Erratum: Self-dual instantons and holomorphic curves

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We correct two mistakes in [2]. The first concerns the exponential decay in the proof of [2, Theorem 7.4] (see Section 2 below) and the second concerns the bubbling argument in the proof of [2, Theorem 9.1] (see Section 3 below).

The analysis deals with the small ε limit of the self-duality equations

$$\begin{aligned} \partial_t A - d_A\Psi + *_s(\partial_s A - d_A\Phi - X_s(A)) &= 0, \\ \partial_t\Phi - \partial_s\Psi - [\Phi,\Psi] + \varepsilon^{-2} * F_A &= 0, \end{aligned} \tag{1}$$

$$A(s+1,t) = f^*A(s,t),\ \Phi(s+1,t) = f^*\Phi(s,t),\ \Psi(s+1,t) = f^*\Psi(s,t), \tag{2}$$

$$\lim_{t\to\pm\infty} A(s,t) = A^\pm(s),\ \lim_{t\to\pm\infty} \Phi(s,t) = \Phi^\pm(s),\ \lim_{t\to\pm\infty} \Psi(s,t) = 0, \tag{3}$$

Here $P \to \Sigma$ is a nontrivial SO(3) bundle over a compact oriented 2-manifold (with area form), $f : P \to P$ is an SO(3)-equivariant lift of an area preserving diffeomorphism $h : \Sigma \to \Sigma$, P_f and Σ_h denote the respective mapping tori, and $*_s$ denotes a family of Hodge $*$-operators on Σ associated to a smooth family of complex structures J_s such that $J_{s+1} = h^*J_s$. $X_s : \mathcal{A}(P) \to \Omega^1(\Sigma, \mathfrak{g}_P)$ denotes a smooth family of Hamiltonian vector fields associated to Hamiltonian functions $H_s : \mathcal{A}(P) \to \mathbb{R}$ that are determined by the holonomy. They are gauge invariant and are smooth with respect to the C^0-topology on $\mathcal{A}(P)$. We have $A(s,t) \in \mathcal{A}(P)$ and $\Phi(s,t), \Psi(s,t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$. The limit connections $a^\pm = A^\pm + \Phi^\pm\, ds \in \mathcal{A}_{\mathrm{flat}}(P_f, H)$ are H-flat as in [1, Proposition 4.4].

For a connection $A \in \mathcal{A}(P)$ with sufficiently small curvature we denote by $H^1_A := \ker\, d_A \cap \ker\, d_A *_s$ the space of harmonic 1-forms in $\Omega^1(\Sigma, \mathfrak{g}_P)$ with respect to the connection A and the Hodge $*$-operator $*_s$, and by $\pi_A : \Omega^1(\Sigma, \mathfrak{g}_P) \to H^1_A$ the projection associated to the Hodge decomposition

$$\Omega^1(\Sigma, \mathfrak{g}_P) = H^1_A \oplus \mathrm{im}\, d_A \oplus \mathrm{im}\, *_s d_A.$$

This is well defined whenever F_A is sufficiently small (in the L^∞-norm). The value of the parameter s is understood from the context. When a connection $A(s) + \Phi(s)\, ds$ on P_f is given we abbreviate $\nabla_s\alpha := \partial_s\alpha(s) + [\Phi(s), \alpha(s)]$.

1 A priori estimates

In preparation for the corrections in the proof of [2, Theorem 9.1] we need a stronger version of [2, Theorem 7.1].

Remark 1.1. The assertion of [2, Theorem 7.1] continues to hold if the hypothesis $\|B_t\|_{L^\infty} + \varepsilon \| C\|_{L^\infty} \le c_0$ is replaced by the weaker inequality

$$\sup_{(s,t)\in\Omega} \|B_t(s,t)\|_{L^2(\Sigma)} + \varepsilon \sup_{(s,t)\in\Omega} \|C(s,t)\|_{L^2(\Sigma)} \le c_0. \tag{4}$$

All the estimates in the proof of [2, Theorem 7.1] continue to hold under this assumption. To see this consider, as an example, the inequality

$$|f_0| \le v_0 + c_1 u_0$$

on page 617. A key term in f_0 is the expression $\langle B_t, *[B_t \wedge C]\rangle$. We can estimate this term by the product $\|B_t\|_{L^2(\Sigma)} \|B_t\|_{L^4(\Sigma)} \|C\|_{L^4(\Sigma)}$ and use the fact that, by (4), the first factor is bounded by c_0. Moreover, the inequality (4) implies $\left\|F_{A(s,t)}\right\|_{L^2(\Sigma)} \le \varepsilon c_0$. This and the Sobolev embedding $W^{1,2}(\Sigma) \hookrightarrow L^4(\Sigma)$ imply uniform estimates of the form

$$\|\phi\|_{L^4(\Sigma)} \le c_1 \|d_A\phi\|_{L^2(\Sigma)},$$

$$\|\alpha\|_{L^4(\Sigma)} \le c_1 \left(\|\alpha\|_{L^2(\Sigma)} + \|d_A\alpha\|_{L^2(\Sigma)} + \|d_A *_s \alpha\|_{L^2(\Sigma)}\right)$$

for all $(s,t) \in \Omega$, $\phi \in \Omega^0(\Sigma, \mathfrak{g}_P)$, and $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$ (see [2, Lemma 7.6]). Applying this to $\phi = C$ and $\alpha = B_t$ we obtain

$$\|B_t\|_{L^4(\Sigma)} \|C\|_{L^4(\Sigma)} \le c_1\sqrt{u_0 v_0}$$

and this leads to the required estimate. The term $\langle B_t, *_s d^2X_s(A)(B_t, B_t)\rangle$ can be estimated by $\|B_t\|^2_{L^4(\Sigma)} \le v_0 + c_2 u_0$. A crucial observation is that the cubic terms in f_0 do not involve derivatives. The arguments in the subsequent steps for the estimates of the higher derivatives are similar (see for example the inequality $|f_1| \le v_1 + c_3^{-1}(\varepsilon^{-1}v_0 + \varepsilon^{-2}u_0)$ on page 618).

