Complex structures, moment maps, and the Ricci form (Extended Version)

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Abstract

The Ricci form is a moment map for the action of the group of exact volume preserving diffeomorphisms on the space of almost complex structures. This observation yields a new approach to the Weil–Petersson symplectic form on the Teichmüller space of isotopy classes of complex structures with real first Chern class zero and nonempty Kähler cone. This extended version of the paper includes a proof of the Bochner–Kodaira–Nakano identity (Appendix B), a brief exposition of Bott–Chern cohomology (Appendix C), and a discussion of the relation between complex structures and differential forms of middle degree (Appendix D).

Contents

1 Introduction

This paper is based on a remark by Simon Donaldson. The remark is that the space of linear complex structures on $\mathbb{R}^{2\mathsf{n}}$ can be viewed as a co-adjoint $\mathrm{SL}(2\mathsf{n},\mathbb{R})$-orbit and hence is equipped with a canonical symplectic form and a Hamiltonian $\mathrm{SL}(2\mathsf{n},\mathbb{R})$-action. Thus, for any volume form ρ on a closed oriented $2\mathsf{n}$-manifold M, the space $\mathscr{J}(M)$ of almost complex structures carries a natural symplectic structure. Following [13], one can then deduce that the action of the group of exact volume preserving diffeomorphisms on $\mathscr{J}(M)$ is a Hamiltonian group action with the Ricci form as a moment map. In the integrable case this yields a new approach to the Weil–Petersson symplectic form on the Teichmüller space of isotopy classes of complex structures with real first Chern class zero and nonempty Kähler cone. Here are the details.

Fix a closed connected oriented $2\mathsf{n}$-manifold M and a positive volume form ρ and denote by $\mathscr{J}(M)$ the space of almost complex structures compatible with the orientation. This space is equipped with a symplectic form

$$\Omega_{\rho,J}(\widehat{J}_1,\widehat{J}_2) := \tfrac{1}{2}\int_M \mathrm{trace}\Big(\widehat{J}_1 J \widehat{J}_2\Big)\rho \qquad \text{for } \widehat{J}_1,\widehat{J}_2\in\Omega^{0,1}_J(M,TM). \tag{1.1}$$

The **Ricci form** $\mathrm{Ric}_{\rho,J}\in\Omega^2(M)$ associated to ρ and J is defined by

$$\mathrm{Ric}_{\rho,J}(u,v) := \tfrac{1}{4}\mathrm{trace}\big((\nabla_u J)J(\nabla_v J)\big) + \tfrac{1}{2}\mathrm{trace}\big(JR^\nabla(u,v)\big) + \tfrac{1}{2}d\lambda^\nabla_J(u,v)$$

for $u,v\in\mathrm{Vect}(M)$, where ∇ is a torsion-free ρ-connection and the 1-form λ^∇_J on M is defined by $\lambda^\nabla_J(u) := \mathrm{trace}\big((\nabla J)u\big)$ for $u\in\mathrm{Vect}(M)$. In the integrable case $\mathbf{i}\mathrm{Ric}_{\rho,J}$ is the curvature of the Chern connection on the canonical bundle associated to the Hermitian structure determined by ρ.

Theorem A (The Ricci Form). *The Ricci form is independent of the choice of the torsion-free ρ-connection ∇ used to define it. It is closed, represents the cohomology class $2\pi c_1(TM,J)$, satisfies $\phi^*\mathrm{Ric}_{\rho,J}=\mathrm{Ric}_{\phi^*\rho,\phi^*J}$ for every diffeomorphism ϕ, and $\mathrm{Ric}_{e^f\rho,J}=\mathrm{Ric}_{\rho,J}+\frac{1}{2}d(df\circ J)$ for all $f\in\Omega^0(M)$. Moreover, the map $J\mapsto 2\mathrm{Ric}_{\rho,J}$ is a moment map for the action of the group $\mathrm{Diff}^{\mathrm{ex}}(M,\rho)$ of exact volume preserving diffeomorphisms on $\mathscr{J}(M)$, i.e. if $t\mapsto J_t$ is a smooth path of almost complex structures on M, then*

$$\frac{d}{dt}\int_M 2\mathrm{Ric}_{\rho,J_t}\wedge\alpha = \tfrac{1}{2}\int_M \mathrm{trace}\Big((\partial_t J_t)J_t(\mathcal{L}_{v_\alpha}J_t)\Big)\rho \tag{1.2}$$

for $t\in\mathbb{R}$ and $\alpha\in\Omega^{2\mathsf{n}-2}(M)$, where $v_\alpha\in\mathrm{Vect}(M)$ is defined by $\iota(v_\alpha)\rho=d\alpha$.

