ETH Zürich

Bachelor's thesis

Curves with many symmetries

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Contents

Introduction

In 1917, Hans Frederik Blichfeldt classified all finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$ in [Bli17]. There are, up to conjugacy, three maximal finite subgroups. They are isomorphic to $\mathrm{PSL}_2(7)$, respectively to A_6 and to a group called the *Hessian group*. We consider the action of these groups on $\mathbb{P}^2$ that is induced by the action of $\mathrm{PGL}_3(\mathbb{C})$.

It is the aim of this Bachelor's thesis to find irreducible projective algebraic curves of small degree that are invariant under one of these groups. Furthermore, we want to study the properties of the curves that we find, such as their automorphism groups, the fields over which they can be defined and the structure of their Jacobian varieties up to isogeny.

In the first section, we review various results about algebraic curves, their automorphism groups and their Jacobian varieties. In the second section, these results are used to find invariant curves and study their properties.

The prerequisites for this thesis are basic algebraic geometry, as it can be found in Chapter 1 of [Har77], and basic representation theory of finite groups. An introduction to the representation theory of finite groups can be found, for example, in [Ser77] or [JL01].

1. Properties of curves

In what follows, we work over the field of complex numbers $\mathbb{C}$.

Notation 1.0.1. We denote the n-dimensional affine space over $\mathbb{C}$ by $\mathbb{A}^n$ and we denote the n-dimensional projective space over $\mathbb{C}$ by $\mathbb{P}^n$. By I_n we denote the $n \times n$ identity matrix over $\mathbb{C}$. Let p be a prime number. Then we denote the finite field with p elements by $\mathbb{F}_p$. We write $\mathbb{Z}_{\geq 0}$ for the non-negative integers and $\mathbb{Z}_{>0}$ for the positive integers. The homogeneous polynomials of degree k in $\mathbb{C}[X_1, \ldots, X_n]$ are denoted by $\mathbb{C}[X_1, \ldots, X_n]_k$. The multiplicative group of $\mathbb{C}$ is denoted by $\mathbb{C}^\times$.

Definition 1.0.2. A *projective algebraic curve* is a closed one-dimensional subvariety of $\mathbb{P}^n$. An *affine algebraic curve* is a closed one-dimensional subvariety of $\mathbb{A}^n$. A projective, respectively affine, algebraic curve is called *plane* if it is a subvariety of $\mathbb{P}^2$, respectively $\mathbb{A}^2$.

Definition 1.0.3. For any homogeneous ideal $I \subset \mathbb{C}[X_0, \ldots, X_n]$ we denote by

$$V(I) := \{[x_0, \ldots, x_n] \in \mathbb{P}^n | \, \forall f \in I : f(x_0, \ldots, x_n) = 0\}$$

the *projective variety defined by I.*

Notation 1.0.4. For any homogeneous $p \in \mathbb{C}[X_1, \ldots, X_n]$ we write $V(p)$ shorthand for $V((p))$.

Definition 1.0.5. For any projective variety $X \subset \mathbb{P}^n$ we denote by $I(X)$ the *homogeneous ideal of X* generated by

$$\{p \in \mathbb{C}\,[X, Y, Z] | \; p \text{ is homogeneous and } \forall P \in X : p(P) = 0\}\,.$$

Fact 1.0.6. *Let C be a projective algebraic curve. Then the homogeneous ideal $I(C)$ is principal. If $I(C) = (p)$ for some homogeneous $p \in \mathbb{C}\,[X, Y, Z]$, then $V(p) = C$. Moreover, the curve C is an irreducible variety if and only if p is irreducible.*

Fact 1.0.7. *For any non-constant homogeneous $p \in \mathbb{C}\,[X, Y, Z]$ the variety $V(p)$ is a plane projective algebraic curve.*

Definition 1.0.8. Let C be a plane projective algebraic curve and let $p \in \mathbb{C}\,[X, Y, Z]$ be homogeneous such that $I(C) = (p)$. Then the *degree* of C is the degree of p.

Theorem 1.0.9 (Weak form of Bézout's theorem)**.** *Let C_1 and C_2 be plane projective algebraic curves. Then $C_1 \cap C_2 \neq \emptyset$.*

Proof. If C_1 and C_2 have an irreducible component in common, this is trivial. Otherwise, the statement follows directly from Corollary 3.10 in [Kun05]. $\square$

1.1. Smoothness, irreducibility and genus

Definition 1.1.1. Let $p \in \mathbb{C}[X,Y,Z]$ be homogeneous and non-zero such that $(p) = I(V(p))$. Then a point $P \in V(p)$ is called a *singular point of* $V(p)$ or *singular* if

$$\frac{\partial p}{\partial X}(P) = \frac{\partial p}{\partial Y}(P) = \frac{\partial p}{\partial Z}(P) = 0.$$

Otherwise, the point P is called a *regular point of* $V(p)$ or *regular*. The curve $V(p)$ is called *regular* or *non-singular* or *smooth* if all of its points are regular. Otherwise, it is called *singular*.

Proposition 1.1.2. *If a projective algebraic curve C is smooth, then it is irreducible.*

Proof. Assume, for contradiction, that C is reducible and let C_1 and C_2 be distinct irreducible components of C. Let $p \in \mathbb{C}[X,Y,Z]$ be homogeneous such that $V(p) = C$. Let $p_1, p_2 \in \mathbb{C}[X,Y,Z]$ be homogeneous such that $I(C_1) = (p_1)$ and $I(C_2) = (p_2)$. Then $p_1|p$ and $p_2|p$. Therefore, by the irreducibility of p_1 and p_2, we have $p = p_1p_2r$ for some homogeneous $r \in \mathbb{C}[X,Y,Z]$. By Theorem 1.0.9, the curves C_1 and C_2 intersect in some point P. By the definitions of p_1 and p_2, we then have $p_1(P) = p_2(P) = 0$. We calculate

$$\frac{\partial p}{\partial X}(P) = \frac{\partial p_1}{\partial X}(P)p_2(P)r(P) + \frac{\partial (p_2r)}{\partial X}(P)p_1(P) = 0.$$

Similarly, we have $\frac{\partial p}{\partial Y}(P) = 0$ and $\frac{\partial p}{\partial Z}(P) = 0$. It follows that $P \in C$ is singular and therefore C is not smooth. This is a contradiction to our assumption that C is smooth. □

Proposition 1.1.3 (Euler's formula)**.** *Let $k \in \mathbb{Z}_{\geq 0}$ and let $p \in \mathbb{C}[X,Y,Z]_k$. Then*

$$X\frac{\partial p}{\partial X} + Y\frac{\partial p}{\partial Y} + Z\frac{\partial p}{\partial Z} = kp.$$

Proof. See Example A.2 in [Kun05]. □

Proposition 1.1.4. *Let $p \in \mathbb{C}[X,Y,Z]$ be homogeneous. If the only solution of*

$$\frac{\partial p}{\partial X} = \frac{\partial p}{\partial Y} = \frac{\partial p}{\partial Z} = 0$$

in $\mathbb{A}^3$ is 0, then $V(p)$ is smooth and $I(V(p)) = (p)$.

Proof. If $I(V(p)) = (p)$ and the assumption of the proposition holds, then p is non-constant and $V(p)$ is smooth by definition. Thus, it remains to show that $I(V(p)) = (p)$. We have $p \in I(V(p))$. Let $q \in \mathbb{C}[X,Y,Z]$ be homogeneous such that $I(V(p)) = (q)$. Then $p = rq$ for non-zero some homogeneous $r \in \mathbb{C}[X,Y,Z]$.
Claim: The polynomial r is constant.

Proof. We have

$$\frac{\partial p}{\partial X} = r\frac{\partial q}{\partial X} + q\frac{\partial r}{\partial X}$$

and similarly for $\frac{\partial p}{\partial Y}$ and $\frac{\partial p}{\partial Z}$. Let $P \in V(p)$. Then $q(P) = 0$ and we have

$$\frac{\partial p}{\partial X}(P) = r(P)\frac{\partial q}{\partial X}(P)$$

and similarly for $\frac{\partial p}{\partial Y}$ and $\frac{\partial p}{\partial Z}$. Since, by assumption, at least one of $\frac{\partial p}{\partial X}(P)$, $\frac{\partial p}{\partial Y}(P)$ or $\frac{\partial p}{\partial Z}(P)$ is non-zero, it follows that $r(P) \neq 0$. Therefore $V(r) \cap V(p) = \emptyset$. It follows from Theorem 1.0.9 and Fact 1.0.7 that r is constant. ☐

In conclusion $(p) = (q) = I(V(p))$. ☐

Definition 1.1.5. For $d \in \mathbb{Z}_{>0}$ we define

$$J_d := \left\{ \underline{\alpha} \in \mathbb{Z}_{\geq 0}^{n+1} \,\middle|\, |\underline{\alpha}|_{\ell_1} = d \right\}.$$

Theorem 1.1.6. *For any $d_0, \ldots, d_n \in \mathbb{Z}_{>0}$ there is a unique polynomial* $\mathrm{Res} \in \mathbb{Z}[u_{i,j}]_{\substack{i \in \{0,\ldots,n\} \\ j \in J_{d_i}}}$ *called* resultant*, such that:*

1. *Let $F_0, \ldots, F_n \in \mathbb{C}[X_0, \ldots, X_n]$ be homogeneous and non-zero with*

   $$F_i = \sum_{\alpha \in J_{d_i}} c_{i,\alpha} \underline{X}^{\underline{\alpha}}$$

   for all $i \in \{0, \ldots, n\}$. Then $F_0, \ldots, F_n$ have a common non-zero root if and only if $\mathrm{Res}(c_{i,j})_{\substack{i \in \{0,\ldots,n\} \\ j \in J_{d_i}}} = 0$.

2. *Suppose that $\forall i \in \{0, \ldots, n\} : F_i = X_i^{d_i}$. Then* $\mathrm{Res}(c_{i,j})_{\substack{i \in \{0,\ldots,n\} \\ j \in J_{d_i}}} = 1$.

3. *The polynomial* Res *is irreducible in* $\mathbb{C}[u_{i,j}]_{\substack{i \in \{0,\ldots,n\} \\ j \in J_{d_i}}}$.

Proof. For a proof see Chapter 13 of [GKZ94] or Theorem 2.3 in [CLO05]. ☐

We will need the previous theorem in order to check if the assumption of Proposition 1.1.4 is true for a certain non-zero homogeneous $p \in \mathbb{C}[X, Y, Z]$. For this purpose, the polynomial Res is evaluated at the coefficients of $\frac{\partial p}{\partial X}$, $\frac{\partial p}{\partial Y}$, $\frac{\partial p}{\partial Z}$ by using Sage [S+14].

Theorem 1.1.7. *If a plane projective algebraic curve C is smooth of degree $k \in \mathbb{Z}_{>0}$, then*

$$\mathrm{genus}(C) = \binom{k-1}{2}.$$

Proof. For a proof see Theorem 14.1 in [Kun05]. ☐

1.2. Riemann surfaces and holomorphic differential forms

Definition 1.2.1. Any one-dimensional complex manifold is called a *Riemann surface.*

The connected compact Riemann surfaces are of special interest to us. The following result is classical:

Theorem 1.2.2. *The following categories are equivalent:*

1. *the category of connected compact Riemann surfaces*
2. *the category of algebraic function fields of one variable over* $\mathbb{C}$ *with the arrows reversed*
3. *the category of smooth irreducible projective algebraic curves over* $\mathbb{C}$

Proof. See Theorem 4.2.9 in [Nam84]. □

Let M be a Riemann surface.

Definition 1.2.3 (See Definitions IV.1.1–IV.1.3 in [Mir95])**.** Let $\{(U_i, \varphi_i : U_i \to V_i)\}_{i \in I}$ be the maximal atlas of M. A *holomorphic 1-form* ω on M is a collection of holomorphic functions $f_i : V_i \to \mathbb{C}$, one for each $i \in I$, such that for any charts $(U_j, \varphi_j : U_j \to V_j)$ and $(U_k, \varphi_k : U_k \to V_k)$ with $j, k \in I$ we have

$$f_k = (f_j \circ T) \cdot \frac{dT}{dz}$$

on $\varphi_k(U_j \cap U_k)$, where $T = \varphi_j \circ \varphi_k^{-1}$ is the transition map. Locally, on (U_i, φ_i) with the local coordinate z, we write

$$\omega = f_i dz.$$

Notation 1.2.4. The complex vector space of holomorphic 1-forms on M is denoted by $H^0(M, \Omega_M)$.

Theorem 1.2.5. *Additionally, suppose that M is connected and compact of genus $g \geq 2$. Then the dimension of $H^0(M, \Omega_M)$ is g.*

Proof. See Proposition III.5.2. in [FK92]. □

Recall that the automorphism group $\mathrm{Aut}(M)$ of M is the group of biholomorphic maps of M onto itself.

Definition 1.2.6. The *canonical representation of* $\mathrm{Aut}(M)$ *on* $H^0(M, \Omega_M)$ is defined by $g\varphi := \varphi \circ g^{-1}$ for $\varphi \in H^0(M, \Omega_M)$ and $g \in \mathrm{Aut}(M)$. It is denoted by $\rho_{H^0(M,\Omega_M)}$. The character of $\rho_{H^0(M,\Omega_M)}$ is denoted by $\chi_{H^0(M,\Omega_M)}$.

Theorem 1.2.7 (Lefschetz Fixed Point Formula)**.** *Suppose that M is compact and connected of genus $g \geq 2$. For any $1 \neq h \in \mathrm{Aut}(M)$ we have*

$$\chi_{H^0(M,\Omega_M)}(h) + \overline{\chi_{H^0(M,\Omega_M)}(h)} = 2 - t,$$

where t is the number of fixed points of h and $\overline{(\cdot)}$ denotes complex conjugation.

Proof. See Corollary V.2.9. in [FK92]. □

1.3. Automorphism groups

In this subsection, let C be a smooth irreducible projective algebraic curve of genus $g \geq 2$. Then, by a result of Schwarz, its automorphism group $\mathrm{Aut}(C)$ is finite. This result can be found, for example, in Corollary V.1.2.2. in [FK92]. But we can say even more about $|\mathrm{Aut}(C)|$:

Theorem 1.3.1 (Hurwitz)**.** *We have*

$$|\mathrm{Aut}(C)| \leq 84(g-1).$$

Proof. See Theorem V.1.3. in [FK92]. □

Theorem 1.3.2. *We have* $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$*.*

Proof. See Example II.7.1.1 in [Har77]. □

Theorem 1.3.3. *Suppose, in addition, that C is plane and of degree ≥ 4. Then any automorphism of C is the restriction of a unique automorphism of $\mathbb{P}^2$.*

Proof. The existence follows from Theorem 5.3.17(3) in [Nam84]. Let $P_1, \ldots, P_4 \in C$ be four points that are in general linear position. Those exist, because every line in $\mathbb{P}^2$ meets C in only finitely many points. Any $T \in \mathrm{PGL}_3(\mathbb{C})$ is uniquely determined by giving the images $P_1, \ldots, P_4$ and the conclusion follows. □

1.4. Quotient curves

In this section, we review the quotient of any smooth irreducible projective algebraic curve, respectively any connected compact Riemann surface, by a finite group.

Definition 1.4.1. Let X and Y be algebraic varieties and let H be a group acting on X. A morphism $p : X \to Y$ is called a *categorial quotient of X by H* if

1. $\forall x \in X \; \forall h \in H : p(h(x)) = p(x)$ and
2. for any algebraic variety Z and any morphism $f : X \to Z$, if $\forall x \in X \; \forall h \in H : f(h(x)) = f(x)$, then f factors uniquely through p. That is, there is a unique morphism $\tilde{f} : Y \to Z$ such that $f = \tilde{f} \circ p$.

If a categorial quotient of X by H exists, it is unique up to a unique isomorphism. In this case, we will sometimes write X/H instead of $p : X \to Y$ or Y.

Theorem 1.4.2. *Let M be a connected Riemann surface and let $H < \mathrm{Aut}(M)$ be a finite group. The topological quotient space M/H can be endowed with the structure of a Riemann surface such that the quotient map $p : M \to M/H$ is holomorphic.*
Moreover, this structure on M/H satisfies the following universal property: let N be a Riemann surface and let $f : M \to N$ be a holomorphic function. If $\forall x \in X \; \forall h \in H : f(h(x)) = f(x)$, then there is a unique holomorphic map $\tilde{f} : M/H \to N$ such that $f = \tilde{f} \circ p$.

Proof. By Theorem III.3.4 in [Mir95], one can endow M/H with the structure of a Riemann surface such that p is holomorphic.
Since f is constant on the orbits of H, there is a unique continuous map $\tilde{f} : M/H \to N$ such that $f = \tilde{f} \circ p$. We need to show that $\tilde{f}$ is holomorphic. Let $x \in M$ and take a chart $\varphi : U \to \mathbb{C}$ of M and a chart $\psi : V \to \mathbb{C}$ of M/H such that $x \in U \subset M$ and $p(x) \in V \subset M/H$. The map p is open by the Open Mapping Theorem since it is non-constant and holomorphic. If $\frac{d}{dz}\psi \circ p \circ \varphi^{-1}\big|_{z=\varphi(x)} \neq 0$, then there is some open neighborhood $\tilde{U} \subset U$ of x such that $\tilde{p} = p|_{\tilde{U}}$ is biholomorphic. Since $p(\tilde{U})$ is open and $\tilde{f}|_{p(\tilde{U})} = f \circ \tilde{p}^{-1}$, it follows that $\tilde{f}$ is holomorphic at x. Otherwise, we have $\frac{d}{dz}\psi \circ p \circ \varphi^{-1}\big|_{z=\varphi(x)} = 0$. Then, since the zeros of a non-constant holomorphic function are isolated, there is an open neighborhood $\tilde{U} \subset U$ of x such that

$$\forall y \in \tilde{U} \setminus \{x\} : \frac{d}{dz}\psi \circ p \circ \varphi^{-1}\Big|_{z=\varphi(y)} \neq 0.$$

It follows that $\tilde{f}$ is holomorphic on $p(\tilde{U}) \setminus \{p(x)\}$. Since p is open, the image $p(\tilde{U})$ is an open neighborhood of $p(x)$. But since $\tilde{f}$ is continuous, the map $\psi \circ \tilde{f}$ is bounded on some neighborhood of $p(x)$ and therefore, by Riemann's theorem on removable singularities, the map $\tilde{f}$ is holomorphic at $p(x)$. □

By Theorem 1.2.2, we can transfer Theorem 1.4.2 from the category of connected compact Riemann surfaces to the category of smooth irreducible projective algebraic curves. On these curves, the quotient we get then satisfies all requirements of Definition 1.4.1 and hence is the categorial quotient.

Definition 1.4.3. Let X be an affine variety and let H be a finite group acting on it. Let $A(X)$ denote the ring of regular functions of X. Then the subring

$$A(X)^H := \{f \in A(X) | \; \forall h \in H \; \forall x \in X : f(hx) = f(x)\}$$

is called the *subring of H-invariants.*

Proposition 1.4.4. *Let X be an affine variety and let H be a finite group acting on it. Let $A(X)$ denote the ring of regular functions of X. Then, the categorial quotient is the morphism of varieties corresponding to the inclusion $A(X)^H \hookrightarrow A(X)$. In particular $A(X/H) = A(X)^H$.*

Proof. See Pages 124-125 in [Har92]. □

Proposition 1.4.5. *Let C be a smooth irreducible projective algebraic curve and let H be a finite group acting on it. Then the categorial quotient C/H is a smooth irreducible projective algebraic curve. Moreover, if $U \subset C$ is an affine H-invariant patch of C, then C/H is the projective completion of U/H.*

Proof. The categorial quotient C/H exists and it is a smooth irreducible projective algebraic curve since the quotient exists for connected compact Riemann surfaces by

Theorem 1.4.2. Let $p : C \to C/H$ be the quotient morphism. The restriction $p|_U : U \to p(U) \cong U/H$ is the quotient morphism for U. From this restriction we can recover p, because $p|_U$ can be uniquely extended to a morphism from the projective completion of U to the projective completion of U/H. □

Theorem 1.4.6. *Let M be a connected compact Riemann surface of genus ≥ 2 and let $H < \mathrm{Aut}(M)$. Denote by $H^0(M, \Omega_M)^H \subset H^0(M, \Omega_M)$ the subspace of points that are fixed by the canonical representation of H. Then $H^0(M, \Omega_M)^H \cong H^0\big(M/H, \Omega_{M/H}\big)$. We have $\dim H^0(M, \Omega_M)^H = \mathrm{genus}(M/H)$.*

Proof. By Proposition V.2.2. in [FK92], we have $H^0(M, \Omega_M)^H \cong H^0\big(M/H, \Omega_{M/H}\big)$. By Corollary V.2.2. in [FK92], we have $\dim H^0(M, \Omega_M)^H = \mathrm{genus}(M/H)$. □

1.5. Elliptic curves

Definition 1.5.1. A pair (C, P) is called an *elliptic curve* if C is a smooth projective algebraic curve of genus 1 and $P \in C$.

Theorem 1.5.2. *Let (C, P) be an elliptic curve. Then, there is a $\lambda \in \mathbb{C}$ such that C is isomorphic as a variety to the plane curve defined by the equation*

$$Y^2 Z = X(X - Z)(X - \lambda Z).$$

*The j-*invariant *of C is defined as*

$$j(C) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}.$$

It depends only on the isomorphism class of C. Any elliptic curves (C, P) and (C', P') are isomorphic as varieties if and only if $j(C) = j(C')$.

Proof. See Theorem IV.4.1. and Proposition IV.4.6. in [Har77]. □

We sometimes omit the point P of an elliptic curve (C, P) when we are only interested in the isomorphism class of C.

1.6. Abelian varieties and Jacobian varieties

Definition 1.6.1. Any projective connected group variety is called an *abelian variety.*

For an introduction to abelian varieties see for example [Mil08].

Proposition 1.6.2. *For any elliptic curve (E, P) there is a unique group structure on E such that E is an abelian variety with identity element P.*

Proof. The curve E is a projective variety. Also E is connected by Theorem VII.2.2. in [Sha13]. For the group structure, see for example Proposition IV.4.8. in [Har77]. □

Let C be a smooth irreducible projective algebraic curve of genus $g > 0$.