Corollary 1.2. *Let $\Omega \subset \mathbb{C}$ be an open set and $K \subset \Omega$ be a compact subset. Then for every constant $c_0 > 0$, there exist constants $\varepsilon_0 > 0$ and $c > 0$ such that the following holds. If $0 < \varepsilon \le \varepsilon_0$ and $\Xi = A + \Phi\, ds + \Psi\, dt$ is a connection on $\Omega \times \Sigma$ that satisfies (1) and (4) then*

$$\|B_t\|_{L^\infty(K\times\Sigma)} + \varepsilon \|C\|_{L^\infty(K\times\Sigma)} \le c\left(\|B_t\|_{L^2(\Omega\times\Sigma)} + \varepsilon \|C\|_{L^2(\Omega\times\Sigma)}\right).$$

Proof. By Remark 1.1, the connection Ξ satisfies (7.4) in [2, page 615]. The assertion follows by taking $p = \infty$. More precisely, (7.4) asserts that

$$\|B_t\|_{L^\infty(K\times\Sigma)} + \varepsilon \|d_A C\|_{L^\infty(K\times\Sigma)} \le c\left(\|B_t\|_{L^2(\Omega\times\Sigma)} + \varepsilon \|C\|_{L^2(\Omega\times\Sigma)}\right).$$

Since $C + \varepsilon^{-2} * F_A = 0$, it follows from (4) that $\|F_A\|_{L^2(\Sigma)} \le \varepsilon c_0$, hence $\|C\|_{L^4(\Sigma)} \le c_1\|d_A C\|_{L^2(\Sigma)}$, hence $\|F_A\|_{L^4(\Sigma)} \le \varepsilon c_2$ and, by [2, Lemma 7.6], $\|C\|_{L^\infty(\Sigma)} \le c_3\|d_A C\|_{L^4(\Sigma)} \le c_4\|d_A C\|_{L^\infty(\Sigma)}$. □

The next a priori estimate is an adaptation of [3, Lemma 9.1] to the present context. It is needed in the bubbling analysis in Section 3.

Lemma 1.3. *There is a constant $\delta_0 > 0$ with the following significance. Let $\Omega \subset \mathbb{R}^2$ be an open set and $K \subset \Omega$ be a compact subset. Then, for every $c_0 > 0$ and every $p \geq 2$, there are positive constants ε_0 and c such that the following holds. If $0 < \varepsilon \leq \varepsilon_0$ and the maps $A : \Omega \to \mathcal{A}(P)$ and $\Phi, \Psi : \Omega \to \Omega^0(\Sigma, \mathfrak{g}_P)$ satisfy (1) and*

$$\|\partial_t A - d_A \Psi\|_{L^\infty(\Omega\times\Sigma)} \leq c_0, \qquad \|F_A\|_{L^\infty(\Omega\times\Sigma)} \leq \delta_0, \tag{5}$$

then

$$\int_K \left(\|F_A\|_{L^2(\Sigma)}^p + \varepsilon^p \|\nabla_s F_A\|_{L^2(\Sigma)}^p + \varepsilon^p \|\nabla_t F_A\|_{L^2(\Sigma)}^p \right) \leq c\varepsilon^{2p}, \tag{6}$$

$$\sup_K \left(\|F_A\|_{L^2(\Sigma)} + \varepsilon \|\nabla_s F_A\|_{L^2(\Sigma)} + \varepsilon \|\nabla_t F_A\|_{L^2(\Sigma)} \right) \leq c\varepsilon^{2-2/p}. \tag{7}$$

The proof uses the following estimate. Denote by $B_r(z) \subset \mathbb{C}$ the open ball of radius r centered at z and abbreviate $B_r := B_r(0)$.

Lemma 1.4 ([3]). *Let $u : B_{R+r} \to \mathbb{R}$ be a C^2-function and $f, g : B_{R+r} \to \mathbb{R}$ be continuous such that*

$$f \leq g + \Delta u, \qquad u \geq 0, \qquad f \geq 0, \qquad g \geq 0.$$

Then

$$\int_{B_R} f \leq \int_{B_{R+r}} g + \frac{4}{r^2} \int_{B_{R+r}\setminus B_R} u.$$

Proof of Lemma 1.3. As in [2, Lemma 7.6] one can show that there exist constants $\delta_0 > 0$ and $c_1 > 0$ such that every $A \in \mathcal{A}(P)$ with $\|F_A\|_{L^\infty(\Sigma)} \leq \delta_0$ satisfies the inequalities

$$\|\phi\| \leq c_1 \|d_A \phi\|,$$

$$\|d_A (*_s dX_s(A)\alpha + \dot{*}_s \alpha)\| \leq c_1 (\|\alpha\| + \|d_A \alpha\| + \|d_A *_s \alpha\|)$$

for $s \in \mathbb{R}$, $\phi \in \Omega^0(\Sigma; \mathfrak{g}_P)$, and $\alpha \in \Omega^1(\Sigma; \mathfrak{g}_P)$. Here and in the following all norms are L^2-norms on Σ.