Proof. See Theorem 2.6. □

The proof of Theorem A is based on the aforementioned observation that the space of linear complex structures is a co-adjoint $\mathrm{SL}(2\mathsf{n},\mathbb{R})$-orbit. Theorem A can then be derived from a general result of Donaldson [13] about the action of the group $\mathrm{Diff}^{\mathrm{ex}}(M,\rho)$ on a suitable space of sections of a fibration over M. In Section 2 we give a direct proof which does not rely on [13]. That the Ricci form is closed and represents 2π times the first Chern class is a consequence of the formula $\mathrm{Ric}_{\rho,J} = \frac{1}{2}\mathrm{trace}(JR^{\widetilde{\nabla}}) + d\lambda_J^\nabla$, where $\omega \in \Omega^2(M)$ is a nondegenerate 2-form compatible with J, ∇ is the Levi–Civita connection of the metric $\omega(\cdot, J\cdot)$, and $\widetilde{\nabla} := \nabla - \frac{1}{2}J\nabla J$. Moreover, $\lambda_J^\nabla = 0$ whenever ω is closed, so one obtains the standard Ricci form in the symplectic case. We emphasize that the dual space of the space of exact divergence-free vector fields is the space of exact 2-forms on M, so one obtains a genuine moment map only for almost complex structures with real first Chern class zero.

Equation (1.2) extends to an identity that holds for all vector fields v. This identity takes the form

$$\int_M \Lambda_\rho(J,\widehat{J}) \wedge \iota(v)\rho = \frac{1}{2}\int_M \mathrm{trace}\Big(\widehat{J}J\mathcal{L}_v J\Big)\rho \tag{1.3}$$

for all $\widehat{J} \in \Omega_J^{0,1}(M,TM)$ and all $v \in \mathrm{Vect}(M)$, where $\Lambda_\rho(J,\widehat{J}) \in \Omega^1(M)$ is defined by $(\Lambda_\rho(J,\widehat{J}))(u) := \mathrm{trace}((\nabla\widehat{J})u + \frac{1}{2}\widehat{J}J\nabla_u J)$ for $u \in \mathrm{Vect}(M)$. Thus Λ_ρ is a 1-form on $\mathscr{J}(M)$ with values in $\Omega^1(M)$. The next theorem shows that the differential of this 1-form is a 2-form on $\mathscr{J}(M)$ with values in $d\Omega^0(M)$.

Theorem B (The one-form Λ_ρ). *Let $v \in \mathrm{Vect}(M)$ and define $f_v \in \Omega^0(M)$ by $f_v\rho := d\iota(v)\rho$. Then, for all $J \in \mathscr{J}(M)$,*

$$\Lambda_\rho(J,\mathcal{L}_v J) = 2\iota(v)\mathrm{Ric}_{\rho,J} - df_v \circ J + df_{Jv}. \tag{1.4}$$

Moreover, if $\mathbb{R}^2 \to \mathscr{J}(M) : (s,t) \mapsto J(s,t)$ is a smooth map, then

$$\partial_s\Lambda_\rho(J,\partial_t J) - \partial_t\Lambda_\rho(J,\partial_s J) + \tfrac{1}{2}d\mathrm{trace}((\partial_s J)J(\partial_t J)) = 0. \tag{1.5}$$

Proof. See Theorem 2.7. □

Theorem C (The Integrable Case). *Let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form and let $J \in \mathscr{J}(M)$ be an integrable almost complex structure. Then $\frac{1}{2\pi}\mathrm{Ric}_{\rho,J}$ is a $(1,1)$-form and represents the first Bott–Chern class of J. Moreover, the first Bott–Chern class of J vanishes if and only if there exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ such that $\mathrm{Ric}_{\rho,\phi^*J} = 0$. If $\mathrm{Ric}_{\rho,J} = \mathrm{Ric}_{\rho,\phi^*J} = 0$ for some orientation preserving diffeomorphism ϕ, then $\phi^*\rho = \rho$.*

Proof. See Theorem 3.1. □

Let $\mathscr{J}_{\mathrm{int},0}(M) \subset \mathscr{J}(M)$ be the space of integrable almost complex structures with real first Chern class zero and nonempty Kähler cone. Then Theorem C shows that the Teichmüller space $\mathscr{T}_0(M) := \mathscr{J}_{\mathrm{int},0}(M)/\mathrm{Diff}_0(M)$ can be identified with the quotient space $\mathscr{T}_0(M,\rho) := \mathscr{J}_{\mathrm{int},0}(M,\rho)/\mathrm{Diff}_0(M,\rho)$, where $\mathscr{J}_{\mathrm{int},0}(M,\rho) := \{J \in \mathscr{J}_{\mathrm{int},0}(M) \,|\, \mathrm{Ric}_{\rho,J} = 0\}$. We emphasize that the quotient group $\mathrm{Diff}_0(M,\rho)/\mathrm{Diff}^{\mathrm{ex}}(M,\rho)$ acts trivially. The space $\mathscr{J}(M)$ carries a complex structure $\widehat{J} \mapsto -J\widehat{J}$ and the symplectic form Ω_ρ in (1.1) is of type $(1,1)$. However, it is not Kähler because the symmetric pairing $\langle \widehat{J}_1, \widehat{J}_2\rangle = \frac{1}{2}\int_M \mathrm{trace}(\widehat{J}_1\widehat{J}_2)\rho$ is indefinite in general. Thus complex submanifolds of $\mathscr{J}(M)$ need not be symplectic. The space $\mathscr{J}_{\mathrm{int},0}(M)$ is an example. Its tangent space at J is the kernel of $\bar\partial_J : \Omega^{0,1}_J(M,TM) \to \Omega^{0,2}_J(M,TM)$. If $\mathrm{Ric}_{\rho,J} = 0$ and $\bar\partial_J\widehat{J} = 0$, then Theorem B implies that there exist unique smooth functions $f = f_{\rho,\widehat{J}}$ and $g = f_{\rho,J\widehat{J}}$ such that