Lemma 1.6.3. *The first homology group of C with $\mathbb{Z}$-coefficients, denoted by $H_1(C,\mathbb{Z})$, can be canonically embedded into $H^0(C,\Omega_C)^*$ by*

$$H_1(C,\mathbb{Z}) \to H^0(C,\Omega_C)^*,$$
$$\gamma \mapsto \left(\omega \mapsto \int_\gamma \omega\right).$$

Proof. See Lemma 11.1.1. in [BL04]. □

Definition 1.6.4. The *Jacobian variety* or *Jacobian* of C is defined as

$$\mathrm{Jac}(C) := H^0(C,\Omega_C)^* / H_1(C,\mathbb{Z}).$$

The Jacobian variety can be endowed with a unique structure of an abelian variety. See, for example, Section 11.1 in [BL04].

Proposition 1.6.5. *The dimension of* $\mathrm{Jac}(C)$ *is* g.

Proof. See Proposition 2.1. in [Mil08]. □

Proposition 1.6.6. *Let $P \in C$ be any point. Then, there is a canonical morphism $f_P : C \to \mathrm{Jac}(C)$ with $f_P(P) = 0$ such that the following universal property is satisfied: let A be an abelian variety and let $g : C \to A$ be a morphism with $g(P) = 0$. Then there exists a unique homomorphism $\tilde{g} : \mathrm{Jac}(C) \to A$ such that $g = \tilde{g} \circ f_P$.*

Proof. For the existence of f_P see Chapter III, Section 2 in [Mil08]. For the universal property see Proposition 6.1. in Chapter III in [Mil08]. □

Definition 1.6.7. A morphism $f : A \to B$ of abelian varieties is called an *isogeny* if $\dim A = \dim B$ and $\ker f$ is finite. If such a morphism $f : A \to B$ exists, we say that A is isogenous to B.

Lemma 1.6.8. *Being isogenous is an equivalence relation on abelian varieties.*

Proof. In Corollary 1.2.7.a) in [BL04] this is shown for complex tori. Since every abelian variety is a complex torus the conclusion follows. □

Theorem 1.6.9 (Poincaré's Complete Reducibility Theorem)**.** *Let X be a an abelian variety and let A be an abelian subvariety. Then there is some abelian subvariety $B \subset X$ such that X is isogenous to $A \times B$.*

Proof. See Theorem 5.3.5. in [BL04]. □

Theorem 1.6.10. *Let X be an abelian variety and let G be a finite group of automorphisms on X. Then, there are simple abelian varieties $A_1, \ldots, A_s$ such that X is isogenous to $A_1^{n_1} \times \cdots \times A_s^{n_s}$ and $A_1^{n_1}, \ldots, A_s^{n_s}$ are G-simple. Moreover, the varieties $A_1^{n_1}, \ldots, A_s^{n_s}$ are unique up to permutation and isogeny.*

Proof. See Propositions 13.5.4. and 13.5.5. in [BL04]. □

Notation 1.6.11. Let $H < \mathrm{Aut}(C)$ and let χ be a character of $\mathrm{Aut}(C)$. Then $\mathrm{Res}_H(\chi)$ denotes restriction of χ to H.

The following fact follows directly from basic character theory. We will need it when we study the Jacobian varieties of the curves that we find.

Fact 1.6.12. *Suppose that* $\mathrm{Aut}(C)$ *is finite. Let* $H < \mathrm{Aut}(C)$ *and let* χ_1 *denote the trivial character of* $\mathrm{Aut}(C)$. *Then* $\dim H^0(C,\Omega_C)^H = 1$ *if and only if*

$$\left\langle \mathrm{Res}_H\left(\chi_{H^0(C,\Omega_C)}\right), \mathrm{Res}_H(\chi_1)\right\rangle = 1.$$

Furthermore, let $\tau : \mathrm{Aut}(C) \to \mathrm{GL}(V)$ *with* $V \subset H^0(C,\Omega_C)$ *be a subrepresentation of* $\rho_{H^0(C,\Omega_C)}$ *and let* ψ *be its character. Then*

$$H^0(C,\Omega_C)^H \subset V \Leftrightarrow \left\langle \mathrm{Res}_H\left(\chi_{H^0(C,\Omega_C)}\right), \mathrm{Res}_H(\chi_1)\right\rangle = \left\langle \mathrm{Res}_H(\psi), \mathrm{Res}_H(\chi_1)\right\rangle.$$

2. Application

2.1. The maximal finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$

Notation 2.1.1. We set

$$\begin{aligned}
\omega &:= e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2},\\
\epsilon &:= e^{4\pi i/9},\\
\gamma &:= \frac{1}{\omega-\omega^2} = \frac{1}{3}\left(\omega^2-\omega\right),\\
\mu_1 &:= \frac{-1+\sqrt{5}}{2},\\
\mu_2 &:= \frac{-1-\sqrt{5}}{2} = -\mu_1^{-1},\\
\beta &:= e^{2\pi i/7}.
\end{aligned}$$

We let $\pi : \mathrm{GL}_3(\mathbb{C}) \to \mathrm{PGL}_3(\mathbb{C})$ denote the quotient homomorphism.

Definition / Proposition 2.1.2.

1. *Let* $\tilde{G_1} < \mathrm{SL}_3(\mathbb{C})$ *be the subgroup generated by*

$$S_1 := \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix}, T := \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, U := \begin{pmatrix}\epsilon&0&0\\0&\epsilon&0\\0&0&\epsilon\omega\end{pmatrix}, V := \gamma\begin{pmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{pmatrix}.$$

   Its order is 648.

2. *Let* $\tilde{G}_2 < \mathrm{SL}_3(\mathbb{C})$ *be the subgroup generated by*

$$F_1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, F_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$F_3 := \frac{1}{2}\begin{pmatrix} -1 & \mu_2 & \mu_1 \\ \mu_2 & \mu_1 & -1 \\ \mu_1 & -1 & \mu_2 \end{pmatrix}, F_4 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}.$$

Its order is 1080*.*

3. *Let* $\tilde{G}_3 < \mathrm{SL}_3(\mathbb{C})$ *be the subgroup generated by*

$$S := \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, R := h\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

where $a := \beta^4 - \beta^3$ *and* $b := \beta^2 - \beta^5$ *and* $c := \beta - \beta^6$ *and*

$$h := -(\beta + \beta^2 + \beta^4 - \beta^6 - \beta^5 - \beta^3)^{-1} = \frac{i}{\sqrt{7}}.$$

Its order is 168*.*

Proof. See Chapter V in [Bli17]. □

Definition 2.1.3. We define

$$G_1 := \pi(\tilde{G}_1), \quad G_2 := \pi(\tilde{G}_2), \quad G_3 := \pi(\tilde{G}_3).$$

The group G_1 is called the *Hessian* group.

The finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$ have been classified in 1917 by Blichfeldt in [Bli17]. Up to conjugacy, only three of them are maximal:

Theorem 2.1.4. *Let G be a maximal finite subgroup of* $\mathrm{PGL}_3(\mathbb{C})$*. Then G is conjugate to either G_1, G_2 or G_3. The groups G_1, G_2 and G_3 have the following properties:*

1. *The group G_1 is of order 216 and is isomorphic to* $\mathbb{F}_3^2 \rtimes \mathrm{SL}_2(\mathbb{F}_3)$*, where* $\mathrm{SL}_2(\mathbb{F}_3)$ *acts naturally on the finite plane* $\mathbb{F}_3^2$*.*
2. *The group G_2 is simple of order 360. It is isomorphic to the alternating group A_6.*
3. *The group G_3 is simple of order 168. It is isomorphic to* $\mathrm{PSL}_2(\mathbb{F}_7)$*. As* $|G_3| = |\tilde{G}_3|$*, we have* $\tilde{G}_3 \cong G_3$*.*

Proof. See Chapter V in [Bli17] and, for the structure of G_1, see Proposition 4.1 in [AD09]. By Proposition 4.14 in [ST00], any simple groups H_1 and H_2 of order 168 are isomorphic. The fact that $\mathrm{PSL}_2(\mathbb{F}_7)$ is simple is stated on Page 145 of [ST00]. □

Corollary 2.1.5. *For every* $i \in \{1, 2\}$*, we cannot find a group* $\bar{G}_i < \mathrm{GL}_3(\mathbb{C})$ *such that* $\pi(\bar{G}_i) = G_i$ *and* $\bar{G}_i \cong G_i$*.*

Proof. As can be seen from the character tables in A.1, the groups G_1 and G_2 do not have faithful 3-dimensional representations. □

2.2. Finding invariant curves

Definition 2.2.1. Let G be a group of automorphisms of $\mathbb{P}^2$ and let C be a plane algebraic curve. The curve C is *invariant under* G or G*-invariant* if $\forall g \in G : gC = C$.

Definition 2.2.2. Let V be a complex vector space. For $k \in \mathbb{Z}_{\geq 0}$, define

$$S^k(V) := V^{\otimes k}/I,$$

where I is the subspace generated by

$$\left\{ v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \,\middle|\, \sigma \in S_k \text{ and } v_1, \ldots, v_k \in V \right\}.$$

The vector space $S^k(V)$ is called the *k-th symmetric power* of V. We denote the image of $v_1 \otimes \cdots \otimes v_k$ in $S^k(V)$ by $v_1 \odot \cdots \odot v_k$.

Fact 2.2.3. *Let $n \in \mathbb{Z}_{>0}$, let $k \in \mathbb{Z}_{\geq 0}$, let V be an n-dimensional complex vector space and let $v_1, \ldots, v_n$ be a basis of V. There is an isomorphism $S^k(V) \to \mathbb{C}[X_1, \ldots, X_n]_k$ given by*

$$v_{i_1} \odot \cdots \odot v_{i_k} \mapsto X_{i_1} \cdots X_{i_k} \text{ for any } i_1, \ldots, i_k \in \{1, \ldots, n\}.$$

In what follows, we will often identify $S^k((\mathbb{C}^n)^*)$ with $\mathbb{C}[X_1, \ldots, X_n]_k$ via this isomorphism, where the basis we choose for $(\mathbb{C}^n)^*$ is the dual of the standard basis of $\mathbb{C}^n$.

Definition 2.2.4. Let $\tilde{G} < \mathrm{GL}_n(\mathbb{C})$ be a subgroup. Then, for any $k \in \mathbb{Z}_{\geq 0}$, we define a representation $S^k\rho^*_{\tilde{G}} : \tilde{G} \to \mathrm{GL}(S^k((\mathbb{C}^n)^*))$ given by

$$g \mapsto (v_1 \odot \cdots \odot v_k \mapsto v_1 \circ g^{-1} \odot \cdots \odot v_k \circ g^{-1}).$$

Notation 2.2.5. Let $\tilde{G} < \mathrm{GL}_n(\mathbb{C})$. We let $S^k\chi^*_{\tilde{G}}$ denote the character of $S^k\rho^*_{\tilde{G}}$.

Fact 2.2.6. *For any $k \in \mathbb{Z}_{\geq 0}$ the representation $S^k\rho^*_{\tilde{G}}$ induces a representation of $\tilde{G}$ on $\mathbb{C}[X_1, \ldots, X_n]_k$ by Fact 2.2.3 and therefore it induces an action of $\tilde{G}$ on the homogeneous elements of $\mathbb{C}[X_1, \ldots, X_n]$. Let $p \in \mathbb{C}[X_1, \ldots, X_n]$ be homogeneous and let $g \in \tilde{G}$. Then for any $P \in \mathbb{C}^n$ the action can be written as $(gp)(P) = p(g^{-1}P)$ by identifying the polynomials p and gp with their induced polynomial functions.*

Definition 2.2.7. Let $\tilde{G} < \mathrm{GL}_n(\mathbb{C})$ and let $p \in \mathbb{C}[X_1, \ldots, X_n]_k$ for some $k \in \mathbb{Z}_{\geq 0}$. If $gp = p$ for all $g \in \tilde{G}$, we call p *invariant under* $\tilde{G}$ or $\tilde{G}$*-invariant*.

In order to calculate $S^k\chi^*_{\tilde{G}}$, we have the following proposition:

Proposition 2.2.8. *Let $\tilde{G} < \mathrm{GL}_n(\mathbb{C})$ be finite, let $g \in \tilde{G}$ and let $k \in \mathbb{Z}_{>0}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of g counted with their algebraic multiplicities. Then*

$$S^k\chi^*_{\tilde{G}}(g) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \prod_{j=1}^{k} \overline{\lambda_{i_j}},$$

where $\overline{(\cdot)}$ denotes complex conjugation.

Proof. Let $v_1, \dots, v_n$ be a basis of eigenvectors of g, such that $g(v_i) = \lambda_i v_i$. This exists because g has finite order. Let $v_1^*, \dots, v_n^*$ be its dual basis. For any $1 \le i_1 \le \dots \le i_k \le n$ we have

$$\begin{aligned} S^k\rho^*_{\tilde{G}}(g)(v^*_{i_1} \odot \dots \odot v^*_{i_k}) &= v^*_{i_1} \circ g^{-1} \odot \dots \odot v^*_{i_k} \circ g^{-1} \\ &= \left(\prod_{j=1}^k \lambda_{i_j}^{-1}\right) v^*_{i_1} \odot \dots \odot v^*_{i_k} = \left(\prod_{j=1}^k \overline{\lambda_{i_j}}\right) v^*_{i_1} \odot \dots \odot v^*_{i_k}. \end{aligned}$$

The last equality follows from the fact that all eigenvalues of g are roots of unity. Therefore

$$\left\{v^*_{i_1} \odot \dots \odot v^*_{i_k} \,\middle|\, 1 \le i_1 \le \dots \le i_k \le n\right\}$$

is a basis of eigenvectors for $S^k\rho^*_{\tilde{G}}(g)$. As $S^k\chi^*_{\tilde{G}}(g)$ is the sum of the eigenvalues of $S^k\rho^*_{\tilde{G}}(g)$, the conclusion follows. □

Definition 2.2.9. Let G be a group and let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation. A vector $v \in \mathbb{C}^n$ is called an *eigenvector of* ρ if v is an eigenvector for all elements of $\rho(G)$. A linear subspace $V \subset \mathbb{C}^n$ is called an *eigenspace of* ρ if all elements of V are eigenvectors of ρ.

Definition 2.2.10. A character $\chi : G \to \mathbb{C}$ of a group G is called *linear* if it has degree 1.

Definition 2.2.11. Let $\tilde{G} < \mathrm{GL}_n(\mathbb{C})$. For any linear character χ of $\tilde{G}$ and any $k \in \mathbb{Z}_{\ge 0}$ we denote by $E^k_\chi \subset S^k((\mathbb{C}^n)^*)$ the maximal eigenspace of $S^k\rho^*_{\tilde{G}}$ such that $\tilde{G}$ acts on E^k_χ by multiplication with χ. That is

$$E^k_\chi := \left\{ v \in S^k((\mathbb{C}^n)^*) \,\middle|\, \forall g \in \tilde{G} : gv = \chi(g)v \right\}.$$

Fact 2.2.12. *Let $\tilde{G} < \mathrm{GL}_n(\mathbb{C})$ and let $k \in \mathbb{Z}_{\ge 0}$. Then any eigenspace of $S^k\rho^*_{\tilde{G}}$ is contained in E^k_χ for some linear character χ of $\tilde{G}$.*

Proposition 2.2.13. *Let $p \in \mathbb{C}[X, Y, Z]$ be non-constant and homogeneous such that $(p) = I(V(p))$ and let $G < \mathrm{PGL}_3(\mathbb{C})$ and $\tilde{G} < \mathrm{GL}_3(\mathbb{C})$ be such that $\pi(\tilde{G}) = G$. Then $V(p)$ is G-invariant if and only if p is an eigenvector of $S^{\deg p}\rho^*_{\tilde{G}}$.*

Proof. We have:

$$\begin{aligned} V(p) \text{ is } G\text{-invariant} &\Leftrightarrow \forall g \in G : gV(p) = V(p) \\ &\Leftrightarrow \forall \tilde{g} \in \tilde{G}\ \forall P \in \mathbb{C}^3 \setminus \{0\} : p(P) = 0 \to (\tilde{g}^{-1}p)(P) = 0 \\ &\overset{(p) \text{ is radical}}{\Leftrightarrow} \forall \tilde{g} \in \tilde{G} : (\tilde{g}^{-1}p) \subset (p) \\ &\overset{\deg p = \deg \tilde{g}p}{\Leftrightarrow} \forall \tilde{g} \in \tilde{G} \exists \lambda \in \mathbb{C} : \tilde{g}p = \lambda p. \end{aligned}$$

□

Notation 2.2.14. For the remainder of this subsection, let $G < \mathrm{PGL}_3(\mathbb{C})$ and $\tilde{G} < \mathrm{GL}_3(\mathbb{C})$ be finite groups such that $\pi(\tilde{G}) = G$. Furthermore, let χ_1 denote the trivial character of $\tilde{G}$.

By Proposition 2.2.13, in order to find the G-invariant curves it is sufficient to find the eigenspaces E_χ^k of $S^k \rho_{\tilde{G}}^*$ for all $k \in \mathbb{Z}_{>0}$ and all linear characters χ of $\tilde{G}$.
We want to study only one G-invariant curve at a time. If a G-invariant curve comes within a family of G-invariant curves, given by an eigenspace of $S^k \rho_{\tilde{G}}^*$ of dimension greater than 1, then it seems natural to study the family as a whole rather than a single member of it. We therefore restrict our attention to the $\dim E_\chi^k = 1$ case for all $k \in \mathbb{Z}_{>0}$ and all linear characters χ of $\tilde{G}$.
Furthermore, we are only interested in irreducible G-invariant curves since a reducible G-invariant curve is the union of irreducible G-invariant curves and curves that are not G-invariant. The latter ones are permuted by the action of G.
To summarize, we are looking for projective algebraic curves that satisfy the following condition:

Condition 2.2.15. Let C be a projective algebraic curve and let $p \in \mathbb{C}[X, Y, Z]_k$ be homogeneous such that $I(C) = (p)$. We require that C is irreducible and that there is a linear character χ of $\tilde{G}$ such that $p \in E_\chi^k$ and $\dim E_\chi^k = 1$.

2.3. Finding the degrees of the eigenvectors

We use the notation from Notation 2.2.14. Let $\chi_1, \ldots, \chi_m$ be the linear characters of $\tilde{G}$ and let $i \in \{1, \ldots, m\}$. We know that $\dim E_{\chi_i}^k$ equals the multiplicity with which χ_i appears in the decomposition into irreducible characters of $S^k \chi_{\tilde{G}}^*$. Therefore, we have

$$\dim E_{\chi_i}^k = \left\langle S^k \chi_{\tilde{G}}^*, \chi_i \right\rangle := \frac{1}{|\tilde{G}|} \sum_{g \in \tilde{G}} S^k \chi_{\tilde{G}}^*(g) \overline{\chi_i(g)}.$$

Thus, in order to find the degrees of the polynomials that are in 1-dimensional eigenspaces of $\tilde{G}$, for every $j \in \{1, \ldots, m\}$, we have to solve the equation

$$\left\langle S^k \chi_{\tilde{G}}^*, \chi_j \right\rangle = 1 \tag{2.3.1}$$

for k.

Definition 2.3.1. A group H is called *perfect* if $H = [H, H]$, where $[H, H]$ is the commutator subgroup of H.

Example 2.3.2. Let H be any non-abelian simple group, then $[H, H] \in \{1, H\}$ since $[H, H] \lhd H$ and therefore $[H, H] = H$, since H is non-abelian. Hence H is perfect. In particular, the groups G_2 and G_3 are perfect.

Proposition 2.3.3. *The groups $\tilde{G}_2$ and $\tilde{G}_3$ are perfect.*

Proof. Firstly, we know that $\tilde{G_3} \cong G_3$ and therefore $\tilde{G_3}$ is perfect. Secondly, we know that $\pi([\tilde{G_2}, \tilde{G_2}]) \supseteq [G_2, G_2] = G_2$. Therefore $[\tilde{G_2} : [\tilde{G_2}, \tilde{G_2}]] \leq 3$. But also $|[\tilde{G_2}, \tilde{G_2}]| > |G_2|$, because otherwise $[\tilde{G_2}, \tilde{G_2}] \cong G_2$, which is impossible by Corollary 2.1.5. Additionally, since $\ker \pi|_{\tilde{G_2}}$ is cyclic of order three and $\ker \pi|_{[\tilde{G_2}, \tilde{G_2}]} \lhd \ker \pi|_{\tilde{G_2}}$, it follows that $\ker \pi|_{[\tilde{G_2}, \tilde{G_2}]} = \ker \pi|_{\tilde{G_2}}$ and hence $[\tilde{G_2}, \tilde{G_2}] = \tilde{G_2}$. □

Proposition 2.3.4. *Any linear character of any perfect group H is trivial.*

Proof. Let χ be a linear character of H. We know that $\chi : H \to \mathbb{C}^\times$ is a homomorphism. But as the abelianization $H/[H, H]$ of H is trivial and $\mathbb{C}^\times$ is abelian, by the universal property of the abelianization, the character χ must be trivial. □

From the character table of G_1 in A.1 we see that G_1 has exactly three distinct linear characters.

Proposition 2.3.5. *The only linear characters of $\tilde{G_1}$ are the lifts of the three linear characters of G_1. All linear characters of $\tilde{G_2}$ and $\tilde{G_3}$ are trivial.*

Proof. In A.4 we find, using GAP, that the only linear characters of $\tilde{G_1}$ are the lifts of the three linear characters of G_1. For $\tilde{G_2}$ and $\tilde{G_3}$ the conclusion follows directly from the previous proposition. □

By calculating explicit formulas for the values of $S^k\chi^*_{\tilde{G_1}}$, $S^k\chi^*_{\tilde{G_2}}$ and $S^k\chi^*_{\tilde{G_3}}$ as done in A.2, we can find every solutions to (2.3.1) for any $\tilde{G} \in \{\tilde{G_1}, \tilde{G_2}, \tilde{G_3}\}$.
In order to exclude some degrees in which the polynomials we find cannot be irreducible, we have the following proposition:

Proposition 2.3.6. *Let $i, j \in \{1, \ldots, m\}$ and $k, l \in \mathbb{Z}_{\geq 0}$ and let $p \in E^k_{\chi_i}$ and $q \in E^l_{\chi_j}$ be homogeneous polynomials. Then $pq \in E^{k+l}_{\chi_i\chi_j}$. In particular, if $\dim E^{k+l}_{\chi_i\chi_j} = 1$ and $p \neq 0 \neq q$, then any $r \in E^{k+l}_{\chi_i\chi_j}$ is reducible.*

Proof. For any $g \in \tilde{G}$ we have $g(pq) = (gp)(gq) = \chi_i(g)\chi_j(g)pq$. □

2.4. Finding the polynomials which are eigenvectors

We use Notation 2.2.14. Recall that $\tilde{G}$ is finite.