Now let A, Φ, Ψ satisy the hypotheses of the lemma and define

$$B_s := \partial_s A - d_A \Phi, \quad B_t := \partial_t A - d_A \Psi, \quad C := \partial_t \Phi - \partial_s \Psi - [\Phi, \Psi]. \tag{8}$$

Then the proof of [2, Theorem 7.1] shows that

$$\begin{aligned} \varepsilon^2 (\nabla_s \nabla_s C + \nabla_t \nabla_t C) &= d_A^{*_s} d_A C - 2 * [B_t \wedge B_t] + *[*_s X_s(A) \wedge B_t] \\ &\quad - * d_A (*_s dX_s(A) B_t + \dot{*}_s B_t). \end{aligned}$$

Hence, with $\Delta := \partial^2/\partial s^2 + \partial^2/\partial t^2$ the standard Laplacian, we have

$$\begin{aligned}\Delta \|C\|^2 &= 2\|\nabla_s C\|^2 + 2\|\nabla_t C\|^2 + 2\langle \nabla_s\nabla_s C + \nabla_t\nabla_t C, C\rangle \\ &= 2\varepsilon^{-4}\|d_A *_s B_t\|^2 + 2\varepsilon^{-4}\|d_A B_t\|^2 + 2\varepsilon^{-2}\|d_A C\|^2 \\ &\quad - 4\varepsilon^{-2}\langle C, *[B_t\wedge B_t]\rangle + 2\varepsilon^{-2}\langle C, *[*_s X_s(A)\wedge B_t]\rangle \\ &\quad - 2\varepsilon^{-2}\langle C, *d_A(*_s dX_s(A)B_t + \dot{*}_s B_t)\rangle \\ &\geq \frac{\delta}{\varepsilon^2}\|C\|^2 - \frac{c}{\varepsilon^2}\|C\|.\end{aligned}$$

The last inequality holds for $\varepsilon \leq \varepsilon_0$, with ε_0 sufficiently small, and suitable positive constants δ and c, depending only on δ_0, c_0, and c_1 (as well as the metrics on Σ and the vector fields X_s). Since $2\Delta\|C\|^p \geq p\|C\|^{p-2}\Delta\|C\|^2$ for $p \geq 2$, this implies

$$\|C\|^p \leq \frac{c}{\delta}\|C\|^{p-1} + \frac{2\varepsilon^2}{p\delta}\Delta\|C\|^p.$$

Using the inequality $ab \leq a^p/p + b^q/q$ with $1/p + 1/q = 1$, $a := c/\delta$ and $b := \|C\|^{p-1}$ we obtain $b^q = \|C\|^p$, and hence

$$\|C\|^p \leq \frac{c^p}{\delta^p} + \frac{2\varepsilon^2}{\delta}\Delta\|C\|^p. \tag{9}$$

By Lemma 1.4, this implies that

$$\int_{B_R(z)}\|C\|^p \leq \frac{\pi(R+r)^2 c^p}{\delta^p} + \frac{8\varepsilon^2}{r^2\delta}\int_{B_{R+r}(z)}\|C\|^p.$$

for every $z \in \mathbb{C}$ and every pair of positive real numbers R and r such that $B_{R+r}(z) \subset \Omega$. Now observe that $\varepsilon^2\|C\| = \|F_A\| \leq \delta_0\mathrm{Vol}(\Sigma)$ and use the last inequality repeatedly, with R replaced by $R+r, R+2r, \ldots, R+(p-1)r$, to obtain the estimate $\int_{B_R(z)}\|C\|^p \leq c_p$ for every $z \in \mathbb{C}$ such that $B_{R+pr}(z) \subset \Omega$. Now choose R and r such that $B_{R+pr}(z) \subset \Omega$ for every $z \in K$. Cover K by finitely many balls of radius R to obtain

$$\int_K \|F_A\|^p = \varepsilon^{2p}\int_K \|C\|^p \leq c_{K,p}\varepsilon^{2p}. \tag{10}$$

It follows from (9) that the function $z \mapsto \|C(z)\|^p + c^p|z-z_0|^2/8\delta^{p-1}\varepsilon^2$ is subharmonic in Ω for every $z_0 \in \mathbb{C}$. Hence, by the mean value inequality and (10), we have

$$\sup_K\|F_A\| = \varepsilon^2\sup_K\|C\| \leq c_{K,p}\varepsilon^{2-2/p} \tag{11}$$

for a suitable constant $c_{K,p}$. It follows from (10) and (11) that every connection $\Xi = A + \Phi\,ds + \Psi\,dt$ on $\Omega \times P$ that satisfies (1) and (5) also satisfies (4) in every compact subset of Ω and hence, by Corollary 1.2, satisfies the hypotheses of [2, Theorem 7.1]. Hence it follows from [2, Theorem 7.1] with $p = \infty$ that, for every open set U with $\mathrm{cl}(U) \subset \Omega$, there is a constant c_U such that every conection Ξ on $\Omega \times P$ that satisfies (1) and (5) also satisfies the estimates

$$\begin{aligned} \varepsilon \left\|\nabla_s B_t\right\|_{L^\infty(U\times\Sigma)} + \varepsilon \left\|\nabla_t B_t\right\|_{L^\infty(U\times\Sigma)} &\le c_U, \\ \varepsilon \left\|C\right\|_{L^\infty(U\times\Sigma)} + \varepsilon^2 \left\|\nabla_s C\right\|_{L^\infty(U\times\Sigma)} + \varepsilon^2 \left\|\nabla_t C\right\|_{L^\infty(U\times\Sigma)} &\le c_U, \\ \left\|C\right\|_{L^2(U\times\Sigma)} + \varepsilon \left\|\nabla_s C\right\|_{L^2(U\times\Sigma)} + \varepsilon \left\|\nabla_t C\right\|_{L^2(U\times\Sigma)} &\le c_U. \end{aligned} \tag{12}$$

Note that the last inequality is equivalent to (6) for $p = 2$.

