal{C}$ (Thm. 2.4).

Assuming that K is a valued field containing $F := \mathrm{Quot}(\mathbf{A})$, we deal with further essential questions concerning the action of inertia on the Tate modules $T_\ell(M)$ of a τ-sheaf M with a good model and characteristic $\iota : \mathrm{Spec}\, K \to \mathcal{C}$:

i) a description of the action of tame inertia, in terms of fundamental characters (Thm. 2.14);
ii) a description of the image of wild inertia (Cor. 2.24), for τ-sheaves of dimension 1.

Chapter 3. With these results, we can now turn back to our original problem. We give a proof (Thm. 3.13) of the following conjecture in the case where the rank r of M is at most 2:

CONJECTURE 3.1. *Let K be a finite extension of F, the function field of $\mathcal{C}$ and M a τ-sheaf over $\mathcal{C}_K$ with characteristic $\iota : \mathrm{Spec}\, K \to \mathcal{C}$, dimension* 1 *and absolute endomorphism ring* $\mathbf{A}$. *The image of the representation $\overline{\rho}_{\mathrm{ad}}$ on the $\kappa_{\mathrm{ad}}[\Gamma_K]$-module $W_{\mathrm{ad}}(M)$* (cf. (14)) *is open in* $\mathrm{GL}_r(\kappa_{\mathrm{ad}})$.

Our theorem applies in particular to Drinfeld modules without complex multiplication of rank 2 (see paragraph **§2**). However, we were determined to avoid any 'purity' assumption, as it seems a better idea to exploit directly the natural relation between τ-sheaves and Galois representations. Here again, just as in the quest for models (cf. paragraph **§3**), it seems to be nothing but a diversion to assume that we are dealing with structures related to t-modules!

For Drinfeld modules, we obtain Conj. 0.21 as a consequence of this Conj. 3.1 using [**Pi2**]; this provides us with a proof if $r \leq 2$ (Thm. 3.20).

Chapter 4. Let R be a discrete valuation ring with function field K. We give a general analog of the classical 'Néron-Ogg-Shafarevič' good reduction criterion on abelian varieties:

THEOREM 4.1. *Let M be a τ-sheaf on $\mathcal{C}_K$ with a characteristic ι and ℓ a closed point of $\mathcal{C}$ such that $\hat{M}_\ell$ is smooth. If the inertia group I_K of K acts trivially on $T_\ell(M)$, then there exists a good model $\mathcal{M}$ over $\mathcal{C}_R$ for M.*

As a consequence, we derive a criterion for trivial reduction (Thm. 4.8). Also, we relate the L-factor of M at a place of bad reduction to the action of Frobenius on the Tate module $T_\ell(M)$ (Thm. 4.12).

Chapter 5. The Galois criterion for trivial reduction can now be applied to shed some new light on uniformizability. Putting $\mathcal{C} = \mathbb{A}^1$, let K be a discretely valued field containing F and with finite residue field k, such that its valuation is infinite with respect to the characteristic $\iota : \mathrm{Spec}\, K \to \mathbb{A}^1$. Extending results by Anderson ([**An1**], Thm. 4) and Pink, we prove

THEOREM 5.13BIS. *For an abelian t-module E with t-motive M, the following statements are all equivalent:*

i) *the abelian t-module E is uniformizable;*
ii) *the uniformization lattice H has* $\mathbf{A}$*-rank r, the rank of E;*

iii) *the analytic τ-sheaf $\widetilde{M}$ on $\mathbb{A}^1_K$ associated to M potentially contains a trivial sub-τ-sheaf of full rank;*
iv) *the τ-sheaf M potentially has a good model $\mathcal{M}$ with trivial reduction;*
v) *the σ-bundle associated to M on the punctured open unit disk around ∞ is trivial;*
vi) *the semistable filtration (cf. (42)) of $\widetilde{M}$ is defined on the whole of $\mathbb{A}^1_K$ and each of its subquotients has trivial reduction;*
vii) *the action of Γ_K on the Tate module $T_\ell(M)$ is potentially trivial, for all closed points ℓ of $\mathcal{C}$; and, last but not least,*
viii) *there exists a closed point ℓ of $\mathcal{C}$ such that the action of Γ_K on the Tate module $T_\ell(M)$ is potentially trivial.*

Chapter 6. Coming back to equation (16), we recall that the correspondence between Tate uniformization and the analytic 'semistable' filtration (17) for a Drinfeld module was crucial in developing a reduction theory for τ-sheaves. We can extend this correspondence to higher dimensional abelian t-motives in the following way:

THEOREM 6.3BIS. *Let K be a complete valued field containing F, whose valuation is finite with respect to the characteristic $\iota : \operatorname{Spec} K \to \mathbb{A}^1$. There exists an antiequivalence between the categories of pure abelian t-modules over K and of pure t-motives over $\mathbb{A}^1_K$, where both categories are endowed with analytic morphisms* (Def. 6.1 & 6.2).

The arguments for this theorem rely on asymptotic estimates for local logarithmic heights on t-modules, which are presented in section 6.V, and weight inequalities induced by nontrivial analytic morphisms (cf. Prop. 6.9). In Thm. 6.16, we work out a further aspect of analytic morphisms of t-modules, namely that of uniformization lattices. Finally, we discuss how, via this theorem, Thm. 1.26 leads to an analytic description of analytic t-modules.

III. Nederlandstalige samenvatting

§1. Het uitgangspunt van dit onderzoeksproject is Serres gevierde resultaat over het beeld van de representaties van de absolute Galoisgroep van een getallenveld op de Tatemodulen van een elliptische kromme:

STELLING 0.20 (Serre ([**Se3**], 1972)). *Zij E een elliptische kromme zonder potentiële complexe multiplicatie, gedefinieerd over een getallenveld K met absolute Galoisgroep Γ_K. Voor elk priemgetal p, beschouwen we het Tatemoduul $T_p(E)$ van E in p en het geassocieerde $\mathbb{Q}_p[\Gamma_K]$-moduul*

$$V_p(E) := \mathbb{Q}_p \otimes T_p(E).$$

Het beeld van de 'adelische' representatie van Γ_K op het gerestringeerde product

$$V_{\mathrm{ad}}(E) := \prod_p{}' V_p(E)$$

is een open deelgroep van $\mathrm{GL}_2(\prod_p' \mathbb{Q}_p)$ (voor de adelische topologie).

De belangrijkste ingrediënten van het bewijs zijn ruwweg de volgende:

i) de Γ_K-modulen $V_p(E)$ vormen een strict compatibel systeem van Galoisrepresentaties waarvoor de Tate- en semisimpliciteitsconjecturen gelden;

ii) een toepassing van p-adische Liegroepentheorie toont aan dat als E geen potentiële complexe multiplicatie bezit, het beeld van de representatie

$$\rho_p : \Gamma_K \to \mathrm{GL}_2(\mathbb{Q}_p),$$

gegeven door de continue actie van Γ_K op $V_p(E)$, open is voor alle p;

iii) we kennen de actie van de gemodereerde inertiegroep op $V_p(E)$;

iv) we hebben een constructie van compatibele systemen van 1-dimensionale Galoisrepresentaties geassocieerd aan Heckekarakters (cfr. [**Se1**]).

Serre bewijst eerst, door het combineren van de resultaten i), iii) en iv) met een classificatie van maximale deelgroepen van de groepen $\mathrm{GL}_2(\mathbb{F}_p)$, het volgende: heeft E geen potentiële complexe multiplicatie, dan is de residuële Galoisrepresentatie

$$\overline{\rho}_p : \Gamma_K \to \mathrm{GL}_2(\mathbb{F}_p),$$

gegeven door het $\mathbb{F}_p[\Gamma_K]$-moduul $E[p]$ van de p-torsiepunten, surjectief voor alle priemgetallen p, op een eindige aantal uitzonderingen na. Stelling 0.20 volgt hieruit, na toepassing van ii) en wat groepentheorie.

§2. We beschouwen een affiene, absoluut irreduciebele gladde kromme $\mathcal{C}$ met constantenveld $\mathbb{F}_q$ en we noteren de ring van globale reguliere functies van $\mathcal{C}$ als $\mathbf{A}$. Analoog aan Stelling 0.20 is er een 'adelisch Mumford-Tate-vermoeden' over het beeld van de adelische representatie geassocieerd aan Drinfeldmodulen (zie ook Conj. 3.17).

CONJECTUUR 0.21 (Pink). *Zij ϕ een Drinfeld-$\mathbf{A}$-moduul van rang r, gedefiniëerd over een globaal functieveld K, zonder potentiële complexe multiplicatie en met generieke karakteristiek. Het beeld van de representatie ρ_Λ van de absolute*

Galoisgroep Γ_K van K op het $F_\Lambda[\Gamma_K]$-moduul $V_\Lambda(\phi)$ (cfr. (13)) *is open voor elke verzameling Λ van gesloten punten van $\mathcal{C}$.*

Pink bewees in [**Pi2**] (Thm. 0.1; 1997) dat dit vermoeden geldt voor elke eindige verzameling Λ. Zijn bewijs is een toepassing van technieken uit de algebraïsche groepentheorie en Serres concept van Frobeniustori.

Het was niet zo moeilijk om in te zien dat Serres ideeën rond de 2-dimensionale representaties geassocieerd met elliptische krommen samen met Pinks resultaat volstaan om Conj. 0.21 te bewijzen in het geval dat de rang van ϕ ten hoogste 2 is. Deze gedachte werd uitgewerkt in [**Ga1**]. Om een dergelijk resultaat voor hogere rang r aan te tonen, zijn meer nieuwe ideeën vereist, in het bijzonder over eindige deelgroepen van algebraïsche groepen (werk van Larsen en Pink [**LP**]) en over de absolute irreducibiliteit van de residuele representaties $\overline{\rho}_\ell$, die wordt bestudeerd in recent onderzoek van Pink en Traulsen.

§3. De rode draad in dit onderzoeksproject is de ambitie om een aantal technieken die in de bovenstaande resultaten worden aangewend, te verruimen van Drinfeldmodulen naar algemene (hoger dimensionale) t-modulen E (cfr. Def. 0.12). In de eerste plaats bestuderen we de actie van inertie op de Tatemodulen geassocieerd met E. Zij K een veld dat $\mathbb{F}_q$ bevat en uitgerust is met een valuatie, dan vereist dit dat we voor het t-moduul E een gunstig model vinden met coëfficiënten in de valuatiering R van R waarvan de reductie modulo het maximale ideaal van R nuttige informatie levert.

Voor Drinfeldmodulen is er een bevredigende theorie van zulke modellen (zie [**Tag2**], §1). Zo weten we, bij voorbeeld, dat er voor elk Drinfeldmoduul ϕ over K potentiëel een zgn. 'stabiel' model bestaat, een Drinfeldmoduul met coëfficiënten in R, isomorf met ϕ en waarvan de reductie $\bar{\phi}$ een Drinfeldmoduul is over het residuveld k. De rang r' van ϕ is mogelijks kleiner dan de rang r van ϕ; het model wordt 'goed' genoemd indien $r = r'$.

Veronderstellen we dat K compleet is, dan zegt Drinfelds propositie over Tate-uniformizatie (Prop. 2.10) het volgende: er bestaat een goed Drinfeldmoduul ϕ' over R, een rooster H in K^{sep} (Def. 6.15), en een exacte rij

$$0 \to H \to \phi' \to \phi \to 0 \tag{19}$$

van rigied analytische ruimtes met een actie van $\mathbf{A}$.

Hieruit volgt dat de inertiegroep I_K van K potentieel unipotent opereert op de Tatemodulen $T_\ell(\phi)$, behalve voor $\ell = \ell'$, waar ℓ' het eventuele residuele characteristieke punt is (cfr. Def. 0.5.iv)). Dit kan men vergelijken met Grothendiecks klassieke monodromiestelling voor p-adische representaties: de inertiegroep van een lokaal veld met residuele karakteristiek $p' \neq p$ opereert potentieel unipotent.

Jammer genoeg is het onmogelijk om deze situatie te extrapoleren naar algemene (abelse) t-modulen. Er bestaat namelijk een 'niet-semistabiel' abels t-moduul (zie Prop. 2.11), waarvoor de actie van de inertiegroep I_K op de Tatemodulen $T_\ell(M)$ niet potentieel unipotent is voor alle $\ell \neq \ell'$.

Wat daarentegen bleek (Stelling 6.22), is dat we de analytische structuur van een Drinfeldmoduul ϕ (zie vgl. (16)) kunnen herinterpreteren als een exacte rij

$$0 \to \hat{N} \to \hat{M}(\phi) \to \hat{M}(\phi') \to 0 \tag{20}$$

van τ-schoven over de rigied analytische affiene rechte $\mathbb{A}^1_K$; hier is $\hat{M}(\phi)$ (resp. $\hat{M}(\phi')$) de analytische τ-schoof geassocieerd aan het t-motief van ϕ (resp. ϕ').

Dit lokte de idee uit om t-modulen (voor een tijdje) aan de kant te zetten en een theorie van modellen voor τ-schoven te ontwikkelen. Deze strategie bleek vruchtbaar en stelde ons in staat om nieuwe inzichten te verkrijgen in enerzijds de Galoisrepresentaties geassocieerd aan τ-schoven, en anderszijds de arithmetische structuur van t-motieven.

§4. Overzicht. We bespreken kort enkele hoofdresultaten uit dit proefschrift. Zij X een irreduciebel Dedekindschema over $\mathbb{F}_q$, met functieveld K, en zij M een τ-schoof over $\mathcal{C}_K$.

Hoofdstuk 1. Is $X = \operatorname{Spec} R$ voor een discrete valuatiering die $\mathbb{F}_q$ bevat, dan kan men een model $\mathcal{M}$ voor M over $\mathcal{C}_X$ definiëren door zijn halm bij het generische punt van de speciale vezel aan te geven. Deze opmerking van L. Lafforgue staat ons toe voor een gegeven τ-schoof M het bestaan te bewijzen (voor algemene X) van

- niet-gedegenereerde modellen $\mathcal{M}$ over $\mathcal{C}_{R'}$ (Def. 1.6), over een eindige separabele uitbreiding R' van R, en
- een maximal model $\mathcal{M}^{\max}$ (Def. 1.12) over $\mathcal{C}_R$ dat een Néroncriterium vervult.

Veronderstellen we dat $X = \operatorname{Spec} R$, waarbij R een complete discrete valuatiering is waarvan het residuveld k algebraïsch is over $\overline{\mathbb{F}}_q$. We bewijzen dat voor elke τ-schoof $\mathcal{M}$ met niet-gedegenereerde reductie, deze reductie *analytisch* kan worden 'gelift' tot een analytische deel-τ-schoof $\mathcal{N}$ van $\mathcal{M}$ met goede reductie (cf. Def. 1.6), tenminste wanneer we $\mathcal{C}$ door een open deelschema vervangen. Hieruit volgt:

STELLING 1.26 (**Analytische semistabiliteit**). *Voor elke τ-schoof $\widetilde{M}$ over $\widetilde{\mathcal{C}}_K$ bestaat er*

- *een niet-leeg open deelschema $\mathcal{C}' \subset \mathcal{C}$,*
- *een eindige separabele uitbreiding R' van R, met breukenveld K', en*
- *een filtratie*

$$0 = \widetilde{N}_0 \subset \widetilde{N}_1 \subset \cdots \subset \widetilde{N}_n = \widetilde{M}|_{\widetilde{\mathcal{C}}'_{K'}} \tag{21}$$

door gesatureerde analytische deel-τ-schoven over $\widetilde{\mathcal{C}}'_{K'}$

zo dat de deelquotienten $\widetilde{M}_i := \widetilde{N}_i / \widetilde{N}_{i-1}$ een goed model bezitten over $\widetilde{\mathcal{C}}'_{R'}$.

Dit veralgemeent het resultaat (19) voor Drinfeldmodulen.

Hoofdstuk 2. Uit het verband tussen ℓ-adische τ-schoven en Galoisrepresentaties (Prop. 0.7), volgt onmiddellijk uit Stelling 1.26 dat voor een τ-schoof M over $\mathcal{C}_K$, de inertiegroep potentieel unipotent opereert op de Tatemodulen $T_\ell(M)$, behalve voor een eindig aantal gesloten punten ℓ van $\mathcal{C}$ (Stelling 2.4).

In de veronderstelling dat K het veld $F = \mathrm{Quot}(\mathbf{A})$ bevat, geven we ook een antwoord op enkele andere essentiële vragen in verband met de actie van inertie op de Tatemodulen $T_\ell(M)$ van een τ-schoof met een goed model en karakteristiek $\iota : \mathrm{Spec}\, K \to \mathcal{C}$:

- i) een beschrijving van de actie van gemodereerde inertie, in termen van fundamentele karakters (Stelling 2.14);
- ii) een beschrijving van het beeld van de wilde inertiegroep (Cor. 2.24), voor τ-schoven met dimensie 1.

Hoofdstuk 3. Op basis van deze resultaten kunnen we nu terugkeren naar ons oorspronkelijke probleem. In Stelling 3.13 bewijzen we het volgende vermoeden in het geval de rang r van M ten hoogste 2 is:

CONJECTUUR 3.1. *Zij K een eindige uitbreiding van F en M een simpele τ-schoof van rang r over $\mathcal{C}_K$ met karakteristiek $\iota : \mathrm{Spec}\, K \to \mathcal{C}$, dimensie 1 en absolute endomorphismenring* $\mathbf{A}$. *Het beeld van de representatie $\bar{\rho}_{\mathrm{ad}}$ van Γ_K op het $\kappa_{\mathrm{ad}}[\Gamma_K]$-moduul $W_{\mathrm{ad}}(\phi)$ (cfr. 14) is open in* $\mathrm{GL}_r(\kappa_{\mathrm{ad}})$.

Deze stelling is in het bijzonder van toepassing voor Drinfeldmodulen zonder potentiële complexe multiplicatie (zie **§2**). Het is evenwel onze opzet geweest om de vereiste van 'puurheid' voor de τ-schoven te vermijden, aangezien het een beter idee lijkt om direct de natuurlijke relatie tussen τ-schoven en Galoisrepresentaties aan te wenden. Het lijkt erop dat het, net zoals in de zoektocht naar modellen (zie **§3**), niet meer dan een omweg is om te veronderstellen dat onze structuren met t-modulen verwant zijn.

Voor Drinfeldmodulen volgt Conj. 0.21, dankzij [**Pi2**], uit Conj. 3.1, wat een bewijs levert voor $r \leq 2$ (cfr. Stelling 3.20).

Hoofdstuk 4. Zij R een discrete valuatiering die $\mathbb{F}_q$ omvat, met perfect residuveld k, en zij K het breukenveld van R. We geven een algemeen analogon voor het bekende Galoiscriterium van Néron-Ogg-Shafarevič voor goede reductie van abelse variëteiten:

STELLING 4.1. *Zij M een τ-schoof over $\mathcal{C}_K$ met een karakteristiek ι en ℓ een gesloten punt van $\mathcal{C}$ zodat $\hat{M}_\ell$ glad is. Is de actie van de inertiegroep I_K op $T_\ell(M)$ triviaal, dan bezit M een goed model over $\mathcal{C}_R$.*

Uit deze stelling kunnen we meteen een Galoiscriterium voor triviale reductie afleiden (Stelling 4.8). Tenslotte leggen we een verband tussen de L-factor van M bij een plaats van slechte reductie voor M en de actie van Frobenius op de Tatemodulen $T_\ell(M)$ (Stelling 4.12).

Hoofdstuk 5. Het Galoiscriterium voor triviale reductie kan worden toegepast om nieuw licht te laten schijnen op Anderson-uniformizatie. Zij $\mathcal{C} = \mathbb{A}^1$ en K een discreet gevalueerd veld dat F omvat, met eindig residuveld k, waarvoor de valuatie met betrekking tot de karakteristiek ι : Spec $K \to \mathbb{A}^1$ oneindig is. Voortbouwend op resultaten van Anderson ([**An1**], Thm. 4) en Pink bewijzen we:

STELLING 5.13BIS. *Voor een abels t-model E met t-motief M zijn de volgende uitspraken equivalent:*

i) *het t-moduul E is uniformizeerbaar;*
ii) *de analytische τ-schoof $\widetilde{M}$ over $\mathbb{A}^1_K$ geassocieerd met M omvat potentieel een triviale deel-τ-schoof met volle rang;*
iii) *de τ-schoof M bezit potentieel een goed model $\mathcal{M}$ met triviale reductie;*
iv) *de σ-bundel geassocieerd aan M over de open eenheidsschijf rond ∞, minus het punt ∞ zelf, is triviaal;*
v) *er bestaat een semistabiele filtratie (cfr. (42)) voor $\widetilde{M}$ die gedefinieerd is over heel $\mathbb{A}^1_K$ en waarvan alle deelquotiënten triviale reductie hebben;*
vi) *de actie van Γ_K op het Tatemoduul $T_\ell(M)$ is potentieel triviaal, voor alle gesloten punten ℓ van $\mathcal{C}$; en tenslotte:*
vii) *er bestaat een gesloten punt ℓ van $\mathcal{C}$ zodat de actie van Γ_K op het Tatemoduul $T_\ell(M)$ potentieel triviaal is.*

Hoofdstuk 6. Tenslotte komen we terug op vgl. (19): het verband tussen Tate-uniformizatie en de analytische semistabiele filtratie (20) voor een Drinfeldmoduul ϕ was van doorslaggevend belang in het ontwikkelen van een reductietheorie voor τ-schoven. We kunnen zo'n correspondentie veralgemenen voor hogerdimensionale pure t-motieven:

STELLING 6.3BIS. *Zij K een compleet gevalueerd veld dat F omvat, en waarvoor de valuatie eindig is m.b.t. de karakteristiek ι :* Spec $K \to \mathbb{A}^1$. *Er bestaat een anti-equivalentie tussen de categorieën van pure abelse t-modulen over K en pure t-motieven over $\mathbb{A}^1_K$, waarbij de morphismen in beide gevallen door analytische homomorphismen zijn gegeven* (Def. 6.1 & 6.2).

Het bewijs van deze Stelling steunt of asymptotische schattingen van een lokale hoogtefunctie voor t-modulen enerzijds (cf. sectie 6.V), en ongelijkheden voor de gewichten van t-motieven waartussen een niet-triviaal analytisch morfisme bestaat (cf. Prop. 6.9). In Stelling 6.16 werken we een verder aspect van analytische morfismen uit, namelijk het opduiken van uniformizatieroosters. Tot slot verklaren we hoe Stelling 1.26 de aanzet geeft tot een analytische beschrijving van t-modulen.

CHAPTER 1

The analytic structure of τ-sheaves

I. Models of τ-sheaves

§1. Models. Let X be an irreducible Dedekind $\mathbb{F}_q$-scheme, i.e. an irreducible smooth one-dimensional scheme over $\mathbb{F}_q$. We denote the function field of X by K. For every τ-sheaf $\mathcal{M}$ on $\mathcal{C}_X$, we denote by $\mathcal{M}_K$ the restriction of $\mathcal{M}$ to the generic fibre $\mathcal{C}_K$.

DEFINITION 1.1. A **model** $\mathcal{M}$ **over** $\mathcal{C}_X$ of a τ-sheaf M on $\mathcal{C}_K$ is a τ-sheaf on $\mathcal{C}_X$ which extends M, i.e. such that $\mathcal{M}_K = M$.

PROPOSITION 1.2. *For any given τ-sheaf M on $\mathcal{C}_X$, there exists a model over $\mathcal{C}_X$.*

PROOF. We choose an extension $\mathcal{M}'$ of the sheaf M to $\mathcal{C}_X$. We can find an invertible sheaf $\mathcal{L}$ on $\mathcal{C}_X$ which is the pullback of an invertible subsheaf of $\mathcal{O}_X$ on X, such that τ extends to a morphism

$$\tau : \sigma^*\mathcal{M}' \to \mathcal{L}^{-1} \otimes \mathcal{M}'.$$

If we put $\mathcal{M} := \mathcal{L} \otimes \mathcal{M}'$ then $\tau(\sigma^*\mathcal{M}) \subset \mathcal{L}^{(q-1)} \otimes \mathcal{M}$, which yields that $\mathcal{M}$ is τ-invariant and hence is a model over $\mathcal{C}_X$ for M. □

Quite often, problems on models can be reduced to the local case, i.e. where the coefficient scheme X equals Spec R, for a discrete valuation ring R, with function field K and residue field k. We now discuss a lemma by Lafforgue which explains how, in this situation, a model can be constructed. We denote by L the rational function field of $\mathcal{C}_R$, and by $\mathcal{O}_\varpi := \mathcal{O}_{\mathcal{C}_R,\varpi}$ the local ring of regular functions at the generic point ϖ of the special fibre $\mathcal{C}_k \hookrightarrow \mathcal{C}_R$.

LEMMA 1.3. *Lafforgue,* [**Laf**] *To give a locally free coherent sheaf $\mathcal{M}$ on $\mathcal{C}_R$ is equivalent to giving its restriction M to $\mathcal{C}_K$ and its stalk $\mathcal{M}_\varpi$ at the point ϖ.*

More precisely, for given M and $\mathcal{M}_\varpi$, we consider the unique largest coherent sheaf $\mathcal{M}$ on $\mathcal{C}_R$ whose restrictions to $\mathcal{C}_K$ and $\mathcal{O}_\varpi$ are M and $\mathcal{M}_\varpi$, respectively. We include a proof of this lemma for future reference:

PROOF. Let M be a sheaf over $\mathcal{C}_K$ and $\mathcal{M}_\varpi$ a free $\mathcal{O}_\varpi$-module, both of same rank r, together with an isomorphism of the r-dimensional L-vector spaces

$$L \otimes_{\mathcal{O}_{\mathcal{C}_K}} M$$

(the stalk of M at the generic point of $\mathcal{C}_K$) and $L \otimes_{\mathcal{O}_\varpi} \mathcal{M}_\varpi$. We need to show that there exists a locally free coherent sheaf $\mathcal{M}$ on $\mathcal{C}_R$, unique up to unique isomorphism, which extends M and $\mathcal{M}_\varpi$.

The stalk $\mathcal{M}_\varpi$ defines a locally free sheaf $\mathcal{M}'$ on a neighborhood U of ϖ which coincides with M on $\mathcal{C}_K \cap U$. Gluing $\mathcal{M}'$ together with M_K, we obtain a locally free sheaf $\mathcal{M}''$ which is defined on an open subscheme $U' := \mathcal{C}_R \setminus Q$, outside a closed set Q of codimension ≥ 2. As $\mathcal{C}_R$ is a surface, Q is a finite set of closed points.

Denoting by j the open embedding $U' \hookrightarrow \mathcal{C}_R$, we consider the push-forward $\mathcal{M} := j_*\mathcal{M}''$. It is shown by Langton ([**Lan**]), using the fact that $\mathcal{C}_R$ is noetherian, that $\mathcal{M}$ is the unique largest coherent and torsion free extension of $\mathcal{M}''$ to $\mathcal{C}_R$. We remark that $\mathcal{M}$ depends functorially upon the data.

Further, Langton proves that, if $i : \mathcal{C}_k \to \mathcal{C}_R$ denotes the closed embedding of the special fibre, $i^*\mathcal{M}$ is torsion free. As $\mathcal{C}_k$ is 1-dimensional and smooth, $i^*\mathcal{M}$ must actually be free. Its rank must be r, because it is so locally on $i^{-1}(U)$. By standard arguments using Nakayama's lemma, it then follows that $\mathcal{M}$ is locally free of rank r. □

COROLLARY 1.4. *To give a τ-sheaf $\mathcal{M}$ on $\mathcal{C}_R$ is equivalent to giving its restriction M to $\mathcal{C}_K$ and its stalk $\mathcal{M}_\varpi$ at ϖ.*

PROOF. By the above lemma, the modules M and $\mathcal{M}_\varpi$ define a unique locally free sheaf $\mathcal{M}$. As M and $\mathcal{M}_\varpi$ are τ-invariant, so is $\mathcal{M}$, again by the above lemma. □

§2. Good and nondegenerate models. For a point x of X, we denote the residue field at x by k_x.

DEFINITION 1.5. Let $\mathcal{M}$ be a τ-sheaf over $\mathcal{C}_X$. For a point x of X, let i_x denote the embedding $\mathcal{C}_{k_x} \mapsto \mathcal{C}_X$. The **reduction of** $\mathcal{M}$ **at** x is defined as the locally free coherent sheaf

$$\overline{\mathcal{M}}_x := i_x^*\mathcal{M} = \mathcal{M} \times_{\mathcal{O}_{\mathcal{C}_X}} \mathcal{O}_{\mathcal{C}_{k_x}}, \tag{22}$$

endowed with the induced homomorphism $\tau : \sigma^*\overline{\mathcal{M}}_x \to \overline{\mathcal{M}}_x$.

DEFINITION 1.6.

i) The τ-sheaf $\mathcal{M}$ is called **good at** x if $\overline{\mathcal{M}}_x$ is a τ-sheaf, i.e.
$$\tau : \sigma^*\overline{\mathcal{M}}_x \to \overline{\mathcal{M}}_x$$
is injective.

ii) The τ-sheaf $\mathcal{M}$ is called **degenerate at** x if $\tau : \sigma^*\overline{\mathcal{M}}_x \to \overline{\mathcal{M}}_x$ is nilpotent, and **nondegenerate** otherwise.

iii) Let $\left(\overline{\mathcal{M}}_{k_x}\right)_1$ denote the maximal τ-sheaf on $\mathcal{C}_{k_x}$ contained in $\overline{\mathcal{M}}_{k_x}$. The rank of $\left(\overline{\mathcal{M}}_{k_x}\right)_1$ is called the **nondegenerate rank** of $\overline{\mathcal{M}}_{k_x}$ at x.

REMARK 1.7. If the residue field k_x is perfect, then σ is an automorphism on $\mathcal{O}_{\mathcal{C}_{k_x}}$. Let H denote the fraction field of this ring. By elementary theory on τ-modules, there exist, for every finite dimensional H-vector space V endowed with a (possibly noninjective) morphism $\tau : \sigma^* V \to V$, a decomposition

$$V = V_1 \oplus V_{\mathrm{nil}}. \tag{23}$$

where V_1 is a τ-sheaf (i.e. τ is injective) and where the action of τ on V_{nil} is nilpotent. For every locally free coherent $\mathcal{O}_{\mathcal{C}_{k_x}}$-module N endowed with a morphism

$$\tau : \sigma^* N \to N,$$

let us denote by V the stalk of N at the generic point of $\mathcal{C}_{k_x}$. Putting $N_1 := V_1 \cap N$, the decomposition (23) then induces an exact sequence

$$0 \to N_1 \to N \to N_{\mathrm{nil}} \to 0. \tag{24}$$

of sheaves on $\mathcal{C}_{k_x}$ endowed with a σ-semilinear morphism τ, where N_1 is a τ-sheaf and the action of N_{nil} is nilpotent.

LEMMA 1.8.

i) *A τ-sheaf $\mathcal{M}$ of $\mathcal{C}_X$ is good at a point x of X if and only if τ is an isomorphism on the stalk $\mathcal{M}_\varpi$ of $\mathcal{M}$ at the generic point ϖ of the special fibre $\mathcal{C}_{k_x}$.*

ii) *Every τ-sheaf $\mathcal{M}$ on $\mathcal{C}_X$ is good outside a finite number of points.*

PROOF. Part i) is obvious from the definition.

We consider the generic point μ of $\mathcal{C}$ and the subscheme $\{\mu\} \times X$ of $\mathcal{C}_X$. As

$$\tau : \sigma^* \mathcal{M} \to \mathcal{M}$$

is an injective homomorphism, its cokernel coker τ is supported on a closed subscheme Y of $\mathcal{C}_X$ of codimension at least 1. Therefore τ is an isomorphism locally at $\{\mu\} \times \{x\}$, for all but a finite number of points x of X. By i), this proves ii). $\square$

Let $X' \to X$ be a finite extension of the coefficient scheme. For a given τ-sheaf $\mathcal{M}$ on $\mathcal{C}_X$, the pullback of $\mathcal{M}$ to $\mathcal{C}_{X'}$ will be denoted by $\mathcal{M}_{X'}$.

PROPOSITION 1.9. *For every τ-sheaf M on $\mathcal{C}_K$, there exists a finite separable extension $X' \to X$ of the coefficient scheme with function field K' and a model $\mathcal{M}$ over $\mathcal{C}_{X'}$ for $M_{K'}$ which is nondegenerate at all closed points of X'.*

PROOF. **a)** *Local case.* (Drinfeld, [**Laf**] Lemme 3) Suppose that $X = \operatorname{Spec} R$, where the ring R is a discrete valuation ring and where x is the unique closed point. We denote by $\hat{\mathcal{O}}_\varpi$ the completion of the local ring of functions $\mathcal{O}_\varpi$ at the generic point ϖ of the special fibre, and by $\hat{F}_\varpi$ its fraction field. We put $V := L \otimes_{\mathcal{O}_{\mathcal{C}_K}} M$ and

$$\hat{V} := \hat{F}_\varpi \otimes_L V.$$

It is shown in [**Laf**] Lemme 3, that there exists a valuation deg on $\hat{V}$ which takes its values in $\frac{1}{e}\mathbb{Z}$, for some $e \in \mathbb{Z}$, and such that

$$\deg \tau(\hat{v}) = q \deg \hat{v},$$

for all $\hat{v} \in \hat{V}$.

Over a finite separable totally ramified extension R' of R, one can assume that deg attains the value 0 on $\hat{V}$. We define a τ-invariant $\mathcal{O}_\varpi$-module by taking

$$\hat{\mathcal{M}}_\varpi := \{\hat{v} \in \hat{V}; \deg \hat{v} \geq 0\}.$$

As deg attains the value 0, $\hat{\mathcal{M}}_\varpi$ is nondegenerate at x. Putting

$$\mathcal{M}_\varpi := \hat{\mathcal{M}}_\varpi \cap V \subset \hat{V},$$

we obtain a τ-module over $\mathcal{O}_\varpi$ which is nondegenerate at x as well, since the module $\mathcal{M}_\varpi$ is dense in $\hat{\mathcal{M}}_\varpi$. By Cor. 1.4, this stalk $\mathcal{M}_\varpi$ at ϖ together with the generic fibre M yield a unique τ-sheaf $\mathcal{M}$ on $\mathcal{C}_R$ with the desired properties.

b) *Global case.* By Lemma 1.8.ii) there exists a nonempty open subscheme $U \subset X$ such that $\mathcal{M}|_{\mathcal{C}_U}$ is good at all closed points of U. From the result in the local case, we obtain, over a finite base extension $X' \to X$, nondegenerate models $\mathcal{M}_\upsilon$ locally at the fibres $\upsilon := \mathcal{C} \times \{u\}$ of the finite number of closed points u of $X \backslash U$. Gluing these with $\mathcal{M}|_U$ defines $\mathcal{M}$. □

LEMMA 1.10. *Every inclusion $i : \mathcal{M}' \hookrightarrow \mathcal{M}$ of τ-sheaves on $\mathcal{C}_X$, yields, for each point x of $\mathcal{C}$, an injective map*

$$\left(\overline{\mathcal{M}}'_{k_x}\right)_1 \hookrightarrow \left(\overline{\mathcal{M}}_{k_x}\right)_1 .$$

PROOF. Taking up the setting of the proof of Prop. 1.9, the inclusion i induces an injective morphism $\mathcal{M}'_\varpi \hookrightarrow \mathcal{M}_\varpi$ of τ-modules over $\mathcal{O}_\varpi$, for the local ring at the generic point of the special fibre $\mathcal{C}_{k_x}$. On $\mathcal{M}_\varpi \supset \mathcal{M}'_\varpi$ we have the valuation deg as before. Any nonzero element $\bar{m}$ of $\left(\overline{\mathcal{M}}'_{k_x}\right)_1$ can be lifted to an element $m \in \mathcal{M}'_\varpi$ such that $\deg m = 0$. This immediately shows that the reduction of m, seen as an element of $\mathcal{M}'_\varpi$ is nonzero. □

REMARK 1.11. *Models for t-modules?* Let E be a t-module over K. A **model** $\mathcal{E}$ for E over X is a t-module over X extending the generic fibre E. In [**Tag2**], Taguchi develops a theory of nondegenerate ('stable') reduction and minimal models for Drinfeld modules.

There is little hope for a fruitful theory of reduction for higher dimensional abelian t-modules. For example, an essential step would be to find a model $\mathcal{E}$ whose reduction $\overline{\mathcal{E}}$ at a closed point x of X is nondegenerate in the following sense: there exists a sub-t-module $\overline{\mathcal{E}}_1$ of $\overline{\mathcal{E}}$ which is abelian.

In Prop. 2.11, we will give an example of an abelian t-module for which a nondegenerate model cannot exist, not even after an extension of the base field K. The arithmetic study of general abelian t-modules should therefore rely on the reduction theory of the associated τ-sheaf $M(E)$.

§3. Maximal models.

DEFINITION 1.12. We say that a model $\mathcal{M}$ for a given τ-sheaf M over $\mathcal{C}_K$ is **maximal** if, for every τ-sheaf $\mathcal{N}$ on $\mathcal{C}_X$, every morphism $f_K : \mathcal{N}_K \to M$ of τ-sheaves on $\mathcal{C}_K$ extends to a morphism $f : \mathcal{N} \to \mathcal{M}$ of τ-sheaves on $\mathcal{C}_X$.

This property is clearly analogous to the Néron mapping property for schemes (cf. [**BLR**]).

PROPOSITION 1.13.

i) *Every τ-sheaf M defined over $\mathcal{C}_K$ admits a maximal model over $\mathcal{C}_X$ which is unique, up to unique isomorphism, and which we denote by $\mathcal{M}^{\max}$.*
ii) *If $\mathcal{M}$ is a good model for M over $\mathcal{C}_X$, then it is maximal.*

PROOF. First, we notice that if the maximal model exists, then it follows from its universal property that it is unique up to unique isomorphism.

a) *Local case.* Suppose that $X = \operatorname{Spec} R$, where R is a discrete valuation ring and where x is the unique closed point. Let $\mathcal{M}_\varpi$ and $\mathcal{M}'_\varpi$ be τ-modules over $\mathcal{O}_\varpi$, the local ring of functions at the generic point ϖ of the special fibre. Note that any uniformizer π of R is a uniformizer of $\mathcal{O}_\varpi$, and that σ acts on π by ${}^\sigma\pi = \pi^q$. If $\mathcal{M}'_\varpi \subset \mathcal{M}_\varpi$, it follows that

$$\text{(25)} \qquad \operatorname{length}_{\mathcal{O}_\varpi}\left(\tau(\sigma^*\mathcal{M}_\varpi)/\tau(\sigma^*\mathcal{M}'_\varpi)\right) = q \cdot \operatorname{length}_{\mathcal{O}_\varpi}\left(\mathcal{M}_\varpi/\mathcal{M}'_\varpi\right).$$

We define the **discriminant** $\Delta(\mathcal{M}_\varpi) \geq 0$ to be the length over $\mathcal{O}_\varpi$ of coker τ:

$$\Delta(\mathcal{M}_\varpi) := \operatorname{length}_{\mathcal{O}_\varpi}\left(\mathcal{M}_\varpi/\tau(\sigma^*\mathcal{M}_\varpi)\right).$$

It follows from (25) that if $\mathcal{M}'_\varpi \subsetneq \mathcal{M}_\varpi$ then

$$\text{(26)} \qquad \Delta(\mathcal{M}'_\varpi) > \Delta(\mathcal{M}_\varpi).$$

We choose a τ-module $\mathcal{M}_\varpi$ which is contained in the stalk V of M at the generic point of $\mathcal{C}_X$ and with minimal discriminant. This $\mathcal{M}_\varpi$ contains all τ-modules over $\mathcal{O}_\varpi$ contained in V: Indeed, suppose there does exist a τ-module $\mathcal{M}'_\varpi$ over $\mathcal{O}_\varpi$ which is not contained in $\mathcal{M}'_\varpi$, then the sum

$$\mathcal{M}'_\varpi + \mathcal{M}_\varpi \supsetneq \mathcal{M}_\varpi,$$

again a τ-invariant projective $\mathcal{O}_\varpi$-module, has a smaller discriminant than the τ-module $\mathcal{M}_\varpi$ by (26).

Take $\mathcal{M}$ to be the τ-sheaf on $\mathcal{C}_R$ defined, following Cor. 1.4, by its generic fibre $\mathcal{M}_K := M$ and its stalk $\mathcal{M}_\varpi$ at ϖ. Let L, V and V' denote the stalk of $\mathcal{O}_{\mathcal{C}_R}$, $\mathcal{M}$ and $\mathcal{N}$, respectively, at the generic point of $\mathcal{C}_R$. For a given morphism

$$f_K : \mathcal{N}_K \to M,$$

we consider its unique extension to a L-linear morphism $f : V' \to V$. The image $f(\mathcal{N}_\varpi) \subset V$ of the stalk $\mathcal{N}_\varpi$ of $\mathcal{N}$ at ϖ is a τ-module over $\mathcal{O}_\varpi$. Hence it must be contained in the maximal τ-module $\mathcal{M}_\varpi$.

We consider the sheaf $f_K(\mathcal{N})$ as a locally free coherent subsheaf of the constant sheaf V. By Cor. 1.4, is completely determined by its generic fibre $f_K(M)$ and stalk $f_K(\mathcal{M}_\varpi)$ at ϖ. Since $f_K(\mathcal{N}_K) \subset M$ and

$$f_K(\mathcal{N}_\varpi) \subset \mathcal{M}_\varpi,$$

we obtain that $f_K(\mathcal{N}) \subset \mathcal{M}$, i.e. f_K extends to a morphism $\mathcal{N} \to \mathcal{M}$. Hence the model $\mathcal{M}$ is maximal, which proves i) in the local case.

If M has a good model $\mathcal{M}$, then $\Delta(\mathcal{M}_\varpi) = 0$ by Lemma 1.8.i), and hence the stalk $\mathcal{M}_\varpi$, as well as the τ-sheaf $\mathcal{M}$, are maximal, which proves ii).

b) *Global case.* Let X be a Dedekind $\mathbb{F}_q$-scheme. Statement ii) follows easily from the local case.

For i), we remark that, by Lemma 1.8.ii), any model $\mathcal{M}'$ on $\mathcal{C}_X$ for M is good at the closed points of an nonempty open subscheme $U \subset X$. Hence $\mathcal{M}'|_U$ is a maximal model for M on $\mathcal{C}_U$, by ii). From the local case, we obtain maximal models $\mathcal{M}_\upsilon$ locally at the fibres $\upsilon := \mathcal{C} \times \{u\}$ of the finite number of closed points u of $X \backslash U$. Gluing these with $\mathcal{M}'|_U$ clearly defines a maximal model $\mathcal{M}$. □

LEMMA 1.14. *Let $X' \to X$ be a finite étale extension of absolutely irreducible Dedekind schemes and let K' denote the function field of X'. A τ-sheaf M on $\mathcal{C}_K$ has a good model over $\mathcal{C}_X$ if and only if $M_{K'}$ has a good model over $\mathcal{C}_{X'}$.*

PROOF. We can assume that $X' \to X$ is Galois. By Prop. 1.13.ii), a good model $\mathcal{M}'$ for $M_{K'}$ is maximal, and, as a consequence, Galois invariant. By Galois descent, the sheaf $\mathcal{M}'$ on $\mathcal{C}_{X'}$ descends to a sheaf $\mathcal{M}$ on $\mathcal{C}_X$, and the morphism

$$\tau : \sigma^*\mathcal{M}' \to \mathcal{M}'$$

yields a morphism $\tau : \sigma^*\mathcal{M} \to \mathcal{M}$. As properties of sheaves descend as well, we obtain that $\tau_{\mathcal{M}}$ is injective (hence $\mathcal{M}$ is a model for M) and that $\mathcal{M}$ is a good model. □

II. Analytic τ-sheaves

§1. Models of analytic τ-sheaves. Let R be a complete discrete valuation $\mathbb{F}_q$-algebra. We denote by K its field of fractions and by k its residue field. Further, let v be its valuation, $|\cdot|_K$ its absolute value and π a uniformizer. Let $\widetilde{\mathcal{C}}_K$ denote the curve $\mathcal{C}_K$, seen as a rigid analytic space in the sense of [**BLR**]. We consider a formal admissible scheme $\widetilde{\mathcal{C}}_R$ which is an R-module for $\widetilde{\mathcal{C}}_K$ (see [**BL**]).

Let $\mathcal{O}_{\widetilde{\mathcal{C}}_K}$ and $\mathcal{O}_{\widetilde{\mathcal{C}}_R}$ denote the structure sheaf of the space $\widetilde{\mathcal{C}}_K$ and the formal scheme $\widetilde{\mathcal{C}}_R$, respectively. The endomorphism σ on $\mathcal{C}_R$ extends in a unique way to an endomorphism on $\widetilde{\mathcal{C}}_K$ and $\widetilde{\mathcal{C}}_R$.