Definition 2.4.1. For any monomial $M \in \mathbb{C}[X, Y, Z]$ we define

$$p_{M,\tilde{G}} := \sum_{g \in \tilde{G}} gM.$$

Proposition 2.4.2. *Suppose that $\dim E^k_{\chi_1} = 1$.*

1. *Let $M \in \mathbb{C}[X, Y, Z]$ be a monomial of degree k. If $p_{M,\tilde{G}} \neq 0$, then* $\operatorname{span}(p_{M,\tilde{G}}) = E^k_{\chi_1}$.

2. Let $n := \dim S^k((\mathbb{C}^3)^*)$ and let $M_1, \ldots, M_n \in \mathbb{C}[X,Y,Z]$ be the monomials of degree k. Then, there is some $i \in \{1, \ldots, n\}$ such that $p_{M_i,\tilde{G}} \neq 0$.

Proof. 1. Since $p_{M,\tilde{G}}$ is $\tilde{G}$-invariant, it is in $E^k_{\chi_1}$. Since $\dim E^k_{\chi_1} = 1$, we have $\operatorname{span}(p_{M,\tilde{G}}) = E^k_{\chi_1}$ if $p_{M,\tilde{G}} \neq 0$.

2. Let $0 \neq p \in E^k_{\chi_1}$ and write $p = \sum_{j=1}^n c_j M_j$ for $c_1, \ldots, c_n \in \mathbb{C}$. We have

$$0 \neq |\tilde{G}|p = \sum_{g \in \tilde{G}} gp = \sum_{j=1}^n c_j \sum_{g \in \tilde{G}} gM_j = \sum_{j=1}^n c_j p_{M_j,\tilde{G}},$$

and the conclusion follows.

□

Therefore, if $\dim E^k_{\chi_1} = 1$ for some $k \in \mathbb{Z}_{>0}$ we can find the $\tilde{G}$-invariant polynomial of degree k by calculating $p_{M,\tilde{G}}$ for different monomials M until we find a non-zero $p_{M,\tilde{G}}$.
Now suppose that χ is a non-trivial linear character of $\tilde{G}$ and that there is some $k \in \mathbb{Z}_{>0}$ with $\dim E^k_\chi = 1$. Let $\tilde{H} := \ker \chi$ and calculate $p_{M,\tilde{H}}$ for the monomials $M \in \mathbb{C}[X,Y,Z]$ of degree k. If $p_{M,\tilde{H}} \neq 0$ for some monomial M, we check if $p_{M,\tilde{H}} \in E^k_\chi$ by letting $\tilde{G}$ act on $p_{M,\tilde{H}}$. If $p_{M,\tilde{H}} \in E^k_\chi$ we are done with the given k and χ.
The calculations to solve (2.3.1) for $\tilde{G} \in \{\tilde{G_1}, \tilde{G_2}, \tilde{G_3}\}$ were done in A.6 using Sage [S+14]. The calculations to find the non-zero polynomials which are in 1-dimensional eigenspaces of $\tilde{G}$ were done using GAP [GAP15] in A.7.
We obtain the following results:

G_1-invariant curves
Let χ_1, χ_2 and χ_3 be the linear characters of $\tilde{G_1}$. By Proposition 2.3.5, they are the lifts of the respective linear characters of G_1 as defined in A.1. We have:

$$\left\langle S^k \chi^*_{\tilde{G_1}}, \chi_1 \right\rangle = 1 \text{ for } k = 9, 12, 21, 24, 33,$$
$$\left\langle S^k \chi^*_{\tilde{G_1}}, \chi_2 \right\rangle = 1 \text{ has no solutions for any } k \in \mathbb{Z}_{>0},$$
$$\left\langle S^k \chi^*_{\tilde{G_1}}, \chi_3 \right\rangle = 1 \text{ for } k = 6, 15, 18, 27, 30, 39.$$

By Proposition 2.3.6 the polynomials of degrees 15, 18, 21, 24, 27, 30, 33 and 39 are reducible. This leaves only the degrees 6, 9 and 12. We find the following polynomial in $E^6_{\chi_3}$:

$$p_1 := X^6 - 10\,X^3Y^3 + Y^6 - 10\,X^3Z^3 - 10\,Y^3Z^3 + Z^6.$$

The curve $V(p_1)$ is irreducible by Proposition 2.4.3 below. The $\tilde{G_1}$-invariant polynomials of degree 9 and 12 are reducible, as can be seen by a computation in GAP. See A.7.

G_2-invariant curves

Let χ_1 be the trivial character of $\tilde{G}_2$. This is the only character of $\tilde{G}_2$ by Proposition 2.3.5. We have:

$$\left\langle S^k \chi^*_{\tilde{G}_2}, \chi_1 \right\rangle = 1 \text{ for } k = 6, 45, 51.$$

By Proposition 2.3.6 the polynomial of degree 51 is reducible. This leaves only the polynomials of degree 6 and 45. We obtain the following $\tilde{G}_2$-invariant:

$$p_2 := X^6 + aX^4Y^2 + bX^2Y^4 + Y^6 + bX^4Z^2 + cX^2Y^2Z^2 + aY^4Z^2 + aX^2Z^4 + bY^2Z^4 + Z^6,$$

where

$$\begin{aligned} a &:= \frac{15}{8} + \frac{15}{8}i\sqrt{3} - \frac{9}{8}\sqrt{5} + \frac{3}{8}i\sqrt{3}\sqrt{5} \\ b &:= \frac{15}{8} - \frac{15}{8}i\sqrt{3} + \frac{9}{8}\sqrt{5} + \frac{3}{8}i\sqrt{3}\sqrt{5} \\ c &:= 15 - 3i\sqrt{3}\sqrt{5}. \end{aligned}$$

The curve $V(p_2)$ is irreducible by Proposition 2.4.3 below. The $\tilde{G}_2$-invariant polynomial of degree 45 is reducible, as can be seen by a computation in GAP. See A.7.

G_3-invariant curves

Let χ_1 be the trivial character of $\tilde{G}_3$. This is the only character of $\tilde{G}_3$ by Proposition 2.3.5. We have:

$$\left\langle S^k \chi^*_{\tilde{G}_3}, \chi_1 \right\rangle = 1 \text{ for } k = 4, 6, 8, 10, 21, 25, 27, 29, 31.$$

By Proposition 2.3.6 the polynomials of degrees 8, 10, 25, 27, 29 and 31 are reducible. This leaves only the polynomials of degree 4, 6 and 21. We obtain the following $\tilde{G}_3$-invariants:

$$\begin{aligned} p_3 &:= XY^3 + X^3Z + YZ^3 \\ p_4 &:= X^5Y + XZ^5 + Y^5Z - 5X^2Y^2Z^2. \end{aligned}$$

The curves $V(p_3)$ and $V(p_4)$ are irreducible by Proposition 2.4.3 below. The $\tilde{G}_3$-invariant polynomial of degree 21 is reducible, as can be seen by a computation in GAP. See A.7.

Proposition 2.4.3. *The projective algebraic curve $V(p_i)$ is smooth and irreducible for any $i \in \{1, \dots, 4\}$.*

Proof. By computing the multipolynomial resultant of the partial derivatives of p_i, as defined in Theorem 1.1.6, we see that $\frac{\partial p_i}{\partial X}$, $\frac{\partial p_i}{\partial Y}$ and $\frac{\partial p_i}{\partial Z}$ have no non-zero common roots. The multipolynomial resultant was computed using Sage [S+14] in A.7. Using Proposition 1.1.4 we see that all of them are smooth and hence irreducible by Proposition 1.1.2 and Fact 1.0.6. □

In summary we have:

Proposition 2.4.4.

1. *For $\tilde{G} = \tilde{G}_1$, the only curve satisfying Condition 2.2.15 is $V(p_1)$.*
2. *For $\tilde{G} = \tilde{G}_2$, the only curve satisfying Condition 2.2.15 is $V(p_2)$.*
3. *For $\tilde{G} = \tilde{G}_3$, the only curves satisfying Condition 2.2.15 are $V(p_3)$ and $V(p_4)$.*

Definition 2.4.5. For any $i \in \{1, \ldots, 4\}$ we define

$$C_i := V(p_i)\,.$$

Proposition 2.4.6. *We have* $\mathrm{Aut}(C_1) \cong G_1$ *and* $\mathrm{Aut}(C_2) \cong G_2$ *and* $\mathrm{Aut}(C_3) \cong \mathrm{Aut}(C_4) \cong G_3$.

Proof. By construction, we have homomorphisms $G_1 \to \mathrm{Aut}(C_1)$ and $G_2 \to \mathrm{Aut}(C_2)$ and $G_3 \to \mathrm{Aut}(C_3)$ and $G_3 \to \mathrm{Aut}(C_4)$, where the homomorphisms are given by the restriction of the automorphisms of $\mathbb{P}^2$ to the curves. By Theorem 1.3.3, any automorphism of any of the curves is the restriction of a unique automorphism of $\mathbb{P}^2$. Thus, the homomorphisms are injective. Furthermore, since G_1, G_2 and G_3 are maximal finite subgroups of $\mathrm{Aut}(\mathbb{P}^2)$ and $\mathrm{Aut}(C_i)$ is finite for $i \in \{1, \ldots, 4\}$, the homomorphisms are surjective, too. □

In the following subsections, we will study the Jacobians of $C_1, \ldots, C_4$ and find the fields over which $C_1, \ldots, C_4$ can be defined.

Notation 2.4.7. For any $\lambda \in \mathbb{C}$ we denote by $\mathrm{Re}(\lambda)$ the real part of λ.

2.5. The curve C_1

The curve C_1 is already defined over $\mathbb{Q}$ because $p_1 \in \mathbb{Q}[X, Y, Z]$.

In what follows, $\chi_1, \ldots, \chi_{10}$ denote the irreducible characters of $\mathrm{Aut}(C_1)$ as defined in the character table of $G_1 \cong \mathrm{Aut}(C_1)$ in A.1.

Lemma 2.5.1. *We have either*

$$\chi_{H^0(C_1,\Omega_{C_1})} = \chi_4 + \chi_9$$

or

$$\chi_{H^0(C_1,\Omega_{C_1})} = \chi_4 + \chi_{10}.$$

Here $\deg \chi_4 = 2$ *and* $\deg \chi_9 = \deg \chi_{10} = 8$ *and* $\chi_9(\cdot) = \overline{\chi_{10}(\cdot)}$ *and* χ_2 *has values in* $\mathbb{Z}$.

Proof. We have $\deg \chi_{H^0(C_1,\Omega_{C_1})} = \deg \rho_{H^0(C_1,\Omega_{C_1})} = \text{genus}(C_1) = 10$ by Theorem 1.2.5 and Theorem 1.1.7. Since the number of fixed points t of any automorphism in $\text{Aut}(C_1)$ is ≥ 0, the Lefschetz Fixed Point Formula Theorem 1.2.7 gives us the following upper bound:

$$\forall g \in \text{Aut}(C_1) \setminus \{1\} : \text{Re}\left(\chi_{H^0(C_1,\Omega_{C_1})}(g)\right) \leq 1.$$

We know that $\chi_{H^0(C_1,\Omega_{C_1})}$ is the sum of irreducible characters of $\text{Aut}(C_1)$, and the above upper bound restricts the combinations of irreducible characters in the decomposition of $\chi_{H^0(C_1,\Omega_{C_1})}$. By looking at the character table of G_1 we find that the only characters of $\text{Aut}(C_1)$ of degree 10 that respect the bound are:

$$\begin{aligned} &\chi_{H^0(C_1,\Omega_{C_1})} = \chi_4 + \chi_8 \\ &\qquad\qquad\text{or} \\ &\chi_{H^0(C_1,\Omega_{C_1})} = \chi_i + \chi_j, \text{ where } i \in \{4,5,6\} \text{ and } j \in \{9,10\}. \end{aligned} \tag{2.5.1}$$

We use the notation from Notation 2.1.1 and Definition / Proposition 2.1.2. In order to further determine $\chi_{H^0(C_1,\Omega_{C_1})}$ we look at the automorphism $g = \pi\left(T^2 S_1 U\right)$. As $T^2 S_1 U$ has the three different eigenvalues 1, ω and ω^2, all its eigenspaces are 1-dimensional and therefore g has at most 3 fixed points on C_1. The fixed points of g in $\mathbb{P}^2$ are

$$\left(1 : \epsilon : \epsilon^8\right), \left(1 : \epsilon^4 : \epsilon^5\right), \left(1 : \epsilon^7 : \epsilon^2\right).$$

By evaluating p_1 at these points, we find that all three of them lie on C_1. By Theorem 1.2.7, we then have

$$\text{Re}\left(\chi_{H^0(C_1,\Omega_{C_1})}(g)\right) = -\frac{1}{2}.$$

By evaluating all possible candidates for $\chi_{H^0(C_1,\Omega_{C_1})}$ in (2.5.1) on g using GAP, we find that either

$$\begin{aligned} &\chi_{H^0(C_1,\Omega_{C_1})} = \chi_4 + \chi_9 \\ &\qquad\qquad\text{or} \\ &\chi_{H^0(C_1,\Omega_{C_1})} = \chi_4 + \chi_{10}. \end{aligned}$$

The claims about the degrees and values of the characters follow directly from the character table. □

Let $\sigma : G_1 \to G_1$ denote complex conjugation. This is a well-defined automorphism since we have for the generators S_1, T, U and V of $\tilde{G}_1$:

$$\overline{T} = T \quad \text{and} \quad \overline{S_1} = S_1^{-1} \quad \text{and} \quad \overline{U} = U^{-1} \quad \text{and} \quad \overline{V} = V^{-1}.$$

The automorphism σ permutes $\chi_4 + \chi_9$ and $\chi_4 + \chi_{10}$, the two possibilities for $\chi_{H^0(C_1,\Omega_{C_1})}$. We will, without loss of generality, restrict us to the case $\chi_{H^0(C_1,\Omega_{C_1})} = \chi_4 + \chi_9$ since the cases only differ by an automorphism of G_1.

Proposition 2.5.2. *The Jacobian* $\mathrm{Jac}(C_1)$ *is isogenous to* $E^8 \times B$*, where* B *is a two-dimensional abelian variety that is not the product of two elliptic curves over the rational numbers and* E *an elliptic curve given by the equation*

$$Y^2Z = X^3 + Z^3$$

with $j(E) = 0$*.*

Proof. Let τ_2 and τ_8 be the irreducible subrepresentations of $\rho_{H^0(C_1,\Omega_{C_1})}$ that correspond to χ_4, respectively to χ_9. Let $H^0(C_1, \Omega_{C_1}) = V_2 \oplus V_8$ be the corresponding decomposition of $H^0(C_1, \Omega_{C_1})$ with $\dim V_2 = 2$ and $\dim V_8 = 8$.
The symmetric group S_3 acts linearly on $\mathbb{C}[X, Y, Z]$ by permuting the variables. Since p_1 is a symmetric polynomial, the curve $C_1 = V(p_1)$ is invariant under the induced action of S_3 on $\mathrm{PGL}_3(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2)$. Let $H < \mathrm{Aut}(C_1)$ denote the group of automorphisms of C_1 that is induced by the action of S_3 on $\mathrm{Aut}(\mathbb{P}^2)$. Using GAP, in A.8, we find that

$$\left\langle \mathrm{Res}_H\left(\chi_{H^0(C_1,\Omega_{C_1})}\right), \mathrm{Res}_H(\chi_1)\right\rangle = 1 = \langle \mathrm{Res}_H(\chi_9), \mathrm{Res}_H(\chi_1)\rangle\,. \tag{2.5.2}$$

By Fact 1.6.12, this implies that $\dim H^0(C_1, \Omega_{C_1})^H = 1$ and $H^0(C_1, \Omega_{C_1})^H \subset V_8$. The categorial quotient $E := C_1/H$ is a smooth projective algebraic curve by Proposition 1.4.5 and has genus 1 by Theorem 1.4.6. By a calculation in A.8, we find that E is isomorphic to the curve defined by

$$Y^2Z = X^3 + Z^3.$$

We have $j(E) = 0$. Let $q : C_1 \to C_1/H$ denote the quotient morphism. The pullback map $q^* : H^0(E, \Omega_E) \to H^0(C_1, \Omega_{C_1})$ is injective since q is surjective. The image of q^* is $H^0(C_1, \Omega_{C_1})^H$. Since τ_8 is irreducible of degree 8, there are $g_1, \ldots, g_8$ in $\mathrm{Aut}(C_1)$ such that

$$V_8 = \bigoplus_{i=1}^{8} g_i(q^*(H^0(E, \Omega_E))). \tag{2.5.3}$$

Choose any point $\tilde{P} \in C_1$ and let $P := q(\tilde{P})$ and consider the elliptic curve E with identity P. By the universal property of the Jacobian $\mathrm{Jac}(C_1)$ in Proposition 1.6.6, there is a unique homomorphism $\tilde{q} : \mathrm{Jac}(C_1) \to E$ such that $q = \tilde{q} \circ f_{\tilde{P}}$, where $f_{\tilde{P}}$ is the unique morphism given by Proposition 1.6.6. Define a morphism

$$\hat{q} := (\tilde{q} \circ g_1, \ldots, \tilde{q} \circ g_8) : \mathrm{Jac}(C_1) \to E^8,$$

where the action of $\mathrm{Aut}(C_1)$ on $\mathrm{Jac}(C_1)$ is induced by the action of $\mathrm{Aut}(C_1)$ on $H^0(C_1, \Omega_{C_1})$. The corresponding pullback map on the holomorphic differentials

$$\hat{q}^* : H^0(E, \Omega_E)^{\oplus 8} \cong H^0\big(E^8, \Omega_{E^8}\big) \to H^0\big(\mathrm{Jac}(C_1), \Omega_{\mathrm{Jac}(C_1)}\big) \cong H^0(C_1, \Omega_{C_1})$$

is just the injection $H^0(E, \Omega_E)^{\oplus 8} \hookrightarrow V_8 \subset H^0(C_1, \Omega_{C_1})$ given by (2.5.3). Therefore $\hat{q}^*$ is injective and it follows that $\hat{q}$ is surjective. It then follows from Theorem 1.6.9 that $\mathrm{Jac}(C_1)$ is isogenous to $E^8 \times B$ for some abelian variety B.
By a calculation of Professor Pink in A.8, the abelian variety B is not isogenous to E'^2 for any elliptic curve E' defined over $\mathbb{Q}$. It might still be isogenous to E'^2 for an elliptic curve E' defined over a finite extension of $\mathbb{Q}$. □

2.6. The curve C_2

The polynomial p_2 that defines C_2 is not defined over $\mathbb{Q}$. Still, it is possible to find a rational equation for C_2:

Theorem 2.6.1. *Every smooth projective algebraic curve C with $|\mathrm{Aut}(C)| = 360$ is projectively equivalent to the* Wiman sextic*, which is the projective algebraic curve defined by*

$$f_6 := 27Z^6 - 135XYZ^4 - 45X^2Y^2Z^2 + 9(X^5 + Y^5)Z + 10X^3Y^3.$$

That is, there is some $T \in \mathrm{PGL}_3(\mathbb{C})$ such that $T(C) = V(f_6)$.

Proof. See Theorem 2.1 in [DIK00]. □

Denote by $\chi_1, \ldots, \chi_7$ the irreducible characters of $G_2 \cong \mathrm{Aut}(C_2)$ as defined in the character table of G_2 in A.1.

Lemma 2.6.2. *The representation $\rho_{H^0(C_2,\Omega_{C_2})}$ is irreducible with $\chi_{H^0(C_2,\Omega_{C_2})} = \chi_7$.*

Proof. We have $\deg \rho_{H^0(C_2,\Omega_{C_2})} = \dim H^0(C_2, \Omega_{C_2}) = \mathrm{genus}(C_2) = 10$ by Theorem 1.2.5 and Theorem 1.1.7. Since the number of fixed points for any automorphism is ≥ 0, the Lefschetz Fixed Point Formula Theorem 1.2.7 gives us the following upper bound:

$$\forall g \in \mathrm{Aut}(C_2) \setminus \{1\} : \mathrm{Re}\left(\chi_{H^0(C_2,\Omega_{C_2})}(g)\right) \leq 1.$$

By looking at the character table of G_2, we find that the only character of G_2 of degree 10 that respects this bound is χ_7. The conclusion follows. □

Proposition 2.6.3. *The Jacobian variety $\mathrm{Jac}(C_2)$ is isogenous to E^{10} where E is the elliptic curve given by the equation*

$$Y^2Z = X^3 + \left(\frac{1053}{2}i\sqrt{15} + \frac{13365}{2}\right)XZ^2 + \left(54675i\sqrt{15} - 172773\right)Z^3.$$

We have

$$j(E) = \frac{3^6 \cdot 5 \cdot 19}{2^7}i\sqrt{15} - \frac{3^3 \cdot 5^2 \cdot 181}{2^7}.$$

Proof. Since $\rho_{H^0(C_2,\Omega_{C_2})}$ is irreducible by Lemma 2.6.2, we have that $\mathrm{Jac}(C_2)$ is $\mathrm{Aut}(C_2)$-simple, where $\mathrm{Aut}(C_2)$ acts on $\mathrm{Jac}(C_2)$ by the action induced by $\rho_{H^0(C_2,\Omega_{C_2})}$. Then, by Theorem 1.6.10, there is a simple abelian variety A and a $k \in \mathbb{Z}_{>0}$ such that $\mathrm{Jac}(C_2)$ is isogenous to A^k. Define a morphism by

$$\begin{aligned} \varphi : C_2 &\to \mathbb{P}^2 \\ (X : Y : Z) &\mapsto \left(X^2 : Y^2 : Z^2\right). \end{aligned}$$

Then $\varphi(C_2) = V(r)$, where

$$r := U^3 + aU^2V + bUV^2 + U^3 + bU^2W + cUVW + aV^2W + aUW^2 + bVW^2 + W^3,$$

and the coefficients a, b and c are as in the definition of p_2 and U, V and W are the coordinates in the codomain of φ. In A.9, using Sage, we obtain the following equation for a curve $E \cong V(r)$ of genus 1:

$$Y^2Z = X^3 + \left(\frac{1053}{2}i\sqrt{15} + \frac{13365}{2}\right)XZ^2 + \left(54675i\sqrt{15} - 172773\right)Z^3.$$

We have

$$j(E) = \frac{3^6 \cdot 5 \cdot 19}{2^7}i\sqrt{15} - \frac{3^3 \cdot 5^2 \cdot 181}{2^7}.$$

Choose any $\tilde{P} \in C_2$ and let $P := \varphi(\tilde{P})$. Consider the elliptic curve E with identity P. Then Proposition 1.6.6 gives us a morphism $f_{\tilde{P}} : C_2 \to \mathrm{Jac}(C_2)$ and a homomorphism $\tilde{\varphi} : \mathrm{Jac}(C_2) \to E$ such that $\varphi = \tilde{\varphi} \circ f_{\tilde{P}}$. By Theorem 1.6.9, it follows that $\mathrm{Jac}(C_2)$ is isogenous to $E \times B$ where B is some abelian variety. Therefore $E \times B$ is isogenous to A^k. But E and A are simple abelian varieties and E is isogenous to a factor of A^k. Therefore A must be isogenous to E. We have $k = 10$, because $\dim E = 1$ and $\dim \mathrm{Jac}(C_2) = \mathrm{genus}(C_2) = 10$. □

2.7. The curve C_3

The curve C_3 is called the *Klein quartic* and was studied in 1879 by Felix Klein in [Kle79]. It is a *Hurwitz surface*, meaning that its automorphism group has the maximal order allowed by Theorem 1.3.1, which is 168 for a curve of genus 3. In fact, the Klein quartic is, up to isomorphism, the only Hurwitz surface of genus 3. Furthermore, there is no Hurwitz surface in genus 2. See, for example, Section 2.2. in [Elk99]. Since $p_3 \in \mathbb{Q}[X, Y, Z]$, the curve C_3 is defined over $\mathbb{Q}$.