$$\Lambda_\rho(J,\widehat{J}) = -df\circ J + dg, \qquad \int_M f\rho = \int_M g\rho = 0. \tag{1.6}$$

This implies that the restriction of the 2-form $\Omega_{\rho,J}$ to $\ker\bar\partial_J$ vanishes on the subspace $\{\mathcal{L}_vJ \,|\, f_v = f_{Jv} = 0\}$. It turns out that Ω_ρ descends to a symplectic form on the Teichmüller space $\mathscr{T}_0(M,\rho) \cong \mathscr{T}_0(M)$ that is independent of ρ. For $J \in \mathscr{J}_{\mathrm{int},0}(M)$ let ρ_J be the volume form with $\mathrm{Ric}_{\rho_J,J} = 0$ and $\int_M \rho_J = V$.

Theorem D (Teichmüller Space). *The formula*

$$\Omega_J(\widehat{J}_1,\widehat{J}_2) := \int_M \Big(\tfrac{1}{2}\mathrm{trace}\big(\widehat{J}_1J\widehat{J}_2\big) - f_1g_2 + f_2g_1\Big)\rho_J, \tag{1.7}$$

for $J \in \mathscr{J}_{\mathrm{int},0}(M)$ and $\widehat{J}_i \in \Omega^{0,1}_J(M,TM)$ with $\bar\partial_J\widehat{J}_i = 0$ and f_i, g_i as in (1.6), defines a symplectic form on the Teichmüller space $\mathscr{T}_0(M)$. It satisfies the naturality condition $\Omega_{\phi^*J}(\phi^*\widehat{J}_1,\phi^*\widehat{J}_2) = \phi^*\Omega_J(\widehat{J}_1,\widehat{J}_2)$ for every $\phi \in \mathrm{Diff}^+(M)$ and thus the mapping class group acts on $\mathscr{T}_0(M)$ by symplectomorphisms.

Proof. See Theorem 4.4. □

Theorem D gives an alternative construction of the Weil–Petersson symplectic form on Calabi–Yau Teichmüller spaces (see [21, 26, 31, 32, 33, 34] for the polarized case and [15, Ch 16] for the symplectic form on $\mathscr{T}_0(M)$ for the K3 surface). The proof relies on Yau's theorem and the observations, for Ricci-flat Kähler manifolds (M,ω,J), that a vector field v is holomorphic if and only if $\iota(v)\omega$ is harmonic (Lemma 3.9), and that the space of $\bar\partial_J$-harmonic 1-forms $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ is invariant under the map $\widehat{J} \mapsto \widehat{J}^*$ (Lemma 3.10).

Associated to the symplectic form (1.7) on $\mathscr{T}_0(M)$ and the complex structure $\widehat{J} \mapsto -J\widehat{J}$ is the symmetric bilinear form

$$\langle \widehat{J}_1, \widehat{J}_2 \rangle = \int_M \Big(\tfrac{1}{2}\mathrm{trace}\big(\widehat{J}_1 \widehat{J}_2\big) - f_1 f_2 - g_1 g_2 \Big) \rho_J. \tag{1.8}$$

This is indefinite in general, so $\mathscr{T}_0(M)$ need not be Kähler. If ω is a Kähler form with $\omega^{\mathsf{n}}/\mathsf{n}! = \rho_J$, then the subspace of self-adjoint harmonic endomorphisms $\widehat{J} = \widehat{J}^* \in \Omega^{0,1}_J(M, TM)$ is positive for (1.8) (and tangent to the Teichmüller space of ω-compatible complex structures). Its symplectic complement is the negative subspace of skew-adjoint harmonic endomorphisms. The 2-form (1.7) defines a symplectic connection on the space $\mathscr{E}_0(M)$ of isotopy classes of Ricci-flat Kähler structures, fibered over the space $\mathscr{B}_0(M)$ of isotopy classes of Kählerable symplectic forms with real first Chern class zero, whose fiber over $[\omega]$ is the space $\mathscr{T}_0(M, \omega)$ of ω-compatible (integrable) complex structures J with $\mathrm{Ric}_{\omega,J} = 0$ modulo $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$.

Theorem E (A Connection). *The projection