DEFINITION 1.15.

i) An **analytic** τ**-sheaf** $(\widetilde{M}, \tau)$ (for short: $\widetilde{M}$) **of rank** $r \geq 1$**, defined on** $\widetilde{\mathcal{C}}_K$ **(resp.** $\widetilde{\mathcal{C}}_R$**),** is a locally free $\mathcal{O}_{\widetilde{\mathcal{C}}_K}$-module (resp. $\mathcal{O}_{\widetilde{\mathcal{C}}_R}$-module) of finite rank r, endowed with an injective morphism

$$\tau : \sigma^*\widetilde{M} \to \widetilde{M}.$$

ii) A **model** $\widetilde{\mathcal{M}}$ **over** $\widetilde{\mathcal{C}}_R$ of an analytic τ-sheaf M on $\widetilde{\mathcal{C}}_K$ is a τ-sheaf on $\widetilde{\mathcal{C}}_R$ which extends $\widetilde{M}$, i.e. such that the generic fibre associated to $\widetilde{\mathcal{M}}$, in the sense of [**Lüt**], is $\widetilde{M}$.

iii) Let $\mathcal{M}$ be a τ-sheaf over $\mathcal{C}_R$. Let i denote the closed embedding of the special fibre $\mathcal{C}_k \hookrightarrow \mathcal{C}_R$. The **reduction of** $\mathcal{M}$ is defined as the locally free coherent sheaf

$$\overline{\mathcal{M}} := i^*\mathcal{M}, \tag{27}$$

endowed with the induced homomorphism $\tau : \sigma^*\overline{\mathcal{M}} \to \overline{\mathcal{M}}$. The τ-sheaf $\widetilde{\mathcal{M}}$ is called **good** if $\overline{\mathcal{M}}$ is a τ-sheaf. The τ-sheaf $\widetilde{\mathcal{M}}$ is called **degenerate** if τ is nilpotent on $\overline{\mathcal{M}}$ and **nondegenerate** otherwise.

iv) A model $\widetilde{\mathcal{M}}$ for a given analytic τ-sheaf $\widetilde{M}$ over $\widetilde{\mathcal{C}}_K$ is **maximal** if, for every τ-sheaf $\widetilde{\mathcal{N}}$ on $\widetilde{\mathcal{C}}_R$, every morphism $f_K : \widetilde{\mathcal{N}}_K \to \widetilde{M}$ of τ-sheaves on $\widetilde{\mathcal{C}}_K$ extends to a morphism $f : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{M}}$ of τ-sheaves on $\widetilde{\mathcal{C}}_R$.

PROPOSITION 1.16. *Let $\widetilde{M}$ be a τ-sheaf on $\widetilde{\mathcal{C}}_K$.*

i) *There exists, over a finite separable extension R' of R, a nondegenerate model $\widetilde{\mathcal{M}}$ on $\widetilde{\mathcal{C}}_R$ for $\widetilde{M}$.*

ii) *There exists a maximal model $\widetilde{\mathcal{M}}^{\max}$ over $\widetilde{\mathcal{C}}_R$ for $\widetilde{M}$, unique up to unique isomorphism.*

iii) *Let M be an algebraic τ-sheaf on $\mathcal{C}_K$, and $\widetilde{M}$ the associated analytic τ-sheaf. The analytic sheaf associated to $\mathcal{M}^{\max}$ is isomorphic to $\widetilde{\mathcal{M}}^{\max}$.*

We postpone the proof of this proposition to §4.

REMARK 1.17. Similarly as for algebraic τ-sheaves (Def. 0.8), we can associate with every analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathcal{C}}_K$ and every closed point ℓ of $\mathcal{C}$, the **Tate module** $T_\ell(\widetilde{M})$.

§2. Explicit construction of $\widetilde{\mathcal{C}}_R$. Suppose that $\widetilde{\mathcal{C}}$ is an open subscheme of $\mathbb{A}^1$. We set $\widetilde{\mathcal{C}} := \mathbb{A}^1 \setminus S$, where $S = \{s_1, \dots, s_\mu\}$ is a finite set of closed points of $\mathbb{A}^1$, and for each $i \le \mu$, let $f_i \in \mathbf{A} \cong \mathbb{F}_q[t]$ be a generator of the ideal defining s_i.

1. *The space* $\widetilde{\mathcal{C}}_K$. An admissible covering of $\widetilde{\mathcal{C}}_K$ is given by the affinoid spaces

$$\Omega_K^n := \left\{ t \in K;\, |t|_K \le |\pi|_K^{-n} \text{ and } |f_i(t)|_K \ge |\pi|_K^n, \text{ for } 1 \le i \le \mu \right\},$$

for positive integers n. Let $K\left\langle t_0, t_1, \dots, t_\mu \right\rangle$ denote the Tate algebra in the variables

$$t_0, \dots, t_\mu,$$

i.e. the ring of formal power series

$$\sum_{\nu \in \mathbb{N}^{\mu+1}} a_\nu t_0^{\nu_0} t_1^{\nu_1} \cdots t_\mu^{\nu_\mu}$$

for which $a_\nu \to 0$ if $\nu \to \infty$.

Consider the ring $K\left\langle t_0^{(n)}, t_1^{(n)}, \dots, t_\mu^{(n)} \right\rangle [t]$ and its ideal

$$\mathcal{I}_K := \left(t_0^{(n)} - \pi^n t,\, t_1^{(n)} \cdot f_1(t) - \pi^n, \dots, t_\mu^{(n)} \cdot f_\mu(t) - \pi^n \right). \tag{28}$$

The affinoid algebra corresponding to Ω_K^n is then

$$W_K^n := K\left\langle t_0^{(n)}, t_1^{(n)}, \dots, t_\mu^{(n)} \right\rangle [t] / \mathcal{I}_K. \tag{29}$$

Further, Ω_K^n is an affinoid subdomain of Ω_K^m, for m greater than n, via the unique homomorphism $W_K^m \to W_K^n$ which leaves the variable t invariant.

We denote by $K\langle\langle t, t_1, \dots, t_\mu \rangle\rangle$ the ring of entire functions in the variables

$$t, t_1, \dots, t_\mu.$$

Setting

$$\mathcal{I}'_K := \left(t_1 \cdot f_1(t) - 1, \dots, t_\mu \cdot f_\mu(t) - 1\right), \tag{30}$$

the ring of global analytic functions on $\widetilde{\mathcal{C}}_K$ is isomorphic to

$$K \left\langle\!\left\langle t, t_1, \dots, t_\mu \right\rangle\!\right\rangle / \mathcal{I}'_K.$$

2. *An R-model for $\widetilde{\mathcal{C}}_K$.* For each n, we consider the admissible R-algebra

$$W_R^n := R\left\langle t_0^{(n)}, t_1^{(n)}, \dots, t_\mu^{(n)} \right\rangle [t]/\mathcal{I}_R, \tag{31}$$

where we define the ideal $\mathcal{I}_R$ of $R\left\langle t_0^{(n)}, t_1^{(n)}, \dots, t_\mu^{(n)} \right\rangle [t]$ as in (28). Following [**BL**], its formal spectrum defines a formal admissible scheme Ω_R^n over R, whose generic fibre, in the sense of [**BL**], is Ω_K^n. For every integer m greater than n, we have, as before, a canonical embedding $\Omega_R^n \hookrightarrow \Omega_R^m$. The direct limit of the Ω_R^n over all integers $n \geq 0$ then yields an analytic R-model $\widetilde{\mathcal{C}}_R$ for $\widetilde{\mathcal{C}}_K$. Defining $\mathcal{I}'_R$ as in (30), its ring of global analytic functions equals

$$R \left\langle\!\left\langle t, t_1, \dots, t_\mu \right\rangle\!\right\rangle / \mathcal{I}'_R. \tag{32}$$

3. *Construction of σ.* For every n, we define unique continuous ring morphisms

$$(\sigma_n)^* : W_K^{qn} \to W_K^n$$

as follows: $(\sigma_n)^*$ restricts to the Frobenius φ on K and acts trivially on the variable t. Thus we obtain homomorphisms $\sigma_n : \Omega_K^n \to \Omega_K^{qn}$, which, upon taking the limit, endow $\widetilde{\mathcal{C}}_K$ with an endomorphism $\sigma = \lim_{\to} \sigma_n$. Similarly, we can endow $\widetilde{\mathcal{C}}_R$ with such an endomorphism.

§3. Lafforgue's Lemma for $\mathcal{C} = \mathbb{A}^1$. We consider the discrete valuation v_R on

$$H^0\left(\widetilde{\mathbb{A}}_R^1, \mathcal{O}_{\widetilde{\mathbb{A}}_R^1}\right) = R\langle\langle t \rangle\rangle$$

given by

$$v_R\left(\sum_{i=0}^{\infty} f_i\, t^i\right) := \inf_i \{v(f_i)\},$$

and extend this valuation to the fraction field $\widetilde{L}$ of $R\langle\langle t\rangle\rangle$. Note that the nonzero σ-invariant functions have zero valuation.

Denoting the valuation ring of v_R inside $\tilde{L}$ by $\widetilde{\mathcal{O}}_\varpi$, we have

$$H^0\left(\widetilde{\mathbb{A}}_K^1, \mathcal{O}_{\widetilde{\mathbb{A}}_K^1}\right) \cap \widetilde{\mathcal{O}}_\varpi = H^0\left(\widetilde{\mathbb{A}}_R^1, \mathcal{O}_{\widetilde{\mathbb{A}}_R^1}\right).$$

To a given $\mathcal{O}_{\widetilde{\mathbb{A}}_R^1}$-module $\widetilde{\mathcal{M}}$, we associate the $\widetilde{\mathcal{O}}_\varpi$-module

$$\widetilde{\mathcal{M}}_\varpi := \widetilde{\mathcal{O}}_\varpi \otimes H^0(\widetilde{\mathbb{A}}_R^1, \widetilde{\mathcal{M}}).$$

Calling $\widetilde{\mathcal{M}}_\varpi$ suggestively, but with some abuse, the **stalk of $\widetilde{\mathcal{M}}$ at the 'point' ϖ** defined by the valuation v_R, we have the following analog of Lafforgue's lemma:

LEMMA 1.18. *To give a locally free coherent sheaf $\widetilde{\mathcal{M}}$ on $\widetilde{\mathbb{A}}_R^1$ is equivalent to giving its restriction $\widetilde{M}$ to $\widetilde{\mathbb{A}}_K^1$ and its stalk $\widetilde{\mathcal{M}}_\varpi$ at ϖ.*

PROOF. Let $\widetilde{M}$ be a sheaf over $\widetilde{\mathbb{A}}^1_K$ and $\mathcal{M}_\varpi$ a free $\widetilde{\mathcal{O}}_\varpi$-module, both of same rank r, together with an isomorphism of the r-dimensional $\widetilde{L}$-vector spaces

$$\widetilde{L} \otimes_{\mathcal{O}_{\widetilde{\mathcal{C}}_K}} M \text{ and } \widetilde{L} \otimes_{\widetilde{\mathcal{O}}_\varpi} \widetilde{\mathcal{M}}_\varpi.$$

The valuation v_R extends, in a unique continuous way, to the fields $\mathrm{Quot}(W_R^n)$ of meromorphic functions on the covering spaces Ω_K^n for $\widetilde{\mathbb{A}}^1_K$. Let us denote by $\widetilde{\mathcal{O}}^n_\varpi$ the associated valuation ring. We can, informally, view $\widetilde{\mathcal{M}}^n_\varpi := \widetilde{\mathcal{O}}^n_\varpi \otimes \widetilde{\mathcal{M}}_\varpi$ as the pullback of $\widetilde{\mathcal{M}}_\varpi$ from ϖ to the 'point' on Ω_R^n defined by the valuation v_R.

We claim that the restriction $\widetilde{M}_K|_{\Omega_K^n}$ together with $\widetilde{\mathcal{M}}^n_\varpi$ defines, for every integer $n \geq 0$, a sheaf $\widetilde{\mathcal{M}}^n$ on Ω_R^n. Indeed, as the spaces Ω_R^n are noetherian, one checks that, mutatis mutandis, the proof for sheaves on algebraic schemes that we gave in Prop. 1.4 can be carried over.

By uniqueness of these sheaves $\widetilde{\mathcal{M}}^n$, it follows that

$$\widetilde{\mathcal{M}}^{n+1}|_{\Omega_R^n} = \widetilde{\mathcal{M}}^n,$$

for all $n \geq 0$. Therefore, we can glue the $\widetilde{\mathcal{M}}^n$ to a unique $\widetilde{\mathcal{M}}$ which satisfies the conditions. $\square$

As in the algebraic case, we immediately draw from this the following corollary

COROLLARY 1.19. *To give a τ-sheaf $\widetilde{\mathcal{M}}$ on $\widetilde{\mathbb{A}}^1_R$ is equivalent to giving its restriction $\widetilde{M}$ to $\widetilde{\mathbb{A}}^1_K$ and its stalk $\widetilde{\mathcal{M}}_\varpi$ at ϖ.*

§4. Proof of Prop. 1.16.

PROOF OF PROP. 1.16. We first prove ii). In the case $\mathcal{C} = \mathbb{A}^1$, we can, as in the proof of Prop. 1.13, use Lemma 1.19 in order to prove the existence of a maximal model. For a general curve $\mathcal{C}$ we proceed as following: Complete $\mathcal{C}$ to a projective curve $\overline{\mathcal{C}}$ and choose a rational function f on $\overline{\mathcal{C}}$ which has poles exactly at the points of $\overline{\mathcal{C}}\backslash\mathcal{C}$. This yields a finite morphism

$$f : \mathcal{C} \to \mathbb{A}^1.$$

The τ-module $\widetilde{M}^\star := f_*\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$ is endowed with a scalar multiplication of the coherent $\mathcal{O}_{\widetilde{\mathbb{A}}^1_R}$-module $f_*(\mathcal{O}_{\widetilde{\mathcal{C}}_R})$. The image of

$$f_*(\mathcal{O}_{\widetilde{\mathcal{C}}_R}) \otimes_{\mathcal{O}_{\widetilde{\mathbb{A}}^1_R}} \widetilde{\mathcal{M}}^\star \to \widetilde{M}^\star \tag{33}$$

is coherent and τ-invariant, and must therefore be a sub-τ-sheaf of the maximal model $\widetilde{\mathcal{M}}^\star$ of $\widetilde{M}^\star$ on $\widetilde{\mathcal{C}}^\star_R$. Hence $\widetilde{\mathcal{M}}^\star$ carries an action of the $\mathcal{O}_{\widetilde{\mathcal{C}}^\star_R}$-algebra $f_*(\mathcal{O}_{\widetilde{\mathcal{C}}_R})$. We now apply:

LEMMA 1.20. *Let $f : \mathcal{C} \to \mathcal{C}^\star$ be a finite morphism of irreducible affine curves, and let us denote by f the induced morphisms $\widetilde{\mathcal{C}}_K \to \widetilde{\mathcal{C}}^\star_K$ and $\widetilde{\mathcal{C}}_R \to \widetilde{\mathcal{C}}^\star_R$ as well. Let $\widetilde{M}$ be a τ-sheaf on $\widetilde{\mathcal{C}}_K$. To give a model $\widetilde{\mathcal{M}}$ for $\widetilde{M}$ on $\widetilde{\mathcal{C}}_R$ is equivalent to giving a model $\widetilde{\mathcal{M}}^\star$ on $\widetilde{\mathcal{C}}^\star_R$ for the τ-sheaf $\widetilde{M}^\star := f_*\widetilde{M}$ on $\widetilde{\mathcal{C}}^\star_K$ which carries an action of the $\mathcal{O}_{\widetilde{\mathcal{C}}^\star_R}$-algebra $f_*\left(\mathcal{O}_{\widetilde{\mathcal{C}}_R}\right)$. We then have*

$$f_*(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}}^\star.$$

Thus we obtain a model $\widetilde{\mathcal{M}}$ of $\widetilde{M}$ over $\widetilde{\mathcal{C}}_R$. As $f_*(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}}^\star$, the model $\widetilde{\mathcal{M}}$ is clearly maximal.

i) Following the same argument as in Prop. 1.9 and using Cor. 1.19, we can prove that, over a finite extension R' of R, there exists a nondegenerate model $\widetilde{\mathcal{M}}'$ over $\widetilde{\mathbb{A}}^1_R$ for every τ-sheaf $\widetilde{M}'$ on $\widetilde{\mathbb{A}}^1_K$. By a proper analogue of Lemma 1.10, the maximal model $\widetilde{\mathcal{M}}'^{\max}$ is still nondegenerate. For a general curve $\mathcal{C}$, statement i) now follows by a reduction to the case $\mathcal{C} = \mathbb{A}^1$ as before.

To prove statement iii), we notice that the stalks of the maximal model $\mathcal{M}^{\max}$ of M on $\mathcal{C}_R$ and of the maximal model $\widetilde{\mathcal{M}}^{\max}$ of $\widetilde{M}$ on $\widetilde{\mathcal{C}}_R$ are both determined by the maximal τ-module over $\hat{\mathcal{O}}_\varpi$ (same notations as in the proof of Lemma 1.9), and, therefore, the analytic sheaf associated to $\mathcal{M}^{\max}$ is isomorphic to $\widetilde{\mathcal{M}}^{\max}$. □

III. Analytic structure of τ-sheaves

§1. Analytic lifts.

THEOREM 1.21. *Let R be a complete discrete valuation $\mathbb{F}_q$-algebra and K its field of fractions. Suppose that the residue field k of R is algebraic over $\mathbb{F}_q$. Let $\widetilde{\mathcal{M}}$ be a τ-sheaf on $\widetilde{\mathcal{C}}_R$ of nondegenerate rank r_1. There exists*

- *a nonempty open subscheme $\mathcal{C}'$ of $\mathcal{C}$ and*
- *a unique maximal good analytic sub-τ-sheaf $\widetilde{\mathcal{M}}_1 \subset \widetilde{\mathcal{M}}|_{\widetilde{\mathcal{C}}'_R}$ of rank r_1.*

This $\widetilde{\mathcal{M}}_1$ is functorial with respect to analytic homomorphisms.

PROOF. **a)** First, suppose that $\mathcal{C} = \mathbb{A}^1$. We denote by $\overline{\mathcal{M}}_1$ the maximal τ-sheaf inside $\overline{\mathcal{M}}$, the reduction of $\widetilde{\mathcal{M}}$ at the closed point x of Spec R, and by $r_1 := \operatorname{rk} \overline{\mathcal{M}}_1$ the nondegenerate rank of $\overline{\mathcal{M}}$. Let $\mathcal{S}_1$ be the support of the cokernel of τ on $\overline{\mathcal{M}}_1$ and S_1 the finite set of closed points $\mathbb{A}^1$ lying below $\mathcal{S}_1$. We set $\mathcal{C}' := \mathbb{A}^1 \setminus S_1$. In the next paragraph, will present a proof of the following proposition:

PROPOSITION 1.22. *Let R be a complete discrete valuation $\mathbb{F}_q$-algebra and K its field of fractions. Suppose that the residue field k of R is algebraic over $\mathbb{F}_q$. For a nondegenerate analytic τ-sheaf $\widetilde{\mathcal{M}}$ on $\widetilde{\mathbb{A}}^1_R$, there exists an analytic sub-τ-sheaf*

$$\widetilde{\mathcal{N}} \subset \widetilde{\mathcal{M}}|_{\widetilde{\mathcal{C}}'_R}$$

such that $\widetilde{\mathcal{N}}$ has good reduction, and such that

$$\overline{\mathcal{N}} \cong \overline{\mathcal{M}}_1|_{\widetilde{\mathcal{C}}'_k}.$$

Let $\widetilde{\mathcal{M}}_1$ be the saturation of $\widetilde{\mathcal{N}}$ in $\widetilde{\mathcal{M}}$. As $\widetilde{\mathcal{M}}_1$ is τ-invariant, it is actually a τ-sheaf on $\widetilde{\mathcal{C}}'_R$. Further, the model $\widetilde{\mathcal{M}}_1$ is good, as $\widetilde{\mathcal{N}}$ is good. Clearly, $\widetilde{\mathcal{M}}_1$ is a maximal good sub-τ-module of $\widetilde{\mathcal{M}}$ on $\widetilde{\mathcal{C}}'_R$, as its rank equals the nondegenerate rank of $\widetilde{\mathcal{M}}$.

b) To prove the theorem for a general curve $\mathcal{C}$, we consider, as in the proof of Prop. 1.16, a finite morphism $f : \mathcal{C} \to \mathbb{A}^1$, whose degree we denote by deg f. By the above results, we obtain a nonempty open subscheme $\mathcal{C}^\star$ of $\mathbb{A}^1$ and a nontrivial maximal good sub-τ-module $\widetilde{\mathcal{M}}_1^\star$ of rank $r_1 \cdot \deg f$ of the τ-sheaf

$$\widetilde{\mathcal{M}}^\star := f_*(\widetilde{\mathcal{M}})|_{\widetilde{\mathcal{C}}^\star_R}$$

on $\widetilde{\mathcal{C}}^{\star}_R$. We put $\mathcal{C}' := f^{-1}(\mathcal{C}^{\star})$. As $\widetilde{\mathcal{M}}^{\star}_1$ is a *maximal*, it is invariant under the action of $f_*(\mathcal{O}_{\widetilde{\mathcal{C}}'_R})$; hence, by Lemma 1.20, we obtain a τ-sheaf $\widetilde{\mathcal{M}}_1$ on $\mathcal{C}'_R$ such that $f_*(\widetilde{\mathcal{M}}_1) = \mathcal{M}^{\star}_1$. Clearly, $\widetilde{\mathcal{M}}_1$ is a good sub-τ-sheaf of the desired rank. As it is saturated in $\widetilde{\mathcal{M}}$, and its rank is the nondegenerate rank of $\widetilde{\mathcal{M}}$, it is clearly a maximal good sub-τ-sheaf.

c) In order to prove the functoriality, let $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}'$ be τ-sheaves on $\mathcal{C}_R$ with associated maximal good sub-τ-sheaves $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}'$. As the question is local, we may assume that the latter are defined on $\mathcal{C}_K$ as well, and we may just as well suppose that all of the above sheaves are free. Let f denote an analytic homomorphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$, and consider the induced homomorphism

$$\bar{f} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}'' := \widetilde{\mathcal{M}}'/\widetilde{\mathcal{M}}'_1.$$

On the reduction $\overline{\mathcal{M}}_1$ of $\mathcal{M}_1$, the action of τ is injective, whereas, by Remark 1.7, we know that τ acts nilpotently on the reduction $\overline{\mathcal{M}}''$ of $\overline{\mathcal{M}}''$.

For integers s, let us denote the matrix representation of $\tau^s_{\widetilde{\mathcal{M}}_1}$ by Δ_s, that of $\tau^s_{\widetilde{\mathcal{M}}''}$ by Δ''_s, and that of $\bar{f}$ by $\mathcal{F}$, with respect to some fixed bases. We have the equation:

$$\mathcal{F} \cdot \Delta_s = \Delta''_s \cdot \mathcal{F}. \tag{34}$$

For every m, there exists an exponent s such that $\Delta''_s \cong 0 \mod \pi^m$, whereas Δ_s is injective mod π^m; hence $\mathcal{F} \cong 0 \mod \pi^m$, for every m, and so $\bar{f}$ is trivial. This shows that

$$f\left(\widetilde{\mathcal{M}}_1\right) \subset \widetilde{M}_2.$$

□

§2. Proof of Prop. 1.22.

PROOF OF PROP. 1.22. **a)** Let us extend $\widetilde{\mathcal{M}}$ by zero to a free $\mathcal{O}_{\widetilde{\mathbb{A}}^1_R}$-module

$$\widetilde{\mathcal{M}}_e = \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{M}}_0$$

of rank r', endowed with a σ-linear endomorphism τ (cf. Remark 0.4). Let

$$\mathbf{m} := (m_1, \ldots, m_{r'})$$

be a global basis for $\widetilde{\mathcal{M}}_e$, and let

$$\Delta \in \mathrm{Mat}_{r' \times r'}\left(H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)\right)$$

be the matrix representation of τ with respect to this basis, i.e $\tau(\mathbf{m}) = \mathbf{m} \cdot \Delta$. The reduction $\overline{\mathbf{m}}$ of $\mathbf{m}$ yields a basis for $\overline{\mathcal{M}}$.

Next, we choose a basis $\overline{\mathbf{m}}' = (\bar{m}'_i)_{1 \le i \le r_1}$ for $H^0\left(\mathbb{A}^1_k, \overline{\mathcal{M}}_1\right)$ and denote by

$$\overline{\Delta}_1 \in \mathrm{Mat}_{r_1 \times r_1}\left(H^0\left(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}\right)\right)$$

the representation of τ with respect to this basis: $\tau(\overline{\mathbf{m}}') = \overline{\mathbf{m}}' \cdot \overline{\Delta}_1$. We express $\overline{\mathbf{m}}'$ with respect to the basis $\overline{\mathbf{m}}$ of $\widetilde{\mathcal{M}}_e$, by means of a matrix

$$\Psi \in \mathrm{Mat}_{r' \times r_1}\left(H^0\left(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}\right)\right)$$

as follows: $\overline{\mathbf{m}}' = \overline{\mathbf{m}} \cdot \Psi$. Comparing the action of τ with respect to this basis, this yields the equation

$$\Psi \cdot (\overline{\Delta})_1 = \overline{\Delta} \cdot {}^{\sigma}\Psi;$$

here $\overline{\Delta}$ is the reduction of Δ, and the action of σ on Ψ is given by that on the entries of Ψ.

The residue field k being algebraic over $\mathbb{F}_q$, there exists a canonical embedding $k \hookrightarrow R$. Let $\hat{\Delta}_1$ denote the canonical lift via $k \hookrightarrow R$ of $\overline{\Delta}_1$ to an $r_1 \times r_1$-matrix with coefficients in $H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)$, and

$$\hat{\Psi} \in \mathrm{Mat}_{r' \times r_1}\left(H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)\right)$$

the lift of the matrix Ψ.

b) The idea is to construct a basis $\mathbf{n}$ for a sub-τ-sheaf $\widetilde{\mathcal{N}} \subset \widetilde{\mathcal{M}}_e$ on $\widetilde{\mathcal{C}}'_R$ such that τ acts by $\hat{\Delta}_1$ on its basis. If we put $\mathbf{n} := \mathbf{m} \cdot Z$, for a matrix

$$Z \in \mathrm{Mat}_{r' \times r_1}\left(H^0\left(\widetilde{\mathcal{C}}'_R, \mathcal{O}_{\widetilde{\mathcal{C}}'_R}\right)\right), \tag{35}$$

this boils down to finding a Z of full rank r_1, which solves the equation

$$Z \cdot \hat{\Delta}_1 = \Delta \cdot {}^{\sigma}Z. \tag{36}$$

Recall that the reduction $Z \cdot \overline{\Delta_1} \cong \overline{\Delta} \cdot {}^{\sigma}Z$ modulo π of this equation has the solution Ψ.

Let D be the adjoint matrix of $\hat{\Delta}_1$ and put $\delta := \det \overline{\Delta}_1$. We set $\hat{\delta} := \det \hat{\Delta}_1$ and $B := ((\hat{\delta})^{-1}\Delta)$. Recalling that we defined S_1 as the set of closed points of $\mathbb{A}^1$ lying below the set of zeros of δ, the matrix B has entries in the ring $H^0\left(\widetilde{\mathcal{C}}'_R, \mathcal{O}_{\widetilde{\mathcal{C}}'_R}\right)$. Equation (36) is now equivalent to

$$Z = B \cdot {}^{\sigma}Z \cdot D. \tag{37}$$

c) Let $\mathbb{A}^1 \setminus \mathcal{C}' = \{s_1, \ldots, s_\mu\}$, and, for each $i \leq \mu$, let $f_i \in \mathbf{A}$ be a generator for the ideal defining s_i. We recall from equation (32), that, putting

$$\mathcal{R} := R\left\langle\!\left\langle t, t_1, \ldots, t_\mu \right\rangle\!\right\rangle,$$

the ring of entire functions on $\widetilde{\mathcal{C}}'$ is isomorphic to

$$\mathcal{R}/\left(t_1 \cdot f_1(t) - 1, \ldots, t_\mu \cdot f_\mu(t) - 1\right).$$

The endomorphism σ on $\widetilde{\mathcal{C}}'_R$ is induced by the endomorphism σ of $\mathcal{R}$ which acts as the Frobenius φ on R and fixes the indeterminates t_i.

We recall that the reduction $Z \cdot \overline{\Delta_1} \cong \overline{\Delta} \cdot {}^{\sigma}Z$ modulo π of equation (37) has the solution Ψ. We can therefore fix a lift $\mathcal{Z}_0$ with coefficients in $\mathcal{R}$ for the matrix $\hat{\Psi}$, as well as lifts $\mathcal{B}$ and $\mathcal{D}$ for B and D such that the reduced matrices $\overline{\mathcal{Z}}_0$, $\overline{\mathcal{B}_0}$ and $\overline{\mathcal{D}}$ satisfy the equation (37) modulo π. We then set

$$\Phi := \pi^{-1}(\mathcal{B} \cdot {}^{\sigma}\mathcal{Z}_0 \cdot \mathcal{D} - \mathcal{Z}_0) \in \mathrm{Mat}_{r' \times r_1}(\mathcal{R}).$$

The equation (37) lifts to

$$\mathcal{Z} = \mathcal{B} \cdot {}^{\sigma}\mathcal{Z} \cdot \mathcal{D}, \tag{38}$$

for a matrix $\mathcal{Z} \in \mathrm{Mat}_{r' \times r_1}(\mathcal{R})$.

Putting $\mathcal{Z} = \mathcal{Z}_0 + \pi \mathcal{Z}_1$, for some matrix $\mathcal{Z}_1$, we obtain the following equation for $\mathcal{Z}_1$:

$$\mathcal{Z}_1 = \Phi + \pi^{q-1} \mathcal{B} \cdot {}^{\sigma}\mathcal{Z}_1 \cdot \mathcal{D}. \tag{39}$$

We postpone the proof of the following lemma:

LEMMA 1.23. *Equation (39) has a unique solution $\mathcal{Z}_1 \in \mathrm{Mat}_{r' \times r_1}(\mathcal{R})$.*

d) Finally, a solution Z of equation (37) is given by the image of the matrix

$$\mathcal{Z} := \mathcal{Z}_0 + \pi \mathcal{Z}_1$$

under the map

$$\mathcal{R} \to H^0\left(\widetilde{\mathcal{C}}'_R, \mathcal{O}_{\widetilde{\mathcal{C}}'_R}\right).$$

As the reduction Z_0 has full rank, so does Z.

The matrix Z now defines a basis $\mathbf{n} = \mathbf{m} \cdot Z$ for a sub-τ-sheaf $\widetilde{\mathcal{N}} \subset \widetilde{\mathcal{M}}_e$ on $\widetilde{\mathcal{C}}'_R$ such that τ acts as $\hat{\Delta}_1$ on its basis; hence clearly $(\overline{\mathcal{M}})_1 \cong \overline{\mathcal{N}}$. As τ is injective on $\widetilde{\mathcal{N}} \subset \widetilde{\mathcal{M}}_e$, we actually have $\widetilde{\mathcal{N}} \subset \widetilde{\mathcal{M}}$. □

Proof of Lemma 1.23. Put

$$\Phi^{(s)} := (\mathcal{B}\, {}^{\sigma}\mathcal{B} \cdots {}^{\sigma^{s-1}}\mathcal{B})\ {}^{\sigma^s}\Phi\ ({}^{\sigma^{s-1}}\mathcal{D} \cdots {}^{\sigma}\mathcal{D}\, \mathcal{D}).$$

As one easily verifies, the unique formal solution to the equation (39) is

$$\mathcal{Z}_1 := \sum_{s=0}^{\infty} \pi^{q^s - 1} \cdot \Phi^{(s)}. \tag{40}$$

Clearly, this formal sum defines a matrix in $\mathrm{Mat}_{r' \times r_1}(R\langle t, t_1, \ldots, t_\mu \rangle)$. It remains to show that the entries $\mathcal{Z}_1$ are in $\mathcal{R}$.

Every element $\rho \in R\langle\langle t, t_1, \ldots, t_\mu \rangle\rangle$ can be expanded as

$$\rho = \sum_{i=0}^{\infty} \rho_i \pi^i,$$

where the ρ_i are polynomials $\in k[t, t_1, \ldots, t_\mu]$. For every $i \geq 0$, we denote by $\deg \rho_i$ the total degree of $\rho_i \in k[t, t_1, \ldots, t_\mu]$; if $\rho_i = 0$ then $\deg \rho_i := -\infty$. For entire functions, we have the following convergence condition:

$$\rho \in \mathcal{R} \Leftrightarrow \frac{\deg \rho_i}{i} \to 0.$$

Further, we put $\mathcal{B} := \sum \mathcal{B}_i \pi^i$, introducing matrices

$$\mathcal{B}_i = ((\mathcal{B}_i)_{kl}) \in \mathrm{Mat}_{r' \times r'}(k[t, t_1, \ldots, t_\mu]).$$

Let us put $\deg \mathcal{B}_i := \max_{(k,l)} \{\deg(\mathcal{B}_i)_{kl}\}$. We do the same for Φ, $\mathcal{D}$, $\mathcal{Z}_1$ and the $\Phi^{(s)}$, and we set

$$d_i := \max\{\deg \mathcal{B}_i, \deg \mathcal{D}_i, \deg \Phi_i\} \geq 0.$$

Since $\mathcal{B}$, $\mathcal{D}$ and Φ have coefficients in $\mathcal{R}$, we have $d_i / i \to 0$. Fixing an $\epsilon > 0$, there exists a $\kappa(\epsilon) \geq 0$ such that $d_i \leq \epsilon\, i + \kappa(\epsilon)$, for all $i \geq 0$.

Note that ${}^{\sigma}\mathcal{B} = \sum({}^{\sigma}\mathcal{B}_i)\pi^{qi}$ and that $\deg {}^{\sigma}\mathcal{B}_i = \deg \mathcal{B}_i \leq d_i$, for all i. Put

$$(41)\quad J := \Big\{(i_0, \dots i_{s-1};\, j_0;\, k_0, \dots, k_{s-1}) \in \mathbb{N}^{2s+1}; \\ (i_0 + qi_1 + \dots q^{s-1}i_{s-1}) + q^s j_0 + (k_0 + \dots + q^{s-1}k_{s-1}) = n\Big\}.$$

We obtain an estimate

$$\begin{aligned}
&\deg \Phi_n^{(s)} = \deg \left(\sum_J (\mathcal{B}_{i_0}\, {}^{\sigma}\mathcal{B}_{i_1} \cdots {}^{\sigma^{s-1}}\mathcal{B}_{i_{s-1}})\ {}^{\sigma^s}\Phi_{j_0}\ ({}^{\sigma^{s-1}}\mathcal{D}_{k_{s-1}} \cdots {}^{\sigma}\mathcal{D}_{k_1}\, \mathcal{D}_{k_0})\right)\\
&\leq \max_J \left(d_{i_0} + d_{i_1} + \dots + d_{i_{s-1}} + d_{j_0} + d_{k_0} + d_{k_1} + \dots d_{k_{s-1}}\right)\\
&\leq \max\left(\epsilon(i_0 + i_1 + \dots + i_{s-1} + j_0 + i_0 + k_1 + \dots k_{s-1}) + (2s+1)\kappa(\epsilon)\right)\\
&\leq \epsilon n + (2s+1)\kappa(\epsilon).
\end{aligned}$$

Finally, $\deg(\mathcal{Z}_1)_n = \deg\left(\sum_{s=0}^{\infty} \Phi^{(s)}_{n-q^s+1}\right)$. Now

$$\deg \Phi^{(s)}_{n-q^s+1} \leq \epsilon n + \left((2s+3)\kappa(\epsilon) - \epsilon(q^s - 1)\right),$$

where $((2s+3)\kappa(\epsilon) - \epsilon(q^s - 1))$ is a function of s which is bounded by a constant $\tilde{\kappa}(\epsilon)$. Therefore $\deg(\mathcal{Z}_1)_n \leq \epsilon n + \tilde{\kappa}(\epsilon)$, which shows that

$$\limsup \frac{\deg(\mathcal{Z}_1)_n}{n} \leq \epsilon.$$

As this holds for all $\epsilon > 0$, it follows that $\deg(\mathcal{Z}_1)_n/n \to 0$, for all $n \geq 0$. Hence the matrix $\mathcal{Z}_1$ has entire coefficients indeed. $\square$

REMARK 1.24. The entries of the matrix Z (cf. 35) which expresses a basis for the τ-sheaf $\widetilde{\mathcal{N}}$ in terms of a basis for $\widetilde{\mathcal{M}}_e$, are holomorphic functions on $\mathcal{C}'_K$ which may have essential singularities at the points of S_1. Indeed, by equation (40), they are entire functions in the variables t, $f_1(t)^{-1}, \dots, f_\mu(t)^{-1}$. In general, one has to allow such 'singularities' to obtain a nice analytic structure for an analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$: see Prop. 2.11 (in view of Def. 1.27).

What can be proved if we do not assume that the residue field k of R is algebraic over $\mathbb{F}_q$. For a nondegenerate analytic τ-sheaf $\widetilde{\mathcal{M}}$ on $\widetilde{\mathbb{A}}^1_R$, suppose that the support of the cokernel of τ on $\overline{\mathcal{M}}_1$ contains a point $t = s$, where we choose some $s \in k$ which is transcendental over $\mathbb{F}_q$. Note that equation (37) has now a factor $(t - s)$ in the denominator. Therefore, if we solve this equation formally (cf. (40)), we obtain a function with poles at the infinite set of points s^{q^i}. But, in the sense of rigid analysis, this defines an analytic function on some small open disks, but not on a 'larger' global rigid analytic space.

Denoting, as before, the completion of the local ring of regular functions $\mathcal{O}_\varpi$ at the generic point ϖ of the special fibre $\mathcal{C}_k$ of $\mathcal{C}_R$ by $\hat{\mathcal{O}}_\varpi$, equation (40) clearly defines an element in $\hat{\mathcal{O}}_\varpi$. Therefore, using similar arguments as in Thm. 1.21, we can deduce (k not necessarily algebraic over $\mathbb{F}_q$):

PROPOSITION 1.25. *For an analytic τ-sheaf $\widetilde{\mathcal{M}}$ on $\widetilde{\mathcal{C}}_R$ with nondegenerate rank r_1, let $\hat{\mathcal{M}}_\varpi$ denote the completion of the stalk $\mathcal{M}_\varpi$ of $\widetilde{\mathcal{M}}$ at ϖ. There exists a unique maximal good sub-τ-module $\hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}}_\varpi$ of rank r_1 and $\hat{\mathcal{M}}_1$ is functorial with respect to $\hat{\mathcal{O}}_\varpi$-linear homomorphisms.*

§3. Analytic semistability theorem.

THEOREM 1.26 (**Analytic semistability theorem**). *Let R be a complete discrete valuation $\mathbb{F}_q$-algebra, K its field of fractions and suppose the residue field k of R is algebraic over $\mathbb{F}_q$. For every analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathcal{C}}_K$, there exists*

- *a nonempty open subscheme $\mathcal{C}' \subset \mathcal{C}$,*
- *a finite separable extension R' of R, with fraction field K', and*
- *a filtration*

$$0 = \widetilde{N}_0 \subset \widetilde{N}_1 \subset \cdots \subset \widetilde{N}_n = \widetilde{M}|_{\widetilde{\mathcal{C}}'_{K'}} \tag{42}$$

of the pullback $\widetilde{M}|_{\widetilde{\mathcal{C}}'_{K'}}$ of $\widetilde{M}$ to $\widetilde{\mathcal{C}}'_{K'}$ by saturated analytic sub-τ-sheaves on $\widetilde{\mathcal{C}}'_{K'}$

such that the subquotients $\widetilde{M}_i := \widetilde{N}_i/\widetilde{N}_{i-1}$ have good models over $\widetilde{\mathcal{C}}'_{R'}$.

PROOF. By Lemma 1.16 and an analytic analogue of Lemma 1.10, there exists a finite extension R_1 of R, with fraction field K_1, such that the maximal model $\widetilde{\mathcal{M}}^{\max}$ for $\widetilde{M}_{K_1}$ is nondegenerate at the closed point of Spec R_1. By Thm. 1.21 this yields a saturated good sub-τ-sheaf $\mathcal{M}_1$ defined over $\widetilde{\mathcal{C}}^1_K$, for a nonempty open subscheme $\mathcal{C}^1$ of $\mathcal{C}$. Let $\widetilde{M}_1$ denote its generic fibre on $\widetilde{\mathcal{C}}^1_K$ and put

$$\widetilde{M}' := \widetilde{M}/\widetilde{M}_1,$$

which is a τ-sheaf on $\mathcal{C}^1_K$.

By induction on the rank, we obtain, for a nonempty open subscheme

$$\mathcal{C}' \subset \mathcal{C}^1$$

and a finite extension R' of R_1 with fraction field K', a filtration

$$0 = \widetilde{N}_1 \subset \widetilde{N}_2 \subset \cdots \subset \widetilde{N}_n = \widetilde{M}'|_{\widetilde{\mathcal{C}}'_{K'}}.$$

□

DEFINITION 1.27. For an analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathcal{C}}_K$, we call a filtration as in Thm. 1.26 a **semistable filtration for** $\widetilde{M}$. We say that $\widetilde{M}$ is **semistable** if there exists semistable filtration $\{\widetilde{N}_i\}$ which is already defined on $\widetilde{\mathcal{C}}_K$.

As a first example, Drinfeld modules are potentially semistable:

PROPOSITION 1.28 (**Analytic structure of Drinfeld modules**). *Let ϕ be a Drinfeld* **A***-module defined over K and M its t-motive, a τ-sheaf on $\mathbb{A}^1_K$. The associated analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$ is potentially semistable.*

PROOF. First, suppose that the valuation of K is infinite with respect to the characteristic ι. As every Drinfeld module is uniformizable, it will follow from Thm. 5.13 that $\widetilde{M}$ potentially has a good model.

For finite valuations, it is shown in Thm. 6.22, that, over a finite field extension K' of K, there exists an exact sequence

$$0 \to \widetilde{N} \to \widetilde{M} \to \widetilde{M}' \to 0$$

of analytic τ-sheaves on $\mathbb{A}^1_{K'}$. In this sequence N is a trivial τ-sheaf (hence potentially good), and $\widetilde{M}'$ is the good reduction t-motive associated to a Drinfeld module ϕ' with a model of good reduction. $\square$

CHAPTER 2

Local Galois representations

Let $\mathcal{C}$ be an absolutely irreducible affine smooth curve with constant field $\mathbb{F}_q$ and function field F. Consider a *complete* discrete valuation ring R whose fraction field K contains F, with residue field k. We denote by π a uniformizer of R and by x the closed point of Spec R, which we will identify with the associated place of K.

The embedding $F \hookrightarrow K$ defines a generic characteristic map

$$\iota : \operatorname{Spec} K \to \mathcal{C}$$

(cf. Def. 0.5), and we put $\ell_x := \iota(x)$, the residual characteristic point. Finally, let Γ_K be the absolute Galois group of K and I_K its inertia subgroup.

Let M be a τ-sheaf over $\mathcal{C}_K$ with characteristic ι. We denote its maximal model over $\mathcal{C}_R$ by $\mathcal{M} := \mathcal{M}^{\max}$ and the reduction of $\mathcal{M}$ at x by $\overline{\mathcal{M}}$. In this chapter, we establish some properties of the Galois modules $V_\ell(M)$ and $W_\ell(M)$ (Def. 0.8) concerning the action of the inertia subgroups:

- analytic *semistability* (Thm. 1.26) implies unipotent action of inertia on $V_\ell(M)$ on all but a finite number of closed points ℓ of $\mathcal{C}$;
- a description of the characters of *tame inertia* acting on $W_\ell(M)$ at $\ell = \ell_x$, for a τ-sheaf M with a good model at x;
- a description of the *image of inertia* on $V_\ell(M)$, for all ℓ, if the τ-sheaf has characteristic ι, dimension 1 and posesses a good model at x.

I. Semistability of Galois representations

If $\mathcal{M}$ is good at x then, for all but a finite number of closed points ℓ of $\mathcal{C}$, the homomorphism $\tau : \sigma^*\overline{\mathcal{M}} \to \overline{\mathcal{M}}$ is an isomorphism locally at the closed point $\xi := \ell \times \operatorname{Spec} k$ of $\mathcal{C}_k$. If τ is an isomorphism locally at ξ, then $\tau : \sigma^*\hat{\mathcal{M}}_\ell \to \hat{\mathcal{M}}_\ell$ is smooth, and so $T_\ell(M)$ is unramified, by Prop. 0.7.