Denote by $\chi_1, \dots, \chi_6$ the irreducible characters of $G_3 \cong \mathrm{Aut}(C_3)$ as defined in the character table of G_3 in A.1.

Lemma 2.7.1. *The representation $\rho_{H^0(C_3,\Omega_{C_3})}$ is irreducible and either $\chi_{H^0(C_3,\Omega_{C_3})} = \chi_2$ or $\chi_{H^0(C_3,\Omega_{C_3})} = \chi_3$.*

Proof. The representation $\rho_{H^0(C_3,\Omega_{C_3})}$ has degree 3 since $\mathrm{genus}(C_3) = 3$. As for the previous curves, we get an upper bound on the real part of $\chi_{H^0(C_3,\Omega_{C_3})}$ by Theorem 1.2.7:

$$\forall g \in \mathrm{Aut}(C_3) \setminus \{1\} : \mathrm{Re}\left(\chi_{H^0(C_3,\Omega_{C_3})}(g)\right) \leq 1.$$

By looking at the character table of G_3, we find that the only characters of G_3 of degree 3 that respect this bound are χ_2 and χ_3. The conclusion follows. □

Proposition 2.7.2. *The Jacobian* $\mathrm{Jac}(C_3)$ *is isogenous to E^3 for the elliptic curve E that is defined by*

$$Y^2Z = X^3 - 8960XZ^2 - 401408Z^3.$$

We have $j(E) = -3375 = -3^3 \cdot 5^3$.

Proof. The representation $\rho_{H^0(C_3,\Omega_{C_3})}$ is irreducible by Lemma 2.7.1. Therefore, the Jacobian $\mathrm{Jac}(C_3)$ is $\mathrm{Aut}(C_3)$-simple, where $\mathrm{Aut}(C_3)$ acts on $\mathrm{Jac}(C_3)$ by the action induced by $\rho_{H^0(C_3,\Omega_{C_3})}$. Then, by Theorem 1.6.10, there is a simple abelian variety A and a $k \in \mathbb{Z}_{>0}$ such that $\mathrm{Jac}(C_3)$ is isogenous to A^k.
The alternating group A_3 acts linearly on $\mathbb{C}[X,Y,Z]$ by permuting the variables. Since p_3 is invariant under this action, the curve $C_3 = V(p_3)$ is invariant under the induced action of A_3 on $\mathrm{PGL}_3(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2)$. Let $H < \mathrm{Aut}(C_3)$ denote the group of automorphisms of C_3 that is induced by the action of A_3 on $\mathrm{Aut}(\mathbb{P}^2)$. In A.10, we calculate that $E := C_3/H$ is a curve of genus 1 that isomorphic to the curve defined by

$$Y^2Z = X^3 - 8960XZ^2 - 401408Z^3,$$

and $j(E) = -3^3 \cdot 5^3$. Let $q : C_3 \to E$ be the quotient morphism. Choose any $\tilde{P} \in C_3$ and let $P := q(\tilde{P})$. Consider the elliptic curve E with identity P. Then Proposition 1.6.6 gives us a morphism $f_{\tilde{P}} : C_3 \to \mathrm{Jac}(C_3)$ and a homomorphism $\tilde{q} : \mathrm{Jac}(C_3) \to E$ such that $q = \tilde{q} \circ f_{\tilde{P}}$. By Theorem 1.6.9, it follows that $\mathrm{Jac}(C_3)$ is isogenous to $E \times B$ where B is some abelian variety. Therefore $E \times B$ is isogenous to A^k. But E and A are simple abelian varieties and E is isogenous to a factor of A^k. Therefore A must be isogenous to E. We have $k = 3$, because $\dim E = 1$ and $\dim \mathrm{Jac}(C_3) = \mathrm{genus}(C_3) = 3$.

□

2.8. The curve C_4

The curve C_4 is defined over $\mathbb{Q}$ because $p_4 \in \mathbb{Q}[X,Y,Z]$.

Let $\chi_1, \dots, \chi_6$ denote the irreducible characters of G_3 as defined in the character table of G_3 in A.1.

Lemma 2.8.1. *We have either*

$$\chi_{H^0(C_4,\Omega_{C_4})} = \chi_2 + \chi_5$$

or

$$\chi_{H^0(C_4,\Omega_{C_4})} = \chi_3 + \chi_5.$$

Also, $\deg \chi_2 = \deg \chi_3 = 3$ *and* $\deg \chi_5 = 7$.

Proof. The representation $\rho_{H^0(C_4,\Omega_{C_4})}$ has degree 10 because $\mathrm{genus}(C_4) = 10$ by Theorem 1.1.7. As for the previous curves, we get an upper bound on the real part of $\chi_{H^0(C_4,\Omega_{C_4})}$ by Theorem 1.2.7:

$$\forall g \in \mathrm{Aut}(C_4) \setminus \{1\} : \mathrm{Re}\left(\chi_{H^0(C_4,\Omega_{C_4})}(g)\right) \leq 1.$$

By looking at the character table of $G_3 \cong \mathrm{Aut}(C_4)$, we find that the only characters of $\mathrm{Aut}(C_4)$ that respect the bound are $\chi_2 + \chi_5$ and $\chi_3 + \chi_5$. The claims about the degrees and values of the characters follow directly from the character table. □

Let $\sigma : G_3 \to G_3$ denote complex conjugation. This is a well-defined automorphism since we have for the generators S, T and R of $\tilde{G}_3$:

$$\overline{T} = T \quad \text{and} \quad \overline{S} = S^{-1} \quad \text{and} \quad \overline{R} = R^{-1}.$$

The automorphism σ permutes $\chi_2+\chi_5$ and $\chi_3+\chi_5$, the two possibilities for $\chi_{H^0\left(C_4,\Omega_{C_4}\right)}$. We will, without loss of generality, restrict us to the case $\chi_{H^0\left(C_4,\Omega_{C_4}\right)} = \chi_2 + \chi_5$ since the cases only differ by an automorphism of G_3.

Proposition 2.8.2. *The Jacobian* $\mathrm{Jac}(C_4)$ *is isogenous to* $E_1^7 \times E_2^3$ *for elliptic curves* E_1 *and* E_2 *where* E_1 *is defined by*

$$Y^2Z = X^3 + \left(\frac{55}{2}i\sqrt{7} + \frac{55}{6}\right)XZ^2 - \left(\frac{145}{3}i\sqrt{7} + \frac{5843}{27}\right)Z^3$$

and E_2 *is defined by*

$$Y^2Z = (X+7Z)\left(X^2 - 7XZ + 14Z^2\right).$$

We have

$$\begin{aligned} j(E_1) &= -\frac{5^3\cdot 11^3}{2^5} - \frac{5^4\cdot 11^3}{2^5\cdot 7}i\sqrt{7}, \\ j(E_2) &= -3^3\cdot 5^3. \end{aligned}$$

Proof. We use the notation from Definition / Proposition 2.1.2. Let $V := (TS^3)(SR)(TS^3)^{-1}$ and $W := (TS)R(TS)^2$ and let $\tilde{H}_1 := \langle W, V^2\rangle < \tilde{G}_3$. Denote by $H_1 < \mathrm{Aut}(C_4)$ the subgroup of automorphisms of C_4 that is induced by $\tilde{H}_1$. Using GAP, in A.11, we find that

$$\left\langle \mathrm{Res}_{H_1}\left(\chi_{H^0\left(C_4,\Omega_{C_4}\right)}\right), \mathrm{Res}_{H_1}(\chi_1)\right\rangle = 1 = \left\langle \mathrm{Res}_{H_1}(\chi_5), \mathrm{Res}_{H_1}(\chi_1)\right\rangle. \tag{2.8.1}$$

Let τ_3 and τ_7 be the irreducible subrepresentations of $\rho_{H^0\left(C_4,\Omega_{C_4}\right)}$ that correspond to χ_2, respectively to χ_5. Let $H^0(C_4,\Omega_{C_4}) = V_3 \oplus V_7$ be the corresponding decomposition of $H^0(C_4,\Omega_{C_4})$ with $\dim V_3 = 3$ and $\dim V_7 = 7$. By Fact 1.6.12, we have that $\dim H^0(C_4,\Omega_{C_4})^{H_1} = 1$ and $H^0(C_4,\Omega_{C_4})^{H_1} \subset V_7$. The categorial quotient $E_1 := C_4/H_1$ is a smooth projective algebraic curve by Proposition 1.4.5 and has genus 1 by Theorem 1.4.6. By a calculation of Professor Pink in A.11, we see that E_1 is defined by

$$(X-5)Y^2 = -\frac{1}{448}\left(-7+5i\sqrt{7}\right)\left(3i\sqrt{7}X + i\sqrt{7} - 8X^2 + 17X + 3\right)\left(i\sqrt{7} - 4X - 1\right).$$

By using Sage, in A.11, we see that a Weierstrass equation of E_1 is

$$Y^2 = X^3 + \left(\frac{55}{2}i\sqrt{7} + \frac{55}{6}\right)X - \frac{145}{3}i\sqrt{7} - \frac{5843}{27}.$$

We get that

$$j(E_1) = -\frac{5^3 \cdot 11^3}{2^5} - \frac{5^4 \cdot 11^3}{2^5 \cdot 7} i\sqrt{7}.$$

Let $q_1 : C_4 \to E_1$ denote the quotient morphism. Pick any point $\tilde{P_1} \in C_4$ and let $P_1 := q_1(\tilde{P_1})$. Consider the elliptic curve E_1 with identity P_1. In a way that is analogous to the proof of Proposition 2.5.2, we get that there is a surjective homomorphism $\tilde{q_1} : \mathrm{Jac}(C_4) \to E_1$ and $g_1, \ldots, g_7 \in \mathrm{Aut}(C_4)$ such that the morphism

$$\hat{q_1} := (\tilde{q_1} \circ g_1, \ldots, \tilde{q_1} \circ g_7) : \mathrm{Jac}(C_4) \to E_1^7$$

is surjective. By Theorem 1.6.9 and Proposition 1.6.5, the Jacobian $\mathrm{Jac}(C_4)$ is then isogenous to $E_1^7 \times B$ for some abelian variety B of dimension 3.

By a calculation in GAP in A.11, we see that there is no $K < \mathrm{Aut}(C_4)$ such that C_4/K is a curve of genus 1 and $H^0(C_4, \Omega_{C_4})^K \subset V_3$. Therefore, we cannot determine the structure of B in a way that is analogous to the way in which we determined the structure of the 7-dimensional factor of $\mathrm{Jac}(C_4)$. Let $\tilde{H_2} := \langle V \rangle < \tilde{G_3}$ and denote by $H_2 < \mathrm{Aut}(C_4)$ the group of automorphisms of C_4 that is induced by $\tilde{H_2}$. By a calculation in GAP in A.11, we find that

$$\left\langle \mathrm{Res}_{H_2}(\chi_{H^0(C_4, \Omega_{C_4})}), \mathrm{Res}_{H_2}(\chi_1) \right\rangle = 2, \tag{2.8.2}$$

and therefore the quotient C_4/H_2 is a smooth projective curve of genus 2 by Proposition 1.4.5 and Theorem 1.4.6. By a calculation of Professor Pink in A.11, there is a surjective morphism $C_4/H_2 \to C'$, where C' is the hyperelliptic curve defined by

$$Y^2 Z^4 = (2X^2 + XZ + Z^2)(4X^4 - 17X^3 Z + 19X^2 Z^2 + 9XZ^3 + Z^4).$$

In the same calculation we see that there is a morphism from this hyperelliptic curve onto an elliptic curve E_2 defined by

$$Y^2 Z = (X + 7Z)\left(X^2 - 7XZ + 14Z^2\right)$$

with $j(E_2) = -3^3 \cdot 5^3$. Therefore, there is a surjective homomorphism $q_2 : C_4 \to E_2$. Pick any $\tilde{P_2} \in C_4$ and let $P_2 := q_2(\tilde{P_2})$. Consider the elliptic curve E_2 with identity P_2. By Proposition 1.6.6, we get a surjective morphism $\tilde{q_2} : \mathrm{Jac}(C_4) \to E_2$. Therefore, by Theorem 1.6.9, the curve E_2 is isogenous to a factor of $\mathrm{Jac}(C_4)$. Note that E_1 and E_2 are not isogenous because the denominators of $j(E_1)$ and $j(E_2)$ have distinct prime factors. Therefore E_2 has to be isogenous to a factor of B, because it cannot be a factor of E_1^7. The group $\mathrm{Aut}(C_4)$ acts on $\mathrm{Jac}(C_4)$ because the contragredient representation $\rho^*_{H^0(C_4, \Omega_{C_4})}$ descends to the quotient $\mathrm{Jac}(C_4) = H^0(C_4, \Omega_{C_4})^* / H_1(C_4, \mathbb{Z})$. Let $A \subset \mathrm{Jac}(C_4)$ be an $\mathrm{Aut}(C_4)$-simple abelian subvariety. Then the preimage of A in $H^0(C_4, \Omega_{C_4})^*$ is the representation space of some subrepresentation of $\rho^*_{H^0(C_4, \Omega_{C_4})}$ of the same dimension as A. Since $\rho^*_{H^0(C_4, \Omega_{C_4})}$ has exactly two irreducible subrepresentations

τ_3^* and τ_7^* of dimension 3 and 7, respectively, by Theorem 1.6.10, the Jacobian $\mathrm{Jac}(C_4)$ is either the power of one simple abelian variety or it is of the form $A_1^{k_1} \times A_2^{k_2}$ where A_1 and A_2 are simple abelian varieties with $\dim A_1^{k_1} = 7$ and $\dim A_2^{k_2} = 3$. But since we already know two non-isogenous simple components of $\mathrm{Jac}(C_4)$ the second case has to be true. It follows that A_1 is isogenous to E_1 and A_2 is isogenous to E_2, because we already know that E_1^7 is a factor of $\mathrm{Jac}(C_4)$. □

A. Appendix

A.1. Character tables

The following table was computed by using GAP [GAP15] using the command

```
Display(CharacterTable(G1tilde/Group(E(3)*IdentityMat(3))));
```

after defining `G1tilde` using the code in A.3. The character table of the Hessian group G_1 is:

Class size	1	12	54	9	8	12	36	24	36	24
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	α_1	1	1	1	$\overline{\alpha_1}$	α_1	α_1	$\overline{\alpha_1}$	$\overline{\alpha_1}$
χ_3	1	$\overline{\alpha_1}$	1	1	1	α_1	$\overline{\alpha_1}$	$\overline{\alpha_1}$	α_1	α_1
χ_4	2	-1	0	-2	2	-1	1	-1	1	-1
χ_5	2	$-\alpha_1$	0	-2	2	$-\overline{\alpha_1}$	α_1	$-\alpha_1$	$\overline{\alpha_1}$	$-\overline{\alpha_1}$
χ_6	2	$-\overline{\alpha_1}$	0	-2	2	$-\alpha_1$	$\overline{\alpha_1}$	$-\overline{\alpha_1}$	α_1	$-\alpha_1$
χ_7	3	0	-1	3	3	0	0	0	0	0
χ_8	8	2	0	0	-1	2	0	-1	0	-1
χ_9	8	α_2	0	0	-1	$\overline{\alpha_2}$	0	$-\overline{\alpha_1}$	0	$-\alpha_1$
χ_{10}	8	$\overline{\alpha_2}$	0	0	-1	α_2	0	$-\alpha_1$	0	$-\overline{\alpha_1}$

Here $\alpha_1 = e^{4\pi i/3}$ and $\alpha_2 = 2e^{2\pi i/3}$.

The character table of $G_2 \cong A_6$ is taken from page 424 in [JL01]:

Class representative	1	(12)(34)	(123)	(123)(456)	(1234)(56)	(12345)	(123456)
Class size	1	45	40	40	90	72	72
χ_1	1	1	1	1	1	1	1
χ_2	5	1	2	-1	-1	0	0
χ_3	5	1	-1	2	-1	0	0
χ_4	8	0	-1	-1	0	α_1	α_2
χ_5	8	0	-1	-1	0	α_2	α_1
χ_6	9	1	0	0	1	-1	-1
χ_7	10	-2	1	1	0	0	0

Here $\alpha_1 = \frac{1-\sqrt{5}}{2}$ and $\alpha_2 = \frac{1+\sqrt{5}}{2}$.

The character table of $G_3 \cong \mathrm{PSL}_2(7)$ is taken from the pages 313 and 318 in [JL01]:

Class size	1	21	42	56	24	24
Order of representatives	1	2	4	3	7	7
χ_1	1	1	1	1	1	1
χ_2	3	-1	1	0	α	$\overline{\alpha}$
χ_3	3	-1	1	0	$\overline{\alpha}$	α
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	-1	1	0	0
χ_6	8	0	0	-1	1	1

Here $\alpha = \frac{-1+i\sqrt{7}}{2}$.

A.2. Values of $S^k\rho^*_{\tilde{G}_1}$, $S^k\rho^*_{\tilde{G}_2}$ and $S^k\rho^*_{\tilde{G}_3}$

We use the notation from Notation 2.1.1 and Definition / Proposition 2.1.2.

Corollary A.2.1. *If $n = 3$, Proposition 2.2.8 simplifies to*

$$S^k\chi^*_{\tilde{G}}(g) = \sum_{i=0}^{k}\sum_{j=0}^{k-i}\overline{\lambda_1^i\lambda_2^j\lambda_3^{k-i-j}}.$$

Let $i \in \{1,2\}$.

Fact A.2.2. *By $\pi_i : \tilde{G}_i \to G_i$ we denote the projection homomorphism. Let C be a conjugacy class of G_i. Then either $\pi_i^{-1}(C)$ is a conjugacy class in $\tilde{G}_i$, or $\pi_i^{-1}(C) = C_1 \sqcup C_2 \sqcup C_3$, where C_1, C_2 and C_3 are conjugacy classes in $\tilde{G}_i$ with $|C_1| = |C_2| = |C_3| = |C|$. In the latter case we have $C_1 = \omega C_2 = \omega^2 C_3$.*

Also, the linear characters of $\tilde{G}_i$ are lifts of linear characters of G_i by Proposition 2.3.5. Therefore, they are constant on the preimages of conjugacy classes of G_i.
Let C be a conjugacy class of G_i such that $\pi_i^{-1}(C) = C_1 \sqcup C_2 \sqcup C_3$, where C_1, C_2 and C_3 are conjguacy classes of $\tilde{G}_i$. Let χ be some linear character of $\tilde{G}_i$. Then, to calculate $\left\langle S^k\chi^*_{\tilde{G}_i}, \chi\right\rangle$, it is sufficient to calculate $S^k\chi^*_{\tilde{G}_i}(C_1)$ for the case when $k = 3n$ for some $n \in \mathbb{Z}_{>0}$. This is because for $k \not\equiv 0 \mod 3$ we have

$$\begin{aligned}\sum_{l=1}^{3}|C_l|S^k\chi^*_{\tilde{G}_i}(C_l)\overline{\chi(C_l)} &= |C_1|\overline{\chi(C_1)}\sum_{l=1}^{3}S^k\chi^*_{\tilde{G}_i}(C_l)\\ &= |C_1|\overline{\chi(C_1)}(1+\omega^k+\omega^{2k})S^k\chi^*_{\tilde{G}_i}(C_1) = 0.\end{aligned}$$

So the terms for C_1, C_2 and C_3 cancel out in $\left\langle S^k\chi^*_{\tilde{G}_i}, \chi\right\rangle$. Also we have

$$S^{3n}\chi^*_{\tilde{G}_i}(C_1) = S^{3n}\chi^*_{\tilde{G}_i}(C_2) = S^{3n}\chi^*_{\tilde{G}_i}(C_3),$$

so it is sufficient to calculate $S^{3n}\chi^*_{\tilde{G}_i}$ for one of C_1, C_2, C_3.

Notation A.2.3. In the following tables for $\tilde{G}_i$ we have the following conventions: if the union of three conjugacy classes C_1, C_2 and C_3 is the preimage of one conjugacy class in G_i, as in Fact A.2.2, only a row for one of the classes is shown and the classes are named $C_{l,1} = C_1$ and $C_{l,2} = C_2$ and $C_{l,3} = C_3$ for some $l \in \mathbb{Z}_{>0}$. If the name of a conjugacy class only has one index in the table, it is the preimage of a conjugacy class of G_i.

A.2.1. Values of $S^k\rho^*_{\tilde{G}_1}$

Denote by χ_1 the trivial character of $\tilde{G}_1$ and by χ_2 and χ_3 the non-trivial linear characters of $\tilde{G}_1$. The following table lists the conjugacy classes of $\tilde{G}_1$, their size, the eigenvalues of the representatives, values of χ_2, χ_3 and $S^k\chi^*_{\tilde{G}_1}$ or $S^{3n}\chi^*_{\tilde{G}_1}$. We use the conventions from Notation A.2.3.