Now consider the function $u : U \to \mathbb{R}$ defined by

$$u(s,t)^2 := \frac{1}{2}\left(\left\|C(s,t)\right\|^2 + \varepsilon^2 \left\|\nabla_s C(s,t)\right\|^2 + \varepsilon^2 \left\|\nabla_t C(s,t)\right\|^2\right)$$

Again all norms are L^2-norms on Σ. In the following we shall assume, for simplicity, that the Hodge $*$-operator $*_s = *$ is independent of s and that $X_s = 0$ for all s. Then, as in the proof of [2, Theorem 7.1], we have

$$\begin{aligned} \Delta u^2 \;=\; & \varepsilon^{-2} \left\|d_A C\right\|^2 + \left\|\nabla_s C\right\|^2 + \left\|\nabla_t C\right\|^2 + \left\|d_A \nabla_s C\right\|^2 + \left\|d_A \nabla_t C\right\|^2 \\ & + \varepsilon^2 \left\|\nabla_s \nabla_s C\right\|^2 + \varepsilon^2 \left\|\nabla_t \nabla_t C\right\|^2 + 2\varepsilon^2 \left\|\nabla_s \nabla_t C\right\|^2 \\ & - 2\varepsilon^2 \langle C, [\nabla_s C, \nabla_t C]\rangle - 2\varepsilon^{-2} \langle C, *[B_t \wedge B_t]\rangle \\ & - 4\langle \nabla_s C, *[B_t \wedge \nabla_s B_t]\rangle - 4\langle \nabla_t C, *[B_t \wedge \nabla_t B_t]\rangle \\ & + \langle d_A \nabla_s C, [B_s, C]\rangle + \langle d_A \nabla_t C, [B_t, C]\rangle \\ & - \langle \nabla_s C, *[B_s \wedge *d_A C]\rangle - \langle \nabla_t C, *[B_t \wedge *d_A C]\rangle. \end{aligned}$$

For ε sufficiently small it follows that

$$\Delta u^2 \ge \frac{\delta}{\varepsilon^2} u^2 - \frac{c}{\varepsilon^2} u$$

with suitable positive constants δ and c. To see this examine the last eight terms in the formula for Δu^2 and use (12). Now it follows as in (9) that

$$u^p \le \frac{c}{\delta} u^{p-1} + \frac{2\varepsilon^2}{p\delta} \Delta u^p$$

for $p \ge 2$. By (11) and (12), we have $u \le c'/\varepsilon$ for some constant c'. Hence we can argue as above to show that, for every compact subset $K \subset U$, there is a constant $c_{K,p} > 0$ such that $\int_K u^p \le c_{K,p}$ and $\sup_K u^p \le c_{K,p}\varepsilon^{-2}$. This proves the lemma. □

2 Exponential decay

The estimate $f'' \ge \rho^2 f$ in [2, page 623] does not follow from the preceding inequalities. To prove it one needs the following refinement of [2, Lemma 7.5]. All norms are understood on $[0,1] \times \Sigma$. Norms without subscript are L^2-norms.

Lemma 2.1. *Assume all H-flat connections on P_f are nondegenerate. Then there are positive constants δ_0, ε_0, and c such that the following holds. If $A+\Phi\,ds$ is a connection on P_f satisfying*

$$\|F_A\|_{L^\infty} + \|\partial_s A - d_A\Phi - X_s(A)\|_{L^\infty} \le \delta_0$$

and $0<\varepsilon\le\varepsilon_0$ then

$$\begin{aligned}\|\alpha\|^2 + \|\phi\|^2 + \|\psi\|^2 \le c\Big(&\|{*_s}\nabla_s\alpha - {*_s}dX_s(A)\alpha - {*_s}d_A\phi - d_A\psi\|^2\\ &+\varepsilon^2\left\|\nabla_s\psi - \varepsilon^{-2}d_A\alpha\right\|^2 + \varepsilon^2\left\|\nabla_s *_s\phi + \varepsilon^{-2}d_A *_s\alpha\right\|^2\Big)\end{aligned} \tag{13}$$

for every infinitesimal connection $\alpha+\phi\,ds$ on P_f and every $\psi\in\Omega^0(\Sigma_h,\mathfrak{g}_{P_f})$.

Proof. Suppose not. Then there are sequences $\varepsilon_\nu\to 0$ and $A_\nu+\Phi_\nu\,ds\in\mathcal{A}(P_f)$ such that $\|F_{A_\nu}\|_{L^\infty}+\|\partial_s A_\nu - d_{A_\nu}\Phi_\nu - X_s(A_\nu)\|_{L^\infty}\to 0$ and (13) does not hold with $c=\nu$, $\varepsilon=\varepsilon_\nu$, $A=A_\nu$, $\Phi=\Phi_\nu$. The estimate (13) is gauge invariant. Hence, by Uhlenbeck's weak compactness theorem [6, 7], we may assume that the sequence $A_\nu+\Phi_\nu\,ds$ is bounded in $W^{1,p}$ (for some $p>3$). Passing to a subsequence, if necessary, we may assume that it converges, weakly in $W^{1,p}$ and strongly in L^∞, to an H-flat connection $A+\Phi\,ds\in\mathcal{A}_{\mathrm{flat}}(P_f,H)$. Since $A+\Phi\,ds$ is nondegenerate there are positive constants ν_0 and c_0 such that

$$\|\alpha_0\| \le c_0\,\|\pi_{A_\nu}\left(\partial_s\alpha_0 + [\Phi_\nu,\alpha_0] - dX_s(A_\nu)\alpha_0\right)\|$$

for every path $\alpha_0(s)\in H^1_{A_\nu(s)}$ such that $\alpha_0(s+1)=f^*\alpha_0(s)$ and every $\nu\ge\nu_0$.