In conclusion, this shows

LEMMA 2.1. *Let R be a discrete valuation ring. If M has a good model on $\mathcal{C}_R$, then $V_\ell(M)$ is unramified for all but a finite number of closed points ℓ of $\mathcal{C}$.*

REMARK 2.2. There exists a converse to this statement, which is stated in Theorem 4.1.

REMARK 2.3. We immediately obtain an analogous statement for analytic τ-sheaves $\widetilde{M}$: If $\widetilde{M}$ has a good model on $\mathcal{C}_R$, then the Tate module $V_\ell(\widetilde{M})$ is unramified for all but a finite number of closed points ℓ of $\mathcal{C}$.

No longer assuming that M has a good model at x, we now prove that inertia acts (almost everywhere) potentially unipotently on the Tate-modules $V_\ell(M)$:

THEOREM 2.4 (**Semistability of Galois representations**). *Suppose that the residue field k of the discrete valuation ring R is algebraic over $\mathbb{F}_q$. For any τ-sheaf M on $\mathcal{C}_K$, there exists a finite separable extension K' of K such that the action of the inertia group $I_{K'}$ on $V_\ell(M)$ is unipotent, for all but a finite number of closed points ℓ of $\mathcal{C}$.*

PROOF. We consider the analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathcal{C}}_K$ associated to M. By the Semistability Theorem 1.26 we obtain an extension K' a filtration of $\widetilde{M}$ with good subquotients which is defined on $\widetilde{\mathcal{C}}_{K'}$, except at a finite number of closed points of $\mathcal{C}$. This proves the theorem, by 2.3. □

REMARK 2.5. Clearly, if the residue field k of R is algebraic over $\mathbb{F}_q$, we also have: For any analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathcal{C}}_K$, there exists a finite extension K' of K such that the action of the inertia group $I_{K'}$ on $V_\ell(\widetilde{M})$ is unipotent, for all but a finite number of closed points ℓ of $\mathcal{C}$.

REMARK 2.6. We compare our result to the classical theory of local Galois representations on $\mathbb{Q}_p$-vector spaces V. If $p \neq p'$, the residual characteristic of the local field K, then inertia I_K acts potentially unipotently on V. The proof of this statement is remarkably elementary (cf. [**SGA7**], Exposé I). However, given an abelian variety with its associated Galois representations, the fact that one can find a finite extension of the base field such that inertia acts unipotently on *all* Tate modules, is deeper (cf. [**SGA7**], Exposé VIII); for motives, it is a consequence of de Jong's theory of alterations of algebraic varieties (cf. [**Ber**], Prop. 6.3.2).

REMARK 2.7. *Semistable τ-sheaves with characteristic.* Let $\widetilde{M}$ be an analytic τ-sheaf on $\widetilde{\mathcal{C}}_K$ with characteristic ι. As the characteristic is generic, the Tate module $T_\ell(\widetilde{M})$ is well defined for all closed points ℓ of $\mathcal{C}$. If M has a good model $\widetilde{\mathcal{M}}$ over $\mathcal{C}_R$, then I_K acts trivial on $V_\ell(\widetilde{M})$ for all ℓ, except for the residual characteristic point ℓ_x, if it exists.

If $\widetilde{M}$ is semistable, then each of the subquotients $\widetilde{M}_i$ in the semistable filtration $\{\widetilde{N}_i\}$ for $\widetilde{M}$ has characteristic ι or is smooth. Hence, inertia I_K potentially acts unipotently on $V_\ell(M)$ for $\ell \neq \ell_x$. This shows that in particular semistable τ-sheaves with characteristic mirror the classical semistability of the Tate modules. This remark will now allow us to spot a non semistable τ-sheaf.

REMARK 2.8. It may be instructive to discuss Thm. 2.4 in the special case of Drinfeld modules. For a Drinfeld module ϕ defined over K with characteristic $\iota^* : \mathbf{A} \hookrightarrow K$, the inertia group I_K acts unipotently on $V_\ell(M)$ for all closed points ℓ of $\mathcal{C}\setminus\{\ell_x\}$, by the theory of uniformization. Indeed, upon replacing K by a finite extension, we may assume that if x is a finite place, then ϕ has stable reduction at x, i.e. there exists a model Φ for ϕ over R whose reduction is a Drinfeld module over k, possibly of lower rank (cf. [**Tag2**]). We then have the following results:

- If Φ has good reduction at x then, for every ideal $\mathcal{I}$ of $\mathbf{A}$, the scheme of $\mathbf{A}$-modules $\Phi[\mathcal{I}]$ is finite flat. For the maximal ideal λ corresponding to ℓ, the inductive system $\{\phi[\lambda^i]\}_{1\leq i}$ forms an λ-divisible scheme of $\mathbf{A}$ modules (see [**Tag2**], §1). The connected-étale decomposition yields an exact sequence

$$0 \to \Phi[\lambda^i]^0 \to \Phi[\lambda^i] \to \Phi[\lambda^i]^{\mathrm{et}} \to 0$$

where $\{\Phi[\lambda^i]^0\}$ (resp. $\{\Phi[\lambda^i]^{\mathrm{et}}\}$) is a connected (resp. étale) λ-divisible group. If $\ell \neq \ell_x$, then $\{\Phi[\lambda^i]\}$ is étale, which shows that the Galois module $V_\ell(\phi)$ is unramified.

- If Φ has Tate reduction over K_x, then

  PROPOSITION 2.9 (Drinfeld [**Dr1**], §7). *There exists a Drinfeld module Φ' of rank $\bar{r}$ defined over R_x and with good reduction at x and a K_x-analytic epimorphism $e_x : \Phi' \to \Phi$ of Drinfeld $\mathbf{A}$-modules. Its kernel $H_x \subset K_x^{\mathrm{sep}}$ is a strictly discrete projective A-module of rank $r - \bar{r}$.*

  This yields, for every nonzero ideal $\mathcal{I}$ of $\mathbf{A}$, a decomposition

  $$0 \to \Phi'[\mathcal{I}] \to \Phi[\mathcal{I}] \to H_x/\mathcal{I} \to 0.$$

  The lattice H_x being strictly discrete, the orbit under Γ_K of any basis is finite; hence the action of (Γ_x) on H_x is finite as well. Passing to the limit, this shows that, potentially, inertia acts unipotently on $V_\ell(\phi)$ if $\ell \neq \iota(x)$.
- For all infinite places x of K, we have

  PROPOSITION 2.10 (Drinfeld [**Dr1**], §3). *There exists a K_x-analytic epimorphism $e_x : \mathbb{G}_a \to \phi$ of A-modules. Its kernel $H_x \subset K_x^{\mathrm{sep}}$ is a strictly discrete projective $\mathbf{A}$-module of rank r.*

  We obtain a Γ_x-invariant isomorphism $\phi[\mathcal{I}] \cong H_x/\mathcal{I}$, where H_x, as above, has a finite Galois action. Hence, potentially, the action of I_x on $V_\ell(\phi)$ is trivial for all ℓ.

II. Example: (not) semistable τ-sheaves

We put $\mathcal{C} := \mathbb{A}^1$ and consider the discrete valuation ring $R := \mathbb{F}_q[[\pi]]$, denoting by K its field of fractions. We define a characteristic map Spec $K \to \mathcal{C}$ by

$$\iota^* : \mathbf{A} \cong \mathbb{F}_q[t] \to K : t \mapsto \theta,$$

for some $\theta \in K$. For every pair $(\alpha, \gamma) \in K^\times \times K$, we define the τ-sheaf $M_\alpha(\gamma)$ on $\mathbb{A}^1_K$ as follows: The underlying sheaf is $\mathcal{O}_{\mathbb{A}^1_K} \oplus \mathcal{O}_{\mathbb{A}^1_K}$, with global basis

$$\mathbf{m} = (m_1, m_2),$$

and we set τ to be given by the matrix representation

$$\tau(\mathbf{m}) = \mathbf{m} \cdot \begin{pmatrix} 0 & \alpha(t-\theta) \\ \alpha(t-\theta) & \gamma t \end{pmatrix}. \tag{43}$$

This $M_\alpha(\gamma)$ is a pure t-motive of rank 2, dimension 2 and weight 1. If we take (m_1, m_2) to be the coordinate functions on $\mathbb{G}_{a,K}^{\oplus 2}$, then the t-module $E_\alpha(\gamma)$ is given by:

$$(t_E - \theta)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \alpha^{-2}\begin{pmatrix} -\gamma\tau & \alpha(\tau - \theta\gamma)\tau \\ \alpha\tau & 0 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}.$$

Let us now put $\theta := \pi + 1 \in R$. Then the valuation v_π is finite with respect to the characteristic ι and the residual characteristic ℓ_x is defined by the ideal $(t-1)$.

PROPOSITION 2.11. *The (analytic τ-sheaf associated with the) t-motive $M_1(\gamma)$ is potentially semistable on $\mathbb{A}^1_K$ if and only if $v(\gamma) \geq 0$.*

PROOF. If $v(\gamma) \geq 0$, the $\mathcal{O}_{\mathbb{A}^1_R}$-module $\widetilde{\mathcal{M}}$ generated by $\mathbf{m}$ is a good model for $M_1(\gamma)$. A fortiori, the $M_1(\gamma)$ is semistable.

Suppose that $v(\gamma) < 0$. Let γ' be a $(q-1)$-st root of γ, and $R' := R[\gamma']$. We remark that the τ-module $\widetilde{\mathcal{M}}$ with basis $\mathbf{m}' := \gamma' \cdot \mathbf{m}$ over $\mathcal{O}_{\mathbb{A}^1_{R'}}$ defines a nondegenerate model for $M := M_1(\gamma)$. Indeed, the representation of τ with respect to $\mathbf{m}'$ is then given by:

$$\tau(\mathbf{m}') = \mathbf{m}' \cdot \begin{pmatrix} 0 & \gamma'(t-\theta) \\ \gamma'(t-\theta) & t \end{pmatrix}.$$

The maximal τ-sheaf $(\overline{\mathcal{M}})_1$ in $\overline{\mathcal{M}}$ is given by $\mathcal{O}_{\mathbb{A}^1} \cdot \bar{m}'_2$, and the action of τ by

$$\tau(\bar{m}'_2) = t \cdot \bar{m}'_2.$$

If we let ℓ_0 be the closed point of $\mathbb{A}^1$ defined by the ideal (t), then by Thm. 1.22, the filtration $(\overline{\mathcal{M}})_1 \subset \overline{\mathcal{M}}$ lifts to a filtration $\mathcal{N}_1 \subset \widetilde{\mathcal{M}}$, where $\mathcal{N}_1$ is a good τ-sheaf defined on $(\mathbb{A}^1 \setminus \{\ell_0\}) \times \operatorname{Spec} K$. Since the quotient $\mathcal{N}_1$ either has characteristic ι or is smooth, the action of inertia on the Tate modules $T_\ell(M)$ is unipotent for all

$$\ell \in \mathbb{A}^1 \setminus \{\ell_x, \ell_0\}$$

(cf. Prop. 1.22 and Remark 2.6). In the following lemma, we will prove that the action of inertia on $T_{\ell_0}(M_1(\gamma))$ is *not* potentially unipotent. As $\ell_0 \neq \ell_x$, we then obtain, by Remark 2.6, that $M_1(\gamma)$ is not semistable. □

LEMMA 2.12. *If $v(\gamma) \leq 0$, then the action of inertia on $T_{\ell_0}(M_1(\gamma))$ is not potentially unipotent.*

PROOF. Put $M := M_1(\gamma)$. By the definition of the functor T_ℓ, we have

$$T_{\ell_0}(M) = \left\{ \mathbf{v} = (v, w) \in \mathrm{Mat}_{1\times 2}(K^{\mathrm{sep}}[[t]]);\ {}^{\sigma}\mathbf{v} = \mathbf{v} \cdot \begin{pmatrix} 0 & t-\theta \\ t-\theta & \gamma t \end{pmatrix} \right\} \tag{44}$$

Expanding v as a power series $\sum_{i=0}^{\infty} v_i t^i \in K^{\mathrm{sep}}[[t]]$, and doing the same for w, we obtain the system of equations

$$\begin{cases} v_i^q + \theta w_i &= w_{i-1} \\ w_i^q + \theta v_i &= v_{i-1} + \gamma w_{i-1}. \end{cases}$$

By substitution, we deduce the following recursion for $w_i \in K^{\mathrm{sep}}$:

$$w_i^{q^2} - \theta^{q+1} w_i = \gamma^q w_{i-1}^q - (\theta^q + \theta) w_{i-1} + w_{i-2}.$$

By induction, it follows that if $w_0 \neq 0$, then $v_\pi(w_i) = \frac{q^i - 1}{q^i(q-1)} v(\gamma)$. This implies that no nonzero solution $\mathbf{v}$ of (0.8) is defined over a finitely ramified extension of K. Hence, there can exist no finite extension K' and no sub-$\mathbb{F}_q[[t]][\Gamma'_K]$-module

$$T \subset T_{\ell_0}(M)$$

such that $I_{K'}$ acts trivially on T_1. In other words, inertia does not act potentially unipotently on $T_{\ell_0}(M)$. $\square$

The fact that we have an example of a pure t-motive which is non potentially semistable, i.e. which has not even potentially a global nondegenerate structure as a τ-sheaf over $\mathbb{A}^1_K$, shows that it is impossible to find a global nondegenerate model for the t-module E, thus leaving little hope for a general reduction theory of t-modules (cf. Remark 1.11).

III. Action of tame inertia

§1. Fundamental characters. Let M be a τ-sheaf on $\mathcal{C}_K$ with characteristic ι and dimension d. We denote by $\bar{\kappa}_\ell$ an algebraic closure of the residue field κ_ℓ of ℓ. Let $W_\ell^{x,\mathrm{ss}}$ be the semisimplification of the $\kappa_\ell[\Gamma_K]$-module $W_\ell(M)$. We know that the action of I_K on $W_\ell^{x,\mathrm{ss}}$ factors through its quotient I_K^t of tame inertia (cf. [**Se3**], §1); hence the representation of

$$\bar{\rho}_\ell : I_K \to \bar{\kappa}_\ell \otimes W_\ell^{x,\mathrm{ss}}$$

is given by a direct sum of characters $\delta_i : I_K \to \bar{\kappa}_\ell^\times$. We want to give a more detailed qualitative description of the characters δ_i of $W_\ell^{x,\mathrm{ss}}$ in case M has a good model at x.

Let κ be a finite extension of $\mathbb{F}_q$ of order q' and degree $[\kappa : \mathbb{F}_q]$. For an integer ρ, let $\kappa^{[\rho]}$ be the extension of κ of degree ρ, inside a fixed algebraic closure $\bar{\kappa}$ of κ.

Definition 2.13 (Serre). Taking a solution π_ρ of $X^{(q')^\rho} - \pi X = 0$, one defines a tame character

$$\zeta_{\kappa,\rho} : I_K \to \left(\kappa^{[\rho]}\right)^\times \subset \bar{\kappa}^\times : \sigma \mapsto \sigma(\pi_\rho)/\pi_\rho.$$

This character, together with its $\mathrm{Gal}\left(\kappa^{[\rho]}/\kappa\right)$-conjugates

$$\zeta_{\kappa,\rho}^{(q')^i}$$

for $i < [\kappa : \mathbb{F}_q]$, is called the **fundamental character of level** ρ for κ (cf. [**Se3**], §1) (this set of fundamental characters is independent of the choice for π_ρ).

Any tame character $I_K \to \left(\kappa^{[\rho]}\right)^\times \subset \bar{\kappa}^\times$ is a power of $\zeta_{\kappa,\rho}$. Let e_ℓ denote the ramification index of a closed point ℓ of $\mathcal{C}$ for the morphism $\iota : \mathrm{Spec}\, R \to \bar{\mathcal{C}}$.

Theorem 2.14. *Let M be a τ-sheaf on $\widetilde{\mathcal{C}}_K$ with characteristic ι and dimension d. If M has a good model at x and ℓ is the closed point of $\mathcal{C}$ lying below x, then the representation $I_K \to \mathrm{Aut}_{\bar{\kappa}_\ell}\left(\bar{\kappa}_\ell \otimes W_\ell^{x,\mathrm{ss}}\right)$ is isomorphic to the direct sum of products of fundamental characters of level at most r with at most $e_\ell \cdot d$ factors.*

The proof will be given in section III.§3.

§2. Simple ramified τ-modules. Denote by R^{ur} the maximal unramified extension of R, by K^{ur} its field of fractions, whose absolute Galois group $\Gamma_{K^{\mathrm{ur}}} = I_K$. Let κ be a finite extension of $\mathbb{F}_q$ of order q', and φ' the Frobenius morphism

$$x \mapsto x^{q'}.$$

By a τ'-module over a κ-algebra B, we will mean a τ-module over B with respect to the finite field κ, i.e. a projective B-module N, endowed with a map

$$\tau' : (\varphi')^* N \to N.$$

There is an antiequivalence between the category of finite $\kappa[I_K]$-modules W and τ'-modules (N, τ') over K^{ur} (cf. (6)). Our aim is to make this antiequivalence more concrete. The ideas below were inspired by the explicit theory of filtered modules which can be found in [**FL**], §6.

Let us be given a rank $\rho \geq 0$ and a map

$$h : \mathbb{Z}/\rho\mathbb{Z} \to \{0, \dots, q'-1\},$$

which is not the constant map with value $q'-1$. We define a simple free τ'-module $\mathcal{N}_\rho(h)$ over R^{ur} as follows: With respect to a basis $\{n_j\}_{j \in \mathbb{Z}/\rho\mathbb{Z}}$, the morphism

$$\tau' : (\sigma')^* \mathcal{N}_\rho(h) \to \mathcal{N}_\rho(h)$$

is given by:

$$\tau'(n_j) := \pi^{h(j)} \cdot n_{j+1}. \tag{45}$$

Put $N_\rho(h) := K^{\mathrm{ur}} \otimes \mathcal{N}_\rho(h)$ and let $W_\rho(h) := T(N_\rho(h))$ be the associated $\kappa[I_K]$-module. This module $W_\rho(h)$ is given by the solutions

$$\mathbf{x} = (x_1, \dots, x_r) \in (K^{\mathrm{sep}})^{\oplus \rho}$$

of the system of equations

$$x_j^{q'} = \pi^{h(j)} \cdot x_{j+1}.$$

The 1-dimensional $\kappa^{[\rho]}$-vector space $W_\rho(h)$ is generated by any nonzero element

$$(x_i)_{i \in \mathbb{Z}/\rho\mathbb{Z}} \in (K^{\mathrm{sep}})^\rho$$

such that x_i is a root of $X^{(q')^\rho} - \pi^{\nu(i)} X = 0$ and

$$x_i = \pi^{h(i-1)} x_{i-1}^{(q')}.$$

Putting

$$\nu = h(0)\,(q')^{\rho-1} + h(1)\,(q')^{\rho-2} + \cdots + h(\rho-1),$$

this shows that inertia I_K acts on it through a character $\zeta_{\kappa,\rho}^{\nu}$, where $\zeta_{\kappa,\rho}$ is a fundamental character of level ρ.

LEMMA 2.15. *Any simple smooth τ'-module N of rank ρ over K^{ur} is isomorphic to $N_\rho(h)$ for some h.*

PROOF. For any map h, the τ-modules $N_\rho(h)$ and N are isomorphic, if and only if the $\kappa[I_K]$-modules $W_\rho(h)$ and W are isomorphic. The action of I_K on W being simple and hence tame, its representation on $\bar{\kappa}^\times \otimes W$ is isomorphic to the direct sum of a character

$$\delta : I_K \to \left(\kappa^{[\rho]}\right)^\times \subset \bar{\kappa}^\times$$

and its $\mathrm{Gal}\left(\kappa^{[\rho]}/\kappa\right)$-conjugates. We have $\delta = \zeta_{\kappa,\rho}^\nu$ for some $0 \le \nu < (q')^\rho - 1$. Writing down a q-expansion

$$\nu = \nu_0\,(q')^{\rho-1} + \nu_1\,(q')^{\rho-2} + \cdots + \nu_{\rho-1},$$

with each ν_i contained in $\{0, \ldots, q'-1\}$, we then define h by $j \mapsto \nu_j$ and consider the τ-module $N_\rho(h)$. As we saw before, the character δ is then a direct summand of the representation of I_K on $W_\rho(h)$, which shows that $W \cong W_\rho(h)$ as $\kappa[I_K]$-modules. $\square$

Remark that these calculations also yield that two τ-modules $N_\rho(h)$ and $N_\rho(h')$ are isomorphic if and only if h is a translate of h'. For any $i \in \mathbb{Z}/\rho\mathbb{Z}$, set

$$\nu(i) := \sum_{j\in[0,\rho-1]} (q')^{\rho-1-j}\, h(i+j)$$

By our assumptions on h, we have $\nu(i) \le (q')^\rho - 1$. Note that $\nu(0) = \nu$ and that

$$\nu(i+1) \equiv q'\,\nu(i) \mod (q')^\rho - 1.$$

LEMMA 2.16. *The model $\mathcal{N}_\rho(h)$ for $N_\rho(h)$ over R^{ur} is maximal.*

PROOF. It suffices to show that the iterate $(\mathcal{N}_\rho(h), (\tau')^\rho)$ of $(\mathcal{N}_\rho(h), \tau')$, with

$$(\tau')^\rho : (\varphi'^*)^\rho \mathcal{N}_\rho(h) \to \mathcal{N}_\rho(h),$$

is a maximal $(\tau')^\rho$-module. It is a direct sum of rank 1 sub-τ^ρ-modules $\mathcal{N}_i$ of the form

$$\tau^\rho(n_i) = \pi^{\nu(i)} \cdot n_i.$$

Let us denote by $\mathcal{N}_i$ (resp. N_i) the τ^ρ-module over R^{ur} (resp. K^{ur}) generated by

$$n_1, \ldots, n_i.$$

As $\nu(1) < q'^\rho - 1$, the τ^ρ-module $\mathcal{N}_1$ is a maximal model for N_1. By induction, we similarly prove that the $\mathcal{N}_i$ are saturated in the maximal model $\mathcal{N}^{\max}$ for M over R^{ur}. Hence $(\mathcal{N}_\rho(h), \tau^\rho)$ is maximal. $\square$

§3. Representations of tame inertia.

PROPOSITION 2.17. *Let N be a τ'-module of rank r over K with a maximal model $\mathcal{N}$ over R such that* $\mathrm{length}_R(\mathrm{coker}\,\tau'_{\mathcal{N}}) = m$. *If ℓ is the closed point of $\mathfrak{C}$ lying below x, then the representation $I_K \to \mathrm{Aut}_{\bar{\kappa}}(\bar{\kappa} \otimes W^{x,\mathrm{ss}})$ is isomorphic to the direct sum of products of fundamental characters of level at most r with at most m factors.*

PROOF. Take a free summand $\mathcal{N}_i$, of rank ρ, in the semisimplification

$$\mathcal{N}^{x,\mathrm{ss}} = \bigoplus \mathcal{N}_i$$

of $\mathcal{N}$; then length(coker $\tau'_{\mathcal{N}_i}$) $\leq m$. The maximal model of $\mathcal{N}_i$ is isomorphic, over R^{ur}, to some $\mathcal{N}_\rho(h)$, by Lemma 2.15 and 2.16. Thus

$$\sum_{j \in \mathbb{Z}/\rho\mathbb{Z}} h(j) \leq \mathrm{length}_R \left(\mathrm{coker}\ \tau'_{\mathcal{N}_\rho(h)}\right) \leq m.$$

This shows that the representation of I_K on the κ-vector space W_* associated to $\mathcal{N}_i$ is isomorphic to the direct sum of products of fundamental characters of level at most r with at most m factors. $\square$

Let M be a τ-sheaf on $\widetilde{\mathcal{C}}_K$ with characteristic ι and dimension d.

PROOF OF THM. 2.14. For places x of K at which M has a good model, the $\kappa_\ell[I_K]$-module $W_\ell(M)$ is determined by the reduction $\overline{M}_\ell$ of M at the point Spec $K^{\mathrm{ur}} \times_{\mathbb{F}_q} \ell$, or, equivalently, by the τ'-module $(\overline{\mathcal{M}}_\ell, \tau')$ over K^{ur}, with

$$\tau' := \tau^{[\kappa_\ell : \mathbb{F}_q]}.$$

As M has characteristic ι and dimension d, it follows that

$$\mathrm{length}_R \left(\mathrm{coker}\ \tau'\right) = e_\ell \cdot d.$$

Proposition 2.17 now concludes the proof. $\square$

IV. Image of the action of inertia

Let M be a τ-sheaf on $\mathcal{C}_K$ with characteristic ι and **dimension** 1, which possesses a good model $\mathcal{M}$ over $\mathcal{C}_R$. By Remark 2.7, inertia acts trivially on the Tate modules $T_\ell(M)$ for $\ell \neq \ell_x := \iota(x)$. In this section, we give a qualitative description of the image of inertia I_K in Aut$(T_\ell(M))$ for $\ell = \ell_x$ (cf. Cor. 2.24).

§1. Connected ℓ-adic τ-sheaves. Consider a closed point ξ of $\mathcal{C}_R$. Let $\hat{\mathcal{O}}_\xi$ denote the completion of the local ring $\mathcal{O}_{\mathcal{C}_R,\xi}$ of regular functions at ξ with respect to its maximal ideal $\mathfrak{m}_\xi$.

DEFINITION 2.18. We call an ℓ-adic τ-sheaf $\hat{\mathcal{M}}_\ell$ on $\hat{\mathcal{C}}_{R,\ell}$ **connected** if its reduction $\overline{\mathcal{M}}_\xi$ at ξ is nilpotent, i.e. $\tau^n : (\sigma^*)^n \overline{\mathcal{M}}_\xi \to \overline{\mathcal{M}}_\xi$ is the zero morphism for some $n > 0$.

LEMMA 2.19. *For every ℓ-adic τ-module $\hat{\mathcal{M}}_\ell$ on $\hat{\mathcal{C}}_{R,\ell}$, there exists an exact sequence*

$$0 \to \hat{\mathcal{M}}_\ell^{\mathrm{et}} \to \hat{\mathcal{M}}_\ell \to \hat{\mathcal{M}}_\ell^0 \to 0$$

of ℓ-adic τ-sheaves over $\hat{\mathcal{C}}_\ell$, where $\hat{\mathcal{M}}_\ell^{\mathrm{et}}$ is smooth and $\hat{\mathcal{M}}_\ell^0$ is connected.

PROOF. The τ-sheaf $\overline{\mathcal{M}}_\xi$ over ξ contains a τ-module $\overline{\mathcal{M}}_1$ (of rank r_1) as a direct summand, such that τ acts nilpotently on

$$\overline{\mathcal{M}}_{\mathrm{nil}} := \overline{\mathcal{M}}_\xi / \overline{\mathcal{M}}_1.$$

We choose a basis $\overline{\mathbf{m}}' := (\bar{m}_i)_{1 \le i \le r_1}$ for $\overline{\mathcal{M}}$, which we then extend to a basis

$$\overline{\mathbf{m}} := (\bar{m}_1, \dots, \bar{m}_r)$$

for $\overline{\mathcal{M}}_\xi$. Any lift $\mathbf{m} = (m_i)$ of $\overline{\mathbf{m}}$ to $\hat{\mathcal{M}}_\ell$ yields a $\hat{\mathcal{O}}_\xi$-basis of the latter. Put

$$\mathbf{m}' := (m_1, \dots, m_{r_1}).$$

Let

$$\overline{\Delta}_1 \in \mathrm{Mat}_{r_1 \times r_1}(k)$$

(resp. $\overline{\Delta} \in \mathrm{Mat}_{r \times r}(\hat{\mathcal{O}}_\xi)$) be the matrix representation of τ with respect to the basis $\overline{\mathbf{m}}_1$ (resp. $\mathbf{m}$), and consider any lift $\hat{\Delta}_1$ of $\overline{\Delta}_1$ with coefficients in $\hat{\mathcal{O}}_\xi$.

We want to construct a basis $\mathbf{n}$ for a sub-τ-module $\hat{\mathcal{M}}_\ell^{\mathrm{et}}$ of rank r_1 of $\hat{\mathcal{M}}_\ell$ such that τ operates as $\hat{\Delta}_1$ with respect to $\mathbf{n}$. Let us express $\mathbf{n}$ in terms of $\mathbf{m}$ as $\mathbf{n} = \mathbf{m} \cdot Z$, for some matrix $Z \in \mathrm{Mat}_{r \times r_1}(\hat{\mathcal{O}}_\xi)$. Comparing the action of τ with respect to $\mathbf{n}$ and $\mathbf{m}$, we obtain the equation

$$Z \cdot \hat{\Delta}_1 = \Delta \cdot {}^{\sigma}Z. \tag{46}$$

We reduce this equation modulo $\mathfrak{m}_\xi$:

$$\bar{Z} \cdot \overline{\Delta}_1 = \overline{\Delta} \cdot {}^{\sigma}\bar{Z},$$

where $\overline{\Delta}$ is the reduction of Δ. A solution for $\bar{Z}$ is given by the matrix which express the basis $\overline{\mathbf{m}}'$ for $\overline{\mathcal{M}}_1$ in terms of $\mathbf{m}$.

As $\hat{\Delta}_1$ is invertible and $\hat{\mathcal{O}}_\xi$ is complete with respect to $\mathfrak{m}_\xi$, we can now determine a solution Z for equation (46) by iteration. The basis $\mathbf{n} : \mathbf{m} \cdot Z$ then generates an saturated étale sub-τ-sheaf $\hat{\mathcal{M}}_\ell^{\mathrm{et}}$ of rank r_1 of $\hat{\mathcal{M}}_\ell$. The quotient $\hat{\mathcal{M}}_\ell^0 := \hat{\mathcal{M}}_\ell / \hat{\mathcal{M}}_\ell^{\mathrm{et}}$ satisfies $\overline{\mathcal{M}}_\xi^0 \cong \overline{\mathcal{M}}_{\mathrm{nil}}$ and is therefore connected. □

§2. Formal 1-dimensional $\mathbf{A}_\ell$-modules. Let $\hat{\mathbb{G}}_{a,R}$ denote the formal additive group over R. Its endomorphism ring $\mathrm{End}(\hat{\mathbb{G}}_{a,R})$ is isomorphic to the skew power series ring $R[[\varphi]]$ generated by the morphism φ and defined by the relation

$$\varphi \cdot f = {}^{\varphi}f \cdot \varphi.$$

DEFINITION 2.20 (Anderson (cf. [**An2**], §3.4)). Let ℓ denote the closed point $\ell_x = \iota(x)$ of $\mathcal{C}$ and λ a uniformizer of $\mathbf{A}_\ell$. Let $\iota^* : \mathbf{A}_\ell \to R$ be a local homomorphism. A **formal 1-dimensional $\mathbf{A}_\ell$-module** $\mathcal{E}$ over R is a continuous homomorphism

$$\mathcal{E} : \mathbf{A}_\ell \to \mathrm{End}(\hat{\mathbb{G}}_{a,R}) \cong R[[\varphi]]$$

such that

$$\begin{aligned} \mathcal{E}(\lambda) &\equiv \iota^*(E) \quad \mathrm{mod}\ R[[\varphi]] \cdot \varphi \cdot R[[\varphi]] \text{ and} \\ \mathcal{E}(\lambda) &\equiv \varphi^s \quad \mathrm{mod}\ \pi \cdot R[[\varphi]], \end{aligned} \tag{47}$$

for some $s \in \mathbb{N}$ (where π denotes a uniformizer of R).

Let us denote the order of the residue field κ_ℓ by q_ℓ. For every i, the kernel

$$\mathcal{E}[\lambda^i] := \ker \mathcal{E}(\lambda^i)$$

is a finite flat scheme of $\mathbf{A}/\lambda^i$-modules of order q_ℓ^{ih} for a integer constant

$$h = s \cdot (\deg \ell)^{-1}$$

which is called the **height** of $\mathcal{E}$. One defines the **Tate module** $T_\ell(\mathcal{E})$ associated to $\mathcal{E}$, a free $\mathbf{A}_\ell$-module with continuous action of Γ_K as

$$T_\ell(\mathcal{E}) = \varprojlim \mathcal{E}[\lambda^i](K^{\mathrm{sep}}).$$

PROPOSITION 2.21 (Anderson, cf. [**An2**], 3.4). *There is an equivalence of categories* $\mathbf{M}$ *between the categories of formal 1-dimensional* $\mathbf{A}_\ell$*-modules and the category of connected 1-dimensional* ℓ*-adic* τ*-sheaves on* $\hat{\mathcal{C}}_{R,\ell}$*, such that*

$$T_\ell(\mathcal{E}) \cong T_\ell(\mathbf{M}(\mathcal{E})).$$

REMARK 2.22. For a Drinfeld $\mathbf{A}$-module ϕ, the results 2.19 and 2.21 on $M(\phi)$ can be obtained by a different method. If λ is a maximal ideal of $\mathbf{A}$, then Taguchi proved in [**Tag2**] that the system $\{\phi[\lambda^n]\}_{n\in\mathbb{N}}$ forms a λ-divisible group. One has a connected-étale exact sequence ([**Tag2**] Remark p. 296) and an equivalence between connected λ-divisible groups and formal $\mathbf{A}$-modules ([**Tag2**] Prop. 1.4).

§3. Congruence subgroups of $\mathrm{GL}_r(\mathbf{A}_\ell)$**.** For a closed point ℓ of $\mathcal{C}$, we consider the free $\mathbf{A}_\ell$-module $T := \mathbf{A}_\ell^{\oplus r}$ of rank r. We set

$$Y := \mathrm{End}_{\kappa_\ell}(T/\lambda) = \mathrm{Mat}_{r\times r}(\kappa_x)$$

and put $U := \mathrm{End}_{\mathbf{A}_\ell}(T) = \mathrm{Mat}_{r\times r}(\mathbf{A}_\ell)$. Putting

$$G := G^0 := \mathrm{Aut}_{\mathbf{A}_\ell}(T_\ell) = \mathrm{GL}_r(\mathbf{A}_\ell), \tag{48}$$

we define, for every $i \geq 1$, the subgroup

$$G^i := 1 + \lambda^i U$$

of G. Finally, for $i \geq 1$, we consider the group $G^{[i]} := G^i/G^{i+1}$, which is isomorphic to Y via

$$\upsilon_i : Y \xrightarrow{\sim} G^{[i]} : y \mapsto 1 + \lambda^i \cdot y.$$

Let $\kappa' := \kappa_\ell^{[r]}$ be an extension of κ_ℓ of degree r and fix an embedding

$$j : \kappa' \hookrightarrow U.$$

The finite group $J := (\kappa')^\times$ acts by conjugation on U. For any integer $i \in \mathbb{Z}/h\mathbb{Z}$, the isotypical component

$$U(i) := \left\{ u \in U;\, u^j = j^{1-q_\ell^i} \cdot u \text{ for all } j \in J \right\}, \tag{49}$$

has $\mathbf{A}_\ell$-rank r (cf. [**Fo1**] and [**Abr**], §1) and we obtain a decomposition of U:

$$U = \bigoplus_{i\in\mathbb{Z}/r\mathbb{Z}} U(i).$$

Likewise, κ' embeds into Y, and we have a decomposition

$$Y := \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} Y(i),$$

where each $Y(i)$ is a 1-dimensional κ'-vector space on which J operates through the character $j \mapsto j^{1-q_\ell^i}$.

For any subgroup H of G, we set, for $i \geq 0$, $H^i := H \cap G^i$ and

$$H^{[i]} := H^i / H^{i+1} \subset G^{[i]}. \tag{50}$$

§4. Image of inertia. Put $\ell := \iota(x)$, and let λ be a uniformizer of $\mathbf{A}_\ell$. We now apply an idea of Fontaine (see [**Fo1**] and [**Abr**]) to give a description of the image of inertia I_K on $T_\ell(\mathcal{E})$, where $\mathcal{E}$ is a formal 1-dimensional $\mathbf{A}_\ell$-module of height h over R. We denote by

$$\rho_\ell : \Gamma_K \to \mathrm{GL}_h(\mathbf{A}_\ell)$$

the Galois representation on $T_\ell(\mathcal{E})$. Fixing an $\mathbf{A}_\ell$-basis for $T := T_\ell(\mathcal{E})$, we use the notations from the previous paragraph.

THEOREM 2.23. *For $\ell := \ell_x = \iota(x)$, let $\mathcal{E}$ be a formal 1-dimensional $\mathbf{A}_\ell$-module of height h over R. If ι is unramified above ℓ, then there exists a function*

$$\nu : \mathbb{Z}/h\mathbb{Z} \to \mathbb{N} \cup \{\infty\}$$

satisfying $\nu(0) = 1$ and

$$\nu(i + j) \leq \nu(i) + \nu(j)$$

such that

$$\rho_\ell(I_K) = J \ltimes \left(1 + \sum_{i \in \mathbb{Z}/h\mathbb{Z}} \lambda^{\nu(i)} \cdot U_\ell(i)\right) \subset \mathrm{GL}_h(\mathbf{A}_\ell). \tag{51}$$

PROOF. For the inertia group I_K, we have an exact sequence

$$1 \to I_K^p \to I_K \to I_K^t \to 1,$$

where I_K^p is the subgroup of wild inertia (higher ramification subgroup), and we can fix a section $I_K^t \to I_K$ of tame inertia. We put $H := \rho_\ell(I_K)$.

1) *Tame inertia.* By assumption λ is a uniformizer of K. The non-trivial λ-torsion points z in

$$\mathcal{E}[\lambda] \cong T/\lambda$$

are roots of an Eisenstein polynomial of degree $q_\ell^h - 1$. As is well known (cf. [**Se3**], §1), it follows from this that the action of I_K factors through tame inertia, that the image $H^{[0]}$ of

$$I_K^t \to \mathrm{Aut}(T/\lambda) \subset Y$$

has order $q_\ell^h - 1$ and can be identified as a group with the multiplicative group J of $\kappa_\ell^{[h]}$.

2) *Wild inertia.* For $i \geq 1$, let us put

$$L^i := K^{\mathrm{ur}}(\mathcal{E}[\lambda^{i+1}]),$$

so we can identify $H^{[i]}$ with $\mathrm{Gal}(L^i/L^{i-1})$. For any point z of

$$\mathcal{E}[\lambda^{i+1}]\backslash\mathcal{E}[\lambda^i] \subset K^{\mathrm{sep}}$$

and any nontrivial $\sigma \in \mathrm{Gal}(L^i/L^{i-1})$, we have $\sigma(z) - z \in \mathcal{E}[\lambda]\backslash\{0\}$. A direct calculation shows that, with respect to the normalized valuation v of $L^i(z)$, we have

$$v(\sigma(z) - z) = q_\ell^{h\cdot i}.$$

By [**Se2**] IV §1, this implies that the subgroup

$$\mathrm{Gal}(L^i(z)/L^i) \subset \mathrm{Gal}(L^i(z)/K^{\mathrm{ur}})$$

is contained in the $\mu(i)$-th higher ramification group $G_{\mu(i)}$ of $\mathrm{Gal}(L^i(z)/K)$, where

$$\mu(i) := q_\ell^{h\cdot i} - 1.$$

But then I_K^t acts trivially by conjugation on

$$\mathrm{Gal}(L^i(z)/L^i),$$

by [**Se2**] IV §2, Prop. 9 (p. 77).

Through the identification $G^{[i]} \cong Y$, for $i \geq 1$, we have a decomposition

$$G^{[i]} = \bigoplus_{j\in\mathbb{Z}/h\mathbb{Z}} Y(j).$$

In the above we found a quotient $\mathrm{Gal}(L^i(z)/L^i)$ of $H^{[i]}$ of order q_ℓ^h which is invariant under conjugation by J, which must hence be isomorphic to

$$Y(0) \subset H^{[i]}.$$

3) *The function* ν. The $\mathbb{Z}[J]$-modules $Y(j)$ are simple for $j \neq 0$. It now follows that $H^{[i]}$ is isomorphic to a direct sum

$$\bigoplus_{j\in J(i)} Y(j),$$

where $J(i)$ is a subset of $\mathbb{Z}/h\mathbb{Z}$ which, by the above, certainly contains 0. We remark that

$$[Y(j), Y(j')] = Y(j + j'),$$

if $(j, j) \neq (0, 0)$ (cf. [**Fo1**] §7). By the commutative diagram

$$\begin{array}{ccccccccc} G^{[i]} & \times & G^{[i']} & \to & G^{[i+i']} & : (g_1, g_2) & \mapsto & g_1g_2g_1^{-1}g_2^{-1} \\ \uparrow & & \uparrow & & \uparrow & & & \\ Y & \times & Y & \to & Y & : (h_1, h_2) & \mapsto & [h_1, h_2] \end{array} \tag{52}$$

this implies that if $j \in J(i)$ and $j' \in J(i')$, then $j+j' \in J(i+i')$ for $(j, j) \neq (0, 0)$. If we now put

$$\nu(j) := \inf\{i \in \mathbb{N};\ j \in J(i)\} \in \mathbb{N} \cup \{\infty\},$$

then ν is the required function. □

Combining Lemma 2.19, Prop. 2.21 and Thm. 2.23, we now conclude:

COROLLARY 2.24. *Let M be a τ-sheaf on $\mathcal{C}_K$ with characteristic ι and* ***dimension*** 1, *which possesses a good model $\mathcal{M}$ over $\mathcal{C}_R$. Suppose that ι is unramified at the point $\ell := \iota(x)$. Then we have*

i) *an exact sequence*

$$0 \to V_\ell(M)^0 \to V_\ell(M) \to V_\ell(M)^{\mathrm{et}} \to 0 \tag{53}$$

of $F_\ell[\Gamma_K]$-modules, where

- *the module $V_\ell(M)^{\mathrm{et}}$ is unramified, and*
- *the image of the action of inertia on $V_\ell(M)^0$ can be described as in Thm. 2.23 above, and*

ii) *an exact sequence*

$$0 \to W_\ell(M)^0 \to W_\ell(M) \to W_\ell(M)^{\mathrm{et}} \to 0 \tag{54}$$

of $\kappa_\ell[\Gamma_x]$-modules where

- *the module $W_\ell(M)^{\mathrm{et}}$ is unramified and*
- *the image of inertia on the h-dimensional vector space $W_\ell(M)^0$ is isomorphic to*

$$\left(\kappa_\ell^{[h]}\right)^\times \subset \mathrm{Aut}_{\kappa_\ell}(W_\ell(M)).$$

CHAPTER 3

The image of global Galois representations

Let $\overline{\mathcal{C}}$ be an absolutely irreducible projective smooth curve with field of constants $\mathbb{F}_q$ and denote its function field by F. Consider the affine curve $\mathcal{C} = \overline{\mathcal{C}} \backslash \{\infty\}$, where ∞ is a fixed closed point of $\overline{\mathcal{C}}$, and put $\mathbf{A} = H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. Let K be a finite extension of $F := \mathrm{Quot}(\mathbf{A})$, and let us denote the morphism Spec $K \to \mathcal{C}$ by ι.

In this chapter, we will study the image of the absolute Galois group Γ_K of K under the residual adelic representation $\overline{\rho}_{\mathrm{ad}}$ associated to a simple τ-sheaf M of rank r over $\mathcal{C}_K$ with characteristic ι and dimension 1. The prominent example to keep in mind is given by the t-motive $M(\phi)$ associated to a Drinfeld module ϕ over $\mathbf{A}$ with coefficients in K and characteristic ι. These τ-sheaves $M(\phi)$ are characterized by the fact that they are pure (cf. Remark 0.18). In what follows, however, we will avoid any 'purity' assumption, as it seems a better idea to exploit directly the natural relation between τ-sheaves and Galois representations.

We will combine techniques adopted from Serre's theory on abelian p-adic representations (cf. [**Se1**]) and his famous theorem Thm. 0.20 on the adelic image of Galois on the torsion of elliptic curves with well known results on the Galois modules $V_\ell(M)$:

- they form a *strictly compatible* system of integral representations;
- the *Tate* and *semisimplicity* conjectures;
- the structure of the *endomorphism* ring;
- properties of the *determinant* Drinfeld module,

Let us call $\mathrm{End}_{\bar{K}}(M)$ the absolute endomorphism ring of M. We recall from equation (14) that $W_{\mathrm{ad}}(M)$ is defined as the product $\prod_\ell W_\ell(M)$ over all closed points ℓ of $\mathcal{C}$. In Thm. 3.13, we will give a proof of the following conjecture in the case $r \leq 2$:

CONJECTURE 3.1. *For a finite extension K of F, let M be a τ-sheaf over $\mathcal{C}_K$ with characteristic ι :* Spec $K \to \mathcal{C}$, *dimension* 1 *and absolute endomorphism ring* $\mathbf{A}$. *The image of the representation $\overline{\rho}_{\mathrm{ad}}$ of Γ_K on the $\kappa_{\mathrm{ad}}[\Gamma_K]$-module $W_{\mathrm{ad}}(M)$ is open in* $\mathrm{GL}_r(\kappa_{\mathrm{ad}})$.