Table A.2.4: Conjugacy classes of $\tilde{G}_1$

Representative	**Class Order**	**Eigenvalues**	χ_2	χ_3	$S^k\chi^*_{\tilde{G}_1}(\cdot)$ **or** $S^{3n}\chi^*_{\tilde{G}_1}(\cdot)$
$C_{1,1} := I_3$	1	$1, 1, 1$	1	1	$S^{3n}\chi^*_{\tilde{G}_1}(C_{1,1}) = \frac{(3n)^2+9n+2}{2}$
$C_2 := T^2$	24	$1, \omega, \omega^2$	1	1	$S^k\chi^*_{\tilde{G}_1}(C_2) = \begin{cases} 1 & \text{if } k \equiv 0 \mod 3 \\ 0 & \text{if } k \equiv 1, 2 \mod 3 \end{cases}$
$C_{3,1} := V^2$	9	$1, -1, -1$	1	1	$S^{3n}\chi^*_{\tilde{G}_1}(C_{3,1}) = \frac{2(3n+1)(-1)^{3n}+1+(-1)^{3n}}{4}$
$C_{4,1} := T^2S_1TV^3$	54	$1, i, -i$	1	1	$S^{3n}\chi^*_{\tilde{G}_1}(C_{4,1}) = \begin{cases} 1 & \text{if } 3n \equiv 0, 1 \mod 4 \\ 0 & \text{if } 3n \equiv 2, 3 \mod 4 \end{cases}$
$C_{5,1} := (S_1T)^2TU$	12	$\epsilon, \epsilon, \epsilon^7$	ω	ω^2	$S^{3n}\chi^*_{\tilde{G}_1}(C_{5,1}) = \frac{\omega^n-(3n+1)\omega^{n+1}}{1-\omega}$
$C_6 := TS_1TU$	72	$1, \omega, \omega^2$	ω	ω^2	$S^k\chi^*_{\tilde{G}_1}(C_6) = S^k\chi^*_{\tilde{G}_1}(C_2)$
$C_{7,1} := S_1^2TUV^2$	36	$-\epsilon, -\epsilon^4, \epsilon^4$	ω	ω^2	$S^{3n}\chi^*_{\tilde{G}_1}(C_{7,1}) = \frac{(-1)^n\omega^n}{1-\omega}\left(\frac{1+(-1)^n\omega}{1+\omega} - \frac{\omega+(-1)^n\omega}{2}\right)$
$C_{8,1} := T^2S_1TU^2$	12	$\epsilon^2, \epsilon^2, \epsilon^5$	ω^2	ω	$S^{3n}\chi^*_{\tilde{G}_1}(C_{8,1}) = \overline{S^{3n}\chi^*_{\tilde{G}_1}(C_{5,1})}$
$C_9 := S_1^2T^2U^2$	72	$1, \omega, \omega^2$	ω^2	ω	$S^k\chi^*_{\tilde{G}_1}(C_9) = S^k\chi^*_{\tilde{G}_1}(C_2)$
$C_{10,1} := S_1TU^2V^2$	36	$\epsilon^8, -\epsilon^8, -\epsilon^2$	ω^2	ω	$S^{3n}\chi^*_{\tilde{G}_1}(C_{10,1}) = \overline{S^{3n}\chi^*_{\tilde{G}_1}(C_{7,1})}$

The columns labeled "Representative", "Class Order", "Eigenvalues", "χ_2" and "χ_3" were computed using GAP in A.5. The values of $S^{3n}\chi^*_{\tilde{G}_1}$ and $S^k\chi^*_{\tilde{G}_1}$ were calculated by hand using Corollary A.2.1. Let $i \in \{1, 2, 3\}$. The following identities, which can be proven by direct calculations, are useful to evaluate $\langle S^k\chi^*_{\tilde{G}_1}, \chi_1\rangle$, $\langle S^k\chi^*_{\tilde{G}_1}, \chi_2\rangle$ and $\langle S^k\chi^*_{\tilde{G}_1}, \chi_3\rangle$:

$$S^{3n}\chi^*_{\tilde{G}_1}(C_{7,i}) + S^{3n}\chi^*_{\tilde{G}_1}(C_{10,i}) = \begin{cases} 2 & 3n \equiv 0, 15 \mod 18 \\ -1 & 3n \equiv 3, 6, 9, 12 \mod 18 \end{cases}$$

$$\chi_2(C_{7,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{7,i}) + \chi_2(C_{10,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{10,i}) = \begin{cases} 2 & 3n \equiv 3,6 \mod 18 \\ -1 & 3n \equiv 0,9,12,15 \mod 18 \end{cases}$$

$$\chi_3(C_{7,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{7,i}) + \chi_3(C_{10,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{10,i}) = \begin{cases} 2 & 3n \equiv 9,12 \mod 18 \\ -1 & 3n \equiv 0,3,6,15 \mod 18 \end{cases}$$

$$S^{3n}\chi^*_{\tilde{G}_1}(C_{5,i}) + S^{3n}\chi^*_{\tilde{G}_1}(C_{8,i}) = \begin{cases} 3n+2 & 3n \equiv 0 \mod 9 \\ -1 & 3n \equiv 3 \mod 9 \\ -3n-1 & 3n \equiv 6 \mod 9 \end{cases}$$

$$\chi_2(C_{5,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{5,i}) + \chi_2(C_{8,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{8,i}) = \begin{cases} -1 & 3n \equiv 0 \mod 9 \\ -3n-1 & 3n \equiv 3 \mod 9 \\ 3n+2 & 3n \equiv 6 \mod 9 \end{cases}$$

$$\chi_3(C_{5,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{5,i}) + \chi_3(C_{8,i})S^{3n}\chi^*_{\tilde{G}_1}(C_{8,i}) = \begin{cases} -3n-1 & 3n \equiv 0 \mod 9 \\ 3n+2 & 3n \equiv 3 \mod 9 \\ -1 & 3n \equiv 6 \mod 9 \end{cases}$$

A.2.2. Values of $S^k\rho^*_{\tilde{G}_2}$

The following table lists the conjugacy classes of $\tilde{G}_2$, their size, the eigenvalues of the representatives and the values of $S^k\chi^*_{\tilde{G}_2}$ or $S^{3n}\chi^*_{\tilde{G}_2}$. Here, $\nu = e^{2\pi i/5}$. We use the notation from Notation A.2.3.

Table A.2.5: Conjugacy classes of $\tilde{G}_2$

Representative	**Class Order**	**Eigenvalues**	$S^k\chi^*_{\tilde{G}_2}(\cdot)$ **or** $S^{3n}\chi^*_{\tilde{G}_2}(\cdot)$
$C_{1,1} := I_3$	1	$1,1,1$	$S^{3n}\chi^*_{\tilde{G}_2}(C_{1,1}) = \frac{(3n)^2+9n+2}{2}$
$C_{2,1} := F_2F_1F_3F_4F_1F_3F_2$	45	$1,-1,-1$	$S^{3n}\chi^*_{\tilde{G}_2}(C_{2,1}) = \frac{2(3n+1)(-1)^{3n}+1+(-1)^{3n}}{4}$
$C_{3,1} := F_2F_1^2F_3F_2F_1F_4F_3F_2F_1$	72	$1,\nu,\nu^4$	$S^{3n}\chi^*_{\tilde{G}_2}(C_{3,1}) = \frac{1}{1-\nu^1}\left(\frac{1-\nu^{-3n-1}}{1-\nu^{-1}} - \nu^{3n+1}\frac{1-\nu^{-6n-2}}{1-\nu^{-2}}\right)$
$C_{4,1} := F_1F_2F_3F_2F_1^2F_4$	72	$1,\nu^2,\nu^3$	$S^{3n}\chi^*_{\tilde{G}_2}(C_{4,1}) = \frac{1}{1-\nu^2}\left(\frac{1-\nu^{-6n-2}}{1-\nu^{-2}} - \nu^{6n+2}\frac{1-\nu^{-12n-4}}{1-\nu^{-4}}\right)$
$C_{5,1} := F_1F_2F_1^2F_3F_2F_4F_3F_1$	90	$1,i,-i$	$S^{3n}\chi^*_{\tilde{G}_2}(C_{5,1}) = \begin{cases} 1 & 3n \equiv 0,1 \mod 4 \\ 0 & 3n \equiv 2,3 \mod 4 \end{cases}$
$C_6 := F_2F_4F_3F_2$	120	$1,\omega,\omega^2$	$S^k\chi^*_{\tilde{G}_2}(C_6) = \begin{cases} 1 & k \equiv 0 \mod 3 \\ 0 & k \equiv 1,2 \mod 3 \end{cases}$
$C_7 := F_2F_1F_3F_2F_1F_4F_1$	120	$1,\omega,\omega^2$	$S^k\chi^*_{\tilde{G}_2}(C_7) = S^k\chi^*_{\tilde{G}_2}(C_6)$

The columns labeled "Representative", "Class Order" and "Eigenvalues" were computed using GAP in A.5. The values of $S^{3n}\chi^*_{\tilde{G}_2}$ and $S^k\chi^*_{\tilde{G}_2}$ were calculated by hand using Corollary A.2.1. Let $i \in \{1,2,3\}$. The following identity, which can be proven by direct calculation, is useful to evaluate $\langle S^k\chi^*_{\tilde{G}_2}, \chi_1\rangle$, where χ_1 is the trivial character of $\tilde{G}_2$:

$$S^{3n}\chi^*_{\tilde{G}_2}(C_{3,i}) + S^{3n}\chi^*_{\tilde{G}_2}(C_{4,i}) = \begin{cases} 2 & 3n \equiv 0,2 \mod 5 \\ 1 & 3n \equiv 1 \mod 5 \\ 0 & 3n \equiv 3,4 \mod 5 \end{cases}$$

A.2.3. Values of $S^k\rho^*_{\tilde{G}_3}$

The following table lists the conjugacy classes of $\tilde{G}_3$, their size, the eigenvalues of the representatives and the values of $S^k\chi^*_{\tilde{G}_3}$. The representatives of the conjugacy classes were found by trying different elements, using the orders of the class representatives from the character table in A.1 and the fact that elements in the same class must have the same eigenvalues.

Table A.2.6: Conjugacy classes of $\tilde{G}_3$

Representative	**Class Order**	**Eigenvalues**	$S^k\chi^*_{\tilde{G}_3}(\cdot)$
I_3	1	$1,1,1$	$S^k\chi^*_{\tilde{G}_3}(I_3) = \frac{k^2+3k+2}{2}$
R	21	$1,-1,-1$	$S^k\chi^*_{\tilde{G}_3}(R) = \frac{2(k+1)(-1)^k+1+(-1)^k}{4}$
T	56	$1,\omega,\omega^2$	$S^k\chi^*_{\tilde{G}_3}(T) = \begin{cases} 1 & k \equiv 0 \mod 3 \\ 0 & k \equiv 1,2 \mod 3 \end{cases}$
RS	42	$1,i,-i$	$S^k\chi^*_{\tilde{G}_3}(RS) = \begin{cases} 1 & k \equiv 0,1 \mod 4 \\ 0 & k \equiv 2,3 \mod 4 \end{cases}$
S	24	β,β^2,β^4	$S^k\chi^*_{\tilde{G}_3}(S) = \frac{1}{1-\beta^2}\left(\beta^{-4k}\frac{1-\beta^{3(k+1)}}{1-\beta^3} - \beta^{-2k+2}\frac{1-\beta^{k+1}}{1-\beta}\right)$
S^{-1}	24	β^3,β^5,β^6	$S^k\chi^*_{\tilde{G}_3}(S^{-1}) = \frac{1}{1-\beta}\left(\beta^{-6k}\frac{1-\beta^{3(k+1)}}{1-\beta^3} - \beta^{-5k+1}\frac{1-\beta^{2(k+1)}}{1-\beta^2}\right)$

The following identity, which can be proven by direct calculation, is useful to evaluate $\langle S^k\chi^*_{\tilde{G}_3}, \chi_1\rangle$, where χ_1 is the trivial character of $\tilde{G}_3$:

$$S^k\chi^*_{\tilde{G}_3}(S) + S^k\chi^*_{\tilde{G}_3}(S^{-1}) = \begin{cases} 2 & k \equiv 0,4 \mod 7 \\ -1 & k \equiv 1,3 \mod 7 \\ -2 & k \equiv 2 \mod 7 \\ 0 & k \equiv 5,6 \mod 7 \end{cases}$$

A.3. Definition of $\tilde{G}_1$, $\tilde{G}_2$ and $\tilde{G}_3$ in GAP

The following code GAP code defines $\tilde{G}_1$, $\tilde{G}_2$ and $\tilde{G}_3$:

Listing A.3.1: DefineGroups.g

```
omega:=E(3);
epsilon:=E(9)^2;
```

```
S1:=[[1,0,0],[0,omega,0],[0,0,omega^2]];
rho:=1/(omega-omega^2);
T:=[[0,1,0],[0,0,1],[1,0,0]];
U:=[[epsilon,0,0],[0,epsilon,0],[0,0,epsilon*omega]];
V:=rho*[[1,1,1],[1,omega,omega^2],[1,omega^2,omega]];
G1tilde:=Group(S1,T,U,V);

F1:=[[0,1,0],[0,0,1],[1,0,0]];
F2:=[[1,0,0],[0,-1,0],[0,0,-1]];
mu1:=(-1+Sqrt(5))/2;
mu2:=(-1-Sqrt(5))/2;
F3:=[[-1,mu2,mu1],[mu2,mu1,-1],[mu1,-1,mu2]]/2;
F4:=[[-1,0,0],[0,0,-omega],[0,-omega^2,0]];
G2tilde:=Group(F1,F2,F3,F4);

beta:=E(7);
S:=[[beta,0,0],[0,beta^2,0],[0,0,beta^4]];
T:=[[0,1,0],[0,0,1],[1,0,0]];
a:=beta^4-beta^3;
b:=beta^2-beta^5;
c:=beta-beta^6;
h:=-(beta+beta^2+beta^4-beta^6-beta^5-beta^3)^(-1);
R:=[[a,b,c],[b,c,a],[c,a,b]]*h;
G3tilde:=Group(S,T,R);
```

A.4. Linear characters of $\tilde{G}_1$

The following GAP code computes and outputs the number of linear characters of $\tilde{G}_1$:

Listing A.4.1: LinearCharactersG1tilde.g

```
Read("DefineGroups.g");
Print("There␣are␣");
Print(Size(LinearCharacters(G1tilde)));
Print("␣different␣linear␣characters␣of␣G1tilde.");
```

The output is:

Listing A.4.2: Output of LinearCharactersG1tilde.g

```
There are 3 different linear characters of G1tilde.
```

A.5. Conjugacy classes of $\tilde{G}_1$ and $\tilde{G}_2$

The following GAP code computes and outputs the data needed for Table A.2.4.

Listing A.5.1: ConjugacyClasses.g

```
Read("DefineGroups.g");

homG1tilde:=EpimorphismFromFreeGroup(G1tilde:names:=["S1","T","U","V"]);
CCG1tilde:=ConjugacyClasses(G1tilde);
```

```
CCG1tildeWords:=List(CCG1tilde,x->PreImagesRepresentative(homG1tilde,
    Representative(x)));
Print("The representatives of the conjugacy classes of G1tilde are: ");
Print(CCG1tildeWords);
Print("\n");
Print("The sizes of the conjugacy classes are: ");
Print(List(CCG1tilde,x->Size(x)));
Print("\n");
Print("The eigenvalues of the representatives of the conjugacy ");
Print("classes with multiplicities are: ");
Print(List(CCG1tilde,x->[List(Eigenspaces(CF(36),Representative(x)),
    y->Dimension(y)),Eigenvalues(CF(36),Representative(x))]));
Print("\n");
LC:=LinearCharacters(G1tilde);
Print("The linear characters evaluated on the conjugacy classes are: ");
Print(List(CCG1tilde,x->[Representative(x)^(LC[2]),
    Representative(x)^(LC[3])]));
Print("\n");

homG2tilde:=EpimorphismFromFreeGroup(G2tilde:names:=["F1","F2","F3","F4"]);
CCG2tilde:=ConjugacyClasses(G2tilde);
CCG2tildeWords:=List(CCG2tilde,x->PreImagesRepresentative(homG2tilde,
    Representative(x)));
Print("The representatives of the conjugacy classes of G2tilde are: ");
Print(CCG2tildeWords);
Print("\n");
Print("The sizes of the conjugacy classes are: ");
Print(List(CCG2tilde,x->Size(x)));
Print("\n");
Print("The eigenvalues of the representatives of the conjugacy ");
Print("classes with multiplicities are: ");
Print(List(CCG2tilde,x->[List(Eigenspaces(CF(60),Representative(x)),
    y->Dimension(y)),Eigenvalues(CF(60),Representative(x))]));
```

A.6. Finding the degrees of the invariant curves

The code in this section solves (2.3.1) for each of the groups $\tilde{G_1}$, $\tilde{G_2}$ and $\tilde{G_3}$ and each of their linear characters.

The following Sage code computes the degrees of the $\tilde{G_1}$-invariant polynomials that are in 1-dimensional eigenspaces of the action of $\tilde{G_1}$ on the homogeneous elements of $\mathbb{C}[X, Y, Z]$:

Listing A.6.1: G1.sage

```
from itertools import product
print """Calculating degrees of tilde(G1)-invariant
polynomials in 1-dimensional eigenspaces"""
var('k')

# The following definitions list the values of the characters of the
# symmetric power representation:
```

```
# The format is [value,n such that value is valid for n mod m,m]
optionsforC1=[[(k^2+3*k+2)/(2*216),0,3]]
optionsforC2C6C9=[[1/27+1/9+1/9,0,3]]
optionsforC3=[[(2*(k+1)+2)/(4*24),0,2],[-2*(k+1)/(4*24),1,2]]
optionsforC4=[[1/4,0,4],[1/4,1,4],[0,2,4],[0,3,4]]
optionsforC5C8=[[(k+2)/18,0,9],[-1/18,3,9],[(-k-1)/18,6,9]]
optionsforC7C10
    =[[2/6,0,18],[2/6,15,18],[-1/6,3,18],[-1/6,6,18],[-1/6,9,18],[-1/6,12,18]]


# We only need to consider k=0 (mod 3) because the inner product is 0 in
# the other cases.

for i1,i2,i3,i4,i5,i6 in product(optionsforC1,optionsforC2C6C9,
optionsforC3,optionsforC4,optionsforC5C8,optionsforC7C10):
    for di in range(1,2): # The range specifies
                          # the dimension of the eigenspaces we look for.
        sols=solve(i1[0]+i2[0]+i3[0]+i4[0]+i5[0]+i6[0]==di,k,solution_dict=
            True)
        for sol in sols:
            if sol[k] in ZZ:
                if (sol[k]>0 and mod(sol[k],i1[2])==i1[1] and
                 mod(sol[k],i2[2])==i2[1] and
                 mod(sol[k],i3[2])==i3[1] and
                 mod(sol[k],i4[2])==i4[1] and
                 mod(sol[k],i5[2])==i5[1] and
                 mod(sol[k],i6[2])==i6[1]):
                    print "="+str(di)+"␣for␣k="+str(sol[k])
print "Done!"
```

Denote by χ_2 the lift of the second linear character of G_1 to $\tilde{G_1}$. The following Sage code computes all $k \in \mathbb{Z}_{>0}$ such that $\dim E^k_{\chi_2} = 1$. None are found.

Listing A.6.2: G1chi2.sage

```
from itertools import product
print "Calculating␣degrees␣of␣polynomials␣in␣1-dimensional␣eigenspaces␣of␣
    chi2"
var('k')

# The following definitions list the values of the characters of the
# symmetric power representation:
# The format is [value,n such that value is valid for n mod m,m]
optionsforC1=[[(k^2+3*k+2)/(2*216),0,3]]
optionsforC2C6C9=[[1/27-1/9,0,3]]
optionsforC3=[[(2*(k+1)+2)/(4*24),0,2],[-2*(k+1)/(4*24),1,2]]
optionsforC4=[[1/4,0,4],[1/4,1,4],[0,2,4],[0,3,4]]
optionsforC5C8=[[(k+2)/18,3,9],[-1/18,6,9],[(-k-1)/18,0,9]]
optionsforC7C10
    =[[2/6,9,18],[2/6,12,18],[-1/6,0,18],[-1/6,3,18],[-1/6,6,18],[-1/6,15,18]]


# We only need to consider k=0 (mod 3) because the inner product is 0 in
# the other cases.
```

```
for i1,i2,i3,i4,i5,i6 in product(optionsforC1,optionsforC2C6C9,
optionsforC3,optionsforC4,optionsforC5C8,optionsforC7C10):
    for di in range(1,2): # The range specifies
                          # the dimension of the eigenspaces we look for.
        sols=solve(i1[0]+i2[0]+i3[0]+i4[0]+i5[0]+i6[0]==di,k,solution_dict=
            True)
        for sol in sols:
            if sol[k] in ZZ:
                if (sol[k]>0 and mod(sol[k],i1[2])==i1[1] and
                 mod(sol[k],i2[2])==i2[1] and
                 mod(sol[k],i3[2])==i3[1] and
                 mod(sol[k],i4[2])==i4[1] and
                 mod(sol[k],i5[2])==i5[1] and
                 mod(sol[k],i6[2])==i6[1]):
                    print "="+str(di)+"␣for␣k="+str(sol[k])
print "Done!"
```

Denote by χ_3 the lift of the third linear character of G_1 to $\tilde{G}_1$. The following Sage code computes all $k \in \mathbb{Z}_{>0}$ such that $\dim E^k_{\chi_3} = 1$.