Now the assertions of [1, Lemmata 7.3 and 7.4] continue to hold for connections $A+\Phi\,ds$ on P_f such that $\|F_A\|_{L^\infty}$ is sufficiently small and the constants in these lemmata depend continuously on $\|\partial_s A - d_A\Phi\|_{L^\infty}$. Since $\|F_{A_\nu}\|_{L^\infty}$ tends to zero, the sequence $\|X_s(A_\nu)\|_{L^\infty}$ is bounded and so is $\|\partial_s A_\nu - d_{A_\nu}\Phi_\nu\|_{L^\infty}$. Hence, by [1, Lemma 7.4], there is a constant $c>0$ such that

$$\begin{aligned}\|\alpha\|^2 \le c\Big(&\|{*_s}\nabla_s\alpha - {*_s}dX_s(A_\nu)\alpha - {*_s}d_{A_\nu}\phi - d_{A_\nu}\psi\|^2\\ &+\varepsilon^2\left\|\nabla_s\psi - \varepsilon^{-2}d_{A_\nu}\alpha\right\|^2 + \varepsilon^2\left\|\nabla_s *_s\phi + \varepsilon^{-2}d_{A_\nu} *_s\alpha\right\|^2\Big)\end{aligned}$$

for every infinitesimal connection $\alpha+\phi\,ds$ on P_f and every $\psi\in\Omega^0(\Sigma_h,\mathfrak{g}_{P_f})$. Here $\nabla_s:=\partial_s+[\Phi_\nu,\cdot]$. Combining this with [1, Lemma 7.3] we find that the connection $A_\nu+\Phi_\nu\,ds$ satisfies (13) for $\nu\ge\nu_0$ and some constant $c>0$. This contradicts our assumption on the sequence $A_\nu+\Phi_\nu\,ds$ and so the lemma is proved. □

Proof of [2, Theorem 7.4]. Let $A+\Phi\,ds+\Psi\,dt$ be a solution of (1-3) and let B_s, B_t, C be given by (8). Assume

$$\varepsilon^{-1}\|F_A\|_{L^\infty(\Sigma_h\times\mathbb{R})} + \|B_t\|_{L^\infty(\Sigma_h\times\mathbb{R})} \le c_0,$$

$$\varepsilon^{-1}\|F_A\|_{L^2(\Sigma_h\times[0,\infty))} + \|B_t\|_{L^2(\Sigma_h\times[0,\infty))} \le \delta.$$

Then, by Corollary 1.2, there is a constant $c_1 > 0$ such that

$$\varepsilon^{-1} \|F_A\|_{L^\infty(\Sigma_h \times \{t\})} + \|\partial_s A - d_A \Phi - X_s(A)\|_{L^\infty(\Sigma_h \times \{t\})} \leq c_1 \delta \tag{14}$$

for $t \geq 1$. Define

$$f(s) := \frac{1}{2} \int_0^1 \left(\|B_t(s,t)\|^2_{L^2(\Sigma,*_s)} + \varepsilon^2 \|C(s,t)\|^2_{L^2(\Sigma,*_s)} \right) dt.$$

Then

$$\begin{aligned} f''(s) &= 2 \|\nabla_s B_t - dX_s(A)B_t - d_A C\|^2 + 2\varepsilon^{-2} \|d_A B_t\|^2 \\ &\quad - 3\langle C, *_s [B_t \wedge B_t]\rangle + \langle *_s d^2 X_s(A)(B_t, B_t), B_t\rangle. \end{aligned}$$

(See [2, page 622].) By (14), the connection $A(\cdot,t) + \Phi(\cdot,t)\, ds \in \mathcal{A}(P_f)$ satisfies the requirements of Lemma 2.1 for $t \geq 1$ and δ sufficiently small. Applying the estimate (13) to the triple $\alpha := B_t$, $\phi := C$, $\psi := 0$ and using the identity $\nabla_s *_s C + \varepsilon^{-2} d_A *_s B_t = 0$, we obtain

$$\|B_t\|^2 + \|C\|^2 \leq c_2 \left(\|\nabla_s B_t - dX_s(A)B_t - d_A C\|^2 + \varepsilon^{-2} \|d_A B_t\| \right).$$

(The mistake in [2] is the factor ε^2 in front of $\|C\|^2$ in this inequality; it can be removed because of the improved inequality in Lemma 2.1.) Combining this with the identity for $f''(s)$ and the fact that $\|B_t\|_{L^\infty} \leq c_1 \delta$ we obtain the desired inequality $f''(t) \geq \rho^2 f(t)$ for $t \geq 1$ and $\rho > 0$ sufficiently small. With this understood the proof proceeds as in [2]. □

3 Bubbling analysis

The assertion in [2, page 634] that the limit connection Ξ_0 represents a **nonconstant** holomorphic sphere $S^2 \to \mathcal{M}(P)$ does not seem to follow from the argument in [2]. A modified bubbling argument will result in a nonconstant holomorphic sphere but only proves a weaker estimate. More precisely, we prove the following theorem instead of [2, Theorem 9.1].

Theorem 3.1. *Let $a^\pm \in \mathcal{A}_{\mathrm{flat}}(P_f, H)$ and assume that either $H \in \mathcal{H}_0^{\mathrm{reg}}$ and $\mu_H(a^-, a^+) \leq 3$, or $\mathcal{CS}_H(a^-) - \mathcal{CS}_H(a^+) < 8\pi^2$. Then there exist positive constants ε_0 and c_0 such that*

$$\varepsilon^{-1} \|F_A\|_{L^\infty} + \|\partial_t A - d_A \Psi\|_{L^\infty} \leq c_0 \tag{15}$$

for every solution A, Φ, Ψ of (1-3) with $0 < \varepsilon \leq \varepsilon_0$.

Remark 3.2. The assertion of [2, Theorem 8.1] continues to hold if the hypothesis (8.1) is replaced by the weaker inequality (15). To see this, replace the last inequality in [2, page 625] by $\|C^\nu\|_{L^p} \leq c\varepsilon_\nu^{2/p-1}$ or, equivalently,

$$\|F_{A_\nu}\|_{L^p} \leq c\varepsilon_\nu^{1+2/p}.$$

For $p = 2$ this follows from the first inequality in [2, page 625, Step 2], for $p = \infty$ it holds by assumption, and for $2 \le p \le \infty$ it follows by interpolation. Now replace the constant ε_ν^2 by $\varepsilon_\nu^{1+2/p}$ in the following places.