I. Global properties

For the finite extension K of F (a field of transcendence degree 1 over $\mathbb{F}_q$), we choose an irreducible projective smooth curve X over $\mathbb{F}_q$ with function field K. For any closed point x of X, which we identify with the associated place of K, we denote the completion of K at x by K_x, its ring of integers by R_x and a uniformizer of R_x by π_x. Let Γ_x be the absolute Galois group of K_x, and I_x its inertia subgroup. The characteristic

$$\iota : \operatorname{Spec} K \to \bar{\mathcal{C}}$$

induced by the embedding $\iota^* : \mathbf{A} \subset F \hookrightarrow K$ extends to a morphism

$$\iota : X \to \bar{\mathcal{C}}.$$

We denote by $\ell_x := \iota(x)$ the closed point of $\bar{\mathcal{C}}$ below a place x of K.

Let M be a τ-sheaf of rank r over $\mathcal{C}_K$ with characteristic map ι and dimension d.

REMARK 3.2. The maximal exterior power $\wedge^{\mathrm{top}} M$ of M is a τ-sheaf of rank 1, with characteristic map ι and dimension d as well. By the tensor compatibility of V_ℓ, we get

$$\wedge^{\mathrm{top}} V_\ell(M) \cong V_\ell(\wedge^{\mathrm{top}} M).$$

If M has dimension 1 then $\wedge^{\mathrm{top}} M$ is pure of weight 1 and therefore isomorphic to the t-motive of a Drinfeld module ϕ of rank 1; then $\wedge^{\mathrm{top}} V_\ell(M)$ is isomorphic to $V_\ell(\phi)$.

§1. Strictly compatible system of representations. The τ-sheaf M admits a maximal model $\mathcal{M}^{\max}$ over $\mathcal{C}_X := X \times \mathcal{C}$ (cf. Prop. 1.13). For a closed point x of X, we let $\overline{\mathcal{M}}_x$ denote the reduction of $\mathcal{M}^{\max}$ at $\operatorname{Spec} k_x \times \mathcal{C}$. For all but a finite set X^{bad} of places x, the model $\mathcal{M}$ is good at x (i.e. its reduction is again a τ-sheaf), by Lemma 1.8; we put $X^{\mathrm{good}} := X \backslash X^{\mathrm{bad}}$.

PROPOSITION 3.3 (Strictly compatible system.). *The system of Galois modules $V_\ell(M)$ is a **strictly compatible** system over $\mathcal{C}$ of **integral** representations with exceptional set X^{bad}, i.e.*

i) *for all closed points ℓ of $\mathcal{C}$, $V_\ell(M)$ is unramified for all places $x \in X^{\mathrm{good}}$ such that $\ell \neq \ell_x$. For such x, the action of a Frobenius substitution* Frob_x *on V_ℓ is well defined;*

ii) *for all $x \in X^{\mathrm{good}}$, the characteristic polynomial*

$$P_x(V_\ell(M); T) := \det(\mathrm{Frob}_x - T \mid V_\ell(M)) \in \mathbf{A}_\ell[T] \tag{55}$$

(which is independent of the choice of Frob_x*) has coefficients in* $\mathbf{A}$ *and is independent of ℓ, for all closed points $\ell \neq \ell_x$ of $\mathcal{C}$.*

PROOF. The cokernel of τ on $\mathcal{M}$ is supported on the graph $\Gamma(\iota)$ of the characteristic $\iota : X \to \mathcal{C}$ in $\mathcal{C}_X$. For any closed point x of $\mathcal{C}^{\mathrm{good}}$, the completion $\hat{\mathcal{M}}_\ell$ of $\mathcal{M}$ at $\operatorname{Spec} R_x \times \{\ell\}$ is smooth for all closed points $\ell \neq \ell_x$ of $\mathcal{C}$. Hence, the Tate module $T_\ell(M)$ is unramified at x for $\ell \neq \ell_x$. Putting $d_x := [k_x : \mathbb{F}_q]$, we define

$$P_x(M; T) := \det\left(\tau^{d_x} - T \mid H^0(\mathcal{C}_{k_x}, \overline{\mathcal{M}}_x)\right) \in \mathbf{A}[T].$$

The following proposition then concludes the proof: □

PROPOSITION 3.4 (Taguchi-Wan, [**TW**], p. 772). *For all closed points x of X^{good} and all closed points $\ell \neq \ell_x$ of $\mathcal{C}$:*

$$P_x(V_\ell(M); T) = P_x(M; T) \in \mathbf{A}[T].$$

REMARK 3.5. Given a Drinfeld module ϕ such that $M = M(\phi)$, the above can just as well be deduced from the reduction theory of Drinfeld modules, explained in [**Dr1**], [**Go2**] and [**Tag2**]. Actually, the set of places at which ϕ has good reduction coincides exactly with the set of closed points of X^{good}; this follows from the Galois criteria of good reduction for both Drinfeld modules and τ-sheaves (cf. [**Tak**], resp. our Thm. 4.1):

$$\begin{gathered}\phi \text{ has good reduction}\\ \Leftrightarrow T_\ell(\phi) \cong T_\ell(M) \text{ is unramified for } \ell \neq \ell_x\\ \Leftrightarrow \mathcal{M} \text{ is good.}\end{gathered}$$

§2. Semistability of Galois representations. As a consequence of Thm. 2.4, we obtain:

PROPOSITION 3.6 (Semistability of Galois representations). *There exists an open subscheme $\mathcal{C}^{\text{sst}}$ of $\mathcal{C}$ and a finite extension K' of K such that, for all closed points ℓ of $\mathcal{C}^{\text{sst}}$ and all places x of K', the action of the inertia group I_x on $V_\ell(M)$ is unipotent if $\ell \neq \ell_x$.*

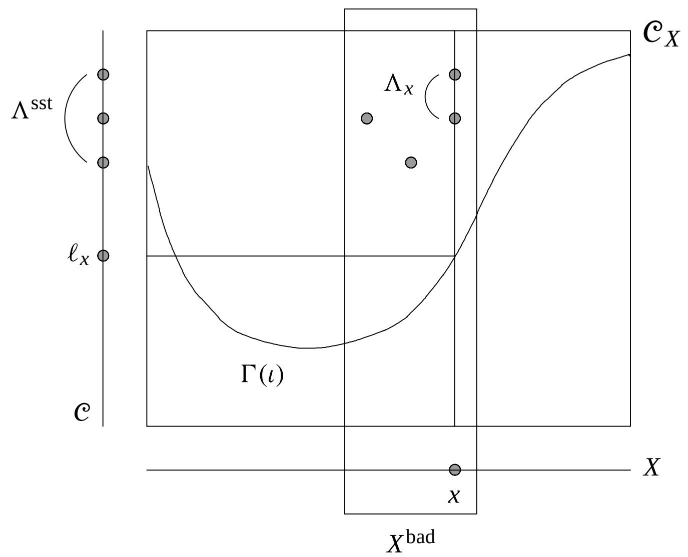


FIGURE 1. At the finite number of places x of bad reduction, we have a finite set Λ_x of closed points of $\mathcal{C}$, such that, if I_x does not acts potentially unipotently on $V_\ell(M)$, then $\ell \in \{\ell_x\} \cup \Lambda_x$. This allows us to define the finite set Λ^{sst}, such that if x is a place of K, and $\ell \notin \{\ell_x\} \cup \Lambda^{\text{sst}}$, then I_x acts potentially unipotently on $V_\ell(M)$.

PROOF. For every place $x \in X^{\text{bad}}$, let Λ_x denote the finite set of closed points defined as $\mathcal{C} \setminus \mathcal{C}'$, where $\mathcal{C}'$ is as in Thm. 1.26. Set

$$\Lambda^{\text{sst}} := \cup_{x \in X^{\text{bad}}} \Lambda_x$$

(cf. Figure 1), and

$$\mathcal{C}^{\mathrm{sst}} := \mathcal{C} \setminus \Lambda^{\mathrm{sst}}.$$

Also, we consider a finite extension K' such that the semistable filtrations are defined over K_x, for every place x of K. □

§3. Tate and semisimplicity conjectures. Taguchi (in the case of Drinfeld modules; cf. [**Tag4**] and [**Tag2**]) and Tamagawa (for general τ-sheaves) [**Tam**]) proved the so-called Tate and semisimplicity conjectures for τ-sheaves for every finitely generated field K:

THEOREM 3.7 (Taguchi, Tamagawa). *For a finitely generated field K containing $\mathbb{F}_q$, let M be a simple τ-sheaf on $\mathcal{C}_K$ K and ℓ a closed point of $\mathcal{C}$ such that the ℓ-adic τ-sheaf $\hat{M}_\ell$ is smooth.*

i) ***Tate conjecture.*** *The map*

$$F_\ell \otimes_{\mathbf{A}} \mathrm{End}_K(M) \to \mathrm{End}_{F_\ell[\Gamma_K]}(V_\ell(M))$$

is an isomorphism.

ii) ***Semisimplicity conjecture.*** *The $F_\ell[\Gamma_K]$-module $V_\ell(M)$ is semisimple.*

§4. The endomorphism ring. The following result holds for any field K containing F:

PROPOSITION 3.8. *For a field K containing F, let M be a simple τ-sheaf on $\mathcal{C}_K$ of characteristic $\iota : \mathrm{Spec}\, K \to \mathcal{C}$ and dimension 1. The ring $\mathrm{End}_K(M)$ of endomorphisms M defined over K is a finitely generated $\mathbf{A}$-algebra of rank at most r^2. If M has dimension 1, then $\mathrm{End}_K(M)$ is commutative.*

PROOF. The ring $E := \mathrm{End}_K(M)$ is an $\mathbf{A}$-algebra via the inclusion $\mathbf{A} \hookrightarrow E$. First of all, if M has rank 1, then one readily sees that $\mathrm{End}_K(M) = \mathbf{A}$.

a) We put $E^0 := \mathrm{End}^0_K(M) := F \otimes_{\mathbf{A}} E$. For every $\alpha \in E$, there exists an $\hat{\alpha}$ such that $\hat{\alpha} \cdot \alpha \in \mathbf{A}$. Indeed, if α is (locally) represented by a matrix B, then its determinant is an element of $\mathbf{A}$, so it suffices for example to consider the endomorphism $\hat{\alpha}$ represented by the adjoint matrix B^{ad}. This proves that E is a torsion free $\mathbf{A}$-module and that E^0 is a division F-algebra.

b) First, we suppose that K is a finitely generated field. Taking a model X of K of finite type over $\mathrm{Spec}\, \mathbb{F}_q$, and a model $\mathcal{M}$ over $\mathcal{C}_X$ for the τ-sheaf M, there exists a closed point x of X at which the reduction $\overline{\mathcal{M}}$ of $\mathcal{M}$ at x is good (straightforward generalization of paragraph **1.I.§2**). Then the stalk $\mathcal{M}_x$ of $\mathcal{M}$ at the closed point of $\mathcal{C}_X$ above x is maximal, by an analog of Lemma 1.13.ii), and therefore $\mathrm{End}_K(M)$-invariant.

Thus we obtain a ring homomorphism

$$j : E \to \mathrm{End}_{k_x}(\overline{\mathcal{M}}),$$

where k_x is the finite residue field at x. Without loss of generality, we may reduce ourselves to the case that $\mathcal{C} = \mathbb{A}^1$, and (upon replacing τ by some power) that $k_x = \mathbb{F}_q$, as we only risk to increase the endomorphism ring. Then $\mathrm{End}_{k_x}(\overline{\mathcal{M}})$ is just the full matrix ring of rank $r \times r$ over $\mathbf{A}$, so for sure it is finitely generated.

We now claim that $E^0 \to F \otimes_{\mathbf{A}} \mathrm{End}_{k_x}(\overline{\mathcal{M}})$ is injective. The ring E^0 being a division F-algebra, it suffices to show that it is not a zero morphism, but this is obvious since it is nonzero on the subring $\mathbf{A} \subset E$ of multiplication-by-a endomorphisms. Thus j is injective, and hence $\mathrm{End}_K(M)$ is finitely generated.

For every closed point ℓ of $\mathcal{C}$, it follows from the Tate conjecture (Thm. 3.7) that the F-algebra $F_\ell \otimes_{\mathbf{A}} E$ embeds into $\mathrm{End}_{F_\ell}(V_\ell(M))$, which is an F_ℓ-algebra of rank r. This shows that E has at most rank r^2.

For an arbitrary field K, we choose an infinite tower of finitely generated fields K_i such that $\cup K_i = K$. For every K_i, the ring $\mathrm{End}_{K_i}(M)$ is finitely generated over $\mathbf{A}$, of rank $\leq r^2$. It follows from the Tate conjecture that the $\mathbf{A}_\ell$-algebra $\mathbf{A}_\ell \otimes_{\mathbf{A}} E$ is saturated in $\mathrm{End}_{\mathbf{A}_\ell}(T_\ell(M))$. Therefore, if, for $K_i \subset K_j$, we have

$$\mathrm{End}_{K_i}(M) \neq \mathrm{End}_{K_j}(M),$$

then this implies that

$$\mathrm{rank}_{\mathbf{A}}\mathrm{End}_{K_i}(M) \leq \mathrm{rank}_{\mathbf{A}}\mathrm{End}_{K_j}(M).$$

As these ranks are bounded by r^2, we obtain a K_j such that

$$\mathrm{End}_{K_j}(M) = \mathrm{End}_K(M),$$

and this proves that $\mathrm{End}_K(M)$ is finitely generated of rank at most r^2.

c) Finally, suppose that M has dimension 1. Endomorphisms by definition commute with τ, so we have a natural map:

$$E \to \mathrm{End}_{\mathcal{O}_{\mathcal{C}_K}}(\mathrm{coker}\,\tau).$$

As coker τ is supported on the closed point γ of $\mathcal{C}_K$ and has rank 1 on the point γ, we can identify the latter with $K = \mathrm{End}_K(K)$. We claim that

$$j : E^0 \to K$$

is a injection. The ring E^0 being a division F-algebra, it suffices to show that j is not a trivial morphism. But this is clear since for any $\alpha \in \mathbf{A} \subset E$, we know, as M has characteristic ι, that $j(\alpha)$ acts on coker τ as $\iota^*(\alpha) \in K$, where ι^* is the embedding $\iota^* : \mathbf{A} \hookrightarrow K$. Hence E^0 is a (commutative) field extension of F. □

II. Image of the residual representations

We first develop a theory of compatible systems of Γ_K-representations associated to Hecke characters, inspired by Serre's theory of abelian p-adic representations (cf. [**Se1**]). This allows us to treat residual representations which decompose absolutely into a direct sum of 1-dimensional characters. As a consequence, we can, following the strategy of [**Se3**], prove Conj. 3.1 in the case $r \leq 2$ (Thm. 3.13).

§1. Serre torus. For a finite separable extension K of F, we choose an irreducible projective smooth curve X over $\mathbb{F}_q$ with function field K. Let $\bar{F}$ denote an algebraic closure of F, and $\bar{\mathbf{A}}$ the integral closure of $\mathbf{A}$ in $\bar{F}$. Let $K_{\mathbb{A}}$ denote the ring of adeles of K.

We choose, for all closed points ℓ of $\mathcal{C}$, an embedding of $\bar{F}$ into an algebraic closure $\bar{F}_\ell$ of F_ℓ, and let $\bar{\mathbf{A}}_\ell$ denote the integral closure of $\mathbf{A}_\ell \in \bar{F}_\ell$. We denote the maximal ideal of $\bar{F}_\ell$ by $\bar{\lambda}$. The residue field of $\bar{F}_\ell$ is isomorphic to $\bar{\kappa}_\ell$.

1. *Elements from class field theory.* We may assume that K is a Galois extension of F. We fix a nonempty finite set S of closed points of X and denote by $S_{\mathcal{C}}$ the finite set of closed points of $\mathcal{C}$ lying below it.

We now consider the product

$$U_S := \prod_{\substack{x \text{ closed pt. of } X \\ x \notin S}} R_x^\times \times \prod_{x \in S} K_x^\times \subset K_{\mathbb{A}}^\times, \tag{56}$$

and, for every closed point ℓ of $\mathcal{C}$, we let

$$U_S^{(\ell)} := \prod_{\substack{x \text{ closed pt. of } X \\ x \notin S \\ \ell_x \neq \ell}} R_x^\times \times \prod_{x \in S} K_x^\times \subset K_{\mathbb{A}}^\times. \tag{57}$$

There is an exact sequence

$$1 \to K^\times/(K^\times \cap U_S) \to K_{\mathbb{A}}^\times/U_S \to C \to 1,$$

where the 'class group' $C = K_{\mathbb{A}}^\times/K^\times U_S$ is a finite abelian group, as S is non-empty (cf. [**Wei**]).

Let $\overline{K^\times U_S^{(\ell)}}$ be the closure of $K^\times U_S^{(\ell)}$ in $K_{\mathbb{A}}^\times$. The Artin reciprocity map of global class field theory induces a continuous isomorphism

$$\omega : K_{\mathbb{A}}^\times/\overline{K^\times U_S^{(\ell)}} \to \operatorname{Gal}\left(K_S^{\mathrm{ab},(\ell)}/K\right),$$

where $K_S^{\mathrm{ab},(\ell)}$ is the maximal abelian extension of K which splits completely at the points of S and is unramified outside the places lying above ℓ.

2. *Serre torus.* We consider the algebraic group $\operatorname{Res}_F^K(\mathbb{G}_{m,K})$, defined for all commutative F-algebras B by

$$\operatorname{Res}_F^K(\mathbb{G}_{m,K})(B) := (B \otimes_F K)^\times.$$

We let $\overline{K^\times \cap U_S}$ be the Zariski closure of $K^\times \cap U_S$ in $\operatorname{Res}_F^K(\mathbb{G}_{m,K})$ and take the quotient group

$$\mathbb{T} := \operatorname{Res}_F^K(\mathbb{G}_{m,K})/\overline{K^\times \cap U_S},$$

an algebraic group defined over F.

Let $\mathbb{S}_{K,S}$ (for short: $\mathbb{S}$) be the push-out over $K^\times/(K^\times \cap U_S)$ of $\mathbb{T}$ and $K_{\mathbb{A}}^\times/U_S$, i.e. the algebraic group with the universal property that, for any algebraic group $\mathbb{S}'$ equipped with morphisms $\mathbb{T} \to \mathbb{S}'$ and $K_{\mathbb{A}}^\times/U_S \to \mathbb{S}'(F)$ such that the diagram

$$\begin{array}{ccc} K^\times/(K^\times \cap U_S) & \to & K_{\mathbb{A}}^\times/U_S \\ \downarrow & & \downarrow \\ \mathbb{T}(F) & \to & \mathbb{S}'(F), \end{array}$$

commutes, there is a unique morphism $\mathbb{S} \to \mathbb{S}'$ through which these maps factor. Serre gave an explicit construction of the torus $\mathbb{S}$ in [**Se1**], II §2.

We now have a commutative diagram

$$
\begin{array}{ccccccc}
1 \to & K^\times/(K^\times \cap U_S) & \to & K_{\mathbb{A}}^\times/U_S & \to & C & \to 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
1 \to & \mathbb{T}(F) & \to & \mathbb{S}(F) & \to & C & \to 1
\end{array}
\tag{58}
$$

For every closed point ℓ of $\mathcal{C}$, consider the composite map

$$\upsilon_\ell : K_{\mathbb{A}}^\times \to K_{\mathbb{A}}^\times/U_S = \mathbb{S}(F) \to \mathbb{S}(F_\ell). \tag{59}$$

On the other hand, we have a continuous map ξ_ℓ defined as the composition of

$$
\begin{tikzcd}
K_{\mathbb{A}}^\times \arrow[d] \arrow[rrd, bend left, "\xi_\ell"] & & \\
\left(\prod_{x|\ell} K_x\right)^\times = \mathrm{Res}_F^K(\mathbb{G}_{m,K})(F_\ell) \arrow[r, phantom] & \arrow[d] & \\
 & \mathbb{T}(F_\ell) \arrow[r] & \mathbb{S}(F_\ell).
\end{tikzcd}
\tag{60}
$$

By the commutativity of diagram (58), it follows that $\upsilon_{\ell|K^\times} = \xi_{\ell|K^\times}$. Thus we obtain, for all ℓ in $\mathcal{C}$, a continuous group homomorphism

$$\Xi_\ell := \upsilon_\ell \cdot \xi_\ell^{-1} : K_{\mathbb{A}}^\times \to \mathbb{S}(F_\ell) \tag{61}$$

The map Ξ_ℓ factors through $K_{\mathbb{A}}^\times/\overline{K^\times U_S^{(\ell)}}$.

Any linear representation $\Phi : \mathbb{S} \to \mathrm{GL}_m$ defined over $\bar{F}$, now yields, for every closed point ℓ of $\mathcal{C}$, a Galois representation Φ_ℓ as follows:

$$
\begin{tikzcd}
\Gamma_K \arrow[r] \arrow[rrdd, bend right, "\Phi_\ell"'] & \mathrm{Gal}\left(K_S^{\mathrm{ab},(\ell)}/K\right) & \\
 & K_{\mathbb{A}}^\times/\overline{K^\times U_S^{(\ell)}} \arrow[u, "\omega"] \arrow[r, "\Xi_\ell"] & \mathbb{S}(\bar{F}_\ell) \arrow[d, "\Phi"] \\
 & & \mathrm{GL}_m(\bar{F}_\ell).
\end{tikzcd}
\tag{62}
$$

We denote by $V_\ell(\Phi)$ the m-dimensional $\bar{F}_\ell[\Gamma_K]$-module $\bar{F}_\ell^{\oplus m}$ with the Γ_K-action given by Φ_ℓ.

The algebraic group $\mathbb{S}$ being a torus, its linear representation Φ can be diagonalized over some finite extension $E \subset \bar{F}$ of F, so $V_\ell(\Phi)$ is semisimple. As any compact subgroup of $\mathrm{GL}_r(E_\ell)$ (where $E_\ell := F_\ell E \subset \bar{F}_\ell$) is contained in a conjugate of $\mathrm{GL}_r(\bar{\mathbf{A}}_\ell \cap E)$, we know that the image of the Galois representation Φ_ℓ is contained in a conjugate of $\mathrm{GL}_r(\bar{\mathbf{A}}_\ell)$.

The system of representations $V_\ell(\Phi)$ is a $\mathcal{C}\backslash S_{\mathcal{C}}$-compatible system of representations. Indeed, for every closed point x of $X\backslash S$ not lying above ℓ, the image of a Frobenius substitution of Frob_x in $\mathrm{Gal}\left(K_S^{\mathrm{ab},(\ell)}/K\right)$ under ω^{-1} is given by a uniformizer $\pi_x \in F_x$. As $\xi_\ell(\pi_x) = 1$, the image of Frob_x under Φ_ℓ is then given by the image in $\mathrm{GL}_m(\bar{\mathbf{A}}_\ell)$ of π_x under the map

$$\begin{tikzcd} K_{\mathbb{A}}^\times \arrow[r] \arrow[rr, bend left, "\upsilon_\ell"] & \mathbb{S}(F) \arrow[r] \arrow[d] & \mathbb{S}(F_\ell) \arrow[d] \\ & \mathrm{GL}_m(\bar{F}) \arrow[r] & \mathrm{GL}_m(\bar{F}_\ell) \end{tikzcd}$$

so its characteristic polynomial (cf. (55))

$$P_x(\Phi; T) := P_x(V_\ell(\Phi); T)$$

has coefficients in $\bar{F}$ is independent of ℓ. As the image of Φ is contained in a conjugate of $\mathrm{GL}(\bar{\mathbf{A}}_\ell)$, the coefficients of $P_x(\Phi; T)$ are contained in $\bar{\mathbf{A}}_\ell$, for every ℓ in $\mathcal{C}\backslash S_{\mathcal{C}}$.

3. *Galois characters associated to characters of* $\mathbb{S}$. Put

$$\Sigma := \mathrm{Aut}_F(K, \bar{F}).$$

Every $\sigma \in \Sigma$ extends to a homomorphism $K \otimes_F \bar{F} \to \bar{F}$ and thus to a morphism

$$[\sigma] : \mathrm{Res}_F^K(\mathbb{G}_{m,K}) \to \mathbb{G}_m$$

defined over $\bar{F}$. These $[\sigma]$'s give a $\mathbb{Z}$-basis for the character group

$$X(\mathrm{Res}_F^K(\mathbb{G}_{m,K})).$$

Furthermore,

$$X(\mathbb{T}) = \left\{ \sum_\sigma n_\sigma[\sigma] \in \mathbb{Z}[\Sigma]; \prod_\sigma \sigma(x)^{n_\sigma} = 1 \text{ for all } x \in K^\times \cap U_S \right\}.$$

The characters of $\mathbb{S}$ sit in the exact sequence:

$$1 \to X(C) \to X(\mathbb{S}) \xrightarrow{j} X(\mathbb{T}) \to 1,$$

where $X(C)$ is the finite group $\mathrm{Hom}\left(C, (\overline{\mathbb{F}}_q)^\times\right)$.

By the universal property of $\mathbb{S}$, if we are given a pair of homomorphisms

$$\left(f, \varphi = \sum_{\sigma \in \Sigma} n_\sigma[\sigma] \right) \in \mathrm{Hom}\left(K_{\mathbb{A}}^\times, \bar{F}^\times\right) \times X(\mathbb{T})$$

satisfying

- $f|_{U_S} = 1$ and
- $f|_{K^\times} = \varphi|_{K^\times}$,

then these maps will factor through a unique character $\Phi \in X(\mathbb{S})$ of $\mathbb{S}$.

Composing Φ with Ξ_ℓ gives a map Φ_ℓ

$$\begin{array}{rclcl} K_{\mathbb{A}}^\times/\overline{K^\times U^\ell} & \to & \mathbb{S}(\bar{F}_\ell) & \to & \bar{F}_\ell^\times \\ x & \mapsto & & & f(\upsilon_\ell(x)) \cdot \varphi(\xi_\ell(x^{-1})), \end{array}$$

and therefore yields a Galois representation $\Gamma_K \to \bar{F}_\ell^\times$ (still denoted by Φ_ℓ) via the isomorphism ω^{-1}.

§2. Characters of the residual representation.

PROPOSITION 3.9. *Let K be a finite separable extension of F and M a τ-sheaf of rank r over $\mathcal{C}_K$ with characteristic ι with dimension* 1. *Suppose we are given*

- *an infinite set $\mathcal{L}$ of closed points ℓ of $\mathcal{C}$ such that $\bar{\kappa}_\ell \otimes W_\ell(M)$ has a Γ_K-invariant subquotient W'_ℓ of $\bar{\kappa}_\ell$-dimension one:*
- *a closed point x_0 of X of degree* 1 *over $\mathbb{F}_q$ at which the maximal model $\mathcal{M}$ on $\mathcal{C}_X$ for M is good;*

Let us denote by χ_ℓ the character

$$\Gamma_K \to \mathrm{Aut}(W'_\ell) \cong \bar{\kappa}_\ell^\times.$$

Then there exists

- *a finite normal field extension K' of K, whose field of constants $K' \cap \overline{\mathbb{F}}_q$ we denote by $\mathbb{F}'$;*
- *a character $\Phi \in X(\mathbb{S}_{K,S})$, for $S := \{x_0\}$, together with its system of 1-dimensional $\bar{F}_\ell[\Gamma_{K'}]$-modules $V_\ell(\Phi)$ (with Galois representation*
$$\Phi_\ell : \Gamma_{K'} \to \mathbf{A}_\ell^\times \subset \bar{F}_\ell^\times);$$
- *a strictly compatible system 1-dimensional $\bar{F}_\ell[\Gamma_{K'}]$-modules $V_\ell(\eta)$, for closed points ℓ of $\mathcal{C}\backslash S_{\mathcal{C}}$, such that the associated Galois characters*
$$\eta_\ell : \Gamma_{K'} \to \mathbf{A}_\ell^\times \subset \bar{F}_\ell^\times$$
factor through $\Gamma_{\mathbb{F}'}$ (the absolute Galois group of $\mathbb{F}'$); and
- *an infinite subset $\mathcal{L}'$ of $\mathcal{L}$*

such that, for all $\ell \in \mathcal{L}'$, we have

$$\chi_\ell \equiv \Phi_\ell \cdot \eta_\ell \mod \bar{\lambda}. \tag{63}$$

PROOF. Let K' be a Galois extension of K such that the semistable filtration for M (see Thm. 1.26) is defined over $\mathcal{C}_{K'}^{\mathrm{sst}}$. Without loss of generality, we may assume that $K = K'$. We may also suppose that $\mathcal{L}$ contains none of the finitely many points of $\mathcal{C}$ which are either

- contained in Λ^{sst}
- ramified in the extension $X \to \bar{\mathcal{C}}$
- equal to $\ell_0 := \iota(x_0)$.

a) *Characters which are 'trivial at x_0'.* The idea is to replace the χ_ℓ by a family of characters ψ_ℓ whose restriction to the decomposition group Γ_{x_0} at x_0 is trivial. Let F' be the splitting field of the polynomial $P_{x_0}(M;T) \in \mathbf{A}[T]$ (cf. Thm. 3.3), and

$$\mathcal{F} := \{f_i\}_{1 \le i \le r}$$

the set of its roots, i.e. the eigenvalues of τ on $\overline{\mathcal{M}}$, or, equivalently, of the Frobenius morphism Frob_{x_0} acting on $T_\ell(M)$ for all $\ell \neq \ell_0$. By Thm. 3.2.3 d), the determinant representation $\det \rho_\ell$ corresponds to the ℓ-adic representation associated with a Drinfeld module. By [**Go2**], we have

$$0 = v_\ell(\det \rho_\ell(\mathrm{Frob}_x)\ \left(= \sum_1^r v_\ell(f_i)\right)$$

if $\ell \neq \ell_x$, and therefore $v_\ell(f_i) = 0$ if $\ell \neq \ell_0$. Let $\bar{\mathcal{C}}'$ be a projective smooth curve with function field F' and $\mathcal{C}'$ the inverse image of $\mathcal{C}$ via the natural map $\bar{\mathcal{C}}' \to \bar{\mathcal{C}}$.

For each $f \in \mathcal{F}$, we can construct a τ-sheaf $\bar{N}_f$ on $\mathcal{C}'$ $(= \mathcal{C}'_{\mathbb{F}_q})$ by taking $\mathcal{O}_{\mathcal{C}'}$ as the underlying sheaf and putting

$$\tau : \sigma^* \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}'} : 1 \mapsto f.$$

For every $\ell \neq \ell_0$, we fix a point $\ell' \in \mathcal{C}'$ above ℓ and an embedding $F'_{\ell'} \hookrightarrow \bar{F}_\ell$. As N_f is smooth at ℓ', the Tate module $T_{\ell'}(N_f)$ of N_f is well defined. The Galois group $\Gamma_{\mathbb{F}_q}$ acts on it through a character

$$\eta_\ell^f : \Gamma_{\mathbb{F}_q} \to (F'_{\ell'})^\times \subset \bar{F}_\ell^\times$$

such that $\eta_\ell^f(\mathrm{Frob}_{x_0}) = f \in F' \subset \bar{F}_\ell^\times$.

As $\mathcal{F}$ is finite, there exists an eigenvalue f and an infinite subset $\mathcal{L}_1$ of $\mathcal{L} \backslash S_{\mathcal{C}}$ such that

$$\chi_\ell(\mathrm{Frob}_{x_0}) \equiv f \bmod \bar{\lambda}$$

for all $\ell \in \mathcal{L}_1$. Then, for all ℓ in $\mathcal{L}_1$, the character

$$\psi_\ell := \chi_\ell \cdot (\eta_\ell^f)^{-1} \mod \bar{\lambda} : \Gamma_k \to \bar{\kappa}_\ell^\times$$

factors through the Galois group an abelian extension of K whose restriction to Γ_{x_0} is trivial, and which is unramified outside the places x_ℓ of K lying above ℓ (by our assumptions, inertia I_x acts unipotently on $V_\ell(M)$ if $\ell = \ell_x$).

Therefore, with the notations of the previous section, the characters ψ_ℓ factor through a diagram

$$\begin{tikzcd} \Gamma_K \arrow[r] & \mathrm{Gal}\left(K_S^{\mathrm{ab},(\ell)}/K\right) & \\ & K_{\mathbb{A}}^\times/\overline{K^\times U_S^{(\ell)}} \arrow[u, "\omega"] \arrow[r] & \bar{\kappa}_\ell^\times. \end{tikzcd}$$

We will denote the map $K_{\mathbb{A}}^\times/\overline{K^\times U_S^{(\ell)}} \to \bar{\kappa}_\ell^\times$ again by ψ_ℓ.

b) *Characters of tame inertia.* Let x denote a place of K above ℓ. By Cor. 2.24, we have an exact sequence of $\kappa_\ell[\Gamma_x]$-modules

$$0 \to W_\ell(M)^0 \to W_\ell(M) \to W_\ell(M)^{\mathrm{et}} \to 0, \tag{64}$$

where the module $W_\ell(M)^{\mathrm{et}}$ is unramified. The image of inertia on the h-dimensional vector space $W_\ell(M)^0$ is isomorphic to $\left(\kappa_\ell^{[h]}\right) \subset \mathrm{Aut}_{\kappa_\ell}(W_\ell(M))$, i.e. a maximal cyclic 'Cartan' subgroup C of order $q_\ell^h - 1$. The image of Γ_x is then contained in the normalizer N of C in $\mathrm{Aut}_{\kappa_\ell}(W_\ell(M))$.

The action of inertia I_x on the subquotient W'_ℓ of $\bar{\kappa}_\ell \otimes_{\kappa_\ell} W_\ell(M)$ is either

i) trivial or
ii) given by a fundamental character $\zeta_{\kappa_\ell,h}$ of level $h \leq r$.

The action induced by N on a 1-dimensional sub-vector space of $W_\ell(M)$ coincides with that of C. Therefore, in case ii), the fact that W'_ℓ is Γ_x-invariant implies that it is fully ramified; hence the residue field k_x must already contain all $q_\ell^h - 1$-st roots of unity, i.e. $\kappa_\ell^{[h]} \subset k_x$.

By assumption, the extension K_x/F_ℓ is unramified. Putting

$$\rho := [k_x : \kappa_\ell],$$

it follows form local class field theory that the fundamental character of level ρ is essentially the inverse of the local norm residue symbol $\omega_\ell : K_x^\times \to \Gamma_x^{\mathrm{ab}}$ (cf. [**Neu**] III, 7.5 p. 63). More precisely:

$$\zeta_\rho \circ \omega : K_x^\times = R_x^\times \times \pi_x^{\mathbb{Z}} \to k_x^\times : x = u \cdot \pi_x^i \mapsto u^{-1} \mod x.$$

For every $h|\rho$, taking the norm maps $N_h : k_x \to \kappa_\ell^{[h]}$ in $\mathbb{Z}[\mathrm{Gal}(k_x/\kappa_\ell)]$, we have the relations $\zeta_{\kappa_\ell,h} = N_h \circ \zeta_{\kappa_\ell,\rho}$.

Let $\bar{x}$ be the unique place of K lying below the place $\bar{\lambda}$ of $\bar{F}$, and $k_{\bar{x}} \subset \bar{\kappa}_\ell$ its residue field. The above observations imply that, under the map

$$\psi_\ell : K_{\mathbb{A}}^\times / \overline{K^\times U_S^{(\ell)}} \to \bar{\kappa}_\ell^\times,$$

the image of any $a_{\bar{x}} \in R_{\bar{x}}^\times \subset K_{\mathbb{A}}^\times$ is given by:

$$a_{\bar{x}} \mapsto \left(a_{\bar{x}}^{-1} \mod \bar{\lambda}\right)^{\mu(\bar{x})},$$

where $\mu(\bar{x}) \subset \mathbb{Z}[\mathrm{Gal}(k_x/\kappa_\ell)]$, and $\mu_{\bar{x}}$ is either zero or equal to N_h, for some h dividing $[k_{\bar{x}} : \kappa_\ell]$.

For any place x of K lying over ℓ, we can identify $\mathrm{Hom}_{\kappa_\ell}(k_x, k_{\bar{x}})$ with

$$\Sigma_x = \mathrm{Hom}_{\kappa_\ell}(K_x, K_{\bar{x}}).$$

Every $\sigma \in \Sigma$ extends uniquely to a $\bar{\sigma} \in \Sigma_{x(\sigma)}$ where $x(\sigma)$ is the unique place of K above ℓ such that $\bar{x} = \sigma x(\sigma)$. With these notations, we obtain, for every

$$a_\ell \in \prod_{x|\ell} R_{\bar{x}}^\times \subset K_{\mathbb{A}}^\times$$

that

$$\psi_\ell : a_\ell \mapsto \left(a_\ell{}^{-1} \mod \bar{\lambda}\right)^{\mu(\ell)}, \tag{65}$$

for some

$$\mu(\ell) \in \mathcal{H} := \left\{\sum_{\sigma\in\Sigma} n_\sigma[\sigma];\ 0 \le n_\sigma \le 1\right\} \subset \mathbb{Z}[\Sigma].$$

As $\mathcal{H}$ is a finite set, there a $\varphi \in \mathcal{H}$ and an infinite subset $\mathcal{L}_2$ of $\mathcal{L}_1$ such that

$$\mu(\ell) = \varphi$$

for all $\ell \in \mathcal{L}_2$.

c) *Characters of* $\mathbb{S}_{K,S}$. We check that φ is a character of $\mathbb{T}$. For every

$$x \in K^\times \cap U_S,$$

we have $\psi_\ell(x) = 1$, as ψ_ℓ factors through $K^\times$. On the other hand, as $x \in U_S \subset K^\times_{\mathbb{A}}$, we obtain from (65) that

$$\psi_\ell(x) \equiv \varphi(x^{-1}) \mod \bar{\lambda}.$$

Thus $\varphi(x) \equiv 1 \mod \bar{\lambda}$, and this for all $\ell \subset \mathcal{L}_2$. As $\mathcal{L}_2$ is infinite, this implies the equality $\varphi(x) = 1$. Hence $\varphi \in X(\mathbb{T})$.

We extend φ to some character $\Phi' = (f, \varphi) \in X(\mathbb{S}_{K,S})$. The character

$$\chi_\ell := \psi_\ell \cdot (\Phi'_\ell)^{-1} \mod \bar{\lambda} : K^\times_{\mathbb{A}} / \overline{K^\times U^\ell} \to \bar{\kappa}^\times_\ell$$

factors through C, so $\chi_\ell \in X(C)$. The $X(C)$ group being finite, we again find an infinite set $\mathcal{L}' \subset \mathcal{L}_2$ of points ℓ with the same character

$$\chi = \chi_\ell \in X(C).$$

Upon replacing Φ' by $\Phi = (\chi \cdot f, \varphi)$, we then obtain, for all $\ell \in \mathcal{L}'$:

$$\Phi_\ell \equiv \psi_\ell \mod \bar{\lambda}.$$

□

§3. Abelian residual representations.

THEOREM 3.10. *Let K be a finite separable extension of F and M a simple τ-sheaf of rank r over $\mathcal{C}_K$ with characteristic ι and dimension 1. For every closed point ℓ, let $W_\ell^{K,\mathrm{ss}}$ denote the semisimplification of the $\bar{\kappa}_\ell^\times[\Gamma_K]$-module*

$$\bar{\kappa}_\ell \otimes W_\ell(M).$$

If $W_\ell^{K,\mathrm{ss}}$ is isomorphic to a direct sum of 1-dimensional Γ_K-representations over $\bar{\kappa}_\ell$ for an infinite set $\mathcal{L}$ of closed points ℓ of $\mathcal{C}$, then the representations $\bar{F}_\ell \otimes V_\ell$ are isomorphic to a direct sum of 1-dimensional Γ_K-representations, for all ℓ of $\mathcal{C}$.

PROOF. **a)** Let x_0 be a closed point of $\mathcal{C}$ at which M has a model with good reduction, and let k_0 denote its finite residue field of degree $d_0 := [k_0 : \mathbb{F}_q]$ with Frobenius endomorphism $\varphi' = \varphi^{d_0}$. We put $F' := k_0 F$. Let $\bar{\mathcal{C}}'$ be the smooth projective curve with function field F' and put $\mathcal{C}' := u^{-1}(\mathcal{C})$ where u is the morphism $\bar{\mathcal{C}}' \to \bar{\mathcal{C}}$. We define a τ'-sheaf M' on $\mathcal{C}'_{k_0}$ as follows: let the underlying sheaf be given by $M' := u^* M$ and put

$$\tau' = u^*(\tau^{d_0}) : \sigma^{d_0 *} M \to M.$$

For every closed point ℓ of $\mathcal{C}$, we have then an isomorphism of Tate modules

$$k_0 \otimes_{\mathbb{F}_q} V_\ell(M) \cong \bigoplus_{\ell' \in u^{-1}(\ell)} V_{\ell'}(M').$$

In particular, if $\bar{F}_\ell \otimes_{F'_{\ell'}} V_{\ell'}(M')$ is isomorphic to a direct sum of 1-dimensional representations then so is $\bar{F}_\ell \otimes_{F_\ell} V_\ell(M)$. Thus we may assume that $k_0 = \mathbb{F}_q$.

b) Let us now write $W_\ell^{K,\mathrm{ss}} = \oplus_{i=1}^r W_\ell^i$, where the action of Γ_K on W_ℓ^i is given by a character χ_ℓ^i. By Prop. 3.9, there exist an infinite subset $\mathcal{L}$, and for every i, compatible systems of 1-dimensional Γ_K-representations $V_\ell(\Phi^i)$ and $V_\ell(\eta^i)$ over $\mathcal{C}\backslash\{\ell_0\}$ such that

$$\chi_\ell^i \equiv \Phi_\ell^i \cdot \eta_\ell^i \mod \bar{\lambda}.$$

We put

$$V'_\ell := \bigoplus_{i=1}^{r} V_\ell(\Phi^i) \otimes V_\ell(\eta^i).$$

This gives a compatible system V' over $\mathcal{C}\backslash\{\ell_0\}$ of integral semisimple Γ_K-representation over $\bar{F}_\ell$.

c) For every x in $X^{\text{good}}\backslash\{x_0\}$, the characteristic polynomials of Frobenius

$$P_x(M;T), \text{ resp. } P_x(V';T) \in \bar{F}(T)$$

are well defined, independent of ℓ and have integral coefficients at every place of $\bar{F}$ above ℓ, for every closed point $\ell \neq \{\ell_0, \ell_x\}$ of $\mathcal{C}$. If the point ℓ is contained in $\mathcal{L}$, we have that $W_\ell^{K,\text{ss}} \cong T'_\ell/\lambda$, and hence:

$$P_x(M;T) \equiv P_x(V';T) \mod \bar{\lambda}$$

This congruence now holds for infinitely many ℓ, which shows that we must have an equality

$$P_x(M;T) = P_x(V';T) \in \bar{F}(T),$$

and this for all but a finite number of places of K.

d) By the Chebotarev density theorem, the Frobenius substitutions are dense in Γ_K. It follows that, for all closed point ℓ of $\mathcal{C}$ and all $\sigma \in \Gamma_K$, that the characteristic polynomial of the action of σ on $V_\ell(M)$ and V'_ℓ coincide. As both systems are systems of semisimple representations (Prop. 3.7 ii), it follows, by the Brauer-Nesbitt theorem, that $V_\ell(M) \cong V'_\ell$ for all ℓ. Hence the Galois module $\bar{F}_\ell \otimes V_\ell(M)$ is a direct sum of 1-dimensional representations. □

§4. Rank 2.