Listing A.6.3: G1chi3.sage

```
from itertools import product
print "Calculating␣degrees␣of␣polynomials␣in␣1-dimensional␣eigenspaces␣of␣
    chi3"
var('k')

# The following definitions list the values of the characters of the
# symmetric power representation:
# The format is [value,n such that value is valid for n mod m,m]
optionsforC1=[[(k^2+3*k+2)/(2*216),0,3]]
optionsforC2C6C9=[[1/27-1/9,0,3]]
optionsforC3=[[(2*(k+1)+2)/(4*24),0,2],[-2*(k+1)/(4*24),1,2]]
optionsforC4=[[1/4,0,4],[1/4,1,4],[0,2,4],[0,3,4]]
optionsforC5C8=[[(k+2)/18,6,9],[-1/18,0,9],[(-k-1)/18,3,9]]
optionsforC7C10
    =[[2/6,3,18],[2/6,6,18],[-1/6,0,18],[-1/6,9,18],[-1/6,12,18],[-1/6,15,18]]


# We only need to consider k=0 (mod 3) because the inner product is 0 in
# the other cases.

for i1,i2,i3,i4,i5,i6 in product(optionsforC1,optionsforC2C6C9,
optionsforC3,optionsforC4,optionsforC5C8,optionsforC7C10):
    for di in range(1,2): # The range specifies
                          # the dimension of the eigenspaces we look for.
        sols=solve(i1[0]+i2[0]+i3[0]+i4[0]+i5[0]+i6[0]==di,k,solution_dict=
            True)
        for sol in sols:
            if sol[k] in ZZ:
                if (sol[k]>0 and mod(sol[k],i1[2])==i1[1] and
                 mod(sol[k],i2[2])==i2[1] and
                 mod(sol[k],i3[2])==i3[1] and
```

```
                  mod(sol[k],i4[2])==i4[1] and
                  mod(sol[k],i5[2])==i5[1] and
                  mod(sol[k],i6[2])==i6[1]):
                     print "="+str(di)+" for k="+str(sol[k])
print "Done!"
```

The following Sage code computes the degrees of the $\tilde{G}_2$-invariant polynomials that are in 1-dimensional eigenspaces of the action of $\tilde{G}_2$ on the homogeneous elements of $\mathbb{C}[X, Y, Z]$:

Listing A.6.4: G2.sage

```
from itertools import product
print """Calculating degrees of tilde(G2)-invariant
polynomials in 1-dimensional eigenspaces"""
var('k');

optionsforC1_1=[[((k^2+3*k+2)/2)/360,0]]
optionsforC2_1=[[(k+2)/16,0],[(k+2)/16,2],[(-k-1)/16,1],[(-k-1)/16,3]]
optionsforC3_1C4_1=[[2/5,0],[2/5,2],[1/5,1],[0,3],[0,4]]
optionsforC5_1=[[1/4,0],[1/4,1],[0,2],[0,3]]
optionsforC6=[[120/1080,0]]
optionsforC7=[[120/1080,0]]

for i1,i2,i3,i4,i5,i6 in product(optionsforC1_1,optionsforC2_1,
    optionsforC3_1C4_1,optionsforC5_1,optionsforC6,optionsforC7):
    for di in range(1,2):
        sols=solve(i1[0]+i2[0]+i3[0]+i4[0]+i5[0]+i6[0]==di,k,
            solution_dict=True)
        for sol in sols:
            if (sol[k] in ZZ and sol[k]>0 and
             mod(sol[k],len(optionsforC2_1))==i2[1] and
             mod(sol[k],len(optionsforC3_1C4_1))==i3[1] and
             mod(sol[k],len(optionsforC5_1))==i4[1] and
             mod(sol[k],3)==0):
                print "="+str(di)+" for k="+str(sol[k])
print "Done!"
```

The following Sage code computes the degrees of the $\tilde{G}_3$-invariant polynomials that are in 1-dimensional eigenspaces of the action of $\tilde{G}_3$ on the homogeneous elements of $\mathbb{C}[X, Y, Z]$:

Listing A.6.5: G3.sage

```
from itertools import product
print """Calculating degrees of tilde(G3)-invariant
polynomials in 1-dimensional eigenspaces"""
var('k');

optionsfor1=[[(k^2+3*k+2)/336,0]]
optionsforR=[[(-k-1)/16,1],[(k+2)/16,0]]
optionsforRS=[[0,2],[0,3],[1/4,0],[1/4,1]]
optionsforT=[[0,1],[0,2],[1/3,0]]
optionsforSSinv=[[2/7,0],[2/7,4],[-1/7,1],[-1/7,3],[-2/7,2],[0,5],[0,6]]
```

```
for i1,i2,i3,i4,i5 in product(optionsfor1,optionsforR,optionsforRS,
    optionsforT,optionsforSSinv):
    for di in range(1,2):
        sols=solve(i1[0]+i2[0]+i3[0]+i4[0]+i5[0]==di,k,
            solution_dict=True)
        for sol in sols:
            if (sol[k] in ZZ and sol[k]>0 and
                mod(sol[k],len(optionsforR))==i2[1] and
                mod(sol[k],len(optionsforRS))==i3[1] and
                mod(sol[k],len(optionsforT))==i4[1] and
                mod(sol[k],len(optionsforSSinv))==i5[1]):
                print "="+str(di)+"␣for␣k="+str(sol[k])
print "Done!"
```

A.7. Finding the invariant curves

The following GAP [GAP15] code computes the invariant curves that satisfy Condition 2.2.15 for the groups $\tilde{G_1}$, $\tilde{G_2}$ and $\tilde{G_3}$. We use the method that is described in 2.4. We only compute curves in the degrees that we found in the previous section of the appendix and that we cannot not rule out by Proposition 2.3.6.

Listing A.7.1: InvariantCurves.g

```
Read("DefineGroups.g");

ActOnPoly:=function(g,p,indets)
return Value(p,indets,g^(-1)*indets);
end;

x:=Indeterminate(Cyclotomics,"x");
y:=Indeterminate(Cyclotomics,"y");
z:=Indeterminate(Cyclotomics,"z");

Print("Invariant␣curves␣for␣G1:\n");

lin:=LinearCharacters(G1tilde);
chi2:=lin[2];
chi3:=lin[3];

CharKer:=KernelOfCharacter(chi2);

Print("We␣get:␣p1=");
p1:=Sum(List(CharKer),g->ActOnPoly(g,x^6,[x,y,z]))/12;
Print("G1tilde␣acts␣on␣p1␣with␣chi3:\n");
Print(Filtered(List(G1tilde,g->ActOnPoly(g,p1,[x,y,z])-(g^chi3)*p1),p->not p
    =0*x));

p9G1:=Sum(List(G1tilde),g->ActOnPoly(g,x^6*y^3,[x,y,z]));
Print("The␣invariant␣curve␣of␣degree␣9␣is␣reducible:\n");
Factors(p9G1);
```

```
p12G1:=Sum(List(G1tilde),g->ActOnPoly(g,x^10*y*z,[x,y,z]));
Print("The␣invariant␣polynomial␣of␣degree␣12␣is␣divisible␣by␣x␣and␣therefore␣
    reducible:");
IsPolynomial(p12G1/x);

Print("\n");
Print("Invariant␣curves␣for␣G2:\n");
p2:=Sum(List(G2tilde),g->ActOnPoly(g,x^6,[x,y,z]))*(2/135);
coeffA:=15/8+(15/8)*E(4)*Sqrt(3)-(9/8)*Sqrt(5)+(3/8)*E(4)*Sqrt(3)*Sqrt(5);
coeffB:=15/8-(15/8)*E(4)*Sqrt(3)+(9/8)*Sqrt(5)+(3/8)*E(4)*Sqrt(3)*Sqrt(5);
coeffC:=15-3*E(4)*Sqrt(3)*Sqrt(5);
Print("p2␣is␣indeed␣of␣the␣form␣given␣in␣Section␣2.2:\n");
x^6+coeffA*x^4*y^2+coeffB*x^2*y^4+y^6+coeffB*x^4*z^2+coeffC*x^2*y^2*z^2+
    coeffA*y^4*z^2+coeffA*x^2*z^4+coeffB*y^2*z^4+z^6-p2;

Print("We␣now␣compute␣the␣invariant␣polynomial␣of␣degree␣45␣(this␣takes␣
    several␣minutes)␣and␣show␣that␣it␣is␣irreducible");
p45G2:=Sum(List(G2tilde),g->ActOnPoly(g,x^41*y^3*z,[x,y,z]));
IsPolynomial(p45G2/x);

Print("\n");
Print("Invariant␣curves␣for␣G3:\n");
p3:=Sum(List(G3tilde),g->ActOnPoly(g,x*y^3,[x,y,z]))/56;
p4:=Sum(List(G3tilde),g->ActOnPoly(g,x^5*y,[x,y,z]))/36;

p21G3:=Sum(List(G3tilde),g->ActOnPoly(g,x^21,[x,y,z]));
Print("The␣invariant␣polynomial␣of␣degree␣21␣is␣reducible␣(the␣factor␣was␣
    found␣using␣Sage):");
IsPolynomial(p21G3/((x^3 - 2*x^2*y - x*y^2 + y^3 - x^2*z + 6*x*y*z - 2*y^2*z
    - 2*x*z^2 - y*z^2 + z^3)));
```

The following Sage [S$^+$14] code verifies the irreducibility of $p_1, \dots, p_4$ by computing the resultant defined in Theorem 1.1.6. Since all resultants are non-zero, the polynomials are irreducible.

Listing A.7.2: Irreducibility.sage

```
K.<sqrtm3>=NumberField(x^2+3)
L.<sqrt5>=K.extension(x^2-5)
R.<X,Y,Z>=PolynomialRing(L)

p1=X^6 - 10*X^3*Y^3 + Y^6 - 10*X^3*Z^3 - 10*Y^3*Z^3 + Z^6

a=15/8+(15/8)*sqrtm3-(9/8)*sqrt5+(3/8)*sqrtm3*sqrt5
b=15/8-(15/8)*sqrtm3+(9/8)*sqrt5+(3/8)*sqrtm3*sqrt5
c=15-3*sqrtm3*sqrt5
p2=X^6+a*X^4*Y^2+b*X^2*Y^4+Y^6+b*X^4*Z^2+c*X^2*Y^2*Z^2+a*Y^4*Z^2+a*X^2*Z^4+b*
    Y^2*Z^4+Z^6

p3=X*Y^3+X^3*Z+Y*Z^3

p4=X^5*Y+X*Z^5+Y^5*Z-5*X^2*Y^2*Z^2

print(R.macaulay_resultant(diff(p1,X),diff(p1,Y),diff(p1,Z)))
```

```
print(R.macaulay_resultant(diff(p2,X),diff(p2,Y),diff(p2,Z)))
print(R.macaulay_resultant(diff(p3,X),diff(p3,Y),diff(p3,Z)))
print(R.macaulay_resultant(diff(p4,X),diff(p4,Y),diff(p4,Z)))
```

A.8. The Jacobian of C_1

The following GAP [GAP15] code verifies (2.5.2):

Listing A.8.1: QuotientCurveGroupC1.g

```
Read("DefineGroups.g");
P1:=-[[1,0,0],[0,0,1],[0,1,0]];
P2:=[[0,1,0],[0,0,1],[1,0,0]];
Htilde:=Group(P1,P2);
IsSubgroup(G1tilde,Htilde);
G1:=G1tilde/Group(E(3)*IdentityMat(3));
hom:=NaturalHomomorphism(G1);;
H:=Image(hom,Htilde);
irr:=Irr(G1);
chiHolDiff:=irr[4]+irr[9];
chi9:=irr[9];
Print(ScalarProduct(RestrictedClassFunction(chiHolDiff,H),TrivialCharacter(H)
    ));
Print(ScalarProduct(RestrictedClassFunction(chi9,H),TrivialCharacter(H)));
```

Using the notation from the proof of Proposition 2.5.2, we calculate an equation of C_1/H: the subring of invariants $\mathbb{C}\left[X,Y,Z\right]^H$ is generated by the elementary symmetric polynomials s_1, s_2 and s_3. We have

$$p_1 = s_1^6 - 6s_1^4 s_2 + 6s_1^3 s_3 + 9s_1^2 s_2^2 + 18 s_1 s_2 s_3 - 12 s_2^3 - 27 s_3^2.$$

Take new coordinates

$$\begin{aligned} X' &:= X + Y + Z, \\ Y' &:= Y, \\ Z' &:= Z. \end{aligned}$$

In these new coordinates we have $s_1 = X'$. After dehomogenizing p_1 by setting $X' = 1$ we get

$$p_1'(Y', Z') = 1 - 6{s'}_2 + 6{s'}_3 + 9{s'}_2^2 + 18{s'}_2{s'}_3 - 12{s'}_2^3 - 27{s'}_3^2,$$

where ${s'}_2$ and ${s'}_3$ are the dehomogenizations of s_2 and s_3. This equation can be readily verified by expanding ${s'}_1, {s'}_2$ and ${s'}_3$. Denote by C_1' the affine curve defined by p_1'. An action of H on C_1' is induced by the action of H on C_1 since affine patch $X' \neq 0$ is H-invariant. We have

$A(C_1')^H = \mathbb{C}[s_2', s_3']/(p_1') \cong \mathbb{C}[X,Y]/(q_0)$ where $q_0 := 1 - 6X + 6Y + 9X^2 + 18XY - 12X^3 - 27Y^2$

as s_2' and s_3' are algebraically independent. We find a nicer form for $V(q_0)$ in Sage:

Listing A.8.2: NormalFormEllipticCurveC1.sage

```
R.<X,Y>=PolynomialRing(QQ)
E=1-6*X+6*Y+9*X^2+18*X*Y-12*X^3-27*Y^2
J=Jacobian(E)
print("J:")
print(J)
E2=Y^2-X^3-1
J2=Jacobian(E2)
print("J2:")
print(J2)
print("The two curves are isomorphic:")
print(J.j_invariant()) # Outputs 0
print(J2.j_invariant()) # Outputs 0
```

Therefore

$$A(C_1')^H \cong \mathbb{C}[X,Y]/(q_1') \text{ where } q_1' := Y^2 - X^3 - 1.$$

By homogenizing and using Proposition 1.4.5, we get the following curve of genus 1 that is isomorphic to C_1/H:

$$q_1 := Y^2Z - X^3 - Z^3.$$

The j-invariant of this elliptic curve is 0.

The following Maple worksheet by Professor Pink shows that the two-dimensional factor of $\mathrm{Jac}(C_1)$ is not isogenous to E'^2 for any elliptic curve E' defined over $\mathbb{Q}$:

```
> restart:
```

Consider the curve C in ℙ^2(ℂ) defined by this polynomial:

```
> F := X^6+Y^6+Z^6-10*X^3*Y^3-10*X^3*Z^3-10*Y^3*Z^3;
```

$$F := X^6 - 10\,X^3\,Y^3 - 10\,X^3\,Z^3 + Y^6 - 10\,Y^3\,Z^3 + Z^6$$

This calculation shows that C is nonsingular:

```
> solve([diff(F,X),diff(F,Y),diff(F,Z)],[X,Y,Z]);
```

$$[[X = 0, Y = 0, Z = 0]]$$

Having degree 6, it therefore has genus 10.

Where does C have good reduction?

Consider a prime p. By symmetry C is smooth over 𝔽_p iff the affine part with Z=1 is smooth.

```
> F1 := subs(Z=1,F);
```

$$F1 := X^6 - 10\,X^3\,Y^3 + Y^6 - 10\,X^3 - 10\,Y^3 + 1$$

So we must find the primes p modulo which the equations

```
> F1;
  F1X := factor(diff(F1,X));
  F1Y := factor(diff(F1,Y));
```

$$X^6 - 10\,X^3\,Y^3 + Y^6 - 10\,X^3 - 10\,Y^3 + 1$$

$$F1X := 6\,X^2\,(X^3 - 5\,Y^3 - 5)$$

$$F1Y := -6\,Y^2\,(5\,X^3 - Y^3 + 5)$$

have no common solution.

Modulo 2 the equation F already factors

```
Factor(F) mod 2;
```

$$(X^3 + Y^3 + Z^3)^2$$

Note that the elliptic curve $X^3 + Y^3 + Z^3$=0 has good reduction at 2.

Modulo 3 the equation already factors

```
> Factor(F) mod 3;
```

$$(X + Y + Z)^6$$

So assume p>3. A common solution in characteristic p with X=0 is one of

```
> F10  := subs(X=0,F1);
  F1X0 := subs(X=0,F1Y);
```

$$F10 := Y^6 - 10\,Y^3 + 1$$

$$F1X0 := -6\,Y^2\,(-Y^3 + 5)$$

```
> ifactor(resultant(F10,F1X0,Y));
```

$$-(2)^{15}\,(3)^9$$

So there is none. By symmetry also none with Y=0. Simplify the remaining equations

```
> G1 := subs([X=U^(1/3),Y=V^(1/3)],F1);
  G2 := subs([X=U^(1/3),Y=V^(1/3)],F1X/X^2/6);
  G3 := subs([X=U^(1/3),Y=V^(1/3)],F1Y/Y^2/6);
```

$$G1 := U^2 - 10\,U\,V + V^2 - 10\,U - 10\,V + 1$$

$$G2 := U - 5\,V - 5$$

$$G3 := -5\,U + V - 5$$

```
> Vsol := solve(G3,V);
```

$$Vsol := 5\,U + 5$$

```
> G1s := factor(subs(V=Vsol,G1));
  G2s := factor(subs(V=Vsol,G2));
```

$$G1s := -12\,(U + 2)\,(2\,U + 1)$$

$$G2s := -24\,U - 30$$

```
> ifactor(resultant(G1s,G2s,U));
```

$$(2)^5\,(3)^5$$

So F1 and F1X and F1Y have no common zeros over any field of characteristic >3.

Conclusion: C has good reduction outside p=2 and p=3.

Count rational points over 𝔽_p for p>3:

```
> PtsC := proc(p)
    nops([msolve(subs(Z=1,F),p)])+
    nops([msolve(subs([Z=0,Y=1],F),p)]);
  end proc:
```

Dito for the elliptic curve

```
> E := X^3+Z^3-Y^2*Z;
```

$$E := X^3 - Y^2 Z + Z^3$$

```
> PtsE := proc(p)
    nops([msolve(subs(Z=1,E),p)])+
    nops([msolve(subs([Z=0,Y=1],E),p)]);
  end proc:
```

The jacobian of C has a factor isogenous to E^8. Suppose the rest is isogenous to E1^2 for an elliptic curve E1. Then E1 must also have good reduction at all p>3. Let PtsE(p) denote the number of 𝔽_p-rational points of E1. Then by the Lefschetz trace formula we have

p+1-PtsC(p) = trace(Frob_p|H^1(C))
p+1-PtsE(p) = trace(Frob_p|H^1(E))
p+1-PtsE1(p) = trace(Frob_p|H^1(E1))

and hence

p+1-PtsC(p) = 8*(p+1-PtsE(p)) + 2*(p+1-PtsE1(p)).

Thus PtsE1(p) is given by

```
> PtsE1 := proc(p)
    p+1 - ( (p+1-PtsC(p)) - 8*(p+1-PtsE(p)) )/2
  end proc:
> for i from 3 to 20 do [ithprime(i),PtsC(ithprime(i)),PtsE
  (ithprime(i)),PtsE1(ithprime(i))] od;
```

[5, 6, 6, 6]
[7, 0, 12, -12]
[11, 12, 12, 12]
[13, 54, 12, 42]
[17, 18, 18, 18]
[19, 72, 12, 78]
[23, 24, 24, 24]
[29, 30, 30, 30]
[31, 0, 36, 0]
[37, 126, 48, 42]
[41, 42, 42, 42]
[43, 0, 36, 54]
[47, 48, 48, 48]
[53, 54, 54, 54]
[59, 60, 60, 60]
[61, 54, 48, 114]
[67, 0, 84, -30]
[71, 72, 72, 72]

For p=7 or 67 we get PtsE1(p)<0, which is a contradiction. Conclusion: The other factor of the jacobian of C is not isogenous to E1^2 for an elliptic curve E1 over ℚ.

By irreducibility it cannot have any elliptic curve E1 as factor; so it is a simple abelian surface. It might still conceivably be isogenous to E1^2 over a finite extension of ℚ, but that seems unlikely.

A.9. The Jacobian of C_2

The following Sage [S+14] code verifies that the curve of genus 1 defined by

$$X^3 + aX^2Y + bXY^2 + X^3 + bX^2Z + cXYZ + aY^2Z + aXZ^2 + bYZ^2 + Z^3 = 0$$

is isomorphic to the curve defined by

$$Y^2Z = X^3 + \left(\frac{1053}{2}i\sqrt{15} + \frac{13365}{2}\right)XZ^2 + \left(54675i\sqrt{15} - 172773\right)Z^3.$$

and calculates the j-invariant. The constant a, b and c are defined as in the definition of p_2 in 2.4.

Listing A.9.1: NormalFormEllipticCurveC2.sage

```
K.<sqrtm3>=NumberField(x^2+3)
L.<sqrt5>=K.extension(x^2-5)
R.<X,Y,Z>=PolynomialRing(L)

a=15/8+(15/8)*sqrtm3-(9/8)*sqrt5+(3/8)*sqrtm3*sqrt5
b=15/8-(15/8)*sqrtm3+(9/8)*sqrt5+(3/8)*sqrtm3*sqrt5
c=15-3*sqrtm3*sqrt5
E=X^3+a*X^2*Y+b*X*Y^2+Y^3+b*X^2*Z+c*X*Y*Z+a*Y^2*Z+a*X*Z^2+b*Y*Z^2+Z^3

J=Jacobian(E)
print(J) #Outputs:
#Elliptic Curve defined by y^2 = x^3 + (1053/2*sqrtm3*sqrt5+13365/2)*x +
#(54675*sqrtm3*sqrt5-172773) over Number Field in sqrt5 with defining#
# polynomial x^2 - 5 over its base field

print(J.j_invariant())
#Outputs: 69255/128*sqrtm3*sqrt5 - 122175/128
```

A.10. The Jacobian of C_3

We use the notation from the proof of Proposition 2.7.2. We calculate an equation for C_3/H.