- In the inequality (8.4) on page 626.
- Replace the inequality $\|A' - A\|_{L^p} \le c_2\varepsilon^2$ by $\|A' - A\|_{L^p} \le c_2\varepsilon^{1+2/p}$ in the middle of page 626.
- In the first two inequalities after (8.9), in the first inequality after (8.10), and in the first inequality in the proof of Step 5 (page 628).
- In the first inequality on page 629 and in the last inequality before (8.11).

The next theorem is a local version on [2, Theorem 8.1]. It is needed in the proof of Theorem 3.1. Let $\Omega_\nu \subset \mathbb{C}$ be an exhausting sequence of open sets and s_ν, $\varepsilon_\nu > 0$, $\delta_\nu > 0$ be seqences of real numbers such that $s_\nu \to s_0$, $\varepsilon_\nu \to 0$, $\delta_\nu \to 0$. Abbreviate $*_{\nu s} := *_{s_\nu + \delta_\nu s}$ and $X_{\nu s} := \delta_\nu X_{s_\nu+\delta_\nu s}$.

Theorem 3.3. *Let $\Xi_\nu = A_\nu + \Phi_\nu\, ds + \Psi_\nu\, dt$ be a sequence of solutions of the equations*

$$\begin{aligned} \partial_t A_\nu - d_{A_\nu}\Psi_\nu + *_{\nu s}(\partial_s A_\nu - d_{A_\nu}\Phi_\nu - X_{\nu s}(A)) &= 0, \\ \partial_t\Phi_\nu - \partial_s\Psi_\nu - [\Phi_\nu, \Psi_\nu] + \varepsilon_\nu^{-2} * F_{A_\nu} &= 0, \end{aligned} \tag{16}$$

on $\Omega_\nu \times P$ such that

$$\sup_\nu \left(\varepsilon_\nu^{-1} \|F_{A_\nu}\|_{L^2(\Omega_\nu\times\Sigma)} + \|\partial_t A_\nu - d_{A_\nu}\Psi_\nu\|_{L^2(\Omega_\nu\times\Sigma)} \right) < \infty, \tag{17}$$

$$\sup_\nu \left(\varepsilon_\nu^{-1} \|F_{A_\nu}\|_{L^\infty(\Omega_\nu\times\Sigma)} + \|\partial_t A_\nu - d_{A_\nu}\Psi_\nu\|_{L^\infty(\Omega_\nu\times\Sigma)} \right) < \infty.$$

Then there is a subsequence, still denoted by Ξ_ν, a sequence of gauge transformations $g_\nu : \Omega_\nu \to \mathcal{G}(P)$, and a connection $\Xi_0 = A_0 + \Phi_0\, ds + \Psi_0\, dt$ on $\mathbb{C} \times P$ such that

$$\partial_t A_0 - d_{A_0}\Psi_0 + *_{s_0}(\partial_s A_0 - d_{A_0}\Phi_0) = 0, \qquad F_{A_0} = 0,$$

$$\lim_{\nu\to\infty} \left(\|g_\nu^* A_\nu - A_0\|_{L^\infty(K\times\Sigma)} + \sup_{(s,t)\in K} \left\|g_\nu^{-1} B_{\nu t} g_\nu - B_{0t}\right\|_{L^2(\Sigma)} \right) = 0$$

for every compact subset $K \subset \mathbb{C}$. Here we denote $B_{\nu t} := \partial_t A_\nu - d_{A_\nu}\Psi_\nu$ and $B_{0t} := \partial_t A_0 - d_{A_0}\Psi_0$.

Proof. We argue as in the proof of [2, Theorem 8.1, Step 3] and use Lemma 1.3 to obtain sharper estimates. More precisely, for every compact subset $K \subset \mathbb{C}$ there is a constant $\nu_K > 0$ such that, for every $(s,t) \in K$ and every $\nu \ge \nu_K$, there is a unique section $\eta_\nu(s,t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ such that

$$F_{A'_\nu} = 0, \qquad A'_\nu := A_\nu + *_{\nu s} d_{A_\nu}\eta_\nu,$$

and

$$\|d_{A_\nu}\eta_\nu\|_{L^\infty(\Sigma)} \le c_1 \|F_{A_\nu}\|_{L^\infty(\Sigma)} \le c_2\varepsilon_\nu. \tag{18}$$

Choose $\Phi'_\nu(s,t), \Psi'_\nu(s,t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ such that

$$d_{A'_\nu} *_{\nu s} \left(\partial_s A'_\nu - d_{A'_\nu}\Phi'_\nu - X_{\nu s}(A'_\nu)\right) = d_{A'_\nu} *_{\nu s} \left(\partial_t A'_\nu - d_{A'_\nu}\Psi'_\nu\right) = 0.$$

Note that the sequence $\Xi'_\nu = A'_\nu + \Phi'_\nu\, ds + \Psi'_\nu\, dt$ depends only on ν and not on the compact set K in question. One proves exactly as in [2, pages 626–627] that the sequence Ξ'_ν satisfies the estimates

$$\begin{aligned} \|\Xi'_\nu - \Xi_\nu\|_{1,p,\varepsilon;K} &\le c_{K,p}\varepsilon_\nu^{1+2/p}, && (19)\\ \|B'_{\nu t}\|_{L^\infty(K\times\Sigma)} &\le c_K, && (20)\\ \|B'_{\nu t} + *_{\nu s}\left(B'_{\nu s} - X_{\nu s}(A'_\nu)\right)\|_{L^p(K\times\Sigma)} &\le c_{K,p}\varepsilon_\nu^{1+2/p}, && (21) \end{aligned}$$

for every compact set $K \subset \mathbb{C}$ and every $p \ge 2$, with suitable positive constants c_K and $c_{K,p}$. In addition we wish to prove the estimate

$$\sup_K \|B'_{\nu t} - B_{\nu t}\|_{L^2(\Sigma)} \le c_K\sqrt{\varepsilon_\nu}. \tag{22}$$

