Motives and Hodge structures over function fields

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1. Motivation: The Mumford-Tate conjecture.

Alexandre Grothendieck's concept of motives was intended as a formal framework combining the many different aspects of algebraic varieties in a single theory. Thus to an algebraic variety X over a number field $K \subset \mathbb{C}$ are associated, among others, the rational mixed Hodge structure $H := H^n(X(\mathbb{C}), \mathbb{Q})$ and the ℓ-adic cohomology group $H^n(X_{\bar{K}}, \mathbb{Q}_\ell) \cong H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ as a a continuous Galois representation. Let $G_\infty \subset \mathrm{Aut}(H)$ be the Hodge group associated to this mixed Hodge structure, and $\Gamma_\ell \subset \mathrm{Aut}(H)(\mathbb{Q}_\ell)$ the image of $\mathrm{Gal}(\bar{K}/K)$. The Mumford-Tate conjecture asserts that Γ_ℓ is commensurable to $G_\infty(\mathbb{Z}_\ell)$, that is, their intersection is open in each of these two groups. The importance and the beauty of this statement lies in the fact that it relates two groups which are constructed in completely different ways and thus reflect very different properties of X, i.e. analytic resp. arithmetic ones.

While there has been some progress on this conjecture when X is an abelian variety, the general case remains completely open. In the analogous case of motives over function fields, however, it is now possible to prove such a conjecture in reasonable generality. The aim of this talk is to explain the necessary theory of Hodge structures and Hodge groups associated to motives over function fields.

My motivation to deal with the function field case is twofold. On the one hand I believe that definitions, theorems, and methods of proof in this area are interesting in themselves and often quite beautiful. On the other hand I hope that the study of function field analogues can provide us eventually with new ideas that re-fertilize the arithmetic over number fields.

2. Drinfeld modules.

In the following we fix a finite field $\mathbb{F}_q$ with q elements, and set $A := \mathbb{F}_q[t]$. This ring will play the role that $\mathbb{Z}$ plays in the number field case. Instead of $\mathbb{Q}$ we work with the rational function field $F := \mathbb{F}_q(t)$, and the completion $\mathbb{R}$ is replaced by $F_\infty := \mathbb{F}_q((t^{-1}))$. In all this theory, these rings may be replaced by finite extensions. As an analogue of $\mathbb{C}$ we take the completion of the algebraic closure of $\mathbb{F}_q((\theta^{-1}))$, denoted $\mathbb{C}_q$. Here θ is a new variable which in this section will be identified with t, but not afterwards. The field $\mathbb{C}_q$ is the basis for non-archimedean analysis in equal characteristic. Note that it has infinite degree over F_∞.

Now consider an A-lattice $\Lambda \subset \mathbb{C}_q$ of rank $r \geq 1$, that is, a discrete A-submodule which is isomorphic to A^r. Let us fix such an isomorphism. One can form the quotient "$\mathbb{C}_q/\Lambda$" in the following sense. One defines formally

$$e_\Lambda(X) := X \cdot \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{X}{\lambda}\right) ,$$

proves that this converges to an $\mathbb{F}_q$-linear power series, i.e., one of the form

$$e_\Lambda(X) = X + e_1 X^q + e_2 X^{q^2} + e_3 X^{q^3} + \ldots ,$$

shows that it converges on all of $\mathbb{C}_q$ and, finally, that the following sequence is exact:

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}_q \xrightarrow{e_\Lambda} \mathbb{C}_q \longrightarrow 0 .$$

Multiplication by t then induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q & \xrightarrow{e_\Lambda} & \mathbb{C}_q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \Phi} & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q & \xrightarrow{e_\Lambda} & \mathbb{C}_q & \longrightarrow & 0 \end{array}$$

and one proves that Φ is a polynomial

$$\Phi(X) = \theta X + \Phi_1 X^q + \ldots + \Phi_r X^{q^r}$$

with $\Phi_r \neq 0$. In this way the quotient "$\mathbb{C}_q/\Lambda$" has been endowed with an algebraic structure over $\mathbb{C}_q$. This object makes up an algebraic Drinfeld module of rank r.