Proposition A.10.1. *Consider the action of A_n on $\mathbb{C}[X_1, \dots, X_n]$ by permutation of the variables. Then*

$$\mathbb{C}[X_1, \dots, X_n]^{A_n} = \mathbb{C}[s_1, \dots, s_n, d_n],$$

where $s_1, \dots, s_n$ are the elementary symmetric polynomials in $X_1, \dots, X_n$ and

$$d_n := \prod_{1 \le i < j \le n} (X_i - X_j).$$

Proof. The elementary symmetric polynomials and d_n are clearly invariant under the action of A_n. Let $p \in \mathbb{C}[X_1, \dots, X_n]^{A_n}$. Let $1 \le i < j \le n$ and let T_{ij} denote the transposition that exchanges i and j. Let

$$p_s := \frac{1}{2}(p + T_{ij}p) \qquad \text{and} \qquad p_a := \frac{1}{2}(p - T_{ij}p).$$

Since T_{ij} has order 2 we get that $T_{ij}p_s = p_s$ and $T_{ij}p_a = -p_a$. It follows that $(X_i - X_j)|p_a$. For any $1 \leq k < l \leq n$ we have $T_{ij}T_{kl}p = p$ since $T_{ij}T_{kl} \in A_n$ and therefore $T_{kl}p = (T_{ij})^{-1}p = T_{ij}p$. It follows that p_s is symmetric and that $(X_k - X_l)|p_a$. It follows that $d_n|p_a$. The polynomial p_a/d_n is symmetric because

$$T_{kl}(p_a/d_n) = (T_{kl}p_a)/(T_{kl}d_n) = -p_a/(-d_n) = p_a/d_n$$

and $\{T_{ab}|\ 1 \leq a < b \leq n\}$ generates S_n. Thus, we can write $p_a = sd_n$ where s is symmetric. We have $p = p_s + p_a = p_s + sd_n$. This finishes the proof since p_s is symmetric. □

Because p_3 is invariant under the action of the alternating group A_3 by permuting variables, we have $p_3 \in \mathbb{C}[s_1, s_2, s_3, d_3]$ by Proposition A.10.1, where s_1, s_2, s_3 and d_3 are defined as in the proposition. Take new coordinates

$$\begin{aligned} X' &:= X + Y + Z. \\ Y' &:= Y, \\ Z' &:= Z. \end{aligned}$$

We now have $s_1 = X'$. The affine patch defined by $X' \neq 0$ is invariant under the action of A_3. Denote by p'_3, s'_2, s'_3, d'_3 the dehomogenizations of p_3, s_2, s_3, d_3 with $X' = 1$. Denote by C'_3 the affine patch of C_3 defined by p'_1. We calculate $A(C'_3)^{A_3} = \mathbb{C}[s'_2, s'_3, d'_3]/(p'_3)$ using Singular [DGPS12]:

Listing A.10.2: QuotientCurveKleinQuartic.sing

```
LIB "finvar.lib";
ring R= 0,(x,y,z),dp;
poly c=(1-y-z)*y^3+(1-y-z)^3*z+y*z^3;
qring S=c;
ideal invar=-y^2-y*z-z^2+y+z,-y^2*z-y*z^2+y*z,2*y^3+3*y^2*z-3*y*z^2-2*z^3-3*y
    ^2+3*z^2+y-z;

ring T=0,(x,y,z),dp;

setring S;
map phi=T,invar;
alg_kernel(phi,T,"kerPhi");

setring T;
print(kerPhi);  //We get:
//kerPhi[1]=40xy-54y2+4xz-2z2-x-7y+z
//kerPhi[2]=2x2-x+y+z
//kerPhi[3]=4374y3+660x2z-324xyz+120xz2+162yz2+20x2+101xy+687y2-430xz+1569yz
    +410z2-160y

ideal ker2=kerPhi[1],kerPhi[2];
// We have ker2==kerPhi :
reduce(ker2,std(kerPhi)); //ker2 is in kerPhi
reduce(kerPhi,std(ker2)); //kerPhi is in ker2
```

We get that

$$A(C_3')^{A_3} = \mathbb{C}[s_2', s_3', d_3']/(p_3') \cong \mathbb{C}[X,Y,Z]/(q_1, q_2),$$

where

$$q_1 := 40XY - 54Y^2 + 4XZ - 2Z^2 - X - 7Y + Z$$
$$q_2 := 2X^2 - X + Y + Z$$

After homogenizing again, by Proposition 1.4.5, we have that

$$E = C_3/H \cong V\big((40XY - 54Y^2 + 4XZ - 2Z^2 - XW - 7YW + ZW, 2X^2 - XW + YW + ZW)\big).$$

Following Section 1.4.3 in [Con99], we find that this intersection of quadrics is isomorphic to the curve of genus 1 defined by the following equation that we obtain by eliminating W:

$$(-X - 7Y + Z)(2X^2) - (-X + Y + Z)(40XY - 54Y^2 + 4XZ - 2Z^2) = 0$$

The following Sage code calculates a Weierstrass form E and its j-invariant:

Listing A.10.3: NormalFormEllipticCurveC3.sage

```
R.<X,Y,Z>=PolynomialRing(QQ)
E=(-X-7*Y+Z)*(2*X^2)-(-X+Y+Z)*(40*X*Y-54*Y^2+4*X*Z-2*Z^2)
J=Jacobian(E)
print(J)#Outputs:
#Elliptic Curve defined by y^2 = x^3 - 8960*x - 401408 over Rational Field

print(J.j_invariant()) # Outputs: -3375
```

We obtain the following equation for E:

$$Y^2Z = X^3 - 8960XZ^2 - 401408Z^3.$$

We have $j(E) = -3^3 \cdot 5^3$.

A.11. The Jacobian of C_4

We verify (2.8.1) using GAP and verify that there is no subgroup $K < \mathrm{Aut}(C_4)$ such that

$$\left\langle \mathrm{Res}_K\left(\chi_{H^0(C_4,\Omega_{C_4})}\right), \mathrm{Res}_K(\chi_1)\right\rangle = 1 = \langle \mathrm{Res}_K(\chi_2), \mathrm{Res}_K(\chi_1)\rangle.$$

Additionally, we verify (2.8.2).

Listing A.11.1: QuotientCurveGroupC4.g

```
Read("DefineGroups.g");
W:=(T*S)*R*(T*S)^2;
V:=(T*S^3)*(S*R)*(T*S^3)^(-1);
H1:=Group(W,V^2);;
```

```
G3:=G3tilde; #G3 is isomorphic to G3tilde
irr:=Irr(G3);
CharHolDiff:=irr[2]+irr[5];
chi5:=irr[5];
Print(ScalarProduct(RestrictedClassFunction(CharHolDiff,H1),TrivialCharacter(
    H1))); # Outputs 1
Print(ScalarProduct(RestrictedClassFunction(irr[5],H1),TrivialCharacter(H1)))
    ; # Outputs 1

subgroups:=List(ConjugacyClassesSubgroups(G3),Representative);
# Since characters are constant on conjugacy classes, it suffices to compute
    the inner products for one subgroup from each conjugacy class of
    subgroups:
Print(List(subgroups,S->[ScalarProduct(RestrictedClassFunction(CharHolDiff,S)
    ,TrivialCharacter(S)),ScalarProduct(RestrictedClassFunction(irr[2],S),
    TrivialCharacter(S))]));
# We see that there is no subgroup with both inner products equal to 1.

H2:=Group(V);;
Print(ScalarProduct(RestrictedClassFunction(CharHolDiff,H2),TrivialCharacter(
    H2))); # Outputs 2
```

We use the notation from the proof of Proposition 2.8.2. The following Maple worksheet by Professor Pink calculates the quotient $E_1 := C_4/H_1$ and a elliptic curve E_2, not isogenous to E_1, that C_4 maps onto:

Consider the simple group of order 168.

According to [Blichfeldt, H. F. Finite collineation groups. University of Chicago Press, Chicago, 1917], §82, p.113, the simple group of order 168 embeds into GL_3(ℂ) by these generators:

```
> alias(zeta=RootOf(X^6+X^5+X^4+X^3+X^2+X+1,X)):
  alpha := zeta^4-zeta^3:
  beta  := zeta^2-zeta^5:
  unprotect('gamma'):
  gamma := zeta^1-zeta^6:
  h := zeta+zeta^2+zeta^4-zeta^3-zeta^5-zeta^6:
  S := [a=zeta*a,b=zeta^2*b,c=zeta^4*c]:
  T := [a=b,b=c,c=a]:
  R := [a=h*(alpha*a+beta*b+gamma*c)/7,
        b=h*(beta*a+gamma*b+alpha*c)/7,
        c=h*(gamma*a+alpha*b+beta*c)/7]:
```

with the relations S^7=T^3=R^2=(RS)^4=1, TST^(-1)=S^4, TR=RT^2:

```
> simplify(subs(R,subs(R,[a,b,c])));
  simplify(subs(R,subs(S,subs(R,subs(S,subs(R,subs(S,subs(R,subs
  (S,[a,b,c])))))))));
  simplify(subs(T,subs(S,subs(T,subs(T,subs(S,subs(S,subs(S,[a,b,
  c]))))))));
  simplify(subs(T,subs(R,subs(T,subs(R,[a,b,c])))));
```

$$[a, b, c]$$
$$[a, b, c]$$
$$[a, b, c]$$
$$[a, b, c]$$

The given representation of dimension 3 is one half of a cuspidal representation of GL(2,7) of dimension 6, and it requires no central extension.

The following equation is invariant under G:

```
> L6 := a^5*b-5*a^2*b^2*c^2+a*c^5+b^5*c;
  simplify(subs(R,L6))-L6;
  simplify(subs(S,L6))-L6;
  simplify(subs(T,L6))-L6;
```

$$L6 := a^5\,b - 5\,a^2\,b^2\,c^2 + a\,c^5 + b^5\,c$$

$$0$$

$$0$$

$$0$$

Consider the elements R and W := (TS)R(TS)^2 and V := (TS^3)(SR)(TS^3)^(-1):

```
> Rop := proc(f)
   simplify(subs(R,f))
  end proc:
  Wop := proc(f)
   simplify(subs(T,subs(S,subs(R,subs(T,subs(S,subs(T,subs(S,f)))
  ))))
  end proc:
  Vop := proc(f)
   simplify(subs(T,subs(S,subs(S,subs(S,subs(S,subs(R,subs(S,subs
  (S,subs(S,subs(S,subs(T,subs(T,f))))))))))))
  end proc:
```

They are non-trivial

```
> Rop([a,b,c]);
  Wop([a,b,c]);
  Vop([a,b,c]);
```

$$\left[-\frac{2}{7}\zeta^5 a-\frac{1}{7}\zeta^5 b+\frac{3}{7}\zeta^5 c-\frac{3}{7}\zeta^4 a+\frac{2}{7}\zeta^4 b+\frac{1}{7}\zeta^4 c-\frac{3}{7}\zeta^3 a+\frac{2}{7}\zeta^3 b+\frac{1}{7}\zeta^3 c\right.$$
$$-\frac{2}{7}\zeta^2 a-\frac{1}{7}\zeta^2 b+\frac{3}{7}\zeta^2 c-\frac{4}{7}a-\frac{2}{7}b-\frac{1}{7}c,\ -\frac{1}{7}\zeta^5 a+\frac{3}{7}\zeta^5 b-\frac{2}{7}\zeta^5 c$$
$$+\frac{2}{7}\zeta^4 a+\frac{1}{7}\zeta^4 b-\frac{3}{7}\zeta^4 c+\frac{2}{7}\zeta^3 a+\frac{1}{7}\zeta^3 b-\frac{3}{7}\zeta^3 c-\frac{1}{7}\zeta^2 a+\frac{3}{7}\zeta^2 b$$
$$-\frac{2}{7}\zeta^2 c-\frac{2}{7}a-\frac{1}{7}b-\frac{4}{7}c,\ \frac{3}{7}\zeta^5 a-\frac{2}{7}\zeta^5 b-\frac{1}{7}\zeta^5 c+\frac{1}{7}\zeta^4 a-\frac{3}{7}\zeta^4 b$$
$$+\frac{2}{7}\zeta^4 c+\frac{1}{7}\zeta^3 a-\frac{3}{7}\zeta^3 b+\frac{2}{7}\zeta^3 c+\frac{3}{7}\zeta^2 a-\frac{2}{7}\zeta^2 b-\frac{1}{7}\zeta^2 c-\frac{1}{7}a-\frac{4}{7}b$$
$$\left.-\frac{2}{7}c\right]$$

$$\left[-\frac{1}{7}\zeta^5 a-\frac{3}{7}\zeta^5 b-\frac{4}{7}\zeta^5 c+\frac{2}{7}\zeta^4 a-\frac{4}{7}\zeta^4 b+\frac{2}{7}\zeta^3 a-\frac{3}{7}\zeta^3 b-\frac{2}{7}\zeta^3 c-\frac{1}{7}\zeta^2 a\right.$$
$$-\frac{3}{7}\zeta^2 c-\frac{2}{7}\zeta b-\frac{3}{7}\zeta c-\frac{2}{7}a-\frac{2}{7}b-\frac{2}{7}c,\ \frac{2}{7}\zeta^5 a-\frac{2}{7}\zeta^5 b+\frac{3}{7}\zeta^5 c$$
$$-\frac{1}{7}\zeta^4 a-\frac{3}{7}\zeta^4 b+\frac{3}{7}\zeta^4 c-\frac{2}{7}\zeta^3 a-\frac{3}{7}\zeta^3 b-\frac{1}{7}\zeta^2 a-\frac{2}{7}\zeta^2 b+\frac{1}{7}\zeta^2 c$$
$$+\frac{2}{7}\zeta a-\frac{1}{7}\zeta c-\frac{4}{7}b+\frac{1}{7}c,\ \frac{2}{7}\zeta^5 b+\frac{3}{7}\zeta^5 c+\frac{1}{7}\zeta^4 a+\frac{1}{7}\zeta^4 b+\frac{1}{7}\zeta^4 c$$
$$+\frac{3}{7}\zeta^3 a+\frac{4}{7}\zeta^3 b+\frac{1}{7}\zeta^3 c-\frac{1}{7}\zeta^2 a+\frac{4}{7}\zeta^2 b+\frac{3}{7}\zeta^2 c+\frac{3}{7}\zeta a+\frac{1}{7}\zeta b+\frac{1}{7}a$$
$$\left.+\frac{2}{7}b-\frac{1}{7}c\right]$$

$$\left[\frac{1}{7}\zeta^5 a+\frac{1}{7}\zeta^5 b+\frac{1}{7}\zeta^5 c-\frac{1}{7}\zeta^4 a+\frac{4}{7}\zeta^4 b+\frac{3}{7}\zeta^4 c+\frac{1}{7}\zeta^3 a+\frac{2}{7}\zeta^3 b-\frac{1}{7}\zeta^3 c\right.$$
$$+\frac{2}{7}\zeta^2 b+\frac{3}{7}\zeta^2 c+\frac{3}{7}\zeta a+\frac{4}{7}\zeta b+\frac{1}{7}\zeta c+\frac{3}{7}a+\frac{1}{7}b,\ \frac{1}{7}\zeta^5 a-\frac{1}{7}\zeta^5 b$$
$$-\frac{2}{7}\zeta^5 c+\frac{4}{7}\zeta^4 a-\frac{1}{7}\zeta^4 b+\frac{2}{7}\zeta^3 a-\frac{1}{7}\zeta^3 c+\frac{2}{7}\zeta^2 a+\frac{2}{7}\zeta^2 b+\frac{2}{7}\zeta^2 c$$
$$+\frac{4}{7}\zeta a-\frac{2}{7}\zeta b+\frac{2}{7}\zeta c+\frac{1}{7}a+\frac{2}{7}b-\frac{1}{7}c,\ \frac{1}{7}\zeta^5 a-\frac{2}{7}\zeta^5 b+\frac{3}{7}\zeta^4 a+\frac{2}{7}\zeta^4 c$$

$$-\frac{1}{7}\zeta^3 a-\frac{1}{7}\zeta^3 b-\frac{1}{7}\zeta^3 c+\frac{3}{7}\zeta^2 a+\frac{2}{7}\zeta^2 b-\frac{2}{7}\zeta^2 c+\frac{1}{7}\zeta a+\frac{2}{7}\zeta b-\frac{1}{7}\zeta c-\frac{1}{7} b+\frac{2}{7} c\Big]$$

and satisfy the relations
R^2=1
W^2=1
V^2=R
(WV)^2=1

```
> Rop(Rop([a,b,c]));
  Wop(Wop([a,b,c]));
  Rop(Vop(Vop([a,b,c])));
  Wop(Vop(Wop(Vop([a,b,c]))));
```

$$[a, b, c]$$
$$[a, b, c]$$
$$[a, b, c]$$
$$[a, b, c]$$

Thus <R,W> is a Klein 4 group, and V has order 4, and together they generate a D4 group H := <V> ⋊ <W>.

To find simpler representations of these elements we change coordinates:

```
> Wfix1 := simplify(solve(Wop([a,b,c])-[a,b,c]));
  Wfix2 := simplify(solve(Wop([a,b,c])+[a,b,c]));
```

$$Wfix1 := \{a = a, b = -\zeta^2 a (\zeta^2 + \zeta + 1), c = \zeta a (\zeta^2 + 1)\}$$

$$Wfix2 := \{a = a, b = -\zeta^5 a - \zeta^5 c - \zeta^4 c - \zeta^2 c - \zeta a - c, c = c\}$$

```
> Vfix1 := simplify(solve(Vop([a,b,c])-[a,b,c]));
  Vfix2 := simplify(solve(Vop([a,b,c])+[a,b,c]));
  Vfix3 := simplify(solve(Vop([a,b,c])-[I*a,I*b,I*c]));
  Vfix4 := simplify(solve(Vop([a,b,c])+[I*a,I*b,I*c]));
```

$$Vfix1 := \{a = a, b = -\zeta^5 a - \zeta^2 a - a, c = \zeta^2 a (\zeta^3 + 1)\}$$

$$Vfix2 := \{a = 0, b = 0, c = 0\}$$

$$Vfix3 := \{a = a, b = a (\zeta^5 - I\zeta^4 + \zeta^4 - I\zeta^3 + \zeta^3 - 2I\zeta^2 + \zeta^2 - 2I\zeta - I), c = -a (\zeta^5 + 2\zeta^4 - 2I\zeta^3 + 2\zeta^3 - 2I\zeta^2 + \zeta^2 - 2I\zeta - 1 - I)\}$$

$$Vfix4 := \{a = a, b = a (\zeta^5 + I\zeta^4 + \zeta^4 + I\zeta^3 + \zeta^3 + 2I\zeta^2 + \zeta^2 + 2I\zeta + I), c = -a (\zeta^5 + 2\zeta^4 + 2I\zeta^3 + 2\zeta^3 + 2I\zeta^2 + \zeta^2 + 2I\zeta - 1 + I)\}$$

```
> basu := subs(a=u,subs(Vfix1,[a,b,c]));
  basv := subs(a=v,subs(Wfix1,[a,b,c]));
  basw := simplify(subs(a=w,subs(Wfix1,Vop([a,b,c]))));
```

$$basu := [u, -\zeta^5 u - \zeta^2 u - u, \zeta^2 u (\zeta^3 + 1)]$$

$$basv := [v, -\zeta^2 v (\zeta^2 + \zeta + 1), \zeta v (\zeta^2 + 1)]$$

$$basw := [w (\zeta^5 + \zeta^3 + \zeta^2 + \zeta + 1), w (\zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1), w (\zeta^2 + \zeta + 1)]$$

```
> mysub := basu+basv+basw:
  mysub := [a=mysub[1],b=mysub[2],c=mysub[3]];
```

$$mysub := [a = w (\zeta^5 + \zeta^3 + \zeta^2 + \zeta + 1) + v + u, b = w (\zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1) - \zeta^2 v (\zeta^2 + \zeta + 1) - \zeta^5 u - \zeta^2 u - u, c = w (\zeta^2 + \zeta + 1) + \zeta v (\zeta^2 + 1) + \zeta^2 u (\zeta^3 + 1)]$$

This change of coordinates transforms the equation L6 into this one:

```
> L6a := collect(simplify(subs(mysub,L6)),[u,v,w]);
```

$$L6a := (-42\zeta^5 - 28\zeta^4 - 28\zeta^3 - 42\zeta^2 - 70) u^6 + ((-105\zeta^5 + 35\zeta^4 - 105\zeta^3 - 70\zeta - 35) v^2 + (-105\zeta^5 + 35\zeta^4 - 105\zeta^3 - 70\zeta - 35) w^2) u^4 + ((70\zeta^5 - 35\zeta^4 + 35\zeta^3 + 35\zeta + 70) v^4 + (-280\zeta^5 + 280\zeta^4 - 140\zeta^3 + 280\zeta^2 - 140\zeta - 140) w^2 v^2 + (70\zeta^5 - 35\zeta^4 + 35\zeta^3 + 35\zeta + 70) w^4) u^2 + (-28\zeta^5 - 42\zeta^4 - 70\zeta^3 - 42\zeta^2 - 28\zeta) v^6 + (-140\zeta^5 - 105\zeta^4 - 70\zeta^3 + 35\zeta^2 + 105\zeta + 35) w^2 v^4 + (-140\zeta^5 - 105\zeta^4 - 70\zeta^3 + 35\zeta^2 + 105\zeta + 35) w^4 v^2 + (-28\zeta^5 - 42\zeta^4 - 70\zeta^3 - 42\zeta^2 - 28\zeta) w^6$$