To see this recall the identities (8.5-7) from [2]. They have the form

$$\begin{aligned} B'_t - B_t &= d_{A'}(\Psi' - \Psi) + *_s d_A\nabla_t\eta + *_s[B_t,\eta],\\ d_A *_s d_A(\Psi' - \Psi) &= d_A *_s B_t - [d_A B_t, \eta] - [F_A, \nabla_t\eta]\\ &\quad -[(A' - A)\wedge([d_A\nabla_t\eta + [B_t,\eta]])] \qquad (23)\\ d_A *_s d_A\nabla_t\eta &= -d_A B_t - [d_A\nabla_t\eta \wedge d_A\eta] - [[B_t,\eta]\wedge d_A\eta]\\ &\quad -2[B_t \wedge *_s d_A\eta] - [d_A *_s B_t, \eta] \end{aligned}$$

Here we have dropped the subscript ν. Since

$$d_A B_t = \nabla_t F_A, \qquad d_A *_s B_t = d_A B_s = \nabla_s F_A$$

we obtain from Lemma 1.3 that, for every compact set $K \subset \mathbb{C}$, there is a constant $c'_K > 0$ such that

$$\sup_K \left(\|d_A B_t\|_{L^2(\Sigma)} + \|d_A *_s B_t\|_{L^2(\Sigma)}\right) \le c'_K\sqrt{\varepsilon}.$$

Hence it follows from (18) and the last equation in (23) that

$$\sup_K \|d_A\nabla_t\eta\|_{L^2(\Sigma)} \le c''_K\sqrt{\varepsilon}.$$

Using this estimate and the second equation in (23) we obtain

$$\sup_K \|d_A(\Psi' - \Psi)\|_{L^2(\Sigma)} \le c'''_K\sqrt{\varepsilon}.$$

Combining the last two estimates with the first equation in (23) we obtain (22). Now Ξ'_ν descends to a sequence

$$\bar{u}'_\nu : K \to \mathcal{M}(P)$$

of approximate holomorphic curves (see (21)) with uniformly bounded derivatives (see (20)). We must prove that the sequence $\bar{u}'_\nu$ is bounded in $W^{2,p}$ for some $p > 2$. By the elliptic bootstrapping analysis for holomorphic curves (see [4, Appendix B]), this is equivalent to a $W^{1,p}$-bound on $\bar\partial_J(\bar{u}'_\nu)$. To obtain such a bound we examine the following formula from [2, page 627]:

$$\begin{aligned} B'_t + *_s(B'_s - X_s(A')) &= *_s \dot{*}_s d_A\eta - [X_s(A), \eta] - *_s(X_s(A') - X_s(A)) \\ &\quad + [(A' - A), \nabla_s\eta] - *_s[(A' - A), \nabla_t\eta] \\ &\quad - d_{A'}(\Psi' - \Psi + \nabla_s\eta) - *_s d_{A'}(\Phi' - \Phi - \nabla_t\eta). \end{aligned} \tag{24}$$

To begin with observe that, by Lemma 1.3, we have estimates of the form

$$\int_K \left(\|d_A B_t\|^p_{L^2(\Sigma)} + \|d_A *_s B_t\|^p_{L^2(\Sigma)} \right) \le c_{K,p}\varepsilon^p.$$

Carrying the argument in the proof of Lemma 1.3 one step further we obtain estimates for the second derivatives of the curvature and hence

$$\int_K \left(\|d_A \nabla_s B_t\|^p_{L^2(\Sigma)} + \|d_A *_s \nabla_s B_t\|^p_{L^2(\Sigma)} \right) \le c_{K,p};$$

similarly for ∇_t. Differentiate the identities in (23) to obtain

$$\int_K \left(\|d_A \nabla_s\nabla_s\eta\|^p_{L^2(\Sigma)} + \|d_A \nabla_t\nabla_t\eta\|^p_{L^2(\Sigma)} + \|d_A \nabla_s\nabla_t\eta\|^p_{L^2(\Sigma)} \right) \le c_{K,p},$$

$$\int_K \left(\|d_A \nabla_s(\Psi' - \Psi)\|^p_{L^2(\Sigma)} + \|d_A \nabla_t(\Psi' - \Psi)\|^p_{L^2(\Sigma)} \right) \le c_{K,p}.$$

Combining these estimates with (24) we obtain

$$\int_K \|\nabla_s(B'_t + *_s(B'_s - X_s(A')))\|^p_{L^2(\Sigma)} \le c_{K,p},$$

and similarly for ∇_t. This is the required $W^{1,p}$-estimate for $\bar\partial_J(\bar{u}'_\nu)$. It follows that $\bar{u}'_\nu$ is bounded in $W^{2,p}$ and hence has a C^1-convergent subsequence. The limit of this subsequence is the required holomorphic curve in $\mathcal{M}(P)$. The assertion of the theorem now follows from (22) and the C^1-convergence of $\bar{u}'_\nu$. □

Proof of Theorem 3.1. Suppose, by contradiction, that there are sequences $\varepsilon_\nu \to 0$ and $\Xi_\nu = A_\nu + \Phi_\nu\, ds + \Psi_\nu\, dt$ such that Ξ_ν satisfies (1-3) with $\varepsilon = \varepsilon_\nu$ and

$$\varepsilon_\nu^{-1} \|F_{A_\nu}\|_{L^\infty} + \|\partial_t A_\nu - d_{A_\nu}\Psi_\nu\|_{L^\infty} \to \infty.$$

For each ν define the energy density $e_\nu : \mathbb{R}^2 \to \mathbb{R}$ by

$$e_\nu(s,t) := \varepsilon_\nu^{-1} \left\| F_{A_\nu(s,t)} \right\|_{L^2(\Sigma)} + \left\| \partial_t A_\nu(s,t) - d_{A_\nu(s,t)} \Psi_\nu(s,t) \right\|_{L^2(\Sigma, *_s)}.$$