PROPOSITION 3.11. *Let K be a finite separable extension of F. For any τ-sheaf M over $\mathcal{C}_K$ of rank 2, with characteristic ι and dimension 1, there exists a finite extension K' of K such that for all but a finite number of closed points ℓ of $\mathcal{C}$ we have either*

- *that $W_\ell^{K',\text{ss}}$ (cf. Thm. 3.10) is a sum of 1-dimensional $\Gamma_{K'}$-representations over $\bar{\kappa}_\ell$, or*
- *that the residual representation*

$$\rho_\ell : \Gamma_{K'} \to \operatorname{Aut}_{\kappa_\ell}(W_\ell(M))$$

is surjective, i.e. $\Omega_\ell := \bar{\rho}_\ell(\Gamma_{K'}) = \operatorname{Aut}_{\kappa_\ell}(W_\ell(M))$

PROOF.

a) We need to list the possibilities for the subgroup Ω_ℓ of $\operatorname{Aut}_{\kappa_\ell}(W_\ell(M))$. If $\mathbb{F}$ is a finite field, any subgroup G of $\mathrm{GL}(r,\mathbb{F})$ whose projection under

$$\Pi : \mathrm{GL}(r,\mathbb{F}) \to \mathrm{PGL}(r,\mathbb{F})$$

is surjective contains $\mathrm{SL}(r, \mathbb{F})$. For $r = 2$, there is Dickson's well known classification of the maximal subgroups of $\mathrm{PGL}(2, \mathbb{F})$:

PROPOSITION 3.12 (Dickson [**Hup**], Thm. 8.27). *Any proper subgroup of the group* $\mathrm{PGL}(2, \mathbb{F})$ *over a finite field* $\mathbb{F}$ *of characteristic* p *is contained in either*

i) *a Borel subgroup;*
ii) *a dihedral group* $\mathcal{D}$ *of order* $2m$*,* m *prime to* p*;*
iii) $\mathrm{PSL}(2, \mathbb{F})$*;*
iv) *a conjugate of the subgroup* $\mathrm{PGL}(2, \mathbb{F}')$ *for some proper subfield* $\mathbb{F}'$ *of* $\mathbb{F}$*; or*
v) *a subgroup isomorphic to one of the groups* $A_4,\ A_5,\ S_4$.

b) We may replace K by a finite separable extension such that the semistable filtration for M (see Thm. 1.26) is defined over $\mathcal{C}_K^{\mathrm{sst}}$. Let Λ be the finite subset of closed points of $\mathcal{C}$, containing the points

- of $\iota(X^{\mathrm{bad}})$,
- of $\mathcal{C}^{\mathrm{sst}}$,
- for which $q_\ell := \#\kappa_\ell \leq 5$, and those
- which ramify in K/F.

Let ℓ be a closed point of $\mathcal{C}\backslash\Lambda$.

Consider a point ℓ in $\mathcal{L}$. As the maximal exterior power $\wedge^2\mathcal{M}^{\max}$ of the maximal model $\mathcal{M}^{\max}$ is good at any place x above ℓ, Cor. 2.24, applied to $r = 1$, tells us that $\det\overline{\rho}_\ell$ is surjective. Therefore, either $\Omega_\ell = \mathrm{GL}(2, \kappa_\ell)$ or $\Pi(\Omega_\ell)$ is a proper subgroup of $\mathrm{PGL}(2, \kappa_\ell)$, where case iii) is then ruled out.

Choosing a splitting of the map $I_x \to I_K^t$, we obtain from Cor. 2.24 that Ω_ℓ contains a cyclic subgroup of order $q_\ell^h - 1$, where $h \leq 2$ is the height of $\hat{\mathcal{M}}_\ell$. This yields that $\Pi(\Omega_\ell)$ contains a maximal (split or non-split) Cartan group, i.e. a maximal cyclic subgroup (of order $q_\ell \pm 1$), $\Pi(C)$ of $\mathrm{PGL}(2, \kappa_\ell)$, which excludes case iv). Also, $\Pi(C)$ is cyclic of order $q_\ell \pm 1 > 5$, whereas the groups in v) only have cycles of order at most 5; hence case v) is also excluded.

If $\Pi(\Omega_\ell)$ is contained in a Borel subgroup - case i) -, then it follows that W^{ss} is a sum of two 1-dimensional Γ_K-representations.

c) So suppose that $\Pi(\Omega_\ell)$ is as in ii). The maximal cyclic subgroup H of order at least 2 inside $\mathcal{D}$ is uniquely determined because $m > 2$ as $2m \geq q_\ell \pm 1 > 5$. Consider the quadratic character

$$\epsilon_\ell : \Gamma_K \xrightarrow{\Pi\circ\overline{\rho}_\ell} \mathcal{D} \to \mathcal{D}/H.$$

We claim that, for all but a finite number of closed points ℓ of $\mathcal{C}\backslash\Lambda$, this character is unramified at all places of K

Let x be a place of K. We distinguish 2 cases:

- $\boxed{\ell_x \notin \Lambda}$. In particular, the τ-sheaf M has a good model at x and x is not ramified in K/F. If $\ell \neq \ell_x$, then the action of I_x on $W_\ell(M)$ is trivial. Suppose that $\ell = \ell_x$. Let h denote the height of $\hat{\mathcal{M}}_\ell$ (cf. Def. 2.20).
  - $\boxed{h = 1}$. From Cor. 2.24, we see that the action via conjugation of the subquotient $\overline{\rho}_\ell(I_x^t) \cong (\kappa_\ell)^\times$ on the p-group $\overline{\rho}_\ell(I_x^p)$, defines a

$\kappa_\ell^\times$-module structure on $\overline{\rho}_\ell(I_x^p)$. As ϵ_ℓ has order 2, and $q_\ell > 5$, it follows that $\epsilon_\ell|_{I_x^p} = 1$. The image C of tame inertia in $\mathrm{PGL}_2(\kappa_\ell)$ is a maximal cyclic subgroup, of order $q_\ell - 1$. Therefore C must coincide with $H \subset \mathcal{D}$ and $\epsilon_\ell|_{I_x} = 1$.

– $\boxed{h = 2}$. Wild inertia acts trivially on $W_\ell(M)$ and the image C of tame inertia in $\mathrm{PGL}_2(\kappa_\ell)$ is a maximal cyclic subgroup, of order $q_\ell + 1$. Therefore C must coincide with $H \subset \mathcal{D}$ and $\epsilon_\ell|_{I_x} = 1$.

- $\boxed{\ell_x \in \Lambda}$. If $\mathcal{M}$ is good at x then I_x acts trivially on $W_\ell(M)$. If $\mathcal{M}$ is not good at x, then there exists an exact sequence

$$0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$$

of good τ-sheaves on $\mathcal{C}_{K_x}^{\mathrm{sst}}$. The reduction $\overline{\mathcal{M}}_1$ of $\mathcal{M}_1$ at x extends to a τ-sheaf on $\mathcal{C}_{k_x}$, which is isomorphic to the maximal τ-sheaf contained in $\overline{\mathcal{M}}$, the reduction of $\mathcal{M}$ at x. The eigenvalue of Frobenius on $T_\ell(\mathcal{M}_1)$ is hence an integral function $\alpha_1 \in \mathbf{A}$, independent on the choice of $\ell \notin \Lambda$. Letting α_2 denote the eigenvalue of Frobenius on $T_\ell(\mathcal{M}_2)$, the action of any Frobenius lift by conjugation group $\overline{\rho}_\ell(I_x^p)$, is given by multiplication by $\beta := \alpha_1/\alpha_2$.

On the other hand, by Remark 3.2, the maximal exterior power

$$\wedge^{\mathrm{top}} T_\ell(M)$$

is isomorphic to $T_\ell(\phi)$, for some Drinfeld $\mathbf{A}$-module ϕ which has good reduction at x, since $\wedge^{\mathrm{top}} T_\ell(M)$ is unramified. By Prop. 3.4 and the fact that $\dim M = 1$, we know that the eigenvalue $\alpha \in \mathbf{A}$ of Frobenius on $T_\ell(\phi)$ satisfies

$$v_{\ell_x}(\alpha) = [k_x : \mathbb{F}_q],$$

whereas $v_{\ell_x}(\alpha_1)$ is an integer multiple of $[k_x : \mathbb{F}_q]$. As we have

$$\alpha = \alpha_1 \cdot \alpha_2,$$

and thus $\alpha_2 \in F$, this yields that $v_{\ell_x}(\beta) \neq 0$. Therefore

$$\beta^2 - 1 \neq 0,$$

and hence $\beta^2 - 1$ is divisible by only a finite number of maximal ideals in $\mathbf{A}$. It follows that, with the exception of a finite number of closed points ℓ of $\mathcal{C}\backslash\Lambda$, the action of Frobenius Frob_x by conjugation on $\overline{\rho}_\ell(I_x^p)$, does not have order 2 mod ℓ. Hence, for all but a finite number of closed points ℓ of $\mathcal{C}$, we have $\epsilon_\ell|_{I_x^p} = 1$.

Let $K^{(\ell)}$ be the field fixed by the kernel of ϵ_ℓ. As the extension $K^{(\ell)}/K$ is unramified at all places of K, there exist only a finite number of possibilities for $K^{(\ell)}$, by Minkowski's theorem. If we call K' the compositum of all these fields, then

$$\epsilon_\ell|_{\Gamma_{K'}} = 1$$

for all $\ell \notin \Lambda$. We conclude that - in case ii) - the group $\overline{\rho}_\ell(\Gamma_{K'}) \subset H$ is abelian, and therefore, $W_\ell^{K',\mathrm{ss}}$ is isomorphic to the sum of two 1-dimensional representations. □

§5. Rank 2: Proof of Conj. 3.1.

THEOREM 3.13. *Let K be a finite extension of F and M a simple τ-sheaf over $\mathcal{C}_K$ of rank at most 2, with characteristic $\iota : \operatorname{Spec} K \to \mathcal{C}$, dimension 1 and absolute endomorphism ring* $\mathbf{A}$. *The image of the representation $\overline{\rho}_{\mathrm{ad}}$ on the $\kappa_{\mathrm{ad}}[\Gamma_K]$-module $W_{\mathrm{ad}}(M)$ is open in $\mathrm{GL}_r(\kappa_{\mathrm{ad}})$.*

PROOF. If $r = 1$, then M potentially has a good model at all places x of K. For every finite place x, the action of inertia on $W_\ell(M)$ is then trivial for $\ell \neq \ell_x$, whereas, by Cor. 2.24, its image under $\overline{\rho}_\ell$ is surjective for $\ell = \ell_x$. This proves the theorem for $r = 1$.

Suppose now that $r = 2$.

a) We may assume that K is a separable extension of F: For any finite extension K of F, then, if M is a τ-sheaf over $\mathcal{C}_K$, the τ-sheaf $M' := (\sigma^c)^*M$ is a τ-sheaf with coefficients in a separable subextension $K' \subset K$ of F, for some integer c. As

$$W_\ell(M) \cong W_\ell(M'),$$

for every closed point ℓ of $\mathcal{C}$, the theorem holds for K/F if and only if it holds for the extension K'/F.

It now follows from Prop. 3.11 either

i) that $\overline{\rho}_\ell$ is surjective for almost all ℓ, or

ii) that $W_\ell^{K',ss}$ is the direct sum of two 1-dimensional Galois representations, for an infinite set $\mathcal{L}$ of closed points ℓ of $\mathcal{C}$. By Thm. 3.10, it follows from this that the $F_\ell[\Gamma_{K'}]$-modules $V_\ell(M)$ are isomorphic to the direct sum of two 1-dimensional Galois representations. However, by the Tate conjecture (Thm. 3.7), this shows that $\operatorname{End}_{K'}(M)$ is larger than $\mathbf{A}$, a contradiction.

b) Let Λ be the finite subset of closed points of $\mathcal{C}$ consisting of

- the points of $\iota(X^{\mathrm{bad}})$,
- those which ramify in K/F, and
- those for which $\Omega_\ell \neq \operatorname{Aut}(W_\ell(M))$.

Let us put $H_\ell^{[0]} := \Omega_{\mathrm{ad}} \cap \mathrm{GL}_r(\kappa_\ell) \subset \mathrm{GL}_r(\kappa_{\mathrm{ad}})$. We claim that

$$H_\ell^{[0]} = \mathrm{GL}_r(\kappa_\ell)$$

for all $\ell \notin \Lambda$, which proves the theorem. For a proof of this claim, we refer to the proof of *'Conj. 3.1 + 3.14 for finite $\Lambda \Rightarrow$ Conj. 3.14 for Λ_{ad}'*, part **c)**, in the next section. □

III. Image of the adelic representation

Let K be a finite extension of F and M a τ-sheaf over $\mathcal{C}_K$. For any finite set Λ of closed points of $\mathcal{C}$, we denote by Γ_Λ the image of the representation

$$\rho_\Lambda : \Gamma_K \to \prod_{\ell \in \Lambda} \operatorname{Aut}_{F_\Lambda}(V_\Lambda(M)), \tag{66}$$

where the $F_\Lambda[\Gamma_K]$-module $V_\Lambda(M)$ was defined in (13).

CONJECTURE 3.14. *Let K be a finite extension of F and M a simple τ-sheaf over $\mathcal{C}_K$, with characteristic $\iota : \operatorname{Spec} K \to \mathcal{C}$, dimension 1 and absolute endomorphism ring $\mathbf{A}$. The image Γ_Λ of the representation ρ_Λ of Γ_K on the module $V_\Lambda(M)$ is open in $\mathrm{GL}_r(F_\Lambda)$, for any set Λ of closed points of $\mathcal{C}$.*

REMARK 3.15. Suppose that Λ is a finite set. In [**Pi2**] Thm. 0.1, Pink proves the above conjecture in the case that M is a t-motive, associated to a Drinfeld module ϕ. After discussions with him, I am convinced that existing methods (beyond the scope of this thesis) would suffice to prove this conjecture. If this is true indeed, then all the results which we will state for Drinfeld modules in what follows, carry over to general M of dimension 1 as well.

We now prove that Conj. 3.1, together with Conj. 3.14 for finite sets Λ, imply Conj. 3.14 for Λ_{ad}, the set of all closed points of $\mathcal{C}$ (and hence, a fortiori, for all Λ).

PROOF OF CONJ. 3.1 + 3.14 FOR FINITE $\Lambda \Rightarrow$ CONJ. 3.14 FOR Λ_{ad}.

a) Let ℓ be a closed point of $\mathcal{C}$. Fixing a basis for $T_\ell(M)$, we put

$$G_\ell := \mathrm{GL}_r(\mathbf{A}_\ell) \cong \mathrm{Aut}_{\mathbf{A}_\ell}(T_\ell(M))$$

and use the notations we introduced in (48) and following. We consider the subgroup

$$H_\ell = \Gamma_{\mathrm{ad}} \cap \mathrm{GL}_r(A_\ell) \subset G_\ell.$$

b) Let Λ be the finite subset of closed points of $\mathcal{C}$ consisting of

- the points of $\iota(X^{\mathrm{bad}})$,
- the points for which $\#\kappa_\ell = 2$,
- those which ramify in K/F, and
- those for which $\Omega_\ell \neq \mathrm{Aut}(W_\ell(M))$ (this is a finite set of points by Conj. 3.1).

By assumption, Conj. 3.14 holds for this set Λ. To show that Γ_{ad} is open in $\mathrm{Aut}_{F_{\mathrm{ad}}}(V_{\mathrm{ad}}(M))$, we need to prove that $H_\ell = G_\ell$ for all $\ell \not\in \Lambda$.

c) For all places x of K above ℓ, M has a good model at x, so the image $\rho_{\mathrm{ad}}(I_x)$ of inertia is contained in H_ℓ. Further, by Cor. 2.24, there is a filtration

$$0 \to W_\ell^0 \to W_\ell(M) \to W_\ell^{\mathrm{et}} \to 0$$

such that I_x acts trivially on the étale quotient W_ℓ^{et} and where the image $\overline{\rho}_\ell(I_x)$ of I_x in $\mathrm{Aut}(W_\ell^0)$ is a maximal non-split torus. In particular, there exists a block matrix

$$B := \begin{pmatrix} \alpha & * \\ 0 & 1 \end{pmatrix} \in H_\ell^{[0]}$$

where α is a scalar matrix in $\mathrm{GL}_h(\kappa_\ell) \cong \mathrm{Aut}(W_\ell^0)$ which is not the identity matrix. The group H_ℓ being closed under conjugation by Γ_{ad}, its quotient $H_\ell^{[0]}$ is closed under conjugation by $\mathrm{GL}_r(\kappa_\ell)$. One easily sees, by considering a product of conjugates of B, that one obtains a unipotent matrix with only one non-zero entry off the diagonal. The $\mathrm{GL}_r(\kappa_\ell)$-conjugates of such an element generate $\mathrm{SL}_r(\kappa_\ell)$. As the image of inertia on the determinant of $\bar{\rho}_\ell$ is surjective (by Remark 3.2, this is a consequence of the case $r = 1$), it follows that $H_\ell^{[0]} = \mathrm{GL}_r(\kappa_\ell)$.

d) Finally, we have to prove that $H^{[k]} \cong Y$ for each $k \geq 1$. By Cor. 2.24, there exists a non-scalar element $h \in H^{[i]}$, for each $i > 1$. From the commutative diagram

$$\begin{array}{ccccccc} G^{[0]} & \times & G^{[i]} & \to & G^{[i]} & : (g, g_1) & \mapsto g^{-1}g_1g \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathrm{GL}_r(\kappa_\ell) & \times & Y & \to & Y & : (g, h) & \mapsto g^{-1}hg, \end{array} \tag{67}$$

we see, as $G^{[0]} = \mathrm{GL}_r(\kappa_\ell)$, that $H^{[i]}$ is closed under conjugation by $\mathrm{GL}_r(\kappa_\ell)$ for every $i \geq 1$.

We then make use of the following lemma, whose elementary proof is omitted:

LEMMA 3.16. *Let L be a field with $\#L > 2$. Every subgroup of* $\mathrm{Mat}_{r\times r}(L)$ *which contains a non-scalar element and is closed under conjugation by* $\mathrm{GL}_n(L)$*, contains* $\mathrm{Mat}^0_{r\times r}(L)$*, the subgroup of matrices of trace 0.*

Notice that, with our notation, $\mathrm{Mat}^0_{r\times r}(L) \cong (\mathrm{SL}_r(\mathbf{A}_\ell))^{[k]} \subset Y$. On the other hand, one sees from Cor. 2.24, that the trace map $\mathrm{Tr} : H^k \to \kappa_\ell$ is surjective, which finally shows that $H^k = Y$, and this for each $k > 1$. This concludes the proof of $H_\ell = G_\ell$. □

IV. The adelic Mumford-Tate conjecture

Let E denote the absolute endomorphism ring $\mathrm{End}_{\bar{K}}(M)$ of M and put

$$E^0 := E \otimes_A F.$$

Set $K(E)$ to be the finite extension of K generated by the coefficients of elements in E and $\Gamma_{K(E)}$ its absolute Galois group. We denote by Γ_{ad} the image of the representation

$$\rho_{\mathrm{ad}} : \Gamma_{K(E)} \to \mathrm{End}_{F_{\mathrm{ad}}}(V_{\mathrm{ad}}(M)) \tag{68}$$

(see equation (14) for the definition of V_{ad}). Finally, there is a natural embedding of E^0 into $\mathrm{Aut}(V_{\mathrm{ad}}(M))$, and we consider its centralizer

$$C_{\mathrm{ad}} \subset \mathrm{Aut}_{F_{\mathrm{ad}}}(V_{\mathrm{ad}}(M)) \tag{69}$$

inside $\mathrm{End}_{F_{\mathrm{ad}}}(V_{\mathrm{ad}}(M))$.

CONJECTURE 3.17. *Let K be a finite extension of F. If M is a simple τ-sheaf over $\mathcal{C}_K$ with characteristic ι and dimension 1, then Γ_{ad} is open in*

$$C_{\mathrm{ad}} \subset \mathrm{Aut}_{F_{\mathrm{ad}}}(V_{\mathrm{ad}}(M)) \,.$$

REMARK 3.18. If M is pure, then it corresponds to the t-motive $M(\phi)$ of a Drinfeld module ϕ. In [**Pi1**], Pink introduced Hodge structures associated to ϕ, and subsequently proved that the Hodge group $G_{H(\phi)}$ of this structure structure is isomorphic to the centralizer C in $\mathrm{GL}_{r,F}$ of $E = \mathrm{End}_{\bar{K}}(M)$. The above conjecture can be seen as the analogue of the **'Grothendieck + Mumford-Tate'** conjecture in the classical theory of motives over number fields:

CONJECTURE 3.19 (Grothendieck-Mumford-Tate, [**Se7**] §9–13, Conj. 11.4?). *Let L be a number field, with absolute Galois group Γ_L. For a motive E defined over L, we denote by $V_p(E)$ the p-adic cohomology of E, and put $\mathbb{Q}_{\mathrm{ad}} := \prod'_p \mathbb{Q}_p$ (the ring of finite adeles of $\mathbb{Q}$) and*

$$V_{\mathrm{ad}}(E) := \prod_p{}' V_p(E).$$

Suppose the associated Mumford-Tate group $G_{M(E)}$ is connected and E is maximal. The image Γ_{ad} of the adelic representation

$$\rho_{\mathrm{ad}} : \Gamma_L \to \mathrm{Aut}_{\mathbb{Q}_{\mathrm{ad}}}(V_{\mathrm{ad}}(E))$$

is open in $G_{M(E)}(\mathbb{Q}_{\mathrm{ad}})$.

Conjecture 3.17 is an immediate consequence of Conj. 3.14, as we now show.

PROOF OF CONJ. 3.14 $\Rightarrow$ CONJ. 3.17. The ideas of this proof originate from [**Pi2**], Thm. 0.2 (cf. p. 408).

Let M be a τ-sheaf satisfying the condition of Conj. 3.17. If we put $\mathbf{A}' := E$ and $F' = E^0$, then by Prop. 3.8, F' is a finite extension of F. We remark that F' has a unique place ∞' above ∞. We put $\mathcal{C}' := \operatorname{Spec} A'$ and consider the finite morphism $f : \mathcal{C}' \to \mathcal{C}$.

By Lemma 1.20, the τ-sheaf M, endowed with an action of $\mathbf{A}'$, induces a τ-sheaf M' on $\mathcal{C}'_K$ such that $f_* M' = M$, denoting the induced morphism $\mathcal{C}'_K \to \mathcal{C}_K$ again by f. However, $\mathcal{C}'$ is not necessarily smooth, so we consider the normalization $\tilde{\mathcal{C}}'$ of $\mathcal{C}'$ and the morphism $\tilde{f} : \tilde{\mathcal{C}}' \to \mathcal{C}'$.

Consider the τ-sheaf

$$M^{\star} := \tilde{f}^* M'$$

on $\tilde{\mathcal{C}}'_K$. We get adjunction morphisms $\tilde{f}_* \tilde{f}^* \to id$ and $id \to \tilde{f}^* \tilde{f}_*$, which are isomorphisms outside the finite set S of singularities of $\mathcal{C}'$. Thus we see that the τ-sheaf $f_* M^{\star}$ on $\mathcal{C}'_K$ is isogenous to M'. As now Tate modules are determined by τ-sheaves up to isogeny (by the Tate conjecture, Thm. 3.7), we can reduce ourselves to the case that $\tilde{\mathcal{C}}' = \mathcal{C}'$.

Upon replacing K by a finite extension, we may assume $E = \mathrm{End}_K(M)$, and hence $\mathrm{End}_K(M') = \mathbf{A}'$. By Conj. 3.14 and the above, this implies, for every closed point ℓ' of $\mathcal{C}'$, that the image of the representation

$$\Gamma_K \to V_{\mathrm{ad}'}(M) := \prod_{\ell'} \mathrm{Aut}(V_{\ell'}(M'))$$

has finite index. Finally, the isomorphism $V_\ell(M) \cong \bigoplus_{\ell'|\ell} V_{\ell'}(M')$ of Tate modules induces an isomorphism

$$C_{\mathrm{ad}} \cong \mathrm{Aut}_{F'_{\mathrm{ad}'}}(V_{\mathrm{ad}'}(M)),$$

which concludes the proof. $\square$

Finally, we can, in the case of Drinfeld modules, conclude the following result, by Thm. 3.13, Remark 3.15 and the above implications:

THEOREM 3.20. *Let K be a finite extension of F. If M is a Drinfeld module defined over K of rank at most 2, with characteristic $\iota^* : \mathbf{A} \hookrightarrow K$, then Γ_{ad} is open in $C_{\mathrm{ad}} \subset \mathrm{End}_{F_{\mathrm{ad}}}(V_{\mathrm{ad}}(\phi))$.*

REMARK 3.21. We remark that if Conj. 3.14 holds for all finite extensions of F, then it also holds for all finitely generated fields K containing F. The same statement follows for Conj. 3.17, and, in particular, Thm. 3.20 holds for all such fields. This can be shown as follows, using the ideas in [**Pi2**] Thm. 1.4.

PROOF. Let K be finitely generated fields containing F and M a τ-sheaf satisfying the condition of Conj. 3.14. By the Tate conjecture (Thm. 3.7) the sub-F_ℓ-algebra $F_\ell\Gamma_\ell$ of $\mathrm{End}_{F_\ell}(V_\ell(M))$ generated by Γ_ℓ is equal to $\mathrm{End}_{F_\ell}(V_\ell(M))$, and this for every closed point ℓ. By lemma [**Pi2**], 1.5, there is an open normal subgroup $\Gamma_1 \subset \Gamma_\ell$ such that for any subgroup $\Omega' \subset \Gamma_\ell$ for which $\Omega'\Gamma_1 = \Gamma_\ell$, we have

$$F_\ell\Omega' = F_\ell\Gamma_\ell$$

(denoting by $F_\ell\Omega'$ the sub-F_ℓ-algebra of $\mathrm{End}_{F_\ell}(V_\ell(M))$ generated by Ω'). Taking the extension $\tilde{K}$ of K fixed by Γ_1, one denotes by $\tilde{X}$ be the normalization of X in $\tilde{K}$ and by π the morphism $\tilde{X} \to X$.

By lemma [**Pi2**], 1.6, there exists a point x of X such that K', the residue field of x, is a finite extension of F, and such that $\pi^{-1}(x)$ is irreducible. Letting Ω'_x be the image of $\Gamma_{K'}$ on $V_\ell(M)$, seen as a subgroup of Γ_ℓ, we then have

$$\Omega'_x\Gamma_1 = \Gamma_\ell.$$

Hence

$$F_\ell\Omega'_x = \mathrm{End}(V_\ell(M)),$$

and therefore the reduction $\overline{M}_x$ of M at x has $\mathrm{End}_{\bar{K}}(\overline{M}_x) = \mathbf{A}$, by the Tate conjecture (Thm. 3.7). Assuming that Conj. 3.14 holds for K', a finite extension of F, we get that the image of $\Gamma_{K'}$ is open in $\mathrm{GL}_r(F_{\mathrm{ad}})$. A fortiori, Γ_{ad} is open in $\mathrm{GL}_r(F_{\mathrm{ad}})$. $\square$

CHAPTER 4

Galois criteria

I. Galois criterion for good reduction

Let R be a complete discrete valuation ring with fraction field K, *perfect* residue field k, uniformizer π and valuation v. Let x denote the closed point of Spec R and Γ_K (resp. I_K) absolute Galois group of K (resp. its inertia subgroup). Let ι denote a morphism Spec $K \to \mathcal{C}$. In this chapter, we propose a τ-sheaf analog for the Néron-Ogg-Shafarevič criterion for good reduction of abelian varieties (cf. [**BLR**], 7.4, Thm. 5):

THEOREM 4.1 (**Galois criterion for good reduction**). *Let M be a τ-sheaf on $\mathcal{C}_K$ with characteristic ι and ℓ a closed point of $\mathcal{C}$ such that $\hat{M}_\ell$ is smooth. If the inertia group I_K of K acts trivially on $T_\ell(M)$, then there exists a good model $\mathcal{M}$ over $\mathcal{C}_R$ for M.*

PROOF. If I_K acts trivially on $T_\ell(M)$, then the $\mathbf{A}_\ell$-module $T_\ell(M)$ yields a representation of $\pi_1(\text{Spec } R)$. Hence, by the correspondence 0.7, the ℓ-adic τ-module $\hat{M}_\ell$ extends to a smooth τ-module $\hat{\mathcal{N}}_\ell$ over $\mathcal{O}_{\hat{\mathcal{C}}_{R,\ell}}$. We then apply the following theorem: □

THEOREM 4.2. *Let M be a τ-sheaf on $\mathcal{C}_K$ with characteristic ι and ℓ a closed point of $\mathcal{C}$ such that $\hat{M}_\ell$ is smooth. If $\hat{M}_\ell$ extends to a τ-sheaf $\hat{\mathcal{N}}_\ell$ on $\hat{\mathcal{C}}_{R,\ell}$ of nondegenerate rank ρ, then there exists a model $\mathcal{M}$ for M on $\mathcal{C}_R$ whose nondegenerate rank is at least ρ.*

The proof of Thm. 4.2 will take up the rest of section **I**.

IDEA/SKETCH OF THE PROOF OF THM. 4.2. As a first approach, let us assume that $\mathcal{C} = \mathbb{A}^1$, that ℓ is a closed point of degree 1 of $\mathbb{A}^1$ and that $\hat{\mathcal{N}}_\ell$ is smooth. In **§5**, we will show how to deal with the general case.

Let t denote a generator for the maximal ideal in $\mathbf{A}$ defining the point ℓ. Let ϖ be the generic point of the special fibre $\mathcal{C}_k$. We can make the following identifications:

$$\begin{aligned} H^0\left(\mathcal{C}_K, \mathcal{O}_{\mathcal{C}_K}\right) &\cong K[t] \\ \mathcal{O}_{\mathcal{C}_{R,\varpi}} &\cong \mathcal{O}_\varpi := R[t]_{(\pi)} \\ \mathcal{O}_{\hat{\mathcal{C}}_{R,\ell}} &\cong K[[t]] \\ \mathcal{O}_{\hat{\mathcal{C}}_{R,\ell}} &\cong R[[t]]. \end{aligned} \tag{70}$$

Let M be a τ-sheaf on $\mathbb{A}^1_K$ with characteristic ι, and let us denote the $K[t]$-module of its global sections by M as well. Supposing the τ-module $\hat{M}_\ell$ over $K[[t]]$ extends to a smooth τ-module $\hat{\mathcal{N}}_\ell$ over $R[[t]]$, the naive idea is to put

$$\mathcal{M} := M \cap \hat{\mathcal{N}}_\ell \subset \hat{M}_\ell.$$

More precisely, we will proceed as follows:

i) The module $\hat{\mathcal{N}}_\ell$ is actually contained in

$$(K \otimes_R R[[t]]) \otimes_{K[t]} M \subset \hat{M}_\ell$$

(Lemma 4.4).

ii) We put $Q := \mathrm{Quot}(R[[t]])$ and denote by B the valuation ring for the valuation v_π in Q (see below for sound definitions). Denoting the stalk of M at the generic point of $\mathcal{C}_K$ by V, the $\mathcal{O}_\varpi$-module

$$\mathcal{N}_\varpi := V \cap \left(B \cdot \hat{\mathcal{N}}_\ell\right) \subset \mathfrak{Q} := Q \otimes_{K(t)} V$$

is free and is of full rank inside V (Lemma 4.5).

iii) Using the fact that $\mathcal{N}_\ell$ is smooth, we show that $\mathcal{N}_\varpi$ is a good τ-module over $\mathcal{O}_\varpi$ (Thm. 4.7).

By Cor. 1.4, the sheaf M together with $\mathcal{N}_\varpi$ define a good model $\mathcal{M}$ over $\mathcal{C}_R$. $\square$

REMARK 4.3. Using the same methods, one can readily establish an analog of Thm. 4.2 and Thm. 4.1 for analytic τ-sheaves.

§1. Notations. We call a monic polynomial

$$h(t) = t^d + \sum_{i=0}^{d-1} h_\nu t^\nu \in R[t]$$

strict if $v(h_\nu) > 0$ for all ν. Every nonzero element $g \in R[[t]]$ has a unique decomposition

$$g = u \cdot \pi^{\nu_g} \cdot \tilde{g}$$

such that $u \in R[[t]]^\times$, $\nu_g \geq 0$ and where $\tilde{g}$ is a strict monic polynomial in $R[t]$ (Weierstraß preparation for $R[[t]]$). We have a valuation v_π on $R[[t]]$ given by $v_\pi(g) := \nu_g$, and extend this valuation to the quotient field Q of $R[[t]]$.

We consider the discrete valuation subring

$$B := \{g \in Q; v_\pi(g) \geq 0\} \subset Q.$$

Its residue field is isomorphic to $k((t))$. Finally, we set $\mathcal{O}_\varpi := R[t]_{(\pi)}$. By the unique factorization in $R[[t]]$, we have

$$\mathcal{O}_\varpi = K(t) \cap B \subset Q.$$

To keep track of all these rings, the following diagram might be useful; we have inclusions from left to right and from top to bottom.

$$
\begin{tikzcd}
R[t] \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, hook] & & R[t]_{(t,\pi)} \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, hook] & & \mathcal{O}_\varpi \arrow[dr, hook] \arrow[dd, hook] & \\
& K[t] \arrow[rr, hook] \arrow[dd, hook] & & K[t]_{(t)} \arrow[rr, hook] \arrow[dd, hook] & & K(t) \arrow[dd, hook] \\
R[[t]] \arrow[rr, hook] \arrow[dr, hook] & & B \cap K[[t]] \arrow[rr, hook] \arrow[dr, hook] & & B \arrow[dr, hook] & \\
& K \otimes_R R[[t]] \arrow[rr, hook] & & Q \cap K[[t]] \arrow[rr, hook] \arrow[d, hook] & & Q \arrow[d, hook] \\
& & & K[[t]] \arrow[rr, hook] & & K((t))
\end{tikzcd}
$$

We extend the Frobenius φ on R to a $\mathbb{F}_q$-linear endomorphism $\sigma := \varphi \otimes id$ on $R[t] = R \otimes \mathbb{F}_q[t]$. This induces in a unique way endomorphisms on all of the aforementioned rings.

§2. A Lemma by Anderson.

LEMMA 4.4. *Let M be τ-sheaf on $\mathbb{A}^1_K$ with characteristic ι and dimension d such that $\hat{M}_\ell$ is smooth at the closed point ℓ of degree 1 of $\mathcal{C}$. Suppose the τ-sheaf $\hat{\mathcal{N}}_\ell$ over $\hat{\mathcal{C}}_{R,\ell}$ is an extension of $\hat{M}_\ell$. The module $\hat{\mathcal{N}}_\ell$ is contained in*

$$(K \otimes_R R[[t]]) \otimes_{K[t]} M \subset \hat{M}_\ell.$$

PROOF. **a)** We want to apply a result of Anderson ([**An2**], Thm. 1). Let $\mathbf{m}$ be a $K[t]$-basis for M. As a locally free $R[[t]]$-module, $\hat{\mathcal{N}}_\ell$ is actually free; let $\mathbf{q}$ be an $R[[t]]$-basis for $\hat{\mathcal{N}}_\ell \subset \hat{M}_\ell$. We express $\mathbf{q}$ in terms of the $K[[t]]$-basis $\mathbf{m}$ for $\hat{M}_\ell$ by means of a matrix $\Psi \in \mathrm{Mat}_{r\times r}(K[[t]])$:

$$\mathbf{q} = \mathbf{m} \cdot \Psi.$$

We have to show that $\Psi \in \mathrm{Mat}_{r\times r}(K \otimes R[[t]])$.

Further, we denote by $\Delta \in \mathrm{Mat}_{r\times r}(K[t])$ and $\hat{\Delta} \in \mathrm{Mat}_{r\times r}(R[[t]])$ the matrix representations of τ on the modules M and $\hat{\mathcal{N}}_\ell$ respectively, i.e. $\tau(\mathbf{m}) = \mathbf{m} \cdot \Delta$ and $\tau(\mathbf{q}) = \mathbf{q} \cdot \hat{\Delta}$. These representations are related to each other by the equation

$$\Delta \cdot {}^{\sigma}\Psi = \Psi \cdot \hat{\Delta}. \tag{71}$$

b) Recall that, for some constant $h \in K^\times$, we have $\det \Delta = h \cdot (t-\theta)^d$ (cf. Example 0.6), where $\iota^* : \mathbf{A} \cong \mathbb{F}_q[t] \to K : t \mapsto \theta$. As $\hat{M}_\ell$ is smooth, we must have $\theta \neq 0$. Let $\tilde{\Delta}$ be the modified adjoint matrix in $\mathrm{Mat}_{r\times r}$ satisfying

$$\tilde{\Delta} \cdot \Delta = (t-\theta)^d.$$

Upon multiplying both sides of equation (71) by $\tilde{\Delta}$, we get:

$$(t-\theta)^d \cdot {}^{\sigma}\Psi = \tilde{\Delta} \cdot \Psi \cdot \hat{\Delta}.$$

c) The equation ${}^{\sigma}z = (t-\theta) \cdot z$ has a nonzero solution $c \in K^{\mathrm{sep}}[[t]]$, as one checks immediately (it is a nonzero element of the Tate module of the Carlitz module, cf. [**Go4**], Ch. 3). The matrix $\Psi' := c^d \cdot \Psi$ then satisfies the equation

$$ {}^{\sigma}Z = \tilde{\Delta} \cdot Z \cdot \hat{\Delta} \tag{72} $$

for $Z \in \mathrm{Mat}_{r\times r}(\bar{K}[[t]])$. We claim that every solution Z for this equation is contained in $\mathrm{Mat}_{r\times r}(K \otimes \bar{R}[[t]])$. Let us write out $Z := \sum_{i=1}^{\infty} Z_i t^i$, introducing matrices $Z_i = (Z_i)_{kl} \in \mathrm{Mat}_{r\times r}(\bar{K})$. For all $i \geq 0$, we set

$$ v(Z_i) := \min_{k,l} \{v\left((Z_i)_{kl}\right)\} \in \mathbb{Z} \cup \{+\infty\}. $$

For $\tilde{\Delta} = (\tilde{\Delta}_{k,l})_{kl}$, considered as a matrix in $\mathrm{Mat}_{r\times r}(K \otimes R[[t]])$, we put

$$ v(\tilde{\Delta}) := \min_{k,l} \left\{ v\left(\tilde{\Delta}_{kl}\right)\right\}; $$

we do the same for $\hat{\Delta}$.

Comparing the coefficients of t^n in equation (72), we get

$$ {}^{\sigma}Z_n = \sum_{i+j+k=n} \tilde{\Delta}_i \cdot Z_j \cdot \hat{\Delta}_k. $$

Thus we see that

$$ q \cdot v(Z_n) \geq v(\tilde{\Delta}) + v(\hat{\Delta}) + \min_{j\leq n} v(Z_j), $$

and it follows by induction on n that

$$ (q-1)v(Z_n) \geq v(\tilde{\Delta}) + v(\hat{\Delta}). $$

This shows that $\Psi' \in \mathrm{Mat}_{r\times r}(K \otimes \bar{R}[[t]])$ indeed.

d) We know distinguish two cases:

i) $\boxed{v(\theta) \leq 0}$. Putting $\zeta := \theta^{-1} \in R$, we rewrite $(t-\theta)$ as

$$ -\zeta^{-1} \cdot (1 - \zeta \cdot t). $$

The power series $c^{-1} \in \bar{K}[[t]]$ satisfies the equation

$$ z = -\zeta^{-1}(1-\zeta\cdot t) \cdot {}^{\sigma}z. $$

An easy calculation shows that the solutions for this equations are contained in $K \otimes \bar{R}[[t]]$. Therefore the matrix

$$ \Psi = c^{-d} \cdot \Psi' $$

is a matrix with coefficients in $K \otimes \bar{R}[[t]]$, and, as it was defined over $K[[t]]$, we may conclude that

$$ \Psi \in \mathrm{Mat}_{r\times r}(K \otimes R[[t]]). $$

ii) $\boxed{v(\theta) > 0}$. We are now in the situation of [**An2**], Thm. 1, p. 52. Consider the matrix $\Psi = c^{-d}\Psi'$, seen as a matrix whose entries are meromorphic functions the open unit disk

$$ \mathcal{D}^0_{\bar{K}} := \{t \in \bar{K};\, v(t) < 1\} $$

(viewed as a rigid analytic space). Following Anderson, one first proves that Ψ has no poles (working over the completion $\mathbf{C}$ of $\bar{K}$, to be more precise). Next, using the fact that Ψ' has entries in $K \otimes \bar{R}[[t]]$ and some estimates on c, one deduces that Ψ has entries in $K \otimes R[[t]]$ indeed.

$\square$

§3. Rational modules defined by formal modules.

LEMMA 4.5. *Let V be an r-dimensional $K(t)$-vector space and put*

$$\mathfrak{Q} := Q \otimes_{K(t)} V.$$

For a given free B-submodule $\mathfrak{B}$ of $\mathfrak{Q}$ of rank r, we define the $\mathcal{O}_\varpi$-module

$$\mathcal{N}_\varpi := V \cap \mathfrak{B}.$$

i) *The $\mathcal{O}_\varpi$-module $\mathcal{N}_\varpi := V \cap \mathfrak{B}$ is free of rank r.*
ii) *The cokernel of $B \cdot \mathcal{N}_\varpi \to \mathfrak{B}$ has finite length as a B-module.*

$$\begin{tikzcd}
\mathcal{O}_\varpi \arrow[dr, hook] \arrow[dd, hook] & \\
& K(t) \arrow[dd, hook] \\
B \arrow[dr, hook] & \\
& Q
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{N}_\varpi \arrow[dr, hook] \arrow[dd, hook] & \\
& V \arrow[dd, hook] \\
\mathfrak{B} \arrow[dr, hook] & \\
& \mathfrak{Q}
\end{tikzcd}$$

PROOF. Let us choose a $K(t)$-basis $\mathbf{v} := (v_1, \ldots, v_r)$ for V. We also fix a B-basis $\mathbf{b} := (b_1, \ldots, b_r)$ for the free module $\mathfrak{B}$ and express $\mathbf{v}$ in terms of $\mathbf{b}$ by means of a matrix $\Omega \in \mathrm{Mat}_{r\times r}(Q)$ as follows: $\mathbf{v} = \mathbf{b} \cdot \Omega$. After dividing $\mathbf{v}$ by a suitable power of π, we can assume that Ω has coefficients in B, so that the elements v_i are contained in $\mathcal{N}_\varpi$.

There exists an $\omega \in \mathbb{Z}$ such that $\pi^\omega \cdot \Omega^{-1}$ has entries in B. Let us write any element $n \in \mathcal{N}_\varpi$, as $n = \mathbf{b} \cdot \Lambda$ with $\Lambda \in \mathrm{Mat}_{r\times 1}(B)$. It then follows that

$$\pi^\omega n = \pi^\omega \mathbf{b} \cdot \Lambda = \mathbf{v} \cdot (\pi^\omega \Omega^{-1}) \cdot \Lambda \subset \mathbf{v} \cdot \mathrm{Mat}_{r\times 1}(B). \tag{73}$$

This shows that $\pi^\omega \mathcal{N}_\varpi$ is contained in the $\mathcal{O}_\varpi$-module generated by $\mathbf{v}$. As $\mathcal{O}_\varpi$ is a noetherian principal ideal domain, the torsion free module $\mathcal{N}_\varpi$ is therefore finitely generated and hence free of rank r. This implies that the cokernel of

$$B \cdot \mathcal{N}_\varpi \to \mathfrak{B}$$

is a torsion module. $\square$

REMARK 4.6. The following example shows that $B \cdot \mathcal{N}_\varpi$ may be strictly smaller than $\mathfrak{B}$. Let α be an element in B with reduction $\bar{\alpha} \in k((t))$. Consider the 2-dimensional $K(t)$-vector space spanned by a basis $\mathbf{v} = (v_1, v_2)$, and the B-module $\mathfrak{B} \subset \mathfrak{Q}$ generated by $\mathbf{b} = (\hat{n}_1, \hat{n}_2)$ such that

$$\mathbf{v} = \mathbf{b} \cdot \begin{pmatrix} 1 & \pi^{-1}\alpha \\ 0 & \pi^{-1} \end{pmatrix}.$$

It is easy to see that, if $\alpha \notin k(t)$, then $V \cap \mathfrak{B}$ is generated by $\mathbf{v}$, such that we have an exact sequence of B-modules

$$0 \to B \cdot \mathcal{N}_\varpi \to \mathfrak{B} \to B/(\pi) \to 0.$$

§4. Nondegenerate formal τ-modules.

THEOREM 4.7. *Let V be τ-module over $K(t)$ and put*

$$\mathfrak{Q} := Q \otimes_{K(t)} V.$$

For a given sub-τ-module $\mathfrak{B}$ over B of $\mathfrak{Q}$ of full rank, we define the $\mathcal{O}_\varpi$-module

$$\mathcal{N}_\varpi := V \cap \mathfrak{B}.$$

The τ-modules $\mathfrak{B}$ over B and $\mathcal{N}_\varpi$ over $\mathcal{O}_\varpi$ have the same nondegenerate rank.