This is indeed invariant under the substitutions corresponding to V and W:

```
> simplify(subs([u=u,v=w,w=-v],L6a)-L6a);
  simplify(subs([u=-u,v=v,w=-w],L6a)-L6a);
```

$$0$$

$$0$$

Simplify coefficients:

```
> L6b := collect(simplify(L6a/coeff(L6a,u,6)),[u,v,w]);
```

$$L6b := u^6 + \left(\left(10\,\zeta^2 + \frac{5}{2} + \frac{15}{2}\,\zeta + \frac{5}{2}\,\zeta^5 + 5\,\zeta^4 + 10\,\zeta^3\right) v^2 + \left(10\,\zeta^2 + \frac{5}{2} + \frac{15}{2}\,\zeta + \frac{5}{2}\,\zeta^5 + 5\,\zeta^4 + 10\,\zeta^3\right) w^2\right) u^4 + \left(\left(\frac{55}{2}\,\zeta^2 - \frac{15}{2} + 5\,\zeta + 20\,\zeta^5 + 40\,\zeta^4 + \frac{85}{2}\,\zeta^3\right) v^4 + \left(-80\,\zeta^5 - 210\,\zeta^4 - 280\,\zeta^3 - 250\,\zeta^2 - 130\,\zeta - 20\right) w^2 v^2 + \left(\frac{55}{2}\,\zeta^2 - \frac{15}{2} + 5\,\zeta + 20\,\zeta^5 + 40\,\zeta^4 + \frac{85}{2}\,\zeta^3\right) w^4\right) u^2 + \left(25\,\zeta^5 + 56\,\zeta^4 + 70\,\zeta^3 + 56\,\zeta^2 + 25\,\zeta\right) v^6 + \left(-105 - \frac{155}{2}\,\zeta^2 - 140\,\zeta + \frac{195}{2}\,\zeta^5 + \frac{225}{2}\,\zeta^4 + 35\,\zeta^3\right) w^2 v^4 + \left(-105 - \frac{155}{2}\,\zeta^2 - 140\,\zeta + \frac{195}{2}\,\zeta^5 + \frac{225}{2}\,\zeta^4 + 35\,\zeta^3\right) w^4 v^2 + \left(25\,\zeta^5 + 56\,\zeta^4 + 70\,\zeta^3 + 56\,\zeta^2 + 25\,\zeta\right) w^6$$

```
> L6b4 := factor(coeff(L6b,u,4));
  L6b2 := factor(coeff(L6b,u,2));
  L6b0 := factor(coeff(L6b,u,0));
```

$$L6b4 := \frac{5}{2}\left(4\,\zeta^2 + 1 + 3\,\zeta + \zeta^5 + 2\,\zeta^4 + 4\,\zeta^3\right)\left(v^2 + w^2\right)$$

$$L6b2 := \frac{5}{2}\left(11\,\zeta^2 - 3 + 2\,\zeta + 8\,\zeta^5 + 16\,\zeta^4 + 17\,\zeta^3\right)\left(2\,\zeta^4\,v^2\,w^2 + 2\,\zeta^2\,v^2\,w^2 + 2\,\zeta\,v^2\,w^2 + v^4 - 4\,v^2\,w^2 + w^4\right)$$

$$L6b0 := \frac{1}{2}\,\zeta\left(25\,\zeta^4 + 56\,\zeta^3 + 70\,\zeta^2 + 56\,\zeta + 25\right)\left(v^2 + w^2\right)\left(5\,\zeta^4\,v^2\,w^2 + 5\,\zeta^2\,v^2\,w^2 + 5\,\zeta\,v^2\,w^2 + 2\,v^4 + 3\,v^2\,w^2 + 2\,w^4\right)$$

```
> t2 := coeff(L6b4,v,2)/5;
  t4 := coeff(L6b2,v,4)/5;
  t6 := coeff(L6b0,v,6);
```

$$t2 := 2\,\zeta^2 + \frac{1}{2} + \frac{3}{2}\,\zeta + \frac{1}{2}\,\zeta^5 + \zeta^4 + 2\,\zeta^3$$

$$t4 := \frac{11}{2}\,\zeta^2 - \frac{3}{2} + \zeta + 4\,\zeta^5 + 8\,\zeta^4 + \frac{17}{2}\,\zeta^3$$

$$t6 := \zeta\left(25\,\zeta^4 + 56\,\zeta^3 + 70\,\zeta^2 + 56\,\zeta + 25\right)$$

```
> simplify(t4/t2^2);
  simplify(t6/t2^3);
```

$$-\zeta\left(\zeta^3 + \zeta + 1\right)$$

$$\zeta^4 + \zeta^2 + \zeta - 2$$

```
> L6c := collect(simplify(subs(u=u*t,L6b)),[u,v,w]);
```

$$L6c := u^6\,t^6 + \left(\left(10\,\zeta^2 + \frac{5}{2} + \frac{15}{2}\,\zeta + \frac{5}{2}\,\zeta^5 + 5\,\zeta^4 + 10\,\zeta^3\right) t^4\,v^2 + \left(10\,\zeta^2 + \frac{5}{2} + \frac{15}{2}\,\zeta + \frac{5}{2}\,\zeta^5 + 5\,\zeta^4 + 10\,\zeta^3\right) w^2\,t^4\right) u^4 + \left(\left(\frac{55}{2}\,\zeta^2 - \frac{15}{2} + 5\,\zeta + 20\,\zeta^5 + 40\,\zeta^4 + \frac{85}{2}\,\zeta^3\right) t^2\,v^4 + \left(-80\,\zeta^5 - 210\,\zeta^4 - 280\,\zeta^3 - 250\,\zeta^2 - 130\,\zeta - 20\right) w^2\,t^2\,v^2 + \left(\frac{55}{2}\,\zeta^2\right.$$

$$- \frac{15}{2} + 5\zeta + 20\zeta^5 + 40\zeta^4 + \frac{85}{2}\zeta^3\Big) w^4 t^2\Big) u^2 + \left(25\zeta^5 + 56\zeta^4 + 70\zeta^3 + 56\zeta^2 + 25\zeta\right) v^6 + \left(-105 - \frac{155}{2}\zeta^2 - 140\zeta + \frac{195}{2}\zeta^5 + \frac{225}{2}\zeta^4 + 35\zeta^3\right) w^2 v^4 + \left(-105 - \frac{155}{2}\zeta^2 - 140\zeta + \frac{195}{2}\zeta^5 + \frac{225}{2}\zeta^4 + 35\zeta^3\right) w^4 v^2 + \left(25\zeta^5 + 56\zeta^4 + 70\zeta^3 + 56\zeta^2 + 25\zeta\right) w^6$$

```
> L6d := subs(t=sqrt(s),L6c);
```

$$L6d := u^6 s^3 + \left(\left(10\zeta^2 + \frac{5}{2} + \frac{15}{2}\zeta + \frac{5}{2}\zeta^5 + 5\zeta^4 + 10\zeta^3\right) s^2 v^2 + \left(10\zeta^2 + \frac{5}{2} + \frac{15}{2}\zeta + \frac{5}{2}\zeta^5 + 5\zeta^4 + 10\zeta^3\right) w^2 s^2\right) u^4 + \left(\left(\frac{55}{2}\zeta^2 - \frac{15}{2} + 5\zeta + 20\zeta^5 + 40\zeta^4 + \frac{85}{2}\zeta^3\right) s v^4 + \left(-80\zeta^5 - 210\zeta^4 - 280\zeta^3 - 250\zeta^2 - 130\zeta - 20\right) w^2 s v^2 + \left(\frac{55}{2}\zeta^2 - \frac{15}{2} + 5\zeta + 20\zeta^5 + 40\zeta^4 + \frac{85}{2}\zeta^3\right) w^4 s\right) u^2 + \left(25\zeta^5 + 56\zeta^4 + 70\zeta^3 + 56\zeta^2 + 25\zeta\right) v^6 + \left(-105 - \frac{155}{2}\zeta^2 - 140\zeta + \frac{195}{2}\zeta^5 + \frac{225}{2}\zeta^4 + 35\zeta^3\right) w^2 v^4 + \left(-105 - \frac{155}{2}\zeta^2 - 140\zeta + \frac{195}{2}\zeta^5 + \frac{225}{2}\zeta^4 + 35\zeta^3\right) w^4 v^2 + \left(25\zeta^5 + 56\zeta^4 + 70\zeta^3 + 56\zeta^2 + 25\zeta\right) w^6$$

```
> L6e := collect(simplify(subs(s=t2,L6d/t2^3)),[u,v,w]);
```

$$L6e := u^6 + \left(5v^2 + 5w^2\right) u^4 + \left(\left(-5\zeta^4 - 5\zeta^2 - 5\zeta\right) v^4 + \left(30\zeta^4 + 30\zeta^2 + 30\zeta + 20\right) w^2 v^2 + \left(-5\zeta^4 - 5\zeta^2 - 5\zeta\right) w^4\right) u^2 + \left(\zeta^4 + \zeta^2 + \zeta - 2\right) v^6 + \left(-5\zeta^4 - 5\zeta^2 - 5\zeta - 10\right) w^2 v^4 + \left(-5\zeta^4 - 5\zeta^2 - 5\zeta - 10\right) w^4 v^2 + \left(\zeta^4 + \zeta^2 + \zeta - 2\right) w^6$$

Maple does not recognize this simplification:

```
> simplify(zeta+zeta^2+zeta^4=(sqrt(-7)-1)/2);
  evalf(zeta+zeta^2+zeta^4-(sqrt(-7)-1)/2);
```

$$\zeta\left(\zeta^3 + \zeta + 1\right) = \frac{1}{2} I\sqrt{7} - \frac{1}{2}$$

$$-2.\ 10^{-10} + 0.\ I$$

So I do it by hand copy and paste:

```
> L6e := u^6+(5*v^2+5*w^2)*u^4+((-5*Zeta^4-5*Zeta^2-5*Zeta)*v^4+
  (30*Zeta^4+30*Zeta^2+30*Zeta+20)*w^2*v^2+(-5*Zeta^4-5*Zeta^2-5*
  Zeta)*w^4)*u^2+(Zeta^4+Zeta^2+Zeta-2)*v^6+(-5*Zeta^4-5*Zeta^2-5*
  Zeta-10)*w^2*v^4+(-5*Zeta^4-5*Zeta^2-5*Zeta-10)*w^4*v^2+(Zeta^4+
  Zeta^2+Zeta-2)*w^6;
```

$$L6e := u^6 + \left(5v^2 + 5w^2\right) u^4 + \left(\left(-5\zeta^4 - 5\zeta^2 - 5\zeta\right) v^4 + \left(30\zeta^4 + 30\zeta^2 + 30\zeta + 20\right) w^2 v^2 + \left(-5\zeta^4 - 5\zeta^2 - 5\zeta\right) w^4\right) u^2 + \left(\zeta^4 + \zeta^2 + \zeta - 2\right) v^6 + \left(-5\zeta^4 - 5\zeta^2 - 5\zeta - 10\right) w^2 v^4 + \left(-5\zeta^4 - 5\zeta^2 - 5\zeta - 10\right) w^4 v^2 + \left(\zeta^4 + \zeta^2 + \zeta - 2\right) w^6$$

```
> L6f := collect(simplify(L6e,[Zeta+Zeta^2+Zeta^4=(sqrt(-7)-1)/2]
  ),[u,v,w]);
```

$$L6f := u^6 + \left(5\,v^2 + 5\,w^2\right)u^4 + \left(\left(-\frac{5}{2}\,\mathrm{I}\,\sqrt{7} + \frac{5}{2}\right)v^4 + \left(15\,\mathrm{I}\,\sqrt{7} + 5\right)w^2\,v^2 + \left(-\frac{5}{2}\,\mathrm{I}\,\sqrt{7} + \frac{5}{2}\right)w^4\right)u^2 + \left(-\frac{5}{2} + \frac{1}{2}\,\mathrm{I}\,\sqrt{7}\right)v^6 + \left(-\frac{5}{2}\,\mathrm{I}\,\sqrt{7} - \frac{15}{2}\right)w^2\,v^4 + \left(-\frac{5}{2}\,\mathrm{I}\,\sqrt{7} - \frac{15}{2}\right)w^4\,v^2 + \left(-\frac{5}{2} + \frac{1}{2}\,\mathrm{I}\,\sqrt{7}\right)w^6$$

Dehomogenize by setting u:=1:

```
> L6fd := subs(u=1,L6f);
```

$$L6fd := 1 + 5\,v^2 + 5\,w^2 + \left(-\frac{5}{2}\,\mathrm{I}\sqrt{7} + \frac{5}{2}\right) v^4 + \left(15\,\mathrm{I}\sqrt{7} + 5\right) w^2\,v^2 + \left(-\frac{5}{2}\,\mathrm{I}\sqrt{7} + \frac{5}{2}\right) w^4 + \left(-\frac{5}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{7}\right) v^6 + \left(-\frac{5}{2}\,\mathrm{I}\sqrt{7} - \frac{15}{2}\right) w^2\,v^4 + \left(-\frac{5}{2}\,\mathrm{I}\sqrt{7} - \frac{15}{2}\right) w^4\,v^2 + \left(-\frac{5}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{7}\right) w^6$$

Now V(v,w) = (w,-v) and W(v,w)=(-v,w).

Calculate subrings of invariants of ℂ[v,w]:
under <V^2> the invariants are ℂ[v^2,vw,w^2].
under <W> the invariants are ℂ[v^2,w].
under <V^2,W> the invariants are ℂ[v^2,w^2].
under <V> the invariants are ℂ[v^2+w^2,v^2w^2,vw(v^2-w^2)].
under <V,W> the invariants are ℂ[v^2+w^2,v^2w^2].

Find equation for the curve divided by <V^2,W>:

```
> solve([v^2+w^2-r,v^2-w^2-s],[v,w]):
  L6fe := simplify(subs(%[],L6fd));
```

$$L6fe := -\frac{1}{2}\,\mathrm{I}\sqrt{7}\,r^3+\mathrm{I}\sqrt{7}\,s^2\,r+\frac{5}{2}\,\mathrm{I}\sqrt{7}\,r^2-5\,\mathrm{I}\sqrt{7}\,s^2-\frac{5}{2}\,r^3+\frac{5}{2}\,r^2+5\,r+1$$

On this V acts by (r,s) ↦ (r,-s) and the further quotient has the equation

```
> L6ff := simplify(subs(s=t^(1/2),L6fe));
```

$$L6ff := -\frac{1}{2}\,\mathrm{I}\sqrt{7}\,r^3+\mathrm{I}\sqrt{7}\,t\,r+\frac{5}{2}\,\mathrm{I}\sqrt{7}\,r^2-5\,\mathrm{I}\sqrt{7}\,t-\frac{5}{2}\,r^3+\frac{5}{2}\,r^2+5\,r+1$$

Solving this for t yields a rational parametrization of this quotient:

```
> factor(solve(L6ff,t));
```

$$-\frac{1}{448}\,\frac{\left(-7+5\,\mathrm{I}\sqrt{7}\right)\left(3\,\mathrm{I}\sqrt{7}\,r+\mathrm{I}\sqrt{7}-8\,r^2+17\,r+3\right)\left(\mathrm{I}\sqrt{7}-4\,r-1\right)}{r-5}$$

So the quotient by <V^2,W> has this equation of genus 1:

```
> s^2=factor(solve(L6ff,t));
```

$$s^2=-\frac{1}{448}\,\frac{\left(-7+5\,\mathrm{I}\sqrt{7}\right)\left(3\,\mathrm{I}\sqrt{7}\,r+\mathrm{I}\sqrt{7}-8\,r^2+17\,r+3\right)\left(\mathrm{I}\sqrt{7}-4\,r-1\right)}{r-5}$$

The quotient by <V> involves three variables
r = v^2+w^2,
x = v^2w^2,
y = vw(v^2-w^2)
satisfying the relation y^2=x(r^2-4x).
Using t = (v^2-w^2)^2 = r^2-4x this quotient is therefore described by the two equations in r, x, y:

```
> rxyeq := y^2-x*(r^2-4*x);
  L6fg := simplify(subs(t=r^2-4*x,L6ff));
```

$$rxyeq := y^2-x\left(r^2-4\,x\right)$$

$$L6fg := \frac{1}{2}\,\mathrm{I}\sqrt{7}\,r^3-4\,\mathrm{I}\sqrt{7}\,r\,x-\frac{5}{2}\,\mathrm{I}\sqrt{7}\,r^2+20\,\mathrm{I}\sqrt{7}\,x-\frac{5}{2}\,r^3+\frac{5}{2}\,r^2+5\,r+1$$

Eliminate x:

```
> solve(L6fg,x);
  L6fh := numer(factor(subs(x=%,rxyeq)));
```

$$-\frac{\frac{1}{56}\,\mathrm{I}\left(\mathrm{I}\sqrt{7}\,r^3-5\,\mathrm{I}\sqrt{7}\,r^2-5\,r^3+5\,r^2+10\,r+2\right)\sqrt{7}}{r-5}$$

$$L6fh := -8\,r^6+30\,r^5-25\,r^4+28\,r^2\,y^2-20\,r^3-280\,r\,y^2-30\,r^2+700\,y^2-10\,r-1$$

```
> L6fi := solve(L6fh,y)[1];
```

$$L6fi := \frac{1}{14}\,\frac{\sqrt{56\,r^6-210\,r^5+175\,r^4+140\,r^3+210\,r^2+70\,r+7}}{r-5}$$

This describes the quotient by a hyperelliptic equation.

```
> L6fj := simplify((14*(r-5)*L6fi)^2/7);
  y2^2 = factor(L6fj);
```

$$L6fj := 8\,r^6-30\,r^5+25\,r^4+20\,r^3+30\,r^2+10\,r+1$$

$$y2^2=\left(2\,r^2+r+1\right)\left(4\,r^4-17\,r^3+19\,r^2+9\,r+1\right)$$

This curve should map onto an elliptic curve. The smallest possible degree of such a map is 2, and such a map of degree 2 exists iff the 6 ramified points satisfy a certain symmetry.

Change coordinates:

```
> L6fk := 512*factor(subs(r=(r1-1)/4,L6fj));
```

$$L6fk := (r1^2+7)\,(r1^4-21\,r1^3+133\,r1^2-63\,r1+14)$$

Find symmetries between the 6 ramified points, beginning with the last 4:

```
> L6fk4 := r1^4-21*r1^3+133*r1^2-63*r1+14;
```

$$L6fk4 := r1^4-21\,r1^3+133\,r1^2-63\,r1+14$$

```
> collect(expand(numer(factor(subs(r1=(a*r1+b)/(c*r1+d),L6fk4)))),
  r1);
```

$$(a^4-21\,a^3\,c+133\,a^2\,c^2-63\,a\,c^3+14\,c^4)\,r1^4+(4\,a^3\,b-21\,a^3\,d-63\,a^2\,b\,c+266\,a^2\,c\,d+266\,a\,b\,c^2-189\,a\,c^2\,d-63\,b\,c^3+56\,c^3\,d)\,r1^3+(6\,a^2\,b^2-63\,a^2\,b\,d+133\,a^2\,d^2-63\,a\,b^2\,c+532\,a\,b\,c\,d-189\,a\,c\,d^2+133\,b^2\,c^2-189\,b\,c^2\,d+84\,c^2\,d^2)\,r1^2+(4\,a\,b^3-63\,a\,b^2\,d+266\,a\,b\,d^2-63\,a\,d^3-21\,b^3\,c+266\,b^2\,c\,d-189\,b\,c\,d^2+56\,c\,d^3)\,r1+b^4-21\,b^3\,d+133\,b^2\,d^2-63\,b\,d^3+14\,d^4$$

```
> subsol := solve([coeffs(collect(%-coeff(%,r1,4)*L6fk4,r1),r1)],
  [a,b,c,d]);
```

$$subsol := \Big[[a=d,b=0,c=0,d=d],\ [a=-d,b=21\,d,c=3\,d,d=d],\ \Big[a=-d,b=-\frac{1}{3}\,d\,(7\,RootOf(35\,\_Z^2+154\,\_Z+67)+2),c=\frac{1}{3}\,RootOf(35\,\_Z^2+154\,\_Z+67)\,d,d=d\Big],\ [a=RootOf(\_Z^4-21\,\_Z^3+133\,\_Z^2-63\,\_Z+14)\,c,b=RootOf(\_Z^4-21\,\_Z^3+133\,\_Z^2-63\,\_Z+14)\,d,c=c,d=d]\Big]$$

```
> i := 2:
  r1sub := factor(subs(subsol[i],(a*r1+b)/(c*r1+d)));
  factor(numer(factor(subs(r1=r1sub,L6fk))))/512/512;
```

$$r1sub := -\frac{r1-21}{3\,r1+1}$$

$$(r1^2+7)\,(r1^4-21\,r1^3+133\,r1^2-63\,r1+14)$$

This substitution has order 2:

```
> factor(subs(r1=r1sub,r1sub));
```

$$r1$$

```
> solve(r1-r1sub);
  r2sub := (3*r1-7)/(r1+3);
  invr2sub := solve(r2=r2sub,r1);
  numer(factor(subs(r1=invr2sub,L6fk)))/1024/32;
```

$$\frac{7}{3},\ -3$$

$$r2sub := \frac{3\,r1-7}{r1+3}$$

$$invr2sub := -\frac{3\,r2+7}{r2-3}$$

$$\left(r2^2+7\right)\left(r2^4-7\,r2^2+14\right)$$

The total substitution from r to r2 is

```
> rsub := factor(subs(r1=invr2sub,(r1-1)/4));
```

$$rsub := -\frac{r2+1}{r2-3}$$

and the resulting polynomial is

```
> L6fl := numer(factor(subs(r=rsub,L6fj)))/64;
```

$$L6fl := \left(r2^2+7\right)\left(r2^4-7\,r2^2+14\right)$$

which is invariant under the symmetry r2 ↦ -r2.

Rename coordinates a:= r2 and b:=y2; so the hyperelliptic curve now has the equation

```
> L6fm := b^2 - subs(r2=a,L6fl);
```

$$L6fm := b^2 - (a^2 + 7)\,(a^4 - 7\,a^2 + 14)$$

and the four automorphisms (a,b) ↦ (±a,±b).

Of these (a,b) ↦ (a,-b) is the hyperelliptic involution with quotient ℙ^1.

The quotient by (a,b) ↦ (-a,b) is the elliptic curve with equation

```
> Ell1 := subs(a=sqrt(c),L6fm);
```

$$Ell1 := b^2 - (c + 7)\,(c^2 - 7\,c + 14)$$

The quotient by (a,b) ↦ (-a,-b) is the elliptic curve with equation

```
> Ell2 := c*factor(subs([a=sqrt(c),b=d/sqrt(c)],L6fm));
```

$$Ell2 := -c^4 + 35\,c^2 + d^2 - 98\,c$$

Determine their j-invariants:

```
> with(algcurves):
> ifactor(j_invariant(Ell1,b,c));
  ifactor(j_invariant(Ell2,c,d));
```

$$-(3)^3\,(5)^3$$

$$-\frac{(5)^6}{(2)^2\,(7)}$$

Since one of them is integral and the other isn't, the elliptic curves are not isogenous.

Also the second one does not have complex multiplication.

Check existing lists to determine whether the first one has complex multiplication.

The following Sage calculates a Weierstrass form of E_1 and $j(E_1)$:

Listing A.11.2: NormalFormEllipticCurveC4.sage

```
K.<sqrtm7>=NumberField(x^2+7)
R.<X,Y>=PolynomialRing(K)
E1=(X-5)*Y^2+(1/448)*(-7+5*sqrtm7)*(3*sqrtm7*X+sqrtm7-8*X^2+17*X+3)*(sqrtm7
    -4*X-1)
J=Jacobian(E1)
print(J) # Outputs:
# Elliptic Curve defined by y^2 = x^3 + (55/2*sqrtm7+55/6)*x +
# (-145/3*sqrtm7-5843/27) over Number Field in sqrtm7
# with defining polynomial x^2 + 7

print(J.j_invariant()) #Outputs:
#-831875/224*sqrtm7 - 166375/32
```

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