By Corollary 1.2, and the time shift invariance of equation (1), this sequence is unbounded. Passing to a subsequence, we may assume that there is a sequence $w_\nu = (s_\nu, t_\nu) \in [0,1] \times \mathbb{R}$ such that $e_\nu(w_\nu) \to \infty$. Applying a time shift, and passing to a further subsequence, we may assume that w_ν converges to $w_0 = (s_0, t_0)$. Using Hofer's lemma ([2, Lemma 9.3]), we may assume that there is a sequence of real numbers $0 < \rho_\nu < 1/2$ such that

$$\sup_{|w - w_\nu| \le \rho_\nu} e_\nu(w) \le 2 e_\nu(w_\nu), \qquad \rho_\nu e_\nu(w_\nu) \to \infty.$$

There are three cases to consider.

Case 1: $\varepsilon_\nu e_\nu(w_\nu) \to \infty$. In this case a nontrivial instanton on S^4 bubbles off. The argument is standard (see [2, pages 630–631]).

Case 2: $\varepsilon_\nu e_\nu(w_\nu) \to 1$. In this case a nontrivial instanton on $\mathbb{C} \times \Sigma$ bubbles off. The bubbling analysis relies on an asymptotic analysis of finite energy solutions of (1) over $\mathbb{C} \times \Sigma$ and on the resulting energy quantization. In [2, pages 632–633] this argument is only sketched. In [3, Proposition 11.1] an analogous argument has been carried out in a situation where the space of connections on P is replaced by a finite dimensional symplectic manifold equipped with a Hamiltonian group action. The adaptation of the proof to the present case is straight forward.

Case 3: $\varepsilon_\nu e_\nu(w_\nu) \to 0$. In this case a nonconstant holomorphic sphere in the moduli space $\mathcal{M}(P) := \mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}(P)$ of flat connections bubbles off. Abbreviate $c_\nu := e_\nu(w_\nu)$ and consider the rescaled sequence

$$\widetilde{A}_\nu(w) := A_\nu(w_\nu + c_\nu^{-1} w),$$

$$\widetilde{\Phi}_\nu(w) := c_\nu^{-1} \Phi_\nu(w_\nu + c_\nu^{-1} w), \qquad \widetilde{\Psi}_\nu(w) := c_\nu^{-1} \Psi_\nu(w_\nu + c_\nu^{-1} w).$$

This triple satisfies (16) and (17) with $\delta_\nu := c_\nu^{-1}$, ε_ν replaced by $\widetilde{\varepsilon}_\nu := \varepsilon_\nu c_\nu$, and $\Omega_\nu := B_{\rho_\nu c_\nu}$. By assumption, we have

$$\left\| \partial_t \widetilde{A}_\nu - d_{\widetilde{A}_\nu} \widetilde{\Psi}_\nu \right\|_{L^2(\Sigma)} + \frac{1}{\widetilde{\varepsilon}_\nu} \left\| F_{\widetilde{A}_\nu} \right\|_{L^2(\Sigma)} = \frac{e_\nu(w_\nu + c_\nu^{-1} w)}{e_\nu(w_\nu)} \le 2 \tag{25}$$

for $|w| \le \rho_\nu c_\nu$ and

$$\left\| \partial_t \widetilde{A}_\nu(0) - d_{\widetilde{A}_\nu(0)} \widetilde{\Psi}_\nu(0) \right\|_{L^2(\Sigma)} + \frac{1}{\widetilde{\varepsilon}_\nu} \left\| F_{\widetilde{A}_\nu(0)} \right\|_{L^2(\Sigma)} = 1. \tag{26}$$

It follows from (25) and Corollary 1.2 that, for every compact subset $K \subset \mathbb{C}$, there are positive constants ν_K and c_K such that, for every $\nu \ge \nu_K$,

$$\left\| \partial_t \widetilde{A}_\nu - d_{\widetilde{A}_\nu} \widetilde{\Psi}_\nu \right\|_{L^\infty(K \times \Sigma)} + \frac{1}{\widetilde{\varepsilon}_\nu} \left\| F_{\widetilde{A}_\nu} \right\|_{L^\infty(K \times \Sigma)} \le c_K. \tag{27}$$

Hence $\widetilde{\Xi}_\nu = \widetilde{A}_\nu + \widetilde{\Phi}_\nu \, ds + \widetilde{\Psi}_\nu \, dt$ satisfies all the requirements of Theorem 3.3. The limit connection Ξ_0 represents a finite energy holomorphic sphere in the symplectic quotient $\mathcal{M}(P)$. We prove that it is nonconstant. Namely, by (27) and Lemma 1.3, we have

$$\lim_{\nu\to\infty} \frac{1}{\widetilde{\varepsilon}_\nu} \left\| F_{\widetilde{A}_\nu(0)} \right\|_{L^2(\Sigma)} = 0.$$

Hence, by Theorem 3.3 and (26),

$$\left\| \partial_t A_0(0) - d_{A_0(0)} \Psi_0(0) \right\|_{L^2(\Sigma)} = \lim_{\nu\to\infty} \left\| \partial_t \widetilde{A}_\nu(0) - d_{\widetilde{A}_\nu(0)} \widetilde{\Psi}_\nu(0) \right\|_{L^2(\Sigma)} = 1.$$

This concludes the discussion of case 3.

Since the bubbling in all three cases results in nontrivial instantons, respectively nonconstant holomorphic spheres, we can argue as in [2, pages 624–625] to obtain a contradiction. Thus the theorem is proved. □

One can now use Theorem 3.1 and the strenthened form of [2, Theorem 8.1] in Remark 3.2 to prove [2, Theorem 9.2].

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