PROOF. **a)** Let $\overline{\mathcal{N}}_\varpi$ denote the reduction of $\mathcal{N}_\varpi$. As k is perfect, we have by Remark 1.7 an exact sequence of $k(t)[\tau]$-modules as follows:

$$0 \to (\overline{\mathcal{N}}_\varpi)_1 \to \overline{\mathcal{N}}_\varpi \to (\overline{\mathcal{N}}_\varpi)_{\mathrm{nil}} \to 0,$$

where $(\overline{\mathcal{N}}_\varpi)_1$ is a τ-module (whose rank we will denote by ρ'), whereas the action of τ on $(\overline{\mathcal{N}}_\varpi)_{\mathrm{nil}}$ is nilpotent. We choose a $k(t)$-basis $(\bar{n}_1, \dots, \bar{n}_{\rho'})$ (resp. $(\bar{n}_{\rho'+1}, \dots, \bar{n}_r)$) for $(\overline{\mathcal{N}}_\varpi)_1$ (resp. $(\overline{\mathcal{N}}_\varpi)_{\mathrm{nil}}$). Finally, we fix a lift

$$\mathbf{n} := (n_1, \dots, n_{\rho'}; n_{\rho'+1}, \dots, n_r)$$

for $(\bar{n}_1, \dots, \bar{n}_r)$ in $\mathcal{N}_\varpi$, which yields an $\mathcal{O}_\varpi$-basis for $\mathcal{N}_\varpi$, and, for every $s > 0$, we denote by $\Delta_s \in \mathrm{Mat}_{r\times r}(\mathcal{O}_\varpi)$ the matrix representation of τ relatively to the basis $\mathbf{n}$, i.e. $\tau^s(\mathbf{n}) = \mathbf{n} \cdot \Delta_s$.

b) We have a similar filtration of the $k((t))[\tau]$-module $\bar{\mathfrak{B}}$ yielding modules $\bar{\mathfrak{B}}_1$ and $\bar{\mathfrak{B}}_{\mathrm{nil}}$. Note that $(\overline{\mathcal{N}}_\varpi)_1$ injects into $\bar{\mathfrak{B}}_1$, by an argument as in Lemma 1.10.

We now assume that $\rho' < \rho$, where ρ is the nondegenerate rank of $\mathfrak{B}$, and want to deduce a contradiction. Let us extend $(\bar{n}_1, \dots, \bar{n}_{\rho'})$ to a $k((t))$-basis

$$\bar{\mathbf{b}}_1 := (\bar{n}_1, \dots, \bar{n}_{\rho'}; \bar{b}_{\rho'+1}, \dots, \bar{b}_\rho)$$

for $\bar{\mathfrak{B}}_1$. For all $s > 0$, let $(\overline{\Delta}_1)_s \in \mathrm{GL}_\rho\big(k((t))\big)$ denote the matrix representation the action of τ^s on $\bar{\mathfrak{B}}_1$ with respect to this basis $\bar{\mathbf{b}}_1$: $\tau^s(\bar{\mathbf{b}}_1) = \bar{\mathbf{b}}_1 \cdot (\overline{\Delta}_1)_s$.

c) On the other hand, we choose a $k((t))$-basis $(\bar{b}_{\rho+1}, \dots, \bar{b}_r)$ for $\bar{\mathfrak{B}}_{\mathrm{nil}}$. Taking some lift $(b_{\rho'+1}, \dots b_r)$ of $(\bar{b}_{\rho'+1}, \dots \bar{b}_r)$ to $\mathfrak{B}$, we obtain a B-basis

$$\mathbf{b} = (n_1, \dots, n_{\rho'}; b_{\rho'+1}, \dots, b_\rho; b_{\rho+1}, \dots b_r)$$

for $\mathfrak{B}$. We denote the B-module spanned by the elements $n_1, \dots, n_{\rho'}$ by $\mathfrak{B}_0$, and further put $\mathfrak{B}_\star := \langle b_{\rho'+1}, \dots, b_\rho\rangle$ and $\mathfrak{B}_2 := \langle b_{\rho+1}, \dots b_r\rangle$.

Notice that a different choice of the elements $(\bar{b}_{\rho'+1}, \dots, \bar{b}_\rho; \bar{b}_{\rho+1}, \dots, \bar{b}_r)$ or their respective lifts would correspond to a basis transformation $\mathbf{b}' := \mathbf{b} \cdot U$ for a matrix in

$$\mathcal{U} := \left\{ \left(\begin{array}{c|c|c} \mathbf{1} & * & * \\ \hline 0 & * & * \\ \hline 0 & \pi * & * \end{array} \right) \right\} \subset \mathrm{GL}_r(B),$$

where the blocks correspond to the composition $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3$.

d) We express $\mathbf{n}$ in terms of $\mathbf{b}$ by means of a matrix

$$\Omega \in \mathrm{GL}_r(Q) \cap \mathrm{Mat}_{r\times r}(B)$$

as follows: $\mathbf{n} = \mathbf{b} \cdot \Omega$. The matrix Ω is then of the form

$$\left\{ \left(\begin{array}{c|c|c} \mathbf{1} & * & * \\ \hline 0 & * & * \\ \hline 0 & * & * \end{array} \right) \right\}.$$

Note that taking a different choice of $\mathbf{b}$ amounts to replacing Ω by $U \cdot \Omega$, for some $U \in \mathcal{U}$. We can find a basis $\mathbf{b}$ such that Ω (and more precisely its $\rho' + 1$-st column) is of the form:

$$\Omega = \begin{array}{c} \qquad\quad \overset{\rho'+1}{\downarrow} \\ \left(\begin{array}{c|c|c|c} \mathbf{1} & Z & * & * \\ \hline 0 & \pi^c & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & W & * & * \end{array} \right) \end{array},$$

where $X \in \mathrm{Mat}_{\rho'\times 1}(B)$, $c \geq 0$ and

$$W \in \pi^c \cdot \mathrm{Mat}_{(r-\rho)\times 1}(B).$$

The blocks correspond to the decomposition $\mathfrak{B} = \mathfrak{B}_0 \oplus \mathfrak{B}_{\star,1} \oplus \mathfrak{B}_{\star,2} \oplus \mathfrak{B}_2$, where we put $\mathfrak{B}_{\star,1} := \langle b_{\rho'+1} \rangle$ and $\mathfrak{B}_{\star,2} := \langle b_{\rho'+1}, \ldots, b_\rho \rangle$; we consider an analogous decomposition for $\mathcal{N}_\varpi$.

e) For any integer s, we will denote by $\hat{\Delta}_s \in \mathrm{Mat}_{r\times r}(B)$ the matrix representing τ^s with respect to $\mathbf{b}$: $\tau^s(\mathbf{b}) = \mathbf{b} \cdot \hat{\Delta}_s$. Observe that the representations $\hat{\Delta}_s$ and Δ_s satisfy the following relation:

$$\hat{\Delta}_s \cdot {}^{\sigma^s}\Omega = \Omega \cdot \Delta_s. \tag{74}$$

We distinguish two cases:

1) Suppose $c > 0$. For $s > 0$, let us write $\hat{\Delta}_s$ as a block matrix of matrices $(\delta \mid \delta' \mid \delta'' \mid \delta''')$ with respect to the decomposition of $\mathfrak{B}$; we do the same for $\Delta_s = (d \mid d' \mid d'' \mid d''')$. The equation (74) can then be rewritten as:

$$(\delta \mid \delta' \mid \delta'' \mid \delta''') \cdot \left(\begin{array}{c|c|c|c} \mathbf{1} & {}^{\sigma^s}Z & * & * \\ \hline 0 & \pi^{q^s c} & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & {}^{\sigma^s}W & * & * \end{array} \right) = \Omega \cdot (d \mid d' \mid d'' \mid d'''). \tag{75}$$

We thus obtain the equations

$$\left\{ \begin{array}{rl} \delta & = \Omega\, d \\ \delta\, {}^{\sigma^s}Z + \pi^{q^s c}(\delta' + \delta'''\, {}^{\sigma^s}W) & = \Omega\, d'. \end{array} \right. \tag{76}$$

Assume that s is large enough so that $v_\pi(\det \Omega) < q^s c$, which implies that

$$\pi^{q^s c} \cdot \Omega^{-1} \in \pi \cdot \mathrm{Mat}_{r\times r}(B).$$

It then follows from (76) that

$$d\, {}^{\sigma^s}Z + (\pi^{q^s c} \cdot \Omega^{-1})(\delta' + \delta'''\, {}^{\sigma^s}W) = d'. \tag{77}$$

We denote by $\bar{d} \in \mathrm{Mat}_{r\times\rho'}(k(t))$ the reduction of d modulo π; similarly, we define $\bar{d}'$. Also, we consider the reductions

$$\bar{\delta} \in \mathrm{Mat}_{r\times\rho'}\big(k((t))\big)$$

and $\bar{Z} \in \mathrm{Mat}_{\rho\times 1}\big(k((t))\big)$ of δ and Z, respectively. Reducing mod π, equation (77) gives

$$\bar{d}\,{}^{\sigma^s}\!\bar{X} = \bar{d}'. \tag{78}$$

As $\bar{\delta}$ has full rank ρ', we deduce from $\bar{\delta} = \bar{\Omega}\cdot\bar{d}$ that $\bar{d}$ has full rank ρ', too. Therefore, the solution $\bar{Z}$ of (78) is unique, namely $\bar{Z}$, and must therefore be algebraic, i.e.

$$\bar{Z} \in \mathrm{Mat}_{\rho\times 1}\big(k(t)\big),$$

as $\bar{d}$ and $\bar{d}'$ have entries in $k(t)$ as well. Finally, let us take the canonical lift Z_0 of $\bar{Z}$ to $\mathcal{O}_\varpi$ and put $Z = Z_0 + \pi Z_1$, for $Z_1 \in \mathrm{Mat}_{\rho'\times 1}(B)$. The element

$$v := \pi^{-1}\,\mathbf{n}\cdot\begin{pmatrix} -Z_0 \\ \hline 1 \\ \hline 0 \\ \hline 0 \end{pmatrix} = \mathbf{b}\cdot\begin{pmatrix} Z_1 \\ \hline \pi^{c-1} \\ \hline 0 \\ \hline \pi^{-1}W \end{pmatrix}$$

is contained both in V and in $\mathfrak{B}$, but not contained in $\mathcal{N}_\varpi$, which obviously contradicts the definition of $\mathcal{N}_\varpi$.

2) If $c = 0$, then we see, denoting by $\bar{\Omega}$ the reduction of Ω modulo π, that the upper left $(\rho'+1)\times(\rho'+1)$-block of $\bar{\Omega}$ has full rank $\rho'+1$. Also, the upper left $\rho\times\rho$ block of $\overline{\Delta}_s$, namely $(\overline{\Delta}_1)_s$ has full rank ρ. We see from this that the matrix

$$\overline{\Delta}_s\cdot{}^{\sigma^s}\bar{\Omega} = \bar{\Omega}\cdot\overline{\Delta}_s$$

has rank at least $\rho'+1$, for all $s > 0$. In particular, the nondegenerate rank of Δ_s is at least $\rho'+1$, which gives a contradiction.

□

§5. Proof of Theorem 4.2.

PROOF OF THEOREM 4.2. **a)** First of all, we can reduce ourselves to the case that $\mathcal{C} = \mathbb{A}^1$. Indeed, for a general curve $\mathcal{C}$, we consider a finite morphism

$$f : \mathcal{C} \to \mathbb{A}^1,$$

and denote the induced morphism $\mathcal{C}_K \to \mathbb{A}^1_K$ (resp. $\mathcal{C}_R \to \mathbb{A}^1_R$) again by f. If $\mathcal{M}^\star$ is the maximal model for $f_*(M)$ on $\mathcal{C}_R$ and $\mathcal{M}$ the maximal model for M on $\mathcal{C}_R$, then $f_*(\mathcal{M}) = \mathcal{M}$, as we already remarked in the proof of Prop. 1.16 (for analytic τ-sheaves). Let ℓ be a closed point of $\mathcal{C}$. If $\hat{\mathcal{N}}_\ell$ has nondegenerate rank ρ on $\hat{\mathcal{C}}_{R,\ell}$, then $f_*(\hat{\mathcal{N}}_\ell)$ has nondegenerate rank at least $\deg f\cdot\rho$. Assuming that the theorem holds for $\mathbb{A}^1$, we then obtain a model with nondegenerate rank at least $\deg f\cdot\rho$, and, by Lemma 1.10, the maximal model $\mathcal{M}^\star$ has at least the same nondegenerate rank. This in turn implies that $\mathcal{M}$ has nondegenerate rank $\geq \rho$.

b) Next, we show that it suffices to prove the result over a finite étale extension R' of R. Suppose that $\hat{\mathcal{N}}_\ell$ has nondegenerate rank ρ on $\hat{\mathcal{C}}_{R,\ell}$, and that we found that the maximal model $\widetilde{\mathcal{M}}'$ of $M_{K'}$ on $\widetilde{\mathcal{C}}_{K'}$ has nondegenerate rank r_1 at least ρ. By Prop. 1.25, this yields a maximal good sub-τ-module $\hat{\mathcal{M}}'_1$ of $\hat{\mathcal{M}}'_\varpi$, the completion of the stalk of $\widetilde{\mathcal{M}}'$ at ϖ, of rank r_1 over $\hat{\mathcal{O}}_\varpi$. This $\hat{\mathcal{M}}_1$ is functorial with respect to $\hat{\mathcal{O}}_\varpi$-linear homomorphisms, and therefore we can, as we did in the proof of Lemma 1.14, apply Galois descent to obtain a good sub-τ-module $\hat{\mathcal{M}}_1$ of $\hat{\mathcal{M}}_\varpi$ of rank r_1. This shows that the nondegenerate rank of the maximal model $\widetilde{\mathcal{M}}$ of $\widetilde{M}$ on $\widetilde{\mathcal{C}}_R$ is at least r_1 (cf. Proof of Lemma 1.9).

c) We now assume that $\mathcal{C} = \mathbb{A}^1$. Let ℓ be a closed point of $\mathbb{A}^1$ of degree s, with residue field κ_ℓ. We may assume the finite field $\kappa_\ell \hookrightarrow R$. Let

$$\{\ell'_1, \ell'_2, \ldots, \ell'_s\}$$

be the set of closed points of lying above ℓ on $\mathbb{A}^1_{\kappa_\ell}$. Let $\hat{\mathcal{C}}_{\kappa_\ell,\ell'_i}$ denote the completion of $\mathbb{A}^1_{\kappa_\ell}$ at ℓ'_i, and put

$$\mathbf{A}'_{\ell'_i} := \mathcal{O}_{\hat{\mathcal{C}}_{\kappa_\ell,\ell'_i}}.$$

As $\mathcal{C}$ is smooth, we have $\mathbf{A}'_{\ell'_i} \cong \kappa_\ell[[\lambda_i]]$, where λ_i denotes a uniformizer at the point ℓ'_i. The ring $R \hat{\otimes}_{\kappa_\ell} \mathbf{A}'_{\ell'_i}$ is then isomorphic to $R[[\lambda_i]]$. Finally

$$\mathcal{O}_{\hat{\mathcal{C}}_{R,\ell}} = R \hat{\otimes}_{\mathbb{F}_q} \mathbf{A}_\ell \cong \prod_i R \hat{\otimes}_{\kappa_\ell} \mathbf{A}'_{\ell'_i} = \prod_i R[[\lambda_i]].$$

Similarly,

$$\mathcal{O}_{\hat{\mathcal{C}}_{K,\ell}} \cong \prod_i K[[\lambda_i]].$$

The endomorphism σ of $\mathcal{O}_{\hat{\mathcal{C}}_{R,\ell}}$ induces morphisms $\sigma : R[[\lambda_i]] \to R[[\lambda_j]]$ for all pairs (i, j) such that ${}^\sigma\ell_i = \ell_j$.

Any $\mathcal{O}_{\hat{\mathcal{C}}_{R,\ell}}$-module $\hat{\mathcal{N}}_\ell$ can be written as a product $\hat{\mathcal{N}}_\ell = \prod_i \hat{\mathcal{N}}_i$, where the $\hat{\mathcal{N}}_i$ are $R[[\lambda_i]]$-modules. If $\hat{\mathcal{N}}_\ell$ is endowed with a σ-linear endomorphism

$$\tau : \sigma^* \hat{\mathcal{N}}_\ell \to \hat{\mathcal{N}}_\ell$$

then τ will induce morphisms

$$\tau : \sigma^* \hat{\mathcal{N}}_i \to \hat{\mathcal{N}}_j$$

if (i, j) satisfies ${}^\sigma\ell_i = \ell_j$ (the same applies to $\mathcal{O}_{\hat{\mathcal{C}}_{K,\ell}}$-modules). Each of the modules $\mathcal{N}_i$ is τ^s-invariant.

Let M and $\mathcal{N}_\ell$ be as in the statement of this theorem. Let us denote the stalk of M at the generic point of $\mathbb{A}^1_K$ by V. Let Q_i, B_i etc. denote the subrings of $K[[\lambda_i]]$ which we defined for $K[[t]]$ in §**§1**.

We consider M as a τ^s-sheaf over $\mathbb{A}^1_K = \mathbb{A}^1_{\kappa_\ell} \otimes_{\kappa_\ell} \operatorname{Spec} K$. Now, upon replacing φ by φ^s, and τ by τ^s, we can apply Lemma 4.4 to the module $\hat{\mathcal{N}}_i \subset \hat{M}_i$ to obtain that

$$\hat{\mathcal{N}}_i \subset (K \otimes R[[\lambda_i]]) \otimes_{K[\lambda_i]} M.$$

We remark that $\mathcal{O}_\varpi = \mathcal{O}_{\mathcal{C}_R,\varpi}$, and define the $\mathcal{O}_\varpi$-modules

$$(\mathcal{N}_\varpi)_i := V \cap (B_i \cdot \hat{\mathcal{N}}_i) \subset (Q_i)_\lambda \otimes_{K(\lambda_i)} V.$$

Again replacing τ by τ^s, we then deduce from Thm. 4.7 that the $(\mathcal{N}_\varpi)_i$ have the same nondegenerate rank as $\hat{\mathcal{N}}_\ell$. Finally, the $\mathcal{O}_{\mathcal{C}_R,\varpi}$-module

$$\mathcal{N}_\varpi := \sum_i (\mathcal{N}_\varpi)_i$$

is τ-invariant and nondegenerate of rank ρ. Hence it defines, by Lemma 1.4, a model $\mathcal{M}$ for M with the desired property. □

II. Galois criterion for trivial reduction

THEOREM 4.8. *Let R be a complete discrete valuation $\mathbb{F}_q$-algebra with field of fractions K and* finite *residue field k. Let M be a τ-sheaf on $\mathcal{C}_K$. The following statements are equivalent:*

i) *There exists a closed point ℓ of $\mathcal{C}$ for which $\hat{M}_\ell$ is smooth such that the $\mathbf{A}_\ell[\Gamma_K]$-module $T_\ell(M)$ is trivial.*
ii) *For every closed point ℓ of $\mathcal{C}$ for which $\hat{M}_\ell$ is smooth, the $\mathbf{A}_\ell[\Gamma_K]$-module $T_\ell(M)$ is trivial.*
iii) *The τ-sheaf M on $\mathcal{C}_K$ has a good model $\mathcal{M}$ such that the reduction $\overline{\mathcal{M}}$ is a trivial τ-sheaf on $\mathcal{C}_k$ (cf. Def. 0.1 iv)).*

PROOF. Clearly ii) $\Rightarrow$ i). If $\mathcal{M}$ is a good τ-sheaf on $\mathcal{C}_R$ such that the reduction $\overline{\mathcal{M}}$ is trivial on $\mathbb{A}^1_k$, then $\hat{\mathcal{M}}_\ell$ is smooth with trivial reduction for every closed point ℓ of $\mathcal{C}$. By the correspondence 0.7, this yields iii) $\Rightarrow$ ii).

It remains to show that i) implies iii). As the action of the inertia group I_K on $T_\ell(M)$ is trivial, the maximal model $\mathcal{M}$ for M on $\mathcal{C}_R$ is good, by Thm. 4.1 (k is perfect). Let us consider its reduction $\overline{\mathcal{M}}$. The theorem now follows from the following proposition: □

PROPOSITION 4.9. *Let k be* finitely generated *field containing $\mathbb{F}_q$. Let M be a τ-sheaf on $\mathcal{C}_k$. The following statements are equivalent:*

i) *There exists a closed point ℓ of $\mathcal{C}$ for which $\hat{M}_\ell$ is smooth such that the $F_\ell[\Gamma_k]$-module $V_\ell(M)$ is trivial.*
ii) *For every closed point ℓ of $\mathcal{C}$ for which $\hat{M}_\ell$ is smooth, the $F_\ell[\Gamma_k]$-module $V_\ell(M)$ is trivial.*
iii) *The τ-sheaf M is trivial (cf. Def. 0.1 iv)).*

PROOF. Clearly iii) $\Rightarrow$ ii) $\Rightarrow$ i). For i) $\Rightarrow$ iii), let N be a trivial τ-sheaf on $\mathcal{C}_k$ of same rank r as M. We use the Tate conjecture: It was stated in Thm. 3.7 for fields of transcendence degree 1 over $\mathbb{F}_q$, but holds for any finitely generated field k (cf. [**Pi2**], Thm. 1.4). For a closed point ℓ of $\mathcal{C}$ such that $\hat{M}_\ell$ is smooth, we have an isomorphism:

$$F_\ell \otimes_{\mathbf{A}} \mathrm{Hom}_k(M, N) \to \mathrm{Hom}_{F_\ell[\Gamma_k]}(V_\ell(N), V_\ell(M)).$$

As both $V_\ell(N) \cong V_\ell(M)$, this yields an isogeny $f : N \to M$. The cokernel of f has finite length, and is supported over finitely many closed points of $\mathcal{C}$. By an induction on the length of coker f, we can find another

$$(N',\ f' : N' \to M)$$

with f' an isomorphism. □

III. Local factors of L-functions

We now show how the above theory explains how the local L-factor of a τ-sheaf (cf. [**Böc**], def. 1.40, and [**TW**]) at a place of bad reduction is related to the action of Frobenius on the associated Galois representations. Let R be a complete discrete valuation $\mathbb{F}_q$-algebra with fraction field K and finite residue field k. Let

$$d_x := [k : \mathbb{F}_q]$$

denote the degree of the closed point x of R. Let M be a τ-sheaf over $\mathcal{C}_K$, $\mathcal{M}$ its maximal model over $\mathcal{C}_R$ and $\overline{\mathcal{M}}$ the reduction of $\mathcal{M}$ at x.

DEFINITION 4.10. We define the **local L-factor for M at x** by

$$L_x(M; Z)^{-1} := \det_{\mathbf{A}}\left(1 - Z^{d_x}\tau^{d_x} \mid H^0(\mathcal{C}_k, \overline{\mathcal{M}})\right) \in \mathbf{A}[Z].$$

Let ℓ be a closed point of $\mathcal{C}$ such that $\hat{M}_\ell$ is smooth. Let

$$H_\ell(M)^{I_K}$$

be the $\mathbf{A}_\ell$-module of invariants of $H_\ell(M)$ (Def. 0.8) under the action of I_K; it is a Γ_K-invariant direct summand of $H_\ell(M)$. The action of $\Gamma_k \cong \Gamma_K/I_K$ on $H_\ell(M)^{I_K}$ is well defined, in particular that of its canonical generator Frob_x which acts as φ^{d_x} on k.

DEFINITION 4.11. We define the **local L-factor for $T_\ell(M)$ at x** by

$$L_x(T_\ell(M); Z)^{-1} := \det\left(1 - Z^{d_x}\mathrm{Frob}_x \mid H_\ell(M)^{I_K}\right) \in \mathbf{A}_\ell[Z].$$

THEOREM 4.12. *Let R be a complete discrete valuation $\mathbb{F}_q$-algebra with fraction field K and finite residue field k and let M be a τ-sheaf over $\mathcal{C}_K$. For all but a finite number of closed points ℓ of $\mathcal{C}$, we have:*

$$L_x(M; Z) = L_x(T_\ell(M); Z).$$

PROOF. **a)** We denote the maximal τ-subsheaf of $\overline{\mathcal{M}}$ on $\mathcal{C}_k$ by $\overline{\mathcal{M}}_1$, and its rank by ρ'. As shown in the proof of Thm. 1.26, we can lift $\overline{\mathcal{M}}_1$ to a saturated analytic sub-τ-sheaf $\widetilde{\mathcal{M}}_1 \subset \widetilde{\mathcal{M}}$ of rank ρ' on $\mathcal{C}'_R$ for some nonempty open subscheme $\mathcal{C}'$ of $\mathcal{C}$. For any closed point ℓ of $\mathcal{C}'$, this yields a saturated ℓ-adic sub-τ-sheaf

$$(\hat{\mathcal{M}}_\ell)_1 \subset \hat{\mathcal{M}}_\ell$$

on $\hat{\mathcal{C}}'_{R,\ell}$. Clearly its reduction $\overline{(\hat{\mathcal{M}}_\ell)_1}$ satisfies

$$\overline{(\hat{\mathcal{M}}_\ell)_1} = \mathcal{O}_{\hat{\mathcal{C}}_{k,\ell}} \otimes_{\mathcal{O}_{\mathcal{C}_k}} (\overline{\mathcal{M}})_1.$$

Outside a finite set S' of closed points of $\mathcal{C}'$, the ℓ-adic τ-sheaf $(\hat{\mathcal{M}}_\ell)_1$ is smooth. We deduce for all closed points $\ell \in \mathcal{C}'' := \mathcal{C} \setminus S'$:

$$\begin{aligned} L_x(M; Z)^{-1} &= \det\left(1 - Z^{d_x}\tau^{d_x} \mid \overline{\mathcal{M}}\right) = \det\left(1 - Z^{d_x}\tau^{d_x} \mid (\overline{\mathcal{M}})_1\right) \\ &= \det\left(1 - Z^{d_x}\tau^{d_x} \mid \mathcal{O}_{\hat{\mathcal{C}}_{k,\ell}} \otimes_{\mathcal{O}_{\mathcal{C}_k}} (\overline{\mathcal{M}})_1\right) \\ &= \det\left(1 - Z^{d_x}\tau^{d_x} \;\middle|\; \overline{(\hat{\mathcal{M}}_\ell)_1}\right) \end{aligned}$$

b) Let ρ be the rank of $H_\ell(M)^{I_K}$. By Prop. 0.7, we deduce the existence of a maximal smooth ℓ-adic τ-sheaf $\hat{\mathcal{N}}_\ell$ of $\hat{\mathcal{M}}_\ell$ on $\hat{\mathcal{C}}_{R,\ell}$ of rank ρ. Let $\overline{\mathcal{N}}_\ell$ denote the reduction of $\hat{\mathcal{N}}_\ell$ to $\hat{\mathcal{C}}_{k,\ell}$. As explained in [**TW**], Cor. 6.2, the same correspondence implies that

$$\det\left(1 - Z^{d_x}\mathrm{Frob} \;\middle|\; H_\ell(M)^{I_K}\right) = \det\left(1 - Z^{d_x}\tau^{d_x} \mid \overline{\mathcal{N}}_\ell\right).$$

We want to show that $\hat{\mathcal{N}}_\ell = (\hat{\mathcal{M}}_\ell)_1$. As the module $\hat{\mathcal{N}}_\ell$ is the maximal smooth sub-τ-sheaf of $\hat{\mathcal{M}}_\ell$ on $\hat{\mathcal{C}}_{R,\ell}$, we have

$$(\hat{\mathcal{M}}_\ell)_1 \subset \hat{\mathcal{N}}_\ell.$$

The τ-sheaf $\hat{\mathcal{M}}_\ell$ has nondegenerate rank ρ and, therefore, by Theorem 4.2, the nondegenerate rank of $\hat{\mathcal{N}}'_\ell$ is at most ρ', which shows that

$$\hat{\mathcal{N}}_\ell = (\hat{\mathcal{M}}_\ell)_1$$

as both are saturated smooth ℓ-adic sub-τ-sheaves on $\hat{\mathcal{C}}_{R,\ell}$ of rank ρ' of $\hat{\mathcal{M}}_\ell$.

Finally, we can deduce that, for all closed points $\ell \in \mathcal{C}''$,

$$\begin{aligned} L_x(T_\ell(M); Z)^{-1} &:= \det\left(1 - Z^{d_x}\mathrm{Frob} \;\middle|\; H_\ell(M)^{I_K}\right) \\ &= \det\left(1 - Z^{d_x}\tau^{d_x} \;\middle|\; \overline{\mathcal{N}}_\ell\right) \\ &= \det\left(1 - Z^{d_x}\tau^{d_x} \mid \overline{(\hat{\mathcal{M}}_\ell)_1}\right) \\ &= L_x(M; Z)^{-1}. \end{aligned} \tag{79}$$

$\square$

CHAPTER 5

Anderson uniformization of t-motives

I. Anderson uniformization

§1. Uniformizable t-modules. Let K be a complete local field which contains the finite field $\mathbb{F}_q$ and a variable θ with $v(\theta) < 0$, where v denotes the valuation of K; in other words, let K be a finite algebraic extension of the field $\mathbb{F}_q((\theta^{-1}))$. Let $|\cdot|$ be the normalized absolute value on K and R its valuation ring. We denote the completion of an algebraic closure of K by $\mathbf{C}$. As we will be dealing with Anderson t-motives in this and the next chapter, we will from now on assume for simplicity that $\mathcal{C} = \mathbb{A}^1$ (cf. section 0.I.**§3**). We define a characteristic map $\iota : \operatorname{Spec} K \to \mathcal{C}$ by means of the ring morphism

$$\iota^* : \mathbf{A} \cong \mathbb{F}_q[t] \to K : t \mapsto \theta.$$

As $v(\theta) < 0$, the valuation v is then infinite with respect to ι (Def. 0.5.iii).

PROPOSITION 5.1 (Anderson, [**An1**]).

i) *For every d-dimensional t-module E over K with characteristic ι, there exists a unique entire $\mathbb{F}_q$-linear map*

(80)
$$e_\infty : \mathbb{G}_{a,K}^{\oplus d} \to \mathbb{G}_{a,K}^{\oplus d},$$

defined over K which induces a morphism $\operatorname{Lie}(E) \to E$ of t-modules and whose derivative $\mathrm{d}e_\infty$ is the identity.

ii) *If E is abelian and has rank r, then the kernel*

$$H := \ker(e_\infty)(\bar{K}) \subset (K^{\mathrm{sep}})^d$$

is a finitely generated sub-$\mathbf{A}$-module of $\operatorname{Lie}(E)(\bar{K}^{\mathrm{sep}})$, which is free of rank at most r and strictly discrete, i.e. every open disk of finite radius contains only a finite number of points of H. (In other words, using Def. 6.15, the t-module H is an $\mathbf{A}$-lattice of rank at most r inside $\operatorname{Lie}(E)(K^{\mathrm{sep}})$.)

In other words, using Def. 6.15, the t-module H is an $\mathbf{A}$-lattice of rank at most r inside $\operatorname{Lie}(E)(K^{\mathrm{sep}})$.

DEFINITION 5.2 (Anderson). If e_∞ is surjective over K^{sep}, then we call E **uniformizable**.

THEOREM 5.3 (Anderson, [**An1**], Thm. 4). *The abelian t-module E is uniformizable if and only if the $\mathbf{A}$-rank of H is r.*

COROLLARY 5.4. *If E is uniformizable, then the Tate module $T_\ell(M)$ is potentially a trivial Γ_K-representation, for every closed point ℓ of $\mathbb{A}^1$.*

PROOF. For any nonzero $a \in A$, the a-torsion module $E[a]$ is isomorphic to

$$H/a \cdot H$$

as an $\mathbf{A}/(a)[\Gamma_K]$-module. Since H is strictly discrete, the orbit of any $\mathbf{A}$-basis for H under Γ_K is finite, and hence, potentially, H is a trivial $\mathbf{A}[\Gamma_K]$-module. This implies that, potentially, for any closed point ℓ of $\mathbb{A}^1$, the $\mathbf{A}_\ell$-module $T_\ell(E) \cong T_\ell(M)$ is a trivial Γ_K-representation. □

§2. Uniformizable analytic τ-sheaves. For an abelian t-module E over K, let $M = M(E)$ be the t-motive associated to E, and $\widetilde{M}$ the analytic τ-sheaf on $\widetilde{\mathbb{A}}^1_K$ obtained from M.

DEFINITION 5.5. An analytic τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$ with characteristic ι is called **uniformizable** if it contains a trivial τ-sheaf of full rank.

We set

$$\mathcal{D}^0_K := \{t \in K;\, |t| \leq 1\},$$

the closed disk of radius 1 around the origin, considered as a rigid analytic space.

LEMMA 5.6. *If a τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$ is uniformizable, then the restriction $\widetilde{M}|_{\mathcal{D}^0_K}$ of $\widetilde{M}$ to $\mathcal{D}^0_K$ is trivial.*

PROOF. Suppose that $\widetilde{M}^\star := \widetilde{M}|_{\mathcal{D}^0_K}$ contains the trivial τ-sheaf $\widetilde{N}$, of same rank. The morphism τ acts via a unit in $\mathcal{O}_{\mathcal{D}^0_K}$ on both $\wedge^{\mathrm{top}}\widetilde{N}$ and $\wedge^{\mathrm{top}}\widetilde{M}^\star$. One easily draws from this the conclusion that the quotient $\widetilde{M}^\star/\widetilde{N}$ is supported on a σ-invariant nontrivial closed subset S of $\mathcal{D}^0_K$. Choosing a line bundle $\mathcal{L}$ whose divisor has support S, we can find a power n such that $\widetilde{M}$ is contained in $\mathcal{L}^{\otimes(-n)} \otimes \widetilde{N}$, which is again a trivial τ-sheaf. By lemma [**An1**], 2.10.6, this implies that $\widetilde{M}^\star$ is trivial. □

LEMMA 5.7. *If the restriction of $\widetilde{M}$ to $\mathcal{D}^0_K$ is trivial, then, for all closed points ℓ of $\mathbb{A}^1$, the Tate module $T_\ell(\widetilde{M})$ is a trivial Γ_K-representation.*

PROOF. Completing $\widetilde{M}$ at $\{\ell\} \otimes \operatorname{Spec} K$ yields a trivial smooth τ-sheaf $\hat{M}_\ell$ on $\hat{\mathcal{C}}_{K,\ell}$. By the correspondence 0.7, this shows that $T_\ell(\widetilde{M})$ is a trivial Γ_K-representation. □

In his paper [**An1**], Anderson showed that the uniformizability of E can be expressed in terms of the t-motive M as follows:

THEOREM 5.8. (Anderson, [**An1**], Thm. 4)) *The abelian t-module E is uniformizable if and only if the analytic τ-sheaf $\widetilde{M}_{\mathcal{D}^0_{\bar{K}}}$ on $\mathcal{D}^0_{\bar{K}}$ is trivial.*

We want to extend this result, by showing that, amongst other criteria, uniformizability of E is equivalent to potential uniformizability for M.

§3. Pink's theory of σ-bundles. A new point of view was introduced by Pink in [**Pi4**]. Taking $z := t^{-1}$ as a local parameter at the point ∞ of $\mathbb{P}^1$, we set

$$\dot{\mathcal{D}}^{\infty}_{\mathbf{C}} := \{z \in \mathbf{C};\ 0 < |z| \ < 1\}$$

to be the punctured open disk of radius 1 around infinity, considered as a rigid analytic space over $\mathbf{C}$.

DEFINITION 5.9 (Pink). A σ**-bundle** $\mathcal{F}$ is a smooth τ-sheaf on $\dot{\mathcal{D}}^{\infty}_{\mathbf{C}}$.

For instance, for each co-prime pair $(u, s) \in \mathbb{Z}^2$ with $s > 0$, consider the sheaf

$$\mathcal{F}_{u,s} := \mathcal{O}^{\oplus s}_{\dot{\mathcal{D}}^{\infty}_{\mathbf{C}}}.$$

We define a σ-linear endomorphism

$$\tau : \sigma^* \mathcal{F}_{u,s} \to \mathcal{F}_{u,s}$$

with respect to the natural basis $\mathbf{e} = (e_i)_{1 \le i \le s}$ of $\mathcal{F}_{u,s}$ by its matrix representation $\tau(\mathbf{e}) = \mathbf{e} \cdot Z_{u,s}$, where

$$Z_{u,s} := \begin{pmatrix} 0 & & 0 & z^{-u} \\ 1 & & 0 & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix} \in \mathrm{Mat}_{s \times s}(\mathbb{F}_q(z)). \tag{81}$$

This endows $\mathcal{F}_{u,s}$ with the structure of a σ-bundle, which is, by definition, **pure of weight** u/s.

If $\widetilde{M}$ is an analytic τ-sheaf on $\widetilde{\mathbb{A}}^1_{\mathbf{C}}$ with characteristic ι, then we notice that, on the annulus

$$\mathcal{A} := \{z \in \mathbf{C};\ |\theta^{-1}| < |z| < 1\} \subset \dot{\mathcal{D}}^{\infty}_{\mathbf{C}},$$

the map

$$\tau : \sigma^* \widetilde{M}|_{\sigma^{-1}(\mathcal{A})} \to \widetilde{M}|_{\mathcal{A}}$$

is an isomorphism.

THEOREM 5.10. (Pink, [**Pi4**]) *There exists a unique σ-bundle $\mathcal{F}(\widetilde{M})$ on $\dot{\mathcal{D}}^{\infty}_{\mathbf{C}}$ such that*

$$\mathcal{F}(\widetilde{M})|_{\mathcal{A}} = \widetilde{M}|_{\mathcal{A}}$$

and $\mathcal{F}(\widetilde{M})$ is isomorphic to a direct sum of copies of pure σ-bundles $\mathcal{F}_{u,s}$.

THEOREM 5.11. (Pink, [**Pi4**]) *An analytic τ-sheaf $\widetilde{M}$ over $\widetilde{\mathbb{A}}^1_{\mathbf{C}}$ is uniformizable if and only if the σ-bundle $\mathcal{F}(\widetilde{M})$ is trivial (i.e. isomorphic to $\mathcal{F}_{0,r}$, for some integer r) on $\dot{\mathcal{D}}^{\infty}_{\mathbf{C}}$.*

II. Main theorem

Let $\widetilde{M}$ be an analytic τ-sheaf on $\widetilde{\mathbb{A}}^1_K$ with characteristic ι. We consider the following statements:

- **U**(K): The analytic τ-sheaf $\widetilde{M}_K$ on $\widetilde{\mathbb{A}}^1_K$ is potentially uniformizable.
- **U**($\bar{K}$): The analytic τ-sheaf $\widetilde{M}_{\bar{K}}$ on $\widetilde{\mathbb{A}}^1_{\bar{K}}$ is uniformizable.
- **U'**(K): The analytic τ-sheaf $\widetilde{M}_{\mathcal{D}^0_K}$ on $\mathcal{D}^0_K$ is potentially trivial.
- **U'**($\mathbf{C}$): The analytic τ-sheaf $\widetilde{M}_{\mathcal{D}^0_{\mathbf{C}}}$ on $\mathcal{D}^0_{\mathbf{C}}$ is trivial.
- $\mathbf{G}_\ell$: There exists a closed point ℓ of $\mathbb{A}^1$ such that the Γ_K-representation $T_\ell(\widetilde{M})$ is potentially trivial.
- **G**: The Γ_K-representation $T_\ell(\widetilde{M})$ is potentially trivial for all closed points ℓ of $\mathbb{A}^1$.
- **R**: The τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$ potentially has a good model $\widetilde{\mathcal{M}}$ such that the reduction $\overline{\mathcal{M}}$ is a trivial τ-sheaf on $\widetilde{\mathbb{A}}^1_k$.
- **R'**: Potentially, the τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$ is semistable and, denoting the subquotients of the semistable filtration for $\widetilde{M}$ by $\widetilde{M}_i$, the reductions $\overline{\mathcal{M}}_i$ of good models $\widetilde{\mathcal{M}}_i$ for $\widetilde{M}_i$ yield trivial τ-sheaves on $\widetilde{\mathbb{A}}^1_k$.
- **P**: The σ-bundle $\mathcal{F}(\widetilde{M})$ is trivial on $\dot{\mathcal{D}}^\infty_{\mathbf{C}}$.

THEOREM 5.12. *For an analytic τ-sheaf on $\widetilde{\mathbb{A}}^1_K$ with characteristic ι, the properties* **U**(K), **U'**(K), $\mathbf{G}_\ell$, **G** *and* **R** *are equivalent.*

Suppose that $\widetilde{M}$ is associated to an abelian t-module E defined over K and consider the following statements:

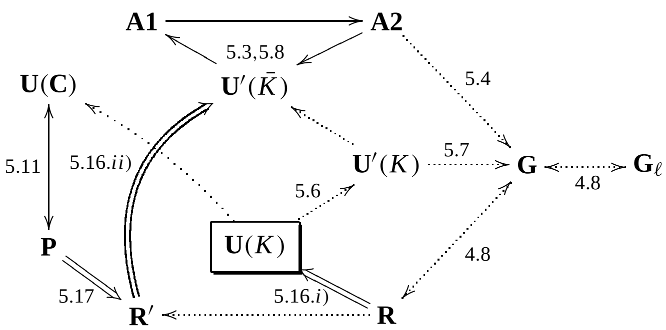
- **A1**: The abelian t-module E is uniformizable.
- **A2**: The **A**-rank of H is r.

THEOREM 5.13 (**Anderson Uniformization**). *Let E be an abelian t-module over K with characteristic ι and $\widetilde{M}$ its associated analytic τ-sheaf on $\widetilde{\mathbb{A}}^1_K$. The properties*

U(K), **U**($\bar{K}$), **U'**(K), **U'**($\mathbf{C}$), **A1**, **A2**, $\mathbf{G}_\ell$, **G**, **R**, **R'** *and* **P**

are equivalent.

OVERVIEW OF THE PROOFS OF THM. 5.12 AND 5.13.
The following diagram illustrates how the mentioned properties are related to each other for a τ-sheaf $\widetilde{M}$, resp. t-motive $\widetilde{M}$. Dotted arrows represent immediate implications or results we already proved and full arrows refer to the theorems by Anderson and Pink. The double arrows correspond to results we will establish in sections **IV** and **V**, thus completing the proof of these theorems. □



III. Example: (not) uniformizable t-motives

In his paper [**An1**], Anderson gave one, rather complicated, example of a not uniformizable abelian t-module. Theorem 5.13 now enables us to give a very simple 1-parameter family of abelian t-modules $E(\gamma)$, where $E(\gamma)$ is uniformizable if and only if γ is contained in the open unit disk around the origin.

We consider the discrete valuation ring $R := \mathbb{F}_q[[\zeta]]$, denote by K its field of fractions and by v its valuation. The valuation on K is then infinite with respect to the characteristic map ι defined by

$$\iota^* : \mathbf{A} \cong \mathbb{F}_q[t] \to K : t \mapsto \theta := \zeta^{-1}.$$

For each $\gamma \in R$, we consider the t-motive $M_\zeta(\gamma)$, and its associated t-module $E_\zeta(\gamma)$, which were introduced in section 2.II.

PROPOSITION 5.14. *The abelian E-module $E_\zeta(\gamma)$ is uniformizable if and only if $v(\gamma) > 0$.*

PROOF. **a)** If $v(\gamma) \geq 0$, the $\mathcal{O}_{\widetilde{\mathbb{A}}^1_R}$-module $\mathcal{M}$ generated by $\mathbf{m}$ is a good model for $M := M_\zeta(\gamma)$. With respect to the basis $\overline{\mathbf{m}}$, the action of τ on the reduction $\overline{\mathcal{M}}$ is given by:

$$\tau(\overline{\mathbf{m}}) = \overline{\mathbf{m}} \cdot \begin{pmatrix} 0 & -1 \\ -1 & \bar{\gamma}\, t \end{pmatrix}.$$

a.i) If $\bar{\gamma} = 0$, then, $\overline{\mathcal{M}}$ is trivial on $\mathbb{A}^1_{\mathbb{F}_{q^2}}$ (cf. property **R**). Hence $E_\zeta(\gamma)$ is uniformizable by Thm. 5.13.

a.ii) The (linear!) endomorphism τ on $\overline{\mathcal{M}}$ satisfies the identity

$$T^2 = \bar{\gamma}\, t\, T + 1.$$

This shows that, if $\bar{\gamma} \neq 0$, no iterate of τ can be the identity on $\overline{\mathcal{M}}$ (as it has non-constant eigenvalues). Repeating the argument in Thm. 4.8, the Γ_k-representation $T_\ell(M)$ cannot be potentially trivial. Hence $E_\zeta(\gamma)$ is not uniformizable in this case (cf. Thm. 5.13, property **G**).

b) If $v(\gamma) < 0$, then one can show, by an argument as in Lemma 2.12, that if ℓ_0 is the closed point of $\mathbb{A}^1$ defined by the ideal (t) in $\mathbf{A} = \mathbb{F}_q[t]$, then the Tate module

$T_{\ell_0}(M)$ is not potentially unramified. Hence, $E_\zeta(\gamma)$ is not uniformizable (cf. Thm. 5.13, property **G**). □

REMARK 5.15. The above example suggests that 'uniformizability' is an 'open' condition with respect to the topology on K: For a uniformizable τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$ with matrix representation Δ, the τ-sheaf obtained by changing the coefficients of Δ by a small amount is still uniformizable. This follows from property **R** of Thm. 5.13. This idea was suggested by [**Pi4**].