Let us now assume that this Drinfeld module is defined over a finitely generated subfield $\mathbb{F}_q(\theta) \subset K \subset \mathbb{C}_q$, that is, that all coefficients $\Phi_i \in K$. For any prime polynomial $\wp(t) \in A$ und any $n \geq 0$ we then have

$$\mathrm{Kern}(\wp(\Phi)^n : \bar{K} \to \bar{K}) = \mathrm{Kern}(\wp(\Phi)^n : \mathbb{C}_q \to \mathbb{C}_q) \cong \wp^{-n}\Lambda/\Lambda \cong (A/\wp^n A)^r .$$

Here $\bar{K}$ denotes the algebraic closure of K in $\mathbb{C}_q$, and the Galois action corresponds to a homomorphism

$$\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_r(A/\wp^n A) .$$

In the limit these homomorphisms fit together to a homomorphism

$$\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_r(A_\wp)$$

and we are interested in its image $\Gamma_\wp$.

We know a priori that all endomorphisms of Φ are defined over a finite extension of K and therefore commute with an open subgroup of $\Gamma_\wp$. Viewing these endomorphisms as subring of the matrix ring

$$\mathrm{End}_{\bar{K}}(\Phi) \cong \{x \in \mathbb{C}_q \mid x\Lambda \subset \Lambda\} \hookrightarrow \mathrm{End}_A(\Lambda) \cong \mathcal{M}_{r\times r}(A) ,$$

we can look at their centralizer

$$G_\infty := \mathrm{Cent}_{\mathrm{GL}_{r,F}}\left(\mathrm{End}_{\bar{K}}(\varphi)\right) .$$

In the generic case $\mathrm{End}_{\bar{K}}(\varphi) = A$ we have, of course, $G_\infty = \mathrm{GL}_{r,F}$; in general G_∞ is a form of $\mathrm{GL}_{r'}$ for $r'|r$.

Satz: *$\Gamma_\wp$ and $G_\infty(A_\wp)$ are commensurable. (see [6])*

This result is a precise analogue of the usual Mumford-Tate conjecture, with one important difference: Here the group G_∞ is defined only ad hoc and does not result from a general theory of Hodge structures. I will now show how to fill this gap.

One central requirement for such a theory is the invariance under tensor products. More precisely: Hodge structures should possess tensor products, and the desired functor associating Hodge structures to (certain) motives should be compatible with tensor products. Tensor products of Drinfeld modules are special cases of Anderson's uniformizable t-motives, so it would be best to have a theory applicable to all of these.

3. Anderson's t-motives.

Anderson made the fundamental and rather subtle observation that the two distinct roles of the variable t, once in the ring of coefficients $A = \mathbb{F}_q[t]$, once as element of the base field $\mathbb{C}_q$, ought to be separated. We have already replaced t by θ in its second meaning. Now take $d \geq 1$ and let $t \in \mathcal{M}_{d\times d}(\mathbb{C}_q)$ be a quadratic matrix whose only eigenvalue is θ, i.e. with

$$(\dagger) \qquad (t-\theta)^n = 0 \qquad \text{for all } n \gg 0.$$

Then we obtain a natural action of A on the vector space $\mathbb{C}_q^d$, and we consider an A-lattice $\Lambda \subset \mathbb{C}_q^d$, discrete and free of finite type over A. In general we cannot write down a series e_Λ as in the preceding section; instead we postulate its existence. For any vector $X \in \mathbb{C}_q^d$ let ${}^\sigma X$ denote taking the q^{th} power in each coefficient. We suppose given a power series

$$e(X) = X + e_1 \cdot {}^\sigma X + e_2 \cdot {}^{\sigma^2} X + \ldots$$

and a polynomial

$$\Phi(X) = \Phi_0 \cdot X + \Phi_1 \cdot {}^\sigma X + \ldots + \Phi_m \cdot {}^{\sigma^m} X$$

with e_i, $\Phi_i \in \mathcal{M}_{d\times d}(\mathbb{C}_q)$ such that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q^d & \xrightarrow{\;e\;} & \mathbb{C}_q^d & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle \Phi} & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}_q^d & \xrightarrow{\;e\;} & \mathbb{C}_q^d & \longrightarrow & 0\,.
\end{array}$$

Essentially this makes up a uniformizable t-motive in the sense of Anderson [1]. (Here I neglect a certain technical assumption. Strictly speaking, the object thus constructed is called a uniformizable t-module, and the term t-motive is reserved for a certain equivalent dual description.) For instance, every Drinfeld module corresponds to a uniformizable t-motive.

Now we reencode the information in the lattice $\Lambda \subset \mathbb{C}_q^d$ in a way that is suited for tensor products. Condition (†) implies a natural map on the right hand side in the sequence

$$0 \longrightarrow \mathfrak{q} \longrightarrow \Lambda \otimes_A \mathbb{C}_q[[t-\theta]] \longrightarrow \mathbb{C}_q^d \longrightarrow 0\,.$$

By Anderson this map is surjective. Thus the main information is contained in the kernel $\mathfrak{q}$, which by construction is a $\mathbb{C}_q[[t-\theta]]$-lattice, i.e., a finitely generated $\mathbb{C}_q[[t-\theta]]$-submodule containing a basis of the vector space $\Lambda \otimes_A \mathbb{C}_q((t-\theta))$.

The second ingredient of the desired Hodge structures is the weight filtration. There is no particular technical difficulty involved in working with mixed objects instead of pure ones. Besides, one reason for this greater generality is the fact that pure t-motives may degenerate into mixed t-motives. If m, n are positive integers, the t-motive is called pure of weight $\mu = -\frac{m}{n}$ if and only if after suitable reparametrization we can write

$$\Phi^n(X) = \ldots + \Phi_{n,m} \cdot {}^{\sigma^m}X$$

with $\det(\Phi_{n,m}) \neq 0$. For example a Drinfeld module of rank r is pure of weight $-\frac{1}{r}$. (My convention differs from Anderson's by a minus sign.) An arbitrary t-motive is called mixed if and only if it possesses an increasing weight filtration $W_\bullet$, indexed by rational numbers, such that each graded piece of weight μ is a pure t-motive of weight μ. If the t-motive is mixed and uniformizable, then each pure constituent is uniformizable, hence the lattice Λ inherits the weight filtration, again denoted by $W_\bullet$.

Now we have collected all the necessary ingredients for the desired Hodge structures. In order to have an F-linear theory we put $H := \Lambda \otimes_A F$, then the uniformizable t-motive up to isogeny determines the data $\underline{H} = (H, W_\bullet, \mathfrak{q})$. The tensor product of two such triples $\underline{H}_i = (H_i, W_\bullet, \mathfrak{q}_i)$ is defined as $(H, W_\bullet, \mathfrak{q})$ with $H := H_1 \otimes_F H_2$, the weight filtration

$$W_\mu(H_1 \otimes_F H_2) := \sum_{\mu_1+\mu_2=\mu} W_{\mu_1}H_1 \otimes_F W_{\mu_2}H_2\,,$$

and $\mathfrak{q} := \mathfrak{q}_1 \otimes_{\mathbb{C}_q[[t-\theta]]} \mathfrak{q}_2$. This definition is compatible with the tensor product of t-motives as defined by Anderson.

4. Mixed $\mathbb{F}_q(t)$-Hodge structures.

Observe that the inclusion $A = \mathbb{F}[t] \hookrightarrow \mathbb{C}_q[[t-\theta]]$ extends naturally to an inclusion $F \subset F_\infty \hookrightarrow \mathbb{C}_q[[t-\theta]]$.

Definition: *A mixed F-pre-Hodge structure $\underline{H} = (H, W_\bullet, \mathfrak{q})$ consists of a finite dimensional F-vector space H, an increasing filtration by F-subspaces $W_\mu H$, indexed by $\mu \in \mathbb{Q}$ and called weight filtration, and a $\mathbb{C}_q[[t-\theta]]$-lattice $\mathfrak{q} \subset H \otimes_F \mathbb{C}_q((t-\theta))$.*

Homomorphisms of such objects are homomorphisms of the underlying F-vector spaces that are compatible with the filtrations and lattices. This category is F-linear but not abelian, so we want to restrict attention to a suitable subcategory. Note also that when $\underline{H}$ comes from a uniformizable t-motive, we have not yet used the discreteness of Λ. This property is related to the following numerical condition. For every F_∞-subspace $H'_\infty \subset H_\infty := H \otimes_F F_\infty$ consider the lattices $\mathfrak{q}' := \mathfrak{q} \cap (H'_\infty \otimes_{F_\infty} \mathbb{C}_q((t-\theta)))$ and $\mathfrak{p}' := H'_\infty \otimes_{F_\infty} \mathbb{C}_q[[t-\theta]]$, and put

$$\deg_{\mathfrak{q}}(H'_\infty) := \dim_{\mathbb{C}_q}\left(\frac{\mathfrak{q}'}{\mathfrak{p}' \cap \mathfrak{q}'}\right) - \dim_{\mathbb{C}_q}\left(\frac{\mathfrak{p}'}{\mathfrak{p}' \cap \mathfrak{q}'}\right).$$

This number measures the size of $\mathfrak{q}'$. On the other hand set

$$\deg_W(H'_\infty) := \sum_{\mu \in \mathbb{Q}} \mu \cdot \dim_{F_\infty} \mathrm{Gr}^W_\mu(H'_\infty).$$

Definition: *A mixed F-pre-Hodge structure $\underline{H} = (H, W_\bullet, \mathfrak{q})$ is called a mixed F-Hodge structure if and only if for every H'_∞ we have*

$$\deg_{\mathfrak{q}}(H') \leq \deg_W(H'),$$

with equality whenever $H' = W_\mu H$ for some $\mu \in \mathbb{Q}$. The full subcategory of all mixed F-Hodge structures is denoted $\mathscr{Hodge}_F$.

A closer look at the pure case shows that this condition is rather similar to the usual semistability condition of vector bundles.

Satz: *$\mathscr{Hodge}_F$ is a neutral tannakian category over F.*

The proof is modeled on similar statements for vector bundles or filtered modules. The hardest part is to show that the semistability condition is invariant under tensor product. The term "neutral" refers to the tautological fiber functor $\underline{H} \mapsto H$. We also have:

Satz: *The above construction defines a tensor functor from the category of uniformizable mixed t-motives over $\mathbb{C}_q$ up to isogeny to the category $\mathscr{Hodge}_F$. This functor is exact, F-linear, fully faithful, and its essential image is closed under taking subquotients.*

The last two statements amount to an analogue of the Hodge conjecture.

5. The Hodge group.

For any object $\underline{H}$ of $\mathscr{H}odge_F$ let $\langle\!\langle \underline{H} \rangle\!\rangle$ denote the smallest abelian full subcategory of $\mathscr{H}odge_F$ that contains $\underline{H}$ and is closed under tensor product, dualization, and subquotients. By general tannakian theory there is a well-defined algebraic subgroup $G_{\underline{H}} \subset \mathrm{Aut}_F(H)$ such that $\langle\!\langle \underline{H} \rangle\!\rangle$ is equivalent to the category of finite dimensional representations of $G_{\underline{H}}$ over F. This group is called the Hodge group of $\underline{H}$. In the case of a Drinfeld module our original expectations are confirmed:

Satz: *If $\underline{H}$ is associated to the Drinfeld module of Section 2, we have $G_{\underline{H}} = G_\infty$.*

More generally, suppose that the coefficients of the t-motive Φ in Section 3 are contained in a finitely generated extension $K \subset \mathbb{C}_q$ of F. As in Section 2 we obtain a Galois representation

$$\mathrm{Gal}(\bar{K}/K) \longrightarrow\!\!\!\!\!\rightarrow \Gamma_\wp \subset \mathrm{Aut}_{A_\wp}(\Lambda \otimes_A A_\wp) \cong \mathrm{GL}_r(A_\wp)\ .$$

Satz: *$\Gamma_\wp$ is commensurable to a Zariski dense subgroup of $G_\infty(A_\wp)$.*

This is proved by combining the above analogue of the Hodge conjecture with a theorem of A. Tamagawa amounting to a strong form of the Tate conjecture for t-motives. I also expect to determine $\Gamma_\wp$ up to commensurability, but the precise statement will be somewhat technical.

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