IV. Models and uniformizability

PROPOSITION 5.16. *For a τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$, we have:*

i) **R** $\Rightarrow$ **U**(K);
ii) **R'** $\Rightarrow$ **U'**$(\bar{K})$.

PROOF. **a)** For i), we assume that $\widetilde{M}$ has a good model $\widetilde{\mathcal{M}}$ and that $\overline{\mathcal{M}}$ is trivial. Choose a global basis $\overline{\mathbf{m}}$ for $\overline{\mathcal{M}}$, the reduction of $\widetilde{\mathcal{M}}$, such that $\tau(\overline{\mathbf{m}}) = \overline{\mathbf{m}}$. Let us extend $\widetilde{\mathcal{M}}$ by zero to a free $\mathcal{O}_{\widetilde{\mathbb{A}}^1_R}$-module

$$\widetilde{\mathcal{M}}_e = \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{M}}_0$$

of rank r' with τ-action. Let $\mathbf{m}$ be a global basis for $\widetilde{\mathcal{M}}_e$, and

$$\Delta \in \mathrm{Mat}_{r' \times r'}\left(H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)\right)$$

the matrix representation of τ with respect to this basis, i.e. $\tau(\mathbf{m}) = \mathbf{m} \cdot \Delta$.

b) We want to construct r independent elements $\mathbf{n} = (n_i)_{1 \le i \le r}$ in

$$H^0(\widetilde{\mathbb{A}}^1_R, \widetilde{\mathcal{M}}_e)$$

which are fixed by τ, and we will do this by lifting the basis $\overline{\mathbf{m}}$ to $\widetilde{\mathcal{M}}$. If we put $\mathbf{n} := \mathbf{m} \cdot Z$, this boils down to finding a matrix

$$Z \in \mathrm{Mat}_{r' \times r}\left(H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)\right)$$

with rank r, which solves the following equation:

$$Z = \Delta \cdot {}^\sigma Z. \tag{82}$$

The reduction $Z = \overline{\Delta} \cdot {}^\sigma Z$ of this equation has the solution

$$Z_0 \in \mathrm{Mat}_{r' \times r}(H^0(\widetilde{\mathbb{A}}^1_k, \mathcal{O}_{\widetilde{\mathbb{A}}^1_k}))$$

expressing the basis $\overline{\mathbf{m}}$ in terms of the reduction of $\mathbf{m}$ in $\overline{\mathcal{M}}$. Let $\hat{Z}_0$ be the canonical lift of Z_0 to $\mathrm{Mat}_{r' \times r}\left(H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)\right)$ which corresponds to the imbedding

$$k \hookrightarrow R.$$

c) Let us set $Z := \hat{Z}_0 + \pi Z_1$ and $B := \pi^{-1}(\Delta \cdot {}^\sigma\hat{Z}_0 - \hat{Z}_0)$, which yields the following equation for Z_1:

$$Z_1 = B + \pi^{q-1}\Delta\, {}^\sigma Z_1.$$

By lemma 1.23, There exists a unique solution

$$Z_1 \in \mathrm{Mat}_{r'\times r}\left(H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)\right)$$

for this equation. The matrix $Z = \hat{Z}_0 + \pi Z_1$ hence solves equation (82). As its reduction $\bar{Z} = Z_0$ has full rank, so does Z. Finally, as τ acts as the identity on each of the basis elements n_i, we see that

$$n_i \in H^0(\widetilde{\mathbb{A}}^1_R, \widetilde{\mathcal{M}}) \subset H^0(\widetilde{\mathbb{A}}^1_R, \widetilde{\mathcal{M}}_e).$$

In conclusion, $\widetilde{M}$ contains the trivial τ-sheaf $\widetilde{N}$ on $\widetilde{\mathbb{A}}^1_K$ generated by the global sections $\mathbf{n}$. This finishes the proof of i).

d) For ii, it follows from i) and Lemma 5.6 that, if $\widetilde{M}$ is semistable such that the reductions $\overline{\mathcal{M}}_i$ of its subquotients $\widetilde{M}_i$ are trivial, then $\widetilde{M}_{\mathcal{D}^0_K}$ is an extension of trivial τ-sheaves. Over the algebraic closure $\bar{K}$, every extension of trivial analytic τ-sheaves is trivial (cf. [**An1**], lemma 2.7.2), which shows that $\widetilde{M}$ satisfies **U'**$(\bar{K})$. □

V. σ-Bundles and uniformizability

PROPOSITION 5.17. *For any τ-sheaf $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_K$, statement* **P** *implies* **R'**.

PROOF. Assume that property **P** holds. Let $\overline{\mathcal{M}}_1$ denote the nondegenerate part of the reduction $\overline{\mathcal{M}}$ of the maximal model $\widetilde{\mathcal{M}}^{\max}$ for $\widetilde{M}$ on $\widetilde{\mathbb{A}}^1_R$.

a) First, let us suppose that $\overline{\mathcal{M}}_1$ is potentially trivial. Upon replacing R by a finite unramified extension, we can, by Prop. 1.22, lift $\overline{\mathcal{M}}_1$ to a sub-τ-sheaf $\widetilde{\mathcal{N}}$ on $\widetilde{\mathbb{A}}^1_R$, with trivial reduction. Consider the saturation $\widetilde{\mathcal{N}}_1$ of $\widetilde{\mathcal{N}}$ in $\widetilde{\mathcal{M}}^{\max}$ and its restriction $\widetilde{N}_1$ to the generic fibre $\widetilde{\mathbb{A}}^1_K$. By Prop. 5.16.i), the analytic τ-sheaf $\widetilde{N}_1$, which is defined over $\widetilde{\mathbb{A}}^1_K$ is uniformizable over $\bar{K}$, a fortiori over $\mathbf{C}$. Considering the quotient τ-sheaf

$$\widetilde{M}' := \widetilde{M}/\widetilde{N}_1$$

over $\widetilde{\mathbb{A}}^1_K$, it is then clear that $\widetilde{M}'_{\mathbf{C}}$ is uniformizable; hence **P** holds for $\widetilde{M}'$. We are thus reduced to proving the proposition in the case where $\overline{\mathcal{M}}_1$ is not potentially trivial.

b) Suppose that $\overline{\mathcal{M}}_1$ is not potentially trivial. Putting $s := [k : \mathbb{F}_q]$, consider the linear endomorphism $T := \tau^s : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_1$. If $(\overline{\mathcal{M}}_1, \tau^s)$ is potentially trivial, then so is $(\overline{\mathcal{M}}_1, \tau)$ (a consequence of Prop. 4.9). It also follows from Thm. 4.9 and the Tate conjecture that if $\overline{\mathcal{M}}_1$ is not trivial then neither is any isogenous τ-sheaf. This implies that T acts nontrivially on V, the stalk of $\overline{\mathcal{M}}_1$ at the generic point of $\mathbb{A}^1_{\bar{k}}$.

Let

$$\mathcal{O}_{\mathbb{P}^1_{\bar{k}},\infty} \cong \bar{k}[[t^{-1}]]$$

be the completion of the local ring of regular functions on $\mathbb{P}^1_{\bar{k}}$ at ∞, with uniformizer $z := t^{-1}$, and F_∞ its field of fractions. For every co-prime pair $(u, s) \in \mathbb{Z}^2$ with $s > 0$, one can define a so-called **pure τ-module $V_{u,s}$ over F_∞ of weight u/s** as follows: taking $F_\infty^{\oplus s}$ as the underlying F_∞-vector space, let the action of τ on the standard basis is given by the matrix $Z_{u,s}$ (cf. (81)).

Consider the characteristic polynomial

$$H(T) := T^h + a_{h-1}T^{h-1} \cdot +a_0 \in k[t][T] \subset F_\infty[T]$$

of T on V. We decompose $F_\infty \otimes_{K(t)} V$ as a direct sum of pure τ^s-modules $V_{u,s}$. The weights that occur in this decomposition are exactly the valuations of the roots of $H(T)$ in F_∞, or, equivalently, the inverses of the slopes of the Newton polygon associated to $H(T)$. This shows that the decomposition of $F_\infty \otimes V$ has at least one component with positive weight, unless all the coefficients of $H(T)$ are constants. But in the latter case, one easily sees that (V, τ^s) is potentially trivial, a contradiction. Hence we may suppose that $F_\infty \otimes V$ contains as a direct summand a pure τ-module V_{u_1,s_1}, with u_1 and $s_1 \geq 1$.

c) Let us extend $\widetilde{\mathcal{M}}$ by zero to a free $\mathcal{O}_{\widetilde{\mathbb{A}}^1_R}$-module

$$\widetilde{\mathcal{M}}_e = \widetilde{\mathcal{M}} \oplus \widetilde{\mathcal{M}}_0$$

of rank r' with τ-action. Let $\mathbf{m}$ be a global basis for $\widetilde{\mathcal{M}}_e$, and

$$\Delta \in \mathrm{Mat}_{r' \times r'}\left(H^0\left(\widetilde{\mathbb{A}}^1_R, \mathcal{O}_{\widetilde{\mathbb{A}}^1_R}\right)\right)$$

the matrix representation of τ with respect to this basis: $\tau(\mathbf{m}) = \mathbf{m} \cdot \Delta$. By lifting the elements $\bar{\mathbf{v}}$ to $\widetilde{\mathcal{M}}_e$, we will construct a σ-bundle $\mathcal{F}' \subset \widetilde{\mathcal{M}}_e$ on $\dot{\mathcal{D}}^\infty_{\mathbf{C}}$, with basis $\mathbf{v}$, such that $\mathcal{F}' \cong \mathcal{F}_{u_1,s_1}$ (cf. (81)).

d) Let $\mathcal{R}$ denote the valuation ring of $\mathbf{C}$. Let us put $\mathbf{v} := \mathbf{m} \cdot Z$. We look for a solution

$$Z \in \mathrm{Mat}_{r' \times r_1}\left(H^0\left(\dot{\mathcal{D}}^\infty_{\mathcal{R}}, \mathcal{O}_{\dot{\mathcal{D}}^\infty_{\mathcal{R}}}\right)\right)$$

of the equation

$$Z \cdot Z_{u_1,s_1} = \Delta \cdot {}^\sigma Z. \tag{83}$$

The reduction

$$Z \cdot Z_{u_1,s_1} = \overline{\Delta} \cdot {}^\sigma Z$$

of this equation has the solution

$$Z_0 \in \mathrm{Mat}_{r' \times r_1}(F_\infty)$$

which represents the F_∞-basis $\bar{\mathbf{v}}$ in terms of the reduction of $\mathbf{m}$ to $\overline{\mathcal{M}}$.

Note that there exists a canonical embedding

$$F_\infty \cong k((z)) \hookrightarrow H^0\left(\dot{\mathcal{D}}^\infty_{\mathcal{R}}, \mathcal{O}_{\dot{\mathcal{D}}^\infty_{\mathcal{R}}}\right).$$

Let $\hat{Z}_0$ denote the canonical lift of Z_0 to

$$\mathrm{Mat}_{r' \times r}(H^0(\dot{\mathcal{D}}^\infty_{\mathcal{R}}, \mathcal{O}_{\dot{\mathcal{D}}^\infty_{\mathcal{R}}})).$$

Starting from $\hat{Z}_0$, we can immediately easily construct via iteration a solution

$$Z = \hat{Z}_0 + \pi Z_1 \in \mathrm{Mat}_{r_1 \times r_1}\left(H^0\left(\dot{\mathcal{D}}^\infty_{\mathcal{R}}, \mathcal{O}_{\mathcal{D}^\infty_{\mathcal{R}}}\right)\right)$$

for (83) which lifts Z_0. As $\bar{Z} = Z_0$ has full rank, the v_i are linearly independent. Further, since $\mathbf{v}$ satisfies $\tau^{s_1}(\mathbf{v}) = t^{u_1}\mathbf{v}$, each of the v_i is clearly contained in

$$H^0(\dot{\mathcal{D}}^\infty_{\mathbf{C}}, \widetilde{\mathcal{M}}) \subset H^0(\dot{\mathcal{D}}^\infty_{\mathbf{C}}, \widetilde{\mathcal{M}}_e).$$

In conclusion, if we set $\mathcal{F}'$ to be the $\mathcal{O}_{\dot{\mathcal{D}}^\infty_{\mathbf{C}}}$-module generated by the global basis $\mathbf{v}$, then $\mathcal{F}'$ is a pure σ-bundle

$$\mathcal{F}' \subset \widetilde{M}|_{\dot{\mathcal{D}}^\infty_{\mathbf{C}}}$$

on $\dot{\mathcal{D}}^\infty_{\mathbf{C}}$ which is isomorphic to $\mathcal{F}_{u_1,s_1}$. The σ-bundle $\mathcal{F}(\widetilde{M})$ being the maximal σ-bundle contained in $\widetilde{M}$, we have $\mathcal{F}' \subset \mathcal{F}(\widetilde{M})$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are pure σ-bundles satisfying $w(\mathcal{F}_1) > w(\mathcal{F}_2)$, then

$$\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0$$

(cf. [**Pi4**]). In the present situation, however, we have $\mathrm{Hom}(\mathcal{F}', \mathcal{F}(\widetilde{M})) \neq 0$ and

$$w(\mathcal{F}') = \frac{u_1}{s_1} > w\left(\mathcal{F}(\widetilde{M})\right) = 0,$$

which gives a contradiction. $\square$

CHAPTER 6

Analytic morphisms of t-motives

Let R be a complete valuation $\mathbb{F}_q$-algebra, with fraction field K, valuation v and residue field k. Like in the previous chapter, we put $\mathcal{C} := \mathbb{A}^1$, and we adopt the notations from Remark 0.3. We fix a characteristic map $\iota : \operatorname{Spec} K \to \mathcal{C}$ by means of an injective $\mathbb{F}_q$-algebra map

$$\iota^* : \mathbf{A} \to K : t \mapsto \theta,$$

for some $\theta \in K$. We will assume that $\theta \in R$, which means that the valuation v is finite with respect to ι (Def. 0.5.iii).

For t-modules E and E', the group of morphisms $\operatorname{Hom}(E, E')$ consists of all algebraic homomorphisms $E \to E'$ which respect the action of $\mathbf{A}$. For t-motives M and M', the group of morphisms $\operatorname{Hom}(M, M')$ consists of $K[t]$-linear homomorphism $M \to M'$ which commute with the action of τ. We recall from Prop. 0.14 that, with the above definitions of morphisms, the categories of abelian t-modules and t-motives are then antiequivalent.

DEFINITION 6.1. For t-modules E and E', the group of **analytic morphisms** $\operatorname{Hom}^{\mathrm{an}}(E', E)$ consists of all rigid analytic entire homomorphisms $E' \to E$ which respect the action of $\mathbf{A}$.

Let $K\langle\langle t\rangle\rangle$ be the ring of entire functions in t. For a t-motive M, put

$$\widetilde{M} := K\langle\langle t\rangle\rangle \otimes_{K[t]} M.$$

We extend τ to a σ-semilinear morphism on $\widetilde{M}$.

DEFINITION 6.2. For t-motives M and M', we define the group of **analytic morphisms** $\operatorname{Hom}^{\mathrm{an}}(M, M')$ as the group of $K\langle\langle t\rangle\rangle$-linear homomorphism $\widetilde{M} \to \widetilde{M}'$ commuting with τ.

We finally remark that all the mentioned groups of morphisms have a natural structure of $\mathbf{A}$-modules. Our main result deals with analytic morphisms of **pure** t-motives (see Def. 6.6):

THEOREM 6.3. *Let E and E' be abelian t-modules whose associated t-motives M and M' are pure of weight w and w', respectively. There exists a natural isomorphism of A-modules*

$$\star : \operatorname{Hom}^{\mathrm{an}}(E', E) \xrightarrow{\sim} \operatorname{Hom}^{\mathrm{an}}(M, M').$$

In other words:

THEOREM 6.3BIS. *There exists an antiequivalence between the categories of pure abelian t-modules over K and of pure t-motives over $\mathbb{A}^1_K$, both endowed with analytic morphisms.*

The first part of this chapter is devoted to the proof of Thm. 6.3. After establishing in §I (Prop. 6.5) that $\star$ is well defined and injective, we prove in section §III (Thm. 6.13) that it is surjective. The arguments for this theorem rely on asymptotic estimates for local logarithmic heights on t-modules, which are presented in section §V, and weight inequalities induced by nontrivial morphisms (cf. §II, Prop. 6.9). In section IV, we work out a further aspect of analytic morphisms of t-modules, namely that of uniformization lattices: see Thm. 6.16.

Analytic morphisms arise naturally in the reduction theory of t-modules and τ-sheaves. In section VI, we recall the Tate uniformization theorem on the analytic structure of Drinfeld modules (i.e. 1-dimensional abelian t-modules) with stable reduction, and show how it gives rise to an analytic filtration of the corresponding t-motive. On the other hand, the theory of analytic semistability (see Thm. 1.26) yields an analytic structure of the t-motive, which should, conjecturally, induce an analytic description of the t-module, as will be explained in section VII.

I. Analyic morphisms of pure t-motives

§1. Topologies on t-motives. Every free $K[t]$-module M of finite rank has a natural topology (**t-topology**), namely that of uniform convergence on any ball of finite radius. Its completion with respect to this topology equals

$$\widetilde{M} = K\langle\langle t\rangle\rangle \otimes_{K[t]} M.$$

On the other hand, the ring $K[\tau]$ has a natural topology of uniform convergence in all balls of finite radius. Identifying τ with the map $x \mapsto x^q$, this topology is that induced by the topology on the polynomial ring $K[x]$ as above via $K[\tau] \subset K[x]$. The completion $K\langle\langle\tau\rangle\rangle$ of $K[\tau]$ with respect to this τ-topology can be identified with the ring of rigid analytic $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_{a,K}$.

For a t-module E, the associated $K[\tau]$-module M is endowed with a natural topology (**τ-topology**) of uniform convergence on all bounded open subsets of $E(K)$. The completion of M with respect to its τ-topology is given the $K\langle\langle\tau\rangle\rangle$-module $\mathrm{Hom}^{\mathrm{an}}(E, \mathbb{G}_{a,K})$ consisting of all rigid analytic entire $\mathbb{F}_q$-linear homomorphism $E \to \mathbb{G}_{a,K}$. This $K\langle\langle\tau\rangle\rangle$-module is isomorphic to

$$M \otimes_{K[\tau]} K\langle\langle\tau\rangle\rangle.$$

Thus a t-motive M is endowed with *two* natural topologies. The following statement shows that the τ-topology is finer than the t-topology.

PROPOSITION 6.4. *There exists a natural injection* $\boxed{M \otimes K\langle\langle\tau\rangle\rangle \hookrightarrow \widetilde{M}}$.

We will give a proof in subsection §§3. Let E and E' be abelian t-modules defined over K, with associated motives M and M'. Every morphism

$$e \in \mathrm{Hom}^{\mathrm{an}}(E', E)$$

induces a $K\langle\langle\tau\rangle\rangle$-linear homomorphism

$$e' : M \otimes K\langle\langle\tau\rangle\rangle \to M' \otimes K\langle\langle\tau\rangle\rangle : m \mapsto m \circ e \tag{84}$$

which commutes with the action of $\mathbf{A}$. By Prop. 6.4, this yields a $K[t]$-linear homomorphism

$$e'' : M \to \widetilde{M}'$$

respecting the action of τ. The homomorphism e'' extends in a unique way to an $K\langle\langle t\rangle\rangle[\tau]$-linear homomorphism $e^\star : \widetilde{M} \to \widetilde{M}'$. Clearly, the map $e \mapsto e^\star$ is injective and $\mathbf{A}$-linear, and we have a commutative diagram

$$\begin{array}{ccccc} M & \subset & M \otimes K\langle\langle\tau\rangle\rangle & \subset & \widetilde{M} \\ & & \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e^\star} \\ M' & \subset & M' \otimes K\langle\langle\tau\rangle\rangle & \subset & \widetilde{M}' \end{array}$$

Thus we have proved

THEOREM 6.5. *Let E and E' be abelian t-modules defined over K, with associated t-motives M and M'. There exists an natural injective $\mathbf{A}$-linear map*

$$\star : \mathrm{Hom}^{\mathrm{an}}(E', E) \to \mathrm{Hom}^{\mathrm{an}}(M, M') : e \mapsto e^\star.$$

§2. Notations. Let E be an abelian t-module of dimension d and rank r, defined over K, with t-motive M. Fixing an isomorphism

$$E \xrightarrow{\sim} \mathbb{G}_{a,K}^{\oplus d},$$

let $\mathbf{x} := (x_1, \cdots, x_d)$ be the corresponding basis of coordinate functions

$$x_i : E \to \mathbb{G}_{a,K};$$

this yields a $K[\tau]$-basis for the t-motive M. On the other hand, we choose a basis $\mathbf{m} = (m_1, \ldots, m_r)$ for the free $K[t]$-module M.

As we will often have to switch from seeing M as a $K[t]$-module to seeing it as a $K[\tau]$-module, we will express the basis $\mathbf{x}$ of coordinate functions for E in terms of $\mathbf{m}$ by means of a matrix $\mathcal{V} \in \mathrm{Mat}_{r\times d}(K[t])$ such that

$$\mathbf{x} = \mathbf{m} \cdot \mathcal{V}.$$

Let us expand every entry $\mathcal{V}_i^j \in K[t]$ of $\mathcal{V}$ as

$$\sum_{\kappa=0}^{\kappa_0} \mathcal{V}_{i,\kappa}^j t^\kappa,$$

putting $\kappa_0 := \deg \mathcal{V} := \max_{j,i} \deg \mathcal{V}_i^j$. We then obtain the formula

$$x_i = \sum_{\kappa=0}^{\kappa_0} \mathcal{V}_{i,\kappa}^j \, m_j \circ t^\kappa, \tag{85}$$

where we have used the ‘Einstein’ *summation convention* for summation over the coordinate indices $i = 1, \ldots, d$, as we will systematically do in the rest of this chapter. Finally, we set $v(\mathcal{V}) := \min_{i,j,\kappa} v(\mathcal{V}_{i,\kappa}^j)$.

Next, we consider the matrix representation of the endomorphisms τ on M. For every integer s, we consider a matrix

$$\Delta^s = \left(\Delta_j^{u,s}\right)_{u,j} \in \mathrm{Mat}_{r\times r}(K[t])$$

such that $\tau(\mathbf{m}) = \mathbf{m}\cdot\Delta^s$. We put $\mu_s := \deg \Delta^s := \max_{j,u} \deg \Delta_j^{u,s}$. Notice that, as

$$\Delta_j^{u,s} := \Delta^1 \cdots {}^{\sigma^{s-1}}(\Delta^1),$$

we get $\mu_s := \deg \Delta^s \leq s\mu_1$; we also note that $\nu_1 \geq 1$. Introducing coefficients $\Delta_{j,\mu}^{u,s} \in K$, we write out

$$\Delta_j^{u,s} = \sum_{\mu=0}^{\mu_s} \Delta_{j,\mu}^{u,s} t^\mu,$$

such that

$$\tau^s \circ m_j = \sum_{\mu=0}^{\mu_s} \Delta_{j,\mu}^{u,s} m_u \circ t^\mu. \tag{86}$$

Also, we set $v(\Delta^s) := \min_{u,j,\mu} v(\Delta_{j,\mu}^{u,s})$.

§3. Proof of Prop. 6.4.

PROOF OF PROP. 6.4. **a)** Notice that an element

$$\sum_{n=0}^{\infty} y_n \tau^n \in K[[\tau]] \subset K[[x]]$$

is contained in $K\langle\langle\tau\rangle\rangle$ if and only if $q^{-n}v(y_n) \to \infty$; this convergence condition is much stronger than that for elements $\sum_{n=0}^{\infty} y^n t^n \in K\langle\langle t\rangle\rangle$, namely

$$n^{-1}v(y_n) \to \infty.$$

We will now show through explicit calculations that, similarly, convergence for τ-topology is a stronger condition than for the t-topology, on a t-motive M.

b) Let us express an element $y \in M \otimes K\langle\langle\tau\rangle\rangle$ in terms of the $K[\tau]$-basis $\mathbf{x}$ of M via $y = \mathcal{F}^c x_c$ (using summation convention), introducing coefficients

$$\mathcal{F}^c \in K\langle\langle\tau\rangle\rangle.$$

Expanding $\mathcal{F}_c$ as $\sum_{n=0}^{\infty} \mathcal{F}_n^c \tau^n$, for some elements $\mathcal{F}_n^c \in K$, we get

$$q^{-n}v(\mathcal{F}_n^c) \to \infty. \tag{87}$$

With these notations, we obtain

$$y = \sum_{n=0}^{\infty} \mathcal{F}_n^c \tau^n \circ x_c.$$

We can, using the notations of (85) and (86), express this formally in terms of the $K[t]$-basis $\mathbf{m}$ and the endomorphism t as follows:

(88)
$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} \mathcal{F}_n^c\, \tau^n \circ \left(\sum_{\kappa=0}^{\kappa_0} \mathcal{V}_{c,\kappa}^j\, m_u \circ t^\kappa \right) \\
&= \sum_{n=0}^{\infty} \sum_{\kappa=0}^{\kappa_0} \sum_{\mu=0}^{n\cdot\mu_1} \mathcal{F}_n^c \left(\mathcal{V}_{c,\kappa}^j \right)^{q^n} \Delta_{j,\mu}^{u,n}\, m_u \circ t^{\kappa+\mu}.
\end{aligned}
$$

Regrouping the terms in this expression, and substituting $\zeta = \mu + \kappa$, we obtain

(89)
$$
y = \sum_{\zeta=0}^{\infty} \left(\sum_{\kappa=0}^{\min(\zeta,\kappa_0)} \sum_{n=\left\lfloor \frac{\zeta-\kappa}{\mu_1} \right\rfloor}^{\infty} \mathcal{F}_n^c \left(\mathcal{V}_{c,\kappa}^j \right)^{q^n} \Delta_{j,\mu}^{u,n} \right) m_u \circ t^\zeta .
$$

What remains to prove is that the coefficients

$$
\mathcal{E}_\zeta^u := \sum_{\kappa=0}^{\min(\zeta,\kappa_0)} \sum_{n=\left\lfloor \frac{\zeta-\kappa}{\mu_1} \right\rfloor}^{\infty} \mathcal{F}_n^c \left(\mathcal{V}_{c,\kappa}^j \right)^{q^n} \Delta_{j,\mu}^{u,n}
$$

are well defined elements of K such that $\sum_{\zeta=0}^{\infty} \mathcal{E}_\zeta^u\, t^\zeta \in K\langle\langle t\rangle\rangle$, for all $u \le r$.

c) Upon replacing the basis $\mathbf{m}$ by a scalar multiple, we may suppose that $v(\Delta^n) \ge 0$ for all $n \ge 1$, and modifying the coordinates on E by a scalar, we can also assume that $v(\mathcal{V}) \ge 0$. Thus it follows from (87) that the series $\mathcal{E}_\zeta^u$ converges to an element in K, for all u and ζ. Moreover, we see that

$$
q^{-\frac{\zeta-\kappa_0}{\mu_1}} v(\mathcal{E}_\zeta^u) \to \infty
$$

for $\zeta \to \infty$. This shows that $\mathcal{F}^c \in K\langle\langle t\rangle\rangle$, and hence $y \in \widetilde{M}$. □

II. Morphisms and weights

We reformulate Def. 0.15:

DEFINITION 6.6 (Anderson [**An1**], 1.9). A t-motive M over $K[t]$ is **pure** of weight w, if there exists an integer z and a $K[[t^{-1}]]$-lattice M_∞ in

$$
V_\infty := M \otimes_{K[t]} K((t^{-1}))
$$

such that

$$
\tau^z((\sigma^z)^* M_\infty) = t^{zw} M_\infty .
$$

We remark that if a pure t-motive has rank r, dimension d and weight w, then

$$
w = \frac{d}{r}.
$$

Let deg denote the natural extension of the degree function on $K[t]$ to $K(t)$. For a matrix $B = (B_{ij})_{i,j} \in \mathrm{Mat}(K(t))$, we put

$$
\deg B = \max_{i,j} \deg B_{ij} .
$$

LEMMA 6.7. *For a pure t-motive M of weight w and rank r, we put, for all integers $s \in zr\mathbb{N}$,*

$$\mathcal{T}^s := t^{-sw}\Delta^s$$

(cf. Def. 6.6 for the definition of z, and cf. (86) for that of Δ). There exists an integer $\delta \geq 0$ such that $\deg \mathcal{T}^s \leq \delta$ *and* $\deg (\mathcal{T}^s)^{-1} \leq \delta$ *for all such s.*

PROOF. With respect to a $K[[t^{-1}]]$-basis $\mathbf{m}$ of M_∞, we have

$$\tau^{sz}(\mathbf{m}') = \mathbf{m}' \cdot (t^{szw}\hat{\mathcal{T}}^{sz}),$$

with $\hat{\mathcal{T}}^{sz} \in \mathrm{GL}_r(K[[t^{-1}]])$ (cf. def. 6.6). Expressing the $K((t^{-1}))$-basis $\mathbf{m}'$ for M_∞ in terms of the fixed basis $\mathbf{m}$ for M by means of the matrix $\gamma \in \mathrm{GL}_r(K((t^{-1})))$ as follows:

$$\mathbf{m}' = \mathbf{m} \cdot \gamma,$$

we obtain:

$$\Delta^{sz} = \gamma \cdot t^{szw}\hat{\mathcal{T}}^{sz} \cdot {}^{\sigma^{sz}}(\gamma^{-1}). \tag{90}$$

As, $\mathcal{T}^{sz} = \gamma \cdot \hat{\mathcal{T}}^{sz} \cdot {}^{\sigma^{sz}}(\gamma^{-1})$, we can take $\delta := 2\max(\deg\gamma, \deg\gamma^{-1})$, and the claim follows. □

Let M and M' be pure t-motives of respective weights w and w', ranks r and r' and constants z and z' as in Def. 6.6. Given a morphism

$$f \in \mathrm{Hom}^{\mathrm{an}}(M, M'),$$

we express f in terms of bases $\mathbf{m}$ and $\mathbf{m}'$ for M and M: let the matrix

$$\mathcal{F} \in \mathrm{Mat}_{r' \times r}(K\langle\langle t\rangle\rangle)$$

satisfy $f(\mathbf{m}) = \mathbf{m}' \cdot \mathcal{F}$. We also put $\mathcal{F} = \sum_{k=0}^{\infty} \mathcal{F}_k t^k$, for matrices

$$\mathcal{F}_k = (\mathcal{F}^u_{j,k})_{u,j} \in \mathrm{Mat}_{r' \times r}(K).$$

Finally, we set

$$v(\mathcal{F}_k) := \min_{u,j} v(\mathcal{F}^u_{j,k})$$

and

$$\hat{v}_k(\mathcal{F}) := \inf_{k' \geq k} v(\mathcal{F}_{k'}). \tag{91}$$

LEMMA 6.8. *Let M and M' be pure t-motives of respective weights w and w'. There exists a constant δ_0, such that for any morphism*

$$f \in \mathrm{Hom}^{\mathrm{an}}(M, M')$$

with matrix representation $\mathcal{F}$, we have, for all s divisible by r, r', z and z', and for all k:

$$\boxed{\hat{v}_k(\mathcal{F}) \geq \min_{j \geq k - s(w'-w) - \delta_0} q^s \hat{v}_j(\mathcal{F})}. \tag{92}$$

PROOF. **a)** As the homomorphism f commutes with τ^s, for all s, we have:

$$\tau^s_{M'} \circ f = f \circ \tau^s_M;$$

in matrix notation (using the notations from (86)):

$$\boxed{\Delta'^s \cdot {}^{\sigma^s}\mathcal{F} = \mathcal{F} \cdot \Delta^s}x. \tag{93}$$

We will now exploit the relations this equation induces on the coefficients of

$$\mathcal{F} = \sum_{k=0}^{\infty} \mathcal{F}_k t^k.$$

b) Recall that $\det \Delta^1 = b(t-\theta)^d$, for some $b \in K^\times$, and that $d = rw$. We put

$$\tilde{\Delta}^1 := (t-\theta)^d (\Delta^1)^{-1} \in \mathrm{Mat}_{r\times r}(K[t])$$

and define $v(\tilde{\Delta}^1)$ just like we defined $v(\Delta^1)$. Note that the Lemma is not affected by replacing the basis $\mathbf{m}$ by a scalar multiple, and therefore we may assume that $v(\tilde{\Delta}^1) \geq 0$. Next, we set $\tilde{\Delta}^s := {}^{\sigma^{s-1}}\!\left(\tilde{\Delta}^1\right) \cdots \tilde{\Delta}^1$. Multiplying on the right by $\tilde{\Delta}^s$ in equation (93) then gives

$$\Delta'^s \cdot {}^{\sigma^s}\mathcal{F} \cdot \tilde{\Delta}^s = \mathcal{F} \cdot (t-\theta)^d \cdots (t-\theta^{q^{s-1}})^d. \tag{94}$$

We expand $(t-\theta)^d \cdots (t-\theta^{q^{s-1}})^d$ as

$$t^{ds} + \sum_{i=0}^{ds-1} \theta_i^s\, t^i,$$

where the coefficients θ_i^s satisfy $v(\theta_i^s) \geq 0$ (here we need the assumption that $v(\theta) \geq 0$, i.e. that the valuation is finite).

Upon replacing the basis $\mathbf{m}'$ by some scalar multiple, we can assume that $v(\Delta'^s) \geq 0$. By Lemma 6.7, $\deg \Delta^s \leq sw + \delta$ and $\deg \Delta'^s \leq sw' + \delta'$. As $\tilde{\Delta}^s$ is, up to a scalar in K, the adjoint matrix of Δ^s, this yields that

$$\deg \tilde{\Delta}^s \leq (r-1)\deg \Delta^s \leq (r-1)(sw+\delta).$$

c) Equation (94) gives

$$\begin{aligned}\left(\sum_{i=0}^{sw'+\delta'} \Delta'^s_i\, t^i\right)\left(\sum_{k=0}^{\infty} {}^{\sigma^s}\mathcal{F}_k t^k\right)\left(\sum_{i=0}^{(r-1)(sw+\delta)} \tilde{\Delta}^s_i\, t^i\right)\\ = \left(\sum_{k=0}^{\infty} \mathcal{F}_k t^k\right)\left(t^{rws} + \sum_{i=0}^{rws-1} \theta_i^s\, t^i\right)\end{aligned} \tag{95}$$

Comparing the coefficients of t^{k+rsw} in this equation yields

$$\sum_{\substack{i+i'+j=k+rsw\\ i' \leq sw'+\delta'\\ i\leq (r-1)(sw+\delta)}} \Delta'^s_{i'} \cdot {}^{\sigma^s}\mathcal{F}_j \cdot \tilde{\Delta}^s_i = \mathcal{F}_k + \sum_{j=1}^{sw} \theta^s_{rws-j}\, \mathcal{F}_{k+j} \tag{96}$$

Putting $\delta_0 := \delta' + (r-1)\delta$, this implies

$$(97) \qquad v(\mathcal{F}_k) \geq \min\left\{ \min_{j\geq k-s(w'-w)-\delta_0} q^s v(\mathcal{F}_j), \min_{j\geq 1} v(\mathcal{F}_{k+j}) \right\}.$$

It follows that $\hat{v}_k(\mathcal{F}) \geq \min_{j\geq k-s(w'-w)-\delta_0} q^s \hat{v}_j(\mathcal{F})$. ☐

PROPOSITION 6.9. *Let M and M' be pure t-motives of weight w and w', respectively.*

i) *If* $w > w'$, *then* $\mathrm{Hom}^{\mathrm{an}}(M, M') = \mathrm{Hom}(M, M') = 0$.
ii) *If* $w = w'$, *then* $\mathrm{Hom}^{\mathrm{an}}(M, M') = \mathrm{Hom}(M, M')$.
iii) *If* $w < w'$, *then* $\mathrm{Hom}(M, M') = 0$.

PROOF. **a)** To show that if $w \neq w'$, then $\mathrm{Hom}(M, M') = 0$, we use the theory of Dieudonné modules, as explained in [**La1**], Appendix B. The definition of purity for M implies, upon passing to the algebraic closure $\bar{K}$, that the τ-module

$$\bar{V}_\infty := M \otimes_{K[t]} \bar{K}((t^{-1}))$$

is isomorphic to a direct sum of modules of the form

$$\bar{K}((t^{-1}))[\tau]/(\tau^z - t^{zw}),$$

with $z \geq 0$. Idem for M replaced by M'. If $w \neq w'$, then the theory shows that $\mathrm{Hom}(V_\infty, V'_\infty) \neq 0$. A fortiori, if $w \neq w'$, then $\mathrm{Hom}(M, M') = 0$.

b) To conclude the proof, we are left to show that if there exists an

$$f \in \mathrm{Hom}^{\mathrm{an}}(M, M') \backslash \mathrm{Hom}(M, M'),$$

then $w < w'$. We let $\mathcal{F}$ be the matrix representation of f and take up the notation from Lemma 6.8. As f is entire, we have $\hat{v}_k(\mathcal{F}) \to \infty$.

- If $w > w'$, then we can choose s such that $-s(w'-w) - \delta' > 0$, and then Lemma 6.8 implies:

  $$\hat{v}_k(\mathcal{F}) \geq \min_{k'>k} \hat{v}_{k'}(\mathcal{F}) = \hat{v}_{k+1}(\mathcal{F})$$

  for k, whereas $\hat{v}_k(\mathcal{F}) \leq hatv_{k+1}(\mathcal{F})$ by definition. Thus we get that $\hat{v}_k(\mathcal{F}) = \hat{v}_{k'}(\mathcal{F})$ for all $k, k' \geq k_0$. As $\hat{v}_k(\mathcal{F}) \to \infty$, this implies that $\mathcal{F}_k = 0$ for all k, which is a contradiction, as $f \neq 0$.
- If $w' = w$, then we obtain, for all $k \geq 0$ and for all $s \geq 1$, that

  $$\hat{v}_k(\mathcal{F}) \geq q^s \hat{v}_{k-\delta'}(\mathcal{F}) > 0.$$

  Letting $s \to \infty$, we deduce for all $k \geq k_0 + \delta'$ that $\mathbb{F}_k = 0$, which is a contradiction, as $f \neq \mathrm{Hom}(M, M')$.

Thus we conclude that $w < w'$. ☐

III. Analytic morphisms of pure t-motives

§1. Local height. With respect to the basis **x** of coordinate functions on E, we define a naive **local (logarithmic) height** $h : E(K) \to \mathbb{R}$ on $E(K)$ as follows:

DEFINITION 6.10.

$$h(P) := \max_{1 \le i \le d} -v(x_i \cdot P)$$

For $a \in \mathbf{A}$ and $P \in E(K)$, let us denote the image of P under the endomorphism $\phi_E(a)$ by $a \cdot P$. Let the standard norm $|\cdot|_\infty$ on $\mathbf{A}$ be defined by

$$|a|_\infty = q^{\deg a}.$$

In section V (see Prop. 6.18 and Prop. 6.19), we will prove the following asymptotic estimates for the height h under the action of $\mathbf{A}$ on $E(K)$:

THEOREM 6.11. *If E is a pure of weight w, then there exist constants $c_1 < 1$ and $c_2 > 1$ and an integer n such that, for all $P \in E(K)$ with $h(P) \gg 0$ and for all nonconstant $a \in \mathbb{F}_q[t^n]$:*

$$\boxed{c_1\, |a|_\infty^{w^{-1}}\, h(P) \le h(a \cdot P) \le c_2\, |a|_\infty^{w^{-1}}\, h(P).} \tag{98}$$

REMARK 6.12. In some situations, it will be useful to replace the pair (E, ϕ_E) by the so-called induced t-module (cf. [**Den**]) consisting of E and ϕ'_E, where

$$\phi'_E : \mathbf{A} \to \mathrm{End}(E) : t \mapsto \phi_E(t^n).$$

We can then assume $n = 1$ in the above formula.

§2. Surjectivity of $\star$. We need to express **m** as well as the endomorphism t in terms of the $K[\tau]$-basis **x**. For this, we introduce, analogously to what we did in (85) and (86), constants λ_0 and $\nu_s \in \mathbb{N}$, for all s, and coefficients $\mathcal{W}^i_{j,\lambda}$, for $\lambda \le \lambda_0$, and $\Theta^{w,s}_{i,\nu}$, for $\nu \le \nu_s$, such that

$$m_j = \sum_{\lambda=0}^{\lambda_0} \mathcal{W}^i_{j,\lambda}\, \tau^\lambda \circ x_i; \tag{99}$$

$$x_i \circ t^s = \sum_{\nu=0}^{\nu^s} \Theta^{w,s}_{i,\nu}\, \tau^\nu \circ x_w. \tag{100}$$

The following proposition, combined with Thm. 6.5, concludes the proof of Thm. 6.3:

THEOREM 6.13. *Let E and E' be abelian t-modules defined over K, with associated motives M and M'. The map*

$$\star : \mathrm{Hom}^{\mathrm{an}}(E', E) \to \mathrm{Hom}^{\mathrm{an}}(M, M')$$

is a bijection.

PROOF. **a)** By Anderson, we know that there is a bijection

$$\mathrm{Hom}(E', E) \cong \mathrm{Hom}(M, M').$$

So let us take

$$f \in \mathrm{Hom}^{\mathrm{an}}(M, M') \backslash \mathrm{Hom}(M, M').$$

We need to show that, for any of the basis coordinate functions x_i, the function $f(x_i) \in \widetilde{M}$ is entire on E'. Indeed, this implies that f restricts to a homomorphism

$$M \otimes_{K[\tau]} K\langle\langle\tau\rangle\rangle \to M' \otimes_{K[\tau]} K\langle\langle\tau\rangle\rangle,$$

hence gives rise to a morphism $e \in \mathrm{Hom}^{\mathrm{an}}(E', E)$ such that $e^\star = f$.

We represent f in terms of bases $\mathbf{m}$ and $\mathbf{m}'$ by means of a matrix $\mathcal{F}$, adopting the same notation as in Lemma 6.8. We want to express the morphism f in terms of the coordinate functions $\mathbf{x}$ and $\mathbf{x}'$. We use the base change formulas (85) and (99) (where we use analogous notations for M', systematically adding a prime), and obtain the following formal equation

$$f(x_i) = \sum_{k=0}^{\infty} \sum_{\kappa,\lambda} \mathcal{V}_{i,\kappa}^{j}\ \mathcal{W}'^{w}_{u,\lambda}\ \mathcal{F}_{j,p}^{u}\ \tau^\lambda \circ x'_w \circ t^{k+\kappa}. \tag{101}$$

Hence, putting

$$\varepsilon(k; P) := \sum_{\kappa,\lambda} \mathcal{V}_{i,\kappa}^{j}\ \mathcal{W}'^{w}_{u,\lambda}\ \mathcal{F}_{j,k}^{u}\ x'_w (t^{k+\kappa} \cdot P)^{q^\lambda}, \tag{102}$$

we need to show that $f(x_i)(P) = \sum_{k=0}^{\infty} \varepsilon^k(P)$ converges for all $P \in E'(K)$. In order to do this, we need estimates on the coefficients $\mathcal{F}_k$ and on the height of $t^{k+\kappa} \cdot P$ in terms of k.

b) Putting $v(\mathcal{F}_k) := \min_{u,j} v(\mathcal{F}_{j,k}^u)$, and $\hat{v}_k(\mathcal{F}) := \inf_{k' \geq k} v_{k'}(\mathcal{F})$ as before, we have, by Lemma 6.8, the inequality

$$\hat{v}_k(\mathcal{F}) \geq \min_{j \geq k - s(w'-w) - \delta_0} q^s \hat{v}_j(\mathcal{F}), \tag{103}$$

which holds for a fixed constant δ_0 and all $s > 0$ divisible by z, r, z' and r. From Prop. 6.9, we know that $w < w'$. Therefore we get, for all such $s \geq 0$ and all

$$k \geq k_0 + s(w' - w) + \delta_0,$$

that

$$\hat{v}_k(\mathcal{F}) \geq q^s \hat{v}_{k - s(w'-w) - \delta_0}(\mathcal{F}). \tag{104}$$

One deduces from this that there exist $v_0, k'_0 > 0$ such that for $k \geq k'_0$:

$$\boxed{v_k \geq q^{\frac{k}{w'-w}} v_0}. \tag{105}$$

c) We may replace E' by an induced t-module without loss of generality, since we are looking for an analytic homomorphism of the underlying group $\mathbb{G}_{a,K}^{\oplus d'}$. Hence, by remark 6.12, we can assume that there exist constants $\eta > 0$ and $c_2 > 1$ such that, for *all* integers s and all $P \in E'(K)$ with $h(P) > \eta$:

$$h(t^s \cdot P) \leq c_2\, q^{s/w'}\, h(P).$$

We can extend this formula to one holding for all P as follows: there exists a constant $c' > 1$, such that, for all integers s and all $P \in E'(K)$:

$$\boxed{h(t^s \cdot P) \le c' q^{s/w'} \max\{h(P), \eta\}}\,. \tag{106}$$

Indeed, either $h(t^{s'} \cdot P) \ge \eta$ for all $s' \le s$ or there exists an $s' \le s$ such that

$$h(t^{s'} \cdot P) \ge \eta$$

and $h(t^i \cdot P) \le \eta$ for $i \le s'$. In the latter case, using (113)

$$h(t^s \cdot P) \le c\, q^{(s-s')/w'} h(t^{s'} \cdot P) \le c\, q^{(s-s')/w'}(-v(\Theta^1) + q^{v_1}\eta) \le c'\, \eta\, q^{s/w'},$$

for a new constant $c' > c_2$.

d) Finally, we can give an estimate for $\varepsilon(k; P)$, for every $P \in E'(K)$. By (106) and (92), we get, for $k \ge k_0'$:

$$\begin{aligned} &v(\varepsilon(k; P)) \\ &\ge v(\mathcal{V}) + v(\mathcal{W}') + v(\mathcal{F}_k) - q^{\lambda_0} h(t^{k+\kappa} \cdot P) \\ &\ge v(\mathcal{V}) + v(\mathcal{W}') + q^{\frac{k}{w'}} \left(v_0\, q^{k\left(\frac{1}{w'-w} - \frac{1}{w'}\right)} - c' q^{\frac{\kappa_0}{w} + \lambda_0} \max\{h(P), \eta\} \right). \end{aligned} \tag{107}$$

Since $0 < w < w'$, one has $\frac{1}{w'-w} > \frac{1}{w'}$. Therefore the expression

$$v_0\, q^{k\left(\frac{1}{w'-w} - \frac{1}{w'}\right)}$$

tends to infinity, and hence so does $v(\epsilon(k; P))$. This shows that the series $f(x_i)$ is indeed an entire function on $P \in E'(K)$. □

QUESTION 6.14. Does Thm. 6.13 hold without the assumption that M and M' are pure?

IV. Uniformization lattices

DEFINITION 6.15. An **A-lattice** H in $E(K)$ is a free finitely generated sub-**A**-module of $E(K)$ which is strictly discrete, i.e. the intersection of H with any open disk of finite radius is finite.

THEOREM 6.16. *Let E and E' be pure t-modules of rank r and r', resp. For every $e \in \mathrm{Hom}^{\mathrm{an}}(E', E)$ is such that $e^\star \in \mathrm{Hom}^{\mathrm{an}}(M, M')$ is surjective, the kernel*

$$H := (\ker\ e)(\bar{K})$$

is an **A**-*lattice in $E'(K^{\mathrm{sep}})$ whose rank h satisfies $h \le r - r'$.*

PROOF. **a)** *H is strictly discrete.* We remark that

$$\mathrm{coker}\ \tau_M$$

is isomorphic to

$$\mathrm{Hom}_K(\mathrm{Lie}(E), K),$$

were $\mathrm{Lie}(E)$ is the Lie-algebra of E. Now $e^\star$ induces a surjective morphism

$$\mathrm{coker}\ \tau_M \ \to \mathrm{coker}\ \tau_{M'}$$

which shows that $\mathrm{Lie}(e) : \mathrm{Lie}(E') \to \mathrm{Lie}(E)$ is injective. By the inverse function theorem, $\ker e$ is therefore a 0-dimensional analytic subvariety of $E \cong \mathbb{A}^d_K$. As affinoid spaces are noetherian, any closed disk contains only a finite number of points of H, whence strict discreteness.

b) *H is torsion free.* By the isomorphism

$$E[a](K) \to \mathrm{Hom}_{K[t][\tau]}(M/aM, \mathrm{Hom}(\mathbf{A}/(a), K))$$

for abelian t-modules (cf. [**An1**]), it follows, for all non-constant $a \in \mathbf{A}$, from

$$M/aM \twoheadrightarrow M'/aM'$$

that

$$E'[a](K) \overset{e}{\hookrightarrow} E[a](K).$$

Hence H is torsion free.

c) *H is finitely generated.* Upon replacing E' by an induced t-module, there exist, by remark 6.12, constants $c_1 < 1$ and $c_2 > 1$ such that, for all integers s and all $P \in E'(K)$ with $h(P) \gg 0$:

$$c_1\, q^{s/w'}\, h(P) \le h(t^s \cdot P) \le c_2\, q^{s/w'}\, h(P), \tag{108}$$

and we can assume that $c_1/c_2 \le q^{1/w'}$. The closed disk

$$\bar{D}_\eta = \{P \in E'(K); h(P) \le \eta\}$$

contains only a finite number μ of elements of H. As H contains no torsion, we have that, for all $P \in H \cap \bar{D}_\eta$, the points $P, t \cdot P, t^2 \cdot P, \dots, t^\mu \cdot P$ are all distinct. This shows that $h(t^{\mu_0} \cdot P) > 0$ for some $\mu_0 \le \mu$, and, a fortiori,

$$h(t^{\mu'} \cdot P) > \eta$$

for all $\mu' > \mu_0$. As it suffices to show that $t^\mu \cdot H$ is finitely generated, we may suppose that $H \cap \bar{D}_\eta = \phi$, hence the estimates of (108) are valid for all $P \in H$.

Since H is torsion free, H injects into $V := \mathbb{F}_q(t) \otimes_{\mathbf{A}} H$. Let V' be a subspace of V of finite dimension α and put $H' := H \cap V'$. We choose an element $P_1 \in H'$ with minimal height. We take recursively, for all $i \le \alpha$, a point P_i with minimal height in $H' \setminus H'_{i-1}$, where we put

$$H'_k := \bigoplus_{1 \le j \le k} \mathbf{A} \cdot P_j,$$

for $k \le i-1$. Put $V'_k := \mathbb{F}_q(t) \cdot H'_k \subset V'$. We will now prove, by induction on i, that $(P_1, \dots, P_k)$ forms a basis for $H' \cap V'_k$. For $k = 1$, the statement is obvious; we assume that it holds for $i \le k-1$.

Suppose that

$$Q \in \left(H' \cap V'_k\right) \setminus H'_k :$$

there exist $\{a_j\}_{1 \le j \le k} \in \mathbf{A}^r$ and a non-constant $b \in \mathbf{A}$ such that b does not divide all of the a_j and such that

$$b \cdot Q = \sum_{1 \le j \le k} a_j \cdot P_j.$$

Upon replacing Q by an element in $Q + H'_k$, we can assume that

$$\deg a_j < \deg b,$$

for all $j \leq k$. By the estimate (108), it follows from

$$h(b \cdot Q) \leq \max_j h(a_j \cdot P_j)$$

that

$$\begin{aligned} c_1 q^{\deg b/w'} h(Q) &\leq h(b \cdot Q) \leq \max_j h(a_j \cdot P_j) \\ &\leq c_2 \max_j q^{\deg a_j/w'} h(P_j) \leq c_2\, q^{(\deg b - 1)/w'} h(P_k). \end{aligned} \tag{109}$$

As we assumed that $c^2 < q^{w^{-1}}$, we obtain $h(Q) < h(P_k)$. However, as

$$Q \notin V'_{k-1},$$

we have $h(Q) \geq h(P_k)$, which gives a contradiction.

In conclusion, since $\dim V' = \alpha = \dim V'_\alpha$, we see that $H'_\alpha = H'$, so H' is finitely generated. Next, we observe that

$$t^{-1}(H')/H' \overset{e}{\hookrightarrow} E[t](K) \cong_{\mathbf{A}} (\mathbf{A}/(t))^r.$$

Further, the kernel of the map $t^{-1}(H')/H' \to H'/t(H')$ is isomorphic to $(\mathbf{A}/(t))^{r'}$. Therefore

$$\mathrm{rk}_{\mathbf{A}/(t)}\, H'/t(H') \leq r - r',$$

which implies that $\mathrm{rk}\, H' \leq r - r'$ and hence $\dim V' \leq r - r'$. We have thus proved, for all finite dimensional subspaces of V, that $\dim V' \leq r - r'$, which shows that V itself is finite dimensional, with $\dim V \leq r - r'$. As H is then finitely generated over $\mathbf{A}$ and torsion free, it is free of rank at most $r - r'$. $\square$

V. Asymptotic bounds on local heights

Let K be a complete valued $\mathbb{F}_q$-field; in this section, we will not assume that the valuation on K is finite with respect to the characteristic ι.

For every $P \in E(K)$, we can deduce from equations (85) to (100):

$$v(x_i(P)) \geq v(\mathcal{V}) + \min_{j,\kappa} v(m_j(t^\kappa \cdot P)); \tag{110}$$

$$v(m_j(P)) \geq v(\mathcal{W}) + \min_{i,\lambda_0} q^\lambda v(x_i(P)); \tag{111}$$

$$q^s v(m_j(P)) \geq v(\Delta^s) + \min_{u,\mu \leq \mu_s} v(m_u(t^\mu \cdot P)); \tag{112}$$

$$v(x_i(t^s \cdot P)) \geq v(\Theta^s) + \min_{w,\nu \leq \nu_s} q^\nu v(x_w(P)). \tag{113}$$

LEMMA 6.17. *Let E be an abelian t-module. For all $\psi > 1$, there exists an integer $n > 0$ such that: for all $P \in E(K)$ with $h(P) \gg 0$ and for all nonconstant $a \in \mathbb{F}_q[t^n]$:*

$$\boxed{h\big(a \cdot P\big) > \psi \cdot h(P)}\,. \tag{114}$$

PROOF. The endomorphism τ obviously satisfies $h(\tau \cdot P) = q \cdot h(P)$. The idea is that, as τ can be expressed in terms of the operator 't' on M (equation (86)), there exists, for each P with $h(P) \gg 0$, a power t^ρ such that $h(t^\rho \cdot P) \sim q \cdot h(P)$. The subtlety is to find a ρ which holds for all P.

a) Combining the three inequalities (110) to (112) gives, for all $s > 0$ and all i:

$$q^s v(x_i \cdot P) \geq \big(q^s v(\mathcal{V}) + v(\Delta^s) + v(\mathcal{W})\big) + \min_{\substack{i' \\ 0\leq\mu\leq\mu_s \\ 0\leq\lambda\leq\lambda_0 \\ 0\leq\kappa\leq\kappa_0}} q^\lambda v(x_{i'}(t^{\mu+\kappa} \cdot P)).$$

Hence $q^s h(P) + q^s v(\mathcal{V}) + v(\Delta^s) + v(\mathcal{W}) \leq \max_{\mu,\lambda,\kappa} q^\lambda h((t^{\mu+\kappa} \cdot P)$.

For a given ψ, let us fix $s \gg 0$ such that $q^s \gg q^{\lambda_0}\psi$. For P with $h(P)$ large enough, we obtain

(115) $$\psi \cdot h(P) \leq \max_{1\leq\rho\leq\mu_s+\kappa_0} h(t^\rho \cdot P).$$

b) Let us put $\rho_0 := \mu_s + \kappa_0$. We deduce from (113) that, for all s,

$$h(t^s \cdot P) \leq -v(\Theta^s) + q^{\nu_0} h(P).$$

In particular, we can find, for P with $h(P) \gg 0$, a constant $\chi_{\rho_0} > 0$ such that

(116) $$\max_{1\leq\rho\leq\rho_0} h(t^\rho \cdot P) \leq \chi_{\rho_0}\, h(P).$$

c) We now take, for each P with $h(P) \gg 0$, a $\rho_P(1)$ with $0 < \rho_P(1) \leq \rho_0$ such that $h(t^{\rho_P(1)} \cdot P) \geq \psi\, h(P)$, and choose recursively, for all $u > 1$, some $\rho_P(1)$ such that

$$\rho_P(u-1) < \rho_P(u) \leq \rho_P(u-1) + \rho_0$$

and for which

$$h(t^{\rho_P(u)} \cdot P) \geq \psi\, h(t^{\rho_P(u-1)} \cdot P) \geq \psi^u h(P).$$

We fix an integer n such that $\psi^{(n/\rho_0)-1} > \chi_{\rho_0}$. For each P, we take the smallest u such that $\rho_P(u) > n$. We then have $\rho_P(u) - n \leq \rho_0$ and $u \geq n/\rho_0$ and we find that

(117) $$\psi^u\, h(P) \leq h(t^{\rho_P(u)} \cdot P) = h\big(t^{(\rho_P(u)-n)} \circ t^n(P)\big) \leq \chi_{\rho_0}\, h(t^n \cdot P),$$

which shows that $h(t^n \cdot P) > \psi\, h(P)$. If a in $\mathbb{F}_q[t^n]$, then

$$h(a \cdot P) = h(t^{\deg a} \cdot P) \geq \psi\, h(P),$$

for P with $h(P) \gg 0$. $\square$

PROPOSITION 6.18. *If E is pure of weight w, then there exists a constant $c_1 < 1$ and an integer n such that, for all $P \in E(K)$ with $h(P) \geq \eta$ and for all nonconstant $a \in \mathbb{F}_q[t^n]$:*

(118) $$\boxed{c_1\, |a|_\infty^{w^{-1}} h(P) \leq h(a \cdot P)}\,.$$

PROOF. Without loss of generality, we may replace the t-module E by an induced module such that, by the previous lemma, for some fixed $\psi > 1$, we have

$$\psi\, h(P) < h(t \cdot P)$$

for $h(P) \gg 0$. The essential idea, expressed by Lemma 6.7, is that, for pure t-modules E and large powers s, the endomorphism $\tau^{sw^{-1}}$ acts, up to lower powers of t, more or less as t^s, and hence $h(t^s \cdot P) \sim q^{sw^{-1}} h(P)$.

a) By lemma 6.7, $\mu_s := \deg \Delta^s \leq sw + \delta$, for a fixed $\delta > 0$ and any integer $s > 0$ divisible by r and z. Hence, using formulas 86 and 99, we get, if $h(P) \gg 0$:

$$\begin{aligned} q^s v(m_j \cdot P) &\geq v(\Delta^s) + \min_{u, 1 \leq \mu \leq sw+\delta} v(m_u(t^\mu \cdot P)) \\ &\geq v(\Delta^s) + v(\mathcal{W}) + \min_{i, \lambda, 1 \leq \mu \leq sw+\delta} q^\lambda v(x_i(t^\mu \cdot P)) \\ &= v(\Delta^s) + v(\mathcal{W}) + q^{\lambda_0}(-h(t^{sw+\delta} \cdot P)) \end{aligned} \tag{119}$$

On the other hand, by (110), $-h(P) \geq v(\mathcal{V}) + \min_{j,\kappa} v(m_j(t^\kappa \cdot P))$, so we get

(120)
$$\begin{aligned} -q^s h(P) &\geq q^s v(\mathcal{V}) + v(\Delta^s) + v(\mathcal{W}) + q^{\lambda_0} \min_{0 \leq \kappa \leq \kappa_0} (-h(t^{sw+\delta+\kappa} \cdot P)) \\ &\geq q^s v(\mathcal{V}) + v(\Delta^s) + v(\mathcal{W}) + q^{\lambda_0} v(\Theta^{\delta+\kappa_0}) - q^{\lambda_0 + \nu_{\delta+\kappa_0}} h(t^{sw} \cdot P). \end{aligned}$$

b) Notice that $v(\Delta^s) \geq (1 + q + \cdots + q^{s-1}) v(\Delta^1)$. Hence, for every small $\epsilon > 0$, we can take $h(P)$ large enough such that for all s:

$$-(q^s v(\mathcal{V}) + v(\Delta^s) + v(\mathcal{W})) \leq \epsilon q^s h(P).$$

We then obtain

$$q^{\lambda_0 + \nu_{\delta+\kappa_0}} h(t^s \cdot P) \geq (1 - \epsilon)(q^s)^{w^{-1}} h(P).$$

Finally, if we set $c_1 := (1-\epsilon)\, q^{-(\lambda_0 + \nu_{\delta+\kappa_0})} < 1$, then, for all s divisible by zr,

$$h\big(t^{sw}(P)\big) \geq c_1\, q^s\, h(P).$$

Therefore, putting $n := zdr$, we obtain for all nonconstant $a \in \mathbb{F}_q[t^{zdr}]$, that

$$h\big(a(P)\big) = h(t^{\deg a} \cdot P) \geq c_1\, |a|_\infty^{w^{-1}}\, h(P).$$

□

PROPOSITION 6.19. *If E is pure of weight w, then there exists a constant $c_2 > 1$ and an integer n such that, for all $P \in E(K)$ with $h(P) \gg 0$ and for all nonconstant $a \in \mathbb{F}_q[t^n]$:*

$$\boxed{h(a \cdot P) \leq c_2\, |a|_\infty^{w^{-1}}\, h(P)}\,. \tag{121}$$

PROOF. The essential idea, using Lemma 6.7, is now to express t^s, for large exponents s, in terms of lower powers of t and the endomorphism τ (cf. equation (86)) and apply the triangle inequality. Let

$$\mathcal{T}^s := t^{-sw} \Delta^s \in \mathrm{Mat}_{r \times r}(K[t, t^{-1}]).$$

a) For an appropriate $\delta' > \delta$, to be fixed later, we approximate

$$t^{\delta'}(\mathcal{T}^s)^{-1} \subset \mathrm{Mat}_{r\times r}(K[t,t^{-1}])$$

by a matrix $\mathcal{A}^s$ with coefficients in $K[t]$ such that

$$\deg(\mathcal{A}^s - t^{\delta'}(\mathcal{T}^s)^{-1}) < 0.$$

If we put

$$\mathcal{B}^s := \Delta^s \mathcal{A}^s - t^{sw+\delta'} = \Delta^s \cdot (\mathcal{A}^s - t^{\delta'}(\mathcal{T}^s)^{-1}) \in \mathrm{Mat}_{r\times r}(K[t]) \tag{122}$$

then $\deg \mathcal{B}^s < sw + \delta$.

b) We expand $\mathcal{A}^s = \sum_{i=0}^{\delta+\delta'} \mathcal{A}_i^s t^i$, introducing matrices

$$\mathcal{A}_i^s = (\mathcal{A}_{j,i}^{u,s})_{u,j} \in \mathrm{Mat}_{r'\times r'}(K)$$

and set

$$v(\mathcal{A}^s) = \min_{u,j,i} v(\mathcal{A}_{j,i}^{u,s}).$$

Similarly, we define $v(\mathcal{B}^s)$, $v(\mathcal{T}^s)$. First, we need an estimate of $v(\mathcal{A}^s)$.

Upon replacing **m** by some scalar multiple, we may suppose that

$$v(\Delta^1) \geq 0$$

and hence that $v(\mathcal{T}^s) = v(\Delta^s) \geq 0$ as well. If $\overline{\mathcal{T}^s}$ is the adjoint of $\mathcal{T}^s$, then $v(\overline{\mathcal{T}^s}) \geq 0$, and

$$(\mathcal{T}^s)^{-1} = (\det \mathcal{T}^s)^{-1}\overline{\mathcal{T}^s}.$$

On the other hand, if we set $\det \Delta^1 = \omega \cdot (t-\theta)^d$ with $v(\omega) \geq 0$, then

$$\det \mathcal{T}^s = \omega^{1+\cdots+q^{s-1}}(1-\theta t^{-1})\cdots(1-\theta^{q^{s-1}}t^{-1}). \tag{123}$$

If we write out $\det(\mathcal{T}^s)^{-1} = \sum_{j=0}^{\infty} d_j t^{-j}$, then, for all n,

$$\min_{1\leq j\leq n} v(d_j) \geq q^s(-v(\omega) + n\min\{v(\theta),0\}). \tag{124}$$

As $\mathcal{A}^s$ is the 'integer part' of

$$t^{\delta'}(\mathcal{T}^s)^{-1} = t^{\delta'}(\det \mathcal{T}^s)^{-1}\overline{\mathcal{T}^s} = \sum_{i=-\delta}^{\infty}\left(\sum_{\substack{-\delta\leq k\\ 0\leq j\\ k+j=i}} d_j(\overline{\mathcal{T}^s})_k\right)t^{-i+\delta'},$$

it follows that

$$v(\mathcal{A}^s) \geq q^s\gamma \text{ and } v(\mathcal{B}^s) \geq q^s\gamma, \tag{125}$$

where $\gamma := (-v(\omega) + (\delta+\delta')\min\{v(\theta),0\})$.

c) The morphism τ^s is represented by $\tau^s(\mathbf{m}) = \mathbf{m}\cdot\Delta^s$, hence, by equation (122):

$$\tau^s(\mathbf{m})\cdot\mathcal{A}^s = \mathbf{m}\cdot(t^{sw+\delta'} + \mathcal{B}^s). \tag{126}$$

Let us set $\mathbf{a} = (a_1, \ldots, a_r) := \tau(\mathbf{m}) \cdot \mathcal{A}^s$ and $\mathbf{b} = (b_1, \ldots, b_r) := \mathbf{m} \cdot \mathcal{B}^s$. As for the right hand side, we will soon (see **e)**) that, upon evaluating the equation

$$\mathbf{a} = \mathbf{m} \circ t^{sw+\delta'} + \mathbf{b}$$

at a point P with $h(P) \gg 0$, the term $\mathbf{m} \circ t^{sw+\delta'}$ is dominant. More precisely:

$$\min_{j, 0 \le \kappa \le \kappa_0} v(m_j(t^{sw+\delta'+\kappa} \cdot P)) < \min_{j, 0 \le \kappa \le \kappa_0} v(b_j(t^{\kappa} \cdot P)), \tag{127}$$

for all j. There we will use the result of the previous lemma to 'bound' the action of the lower powers of t. It follows from (127) that

$$\min_{j,\kappa} v(a_j(t^{\kappa} \cdot P)) = \min_{j,\kappa} v(m_j(t^{sw+\delta'+\kappa} \cdot P)). \tag{128}$$

We now calculate

$$\begin{aligned} v(a_j(P)) &= v\left(\sum_{\mu=0}^{\delta'} \mathcal{A}_{j,\mu}^{u,s}\, \tau^s \circ m_u(t^{\mu} \cdot P)\right) \\ &\ge v(\mathcal{A}^s) + \min_{u, 0 \le \mu \le \delta'} q^s v\left(m_u(t^{\mu} \cdot P)\right) \\ &\ge q^s \gamma + q^s v(\mathcal{W}) - q^{s+\lambda_0} h(t^{\delta'} \cdot P) \end{aligned} \tag{129}$$

d) On the other hand, combining (110) and (129), we get

$$\begin{aligned} -h(t^{sw+\delta'} \cdot P) &\ge v(\mathcal{V}) + \min_{j,\kappa} v(m_j(t^{sw+\delta'+\kappa} \cdot P)) \\ &= v(\mathcal{V}) + \min_{j,\kappa} v(a_j(t^{\kappa} \cdot P)) \\ &\ge v(\mathcal{V}) + q^s v(\mathcal{W}) + q^s \gamma - q^{s+\lambda_0} \max_{0 \le \kappa \le \kappa_0} h(t^{\delta'+\kappa} \cdot P). \end{aligned} \tag{130}$$

If we take $h(P)$ big enough for a fixed small $\epsilon > 0$, then it follows that

(131)

$$\begin{aligned} h(t^{sw} \cdot P) \le h(t^{sw+\delta'} \cdot P) &\le q^{s+\lambda_0} h(t^{\delta'+\kappa_0} \cdot P) - (v(\mathcal{V}) + q^s v(\mathcal{W}) + q^s \gamma) \\ &\le q^s \left((1-\epsilon) q^{\lambda_0} h(t^{\delta'+\kappa_0} \cdot P)\right) \\ &\le \left((1-\epsilon)\chi_{\delta'+\kappa_0} q^{\lambda_0}\right) q^s h(P); \end{aligned}$$

we recall that the constant $\chi_{\delta'+\kappa_0}$ was introduced in (116). If we put

$$c_2 := \left((1-\epsilon)\chi_{\delta'+\kappa_0} q^{\lambda_0}\right),$$

then this proves that

$$h(t^{sw} \cdot P) \le c_2\, (q^s) h(P),$$

for all s divisible by z and r. Finally, if we put $n = zdr$, then, for all nonconstant $a \in \mathbb{F}_q[t^n]$,

$$h(a \cdot P) = h(t^{\deg a} \cdot P) \le c_2\, |a|_{\infty}^{w^{-1}} h(P).$$

e) It remains to prove formula 127. Taking $h(P) \gg 0$, we calculate, using the estimates (125) and (99):

$$\begin{aligned} v(b_j(P)) &= v\left(\sum_{\mu=0}^{sw+\delta} \mathcal{B}_{j,\mu}^{v,s}\, m_v(t^\mu \cdot P)\right) \\ &\geq v(\mathcal{B}^s) + \min_{v,0\leq\mu\leq sw+\delta} v(m_v(t^\mu \cdot P)) \\ &\geq q^s\gamma + v(\mathcal{W}) - q^{\lambda_0} h(t^{sw+\delta} \cdot P) \\ &\geq -(1-\epsilon) q^{\lambda_0} h(t^{sw+\delta} \cdot P) \end{aligned} \tag{132}$$

Thus

$$v\left(b_j(t^\kappa \cdot P)\right) \geq -(1-\epsilon) q^{\lambda_0} h(t^{sw+\delta+\kappa} \cdot P),$$

for all $\kappa \geq 0$. By lemma 6.18, we can find a $\delta' \gg \delta$ such that, if

$$c_1 q^{(\delta'-\delta-\kappa_0)/w} > q^{\lambda_0},$$

then

$$q^{\lambda_0} h(t^{sw+\delta} \cdot P) > h(t^{sw+\delta'+\kappa})$$

for all $\kappa \leq \kappa_0$. Thus

$$\min_{j,\kappa} v\left(b_j(t^\kappa \cdot P)\right) > -(1-\epsilon) h(t^{sw+\delta'+\kappa} \cdot P).$$

Finally, if $h(P) \gg 0$, then

$$\min_{j,\kappa} v(m_j \circ t^{sw+\delta'+\kappa} \cdot P) \leq -h(t^{sw+\delta'} \cdot P) - v(\mathcal{V}) \leq -(1-\epsilon) h(t^{sw+\delta'} \cdot P)$$

and hence $\min_{j,\kappa} v(m_j \circ t^{sw+\delta'+\kappa} \cdot P) < \min_{j,\kappa}\left(b_j(t^\kappa \cdot P)\right)$ indeed. □

REMARK 6.20. **a)** Suppose that E is a d-dimensional t-module such that, for some n, we have, with respect to the coordinate basis **x**:

$$t^n := \sum_{i=0}^{m} \Phi_i \tau^i,$$

where $\Phi_i \in \mathrm{Mat}_{d\times d}(K)$ and Φ_m invertible; the t-module E is then pure, and hence abelian. As explained in [**Den**] in the case of global heights, this implies the existence of a constant $c_a > 0$, depending on $a \in \mathbf{A}$, and $\eta > 0$ such that

$$|a|_\infty^{w^{-1}} h(P) - c_a \leq h\big(a(P)\big) \leq |a|_\infty^{w^{-1}} h(P) + c_a$$

for $h(P) > \eta$. This allows us to define a unique canonical local height function

$$\hat{h}(P) := \lim_{\to} q^{-sw^{-1}} h(t^s \cdot P)$$

which satisfies $\hat{h}\big(a(P)\big) = |a|_\infty^{w^{-1}} \hat{h}(P)$ for all $a \in \mathbb{F}_q[t^n]$, and such that $\hat{h} - h$ is bounded for $h(P) > \eta$.

b) In general, however, it is not possible to define a canonical height $\hat{h}$ such that $\hat{h} - h$ is bounded. Consider for example the 2-dimensional pure t-motive E of weight one given by:

$$(\phi_E(t) - \theta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \tau & \theta\tau^\mu \\ 0 & \tau \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $v(\theta) = 1$ and $\mu \geq 1$. For points $P_x := \begin{pmatrix} 0 \\ x \end{pmatrix}$, with $v(x) < 0$, we obtain that

$$h(t^s(P_x)) = q^{\mu+s}h(P_x) - 1,$$

for all s. Therefore, the constant c_2 in (98) is at least q^μ, and, if $\hat{h}$ is defined, then the function $\hat{h} - h$ cannot be bounded.

VI. Semistability of Drinfeld modules

Upon replacing K by a finite totally ramified extension, every Drinfeld module ϕ defined over K is isomorphic to a Drinfeld module Φ with coefficients in R and which is **stable** in the following sense: it has coefficients in R, and its reduction $\bar{\Phi}$ modulo the maximal ideal of R is a Drinfeld module over the residue field k, of rank $r' \leq r$. A Drinfeld module ϕ is called **good** if it is stable and $r' = r$.

PROPOSITION 6.21 (Tate uniformization (Drinfeld, [**Dr1**] §5)).
For every stable Drinfeld module Φ, there exists a unique good Drinfeld module Φ' with rank r' and a unique non-trivial morphism

$$e_\Phi \in \mathrm{Hom}^{\mathrm{an}}(\Phi', \Phi)$$

such that e_Φ is the identity on the Lie algebra $\mathrm{Lie}(\mathbb{G}_{a,K})$.

By the theory of entire analytic functions on $\mathbb{A}^1$, the analytic map e is surjective over K^{sep}. The kernel $H := \ker(e)(\bar{K})$ is an **A**-lattice in $\Phi'(K)$; its rank is exactly equal to $r - r'$ (see Drinfeld, [**Dr1**] §5, and compare with Theorem 6.16).

Let us denote by M resp. M' the t-motives associated to Φ and Φ'. By Theorem 6.3, the morphism e induces a morphism $e^\star_\phi \in \mathrm{Hom}^{\mathrm{an}}(M, M')$.

THEOREM 6.22. *The Tate uniformization morphism $e^\star_\phi \in \mathrm{Hom}^{\mathrm{an}}(M, M')$ induces an exact sequence*

$$0 \to \widetilde{N} \to \widetilde{M} \xrightarrow{e^\star} \widetilde{M}' \to 0 \tag{133}$$

of τ-modules over $K\langle\langle t\rangle\rangle$, where the τ-module $\widetilde{N}$ is trivial over a finite extension of K.

We recall that an analytic τ-module is called **trivial** if it is isomorphic to a direct sum of copies of $(K\langle\langle t\rangle\rangle, \tau)$.

PROOF. **a)** *Surjectivity of $e^\star$.* As finitely generated ideals in $K\langle\langle t\rangle\rangle$ are principal, the kernel $\widetilde{N} := \ker e^\star$ is finitely generated (hence free) by – a proper analog of

– the structure theorem for finitely generated modules over a principal ideal domain. For each non-constant $a \in \mathbf{A}$, the sequence

$$0 \to H \to \Phi'(\bar{K}) \to \Phi(\bar{K}) \to 0$$

induces the following short exact sequence of $(\mathbf{A}/a)[\Gamma_K]$-modules:

$$0 \to \Phi'[t] \to \Phi[t] \to H/a \cdot H \to 0.$$

For every maximal ideal ℓ of $\mathbf{A}$, let $\mathbf{A}_\ell$ denote the ℓ-adic completion of $\mathbf{A}$; idem for $K[t]_\ell$ and H_ℓ. We obtain an exact sequence of the Tate modules

$$0 \to T_\ell(\Phi') \to T_\ell(\Phi) \to H_\ell \to 0.$$

We set $\hat{M}_\ell := K[t]_\ell \otimes_{K\langle\langle t\rangle\rangle} \widetilde{M}$ (idem for $\widetilde{M}'$ and $\widetilde{N}$); by the contravariant correspondence between Galois representations and smooth ℓ-adic τ-modules (Prop. 0.7), this yields

$$0 \to \hat{N}_\ell \to \hat{M}_\ell \to \hat{M}'_\ell \to 0. \tag{134}$$

In particular, this shows that the sub-τ-module $e^\star(\widetilde{M})$ of $\widetilde{M}'$ over $K\langle\langle t\rangle\rangle$ has the same rank r'. As finitely generated ideals in $K\langle\langle t\rangle\rangle$ are principal, there exists an $\alpha \in K\langle\langle t\rangle\rangle$ such that

$$\wedge^{r'} e^\star(\widetilde{M}) = \alpha \cdot \wedge^{r'} \widetilde{M}'.$$

Now $\det \tau_{e^\star(\widetilde{M})}$ divides $\det \tau_{\widetilde{M}}$, which is, up to a unit in $K^\times$, equal to

$$\det \tau_{\widetilde{M}'} \sim (t - \theta).$$

Hence it follows from the equation

$${}^\sigma\!\alpha \cdot \left(\det \tau_{e^\star(\widetilde{M})}\right) = \alpha \cdot \left(\det \tau_{\widetilde{M}'}\right)$$

that α, up to a unit, is in $\mathbf{A}$. By (134), we conclude that α is not contained in any maximal ideal of $\mathbf{A}$, and hence $e^\star$ is a surjective analytic morphism.

b) *N is potentially trivial.* As H is strictly discrete, the action of Γ_K on H, hence on H_ℓ as well, is finite. Upon replacing K by a finite extension, we may assume that this action is trivial, i.e. $H \subset K$. If the residue field k is finite, then, by (an analytic version of) the Galois criterion for trivial reduction (Thm.4.8), the τ-sheaf obtained from $\widetilde{N}$ has good trivial reduction, as $T_\ell(\widetilde{N}) = H_\ell$ is trivial. By the analytic lifting theorem [**Ga3**], Thm. 2.3, this implies that $\widetilde{N}$ contains a trivial τ-module. Arguing with the determinant, as above, we conclude that this trivial sub-τ-module is in fact saturated.

c) *N is potentially trivial* (II). We now give a proof in the general case (k not necessarily finite). We recall (cf. [**Dr1**], §5) that, for a Drinfeld module ϕ' defined over K, every $\mathbf{A}$-lattice H in $\phi'(K^{\mathrm{sep}})$ defines a unique Drinfeld module ϕ_1 over K and a unique morphism $e_H \in \mathrm{Hom}^{\mathrm{an}}(\phi', \phi_1)$ defined over K with kernel $(\ker e_H)(K^{\mathrm{sep}}) = H$. More precisely, e_H is given by

$$e_H(x) = x \cdot \prod_{Q \in H\setminus\{0\}} \left(1 - \frac{x}{Q}\right).$$

Let now $(P_i)_{1\le i\le r-r'}$ be a successively minimal basis for $H_0 := H$ (cf. [**Tag2**], §4). We consider the free direct summand $J_1 := \mathbf{A}\cdot P_1 \subset H$ of rank 1, and put

$$H_1' := \oplus_{2\le i\le r-r'}\mathbf{A}\cdot P_i.$$

Let Φ_1 be the Drinfeld module and $u_1 \in \mathrm{Hom}^{\mathrm{an}}(\Phi', \Phi_1)$ the morphism associated to J_1. By the minimality of the base (P_i), we have

$$v(Q+Q') = \min\{v(Q), v(Q')\}$$

for $Q \in J_1$ and $Q' \in H_1'$. Since

$$u_1(x) = x\cdot \prod_{Q\in J_1\setminus\{0\}}\left(1-\frac{x}{Q}\right),$$

we obtain $v(u_1(Q')) \le v(Q')$, for all $Q' \in H_1'$. This shows that the free sub-A-module

$$H_1 := u_1(H) = u_1(H_1')$$

of $\phi'(K)$ is strictly discrete, and hence it is a lattice, of rank $r-r'-1$. The associated entire map $e_1 := e_{H_1}$ satisfies $e = e_1\circ u_1$, and hence yields the morphism

$$e_2 : \Phi_1 \to \Phi.$$

By induction, we thus construct Drinfeld modules Φ_i of rank $r'+i$, for every i satisfying $1\le i\le r-r'$ (with $\Phi_0 := \Phi'$ and $\Phi_{r-r'} := \Phi$), together with surjective morphisms

$$u_i \in \mathrm{Hom}^{\mathrm{an}}(\Phi_{i-1}, \Phi_i),$$

whose kernels are A-lattices in $\phi_{i-1}(K)$ or rank 1. Let $\widetilde{M}_i$ denote the respective analytic t-motives. By Thm. 6.3, the u_i induce surjective morphisms

$$u_i^\star \in \mathrm{Hom}^{\mathrm{an}}(\widetilde{M}_i, \widetilde{M}_{i-1}).$$

As one sees from the determinant, each subquotient $\widetilde{N}_i = \ker u_i^\star$ is a trivial τ-module over $K\langle\langle t\rangle\rangle$. Consequently, $\widetilde{N}$ is an extension of trivial τ-modules over $K\langle\langle t\rangle\rangle$. Every such extension is analytically trivial over the $K^{\mathrm{sep}}\langle\langle t\rangle\rangle$ (cf. [**An1**], lemma 2.7.2). Now $T_\ell(\widetilde{N}) = H_\ell$ is a trivial representation for all ℓ, and hence $\hat{N}_{(t)}$ is trivial over $K[[t]]$. This implies that $\widetilde{N}$ is actually already trivial over $K\langle\langle t\rangle\rangle$. □

VII. Tate uniformization of pure t-motives

Let E be an abelian t-module with associated t-motive M. In this section, we want to raise some questions concerning the analytic structure of E. Let

$$0 = \widetilde{N}_0 \subset \widetilde{N}_1 \subset \ldots \subset \widetilde{N}_{s-1} \subset \widetilde{N}_s = \widetilde{M} \tag{135}$$

be a filtration of $\widetilde{M}$ by saturated sub-τ-modules of $\widetilde{M}$ over $K\langle\langle t\rangle\rangle$.

The key example is provided by the filtration (133) of t-motives associated with Drinfeld modules, which we discussed in the previous chapter (Thm. 6.22). For general t-motives, such filtrations arise from the reduction theory of τ-sheaves, cf. Thm. 1.26.

We want to study how the analytic structure of a t-motive M can give rise to an analytic description of the t-module E. Unfortunately, the only well-understood example is given by Drinfeld modules; not even for tensor products of such t-modules do we have enough arguments to work out the ideas we sketch below. In spite of their very speculative nature, I think that these ideas, supported by analogies with the theory of Anderson uniformization (cf. [**An1**] Thm. 4, [**Ga3**] §4), can help to give some insight into this matter.

a) We cannot resist the temptation to call a trivial τ-module over $K[t]$ a (pure) t**-motive of weight** 0. A first question is:

QUESTION 6.23. Are the subquotients in such a filtration induced by algebraic τ-modules, t-motives, pure t-motives even?

We now suppose there is a filtration $\{\widetilde{N}_i\}$ for $\widetilde{M}$ such that all subquotients are pure t-motives. For $0 < i \leq n$, we put $\widetilde{M}_i := \widetilde{M}/\widetilde{N}_{i-1}$ and

$$\widetilde{M}_i' := \widetilde{N}_i/\widetilde{N}_{i-1}.$$

Let w_i, r_i, $d_i = w_i \cdot r_i$ denote the weight, rank and dimension of the pure t-motive M_i inducing $\widetilde{M}_i$, and, if $w_i > 0$, let E_i be the associated t-module; idem for $\widetilde{M}_i'$ with associated M_i', w_i', r_i', d_i and, if $w_i' > 0$, E_i'.

b) From Thm. 6.9 we can extract some information on the weights. The exact sequences

$$0 \to \widetilde{M}_i' \to \widetilde{M}_i \to \widetilde{M}_{i+1} \to 0$$

imply, by that $w_i' \leq w_i \leq w_{i+1}$; in particular, it follows that

$$0 < w = w_1 \leq w_i,$$

for all i. On the other hand, the exact sequences

$$0 \to \widetilde{M}_i' \to \widetilde{N}_{i+1}/\widetilde{N}_{i-1} \to \widetilde{M}_{i+1}' \to 0 \tag{136}$$

yield that $w_i' \leq w_{i+1}'$. for every i. In particular, there exists an m with $0 \leq m \leq s$ such that $w_i' = 0$ if and only if $i \leq m$.

c) By Theorem 6.3, the surjective homomorphism $\widetilde{M}_i \to \widetilde{M}_{i+1}$ induces an entire analytic morphism $e_i \in \mathrm{Hom}^{\mathrm{an}}(E_{i+1}, E_i)$. Let H_i denote the kernel of e_i, which, by Thm. 6.16, is an $\mathbf{A}$-lattice in $E_{i+1}(K^{\mathrm{sep}})$, whose rank h satisfies

$$0 \leq h \leq r - r'.$$

The exact sequence (136) induces also, for every maximal ideal ℓ of $\mathbf{A}$, an exact sequence of $\mathbf{A}_\ell[\Gamma_K]$-modules

$$0 \to T_\ell(M_{i+1}) \to T_\ell(M_i) \to T_\ell(M_i') \to 0. \tag{137}$$

As the t-module E can be defined over a finitely generated extension K of $\mathbb{F}_q(t)$, and we can hence assume that K is not algebraically closed, this provides useful information.

d) Suppose that $i \leq m$. From (137) it follows, considering ℓ-torsion modules for some maximal ideal ℓ of $\mathbf{A}$ that

$$0 \to E_{i+1}[\ell] \to E_i[\ell] \to W \to 0,$$

where W is a trivial $(\mathbf{A}/\ell)[\Gamma_K]$ module $(w_i' = 0)$. This suggests, in analogy with Tate uniformization of Drinfeld modules, that W should be associated to the lattice H_i by $W \cong H_i/\ell H_i$. That triggers the following suggestion:

QUESTION 6.24. Is the complex $0 \to H_i \to E_{i+1}(K^{\text{sep}}) \to E_i(K^{\text{sep}}) \to 0$ exact?

REMARK 6.25. If we suppose that this complex is exact, i.e.

$$E_{i+1}(K^{\text{sep}}) \to E_i(K^{\text{sep}})$$

is surjective, then the exact sequence

$$0 \to H_i \to E_{i+1}(K^{\text{sep}}) \to E_i(K^{\text{sep}}) \to 0$$

induces, for all $a \in \mathbf{A}$, the short exact sequence

$$0 \to E'[a] \xrightarrow{e} E[a] \to H/a \cdot H \to 0$$

of $(\mathbf{A}/a)$-modules; and this then shows that H has full rank $r - r'$.

e) If $i > m$, then the sequence $\widetilde{M}_i' \to \widetilde{M}_i \to \widetilde{M}_{i+1}$ induces, by Thm. 6.3, a complex

$$0 \to E_{i+1}(K^{\text{sep}}) \to E_i(K^{\text{sep}}) \to E_i'(K^{\text{sep}}) \to 0. \tag{138}$$

The exact sequence

$$0 \to T_\ell(E_{i+1}) \to T_\ell(E_i) \to T_\ell(E_i') \to 0,$$

which comes from (137), suggests the question:

QUESTION 6.26. Is the complex (138) exact?

Suppose the answer to the above questions could be proved to hold, then we would obtain a notion of 'Tate uniformizability', an analytic structure of E, which we can axiomatize as follows:

DEFINITION 6.27. A **Tate uniformization** of E consists of
1) an n-tuple $(E := E_1, E_2 \dots, E_n)$ of abelian t-modules,
2) entire morphisms $e_i : E_{i+1} \to E_i$, for $1 \le i \le n-1$,
3) an integer $m \le n$, and, for $i \le m$, an $\mathbf{A}$-lattice H_i of $E_{i+1}(K)$,
4) an $(n-m-1)$-tuple $(E'_{m+1}, \dots, E'_{n-1})$ of abelian t-modules and
5) entire morphisms $e_i' : E_i \to E_i'$, for $m+1 \le i \le n-1$,
such that the following sequences are exact:

$$\begin{gathered} 0 \to H_1 \to E_2 \to E_1 \to 0 \\ \vdots \\ 0 \to H_m \to E_{m+1} \to E_m \to 0 \\ 0 \to E_{m+2} \to E_{m+1} \to E'_{m+1} \to 0 \\ \vdots \\ 0 \to E_n \to E_{n-1} \to E'_{n-1} \to 0. \end{gathered} \tag{139}$$

With his discovery of mathematics,
Galois became absorbed and neglected his other courses.
Before enrolling in M. Vernier's class, typical comments about him had been:
Religious duties – Good
Conduct – Good
Disposition – Happy
Work – Sustained
Progress – Marked
Character – Good, but singular

After a trimester in M. Vernier's class, the comments were:
Religious duties – Good
Conduct – Passable
Disposition – Happy
Work – Inconstant
Progress – Not very satisfactory
Character – Closed and original

The words "singular", "bizarre", "original" and "closed" would appear more and more frequently during the course of Galois's career at Louis-le-Grand.
P. Dupuy (cf. [**Rot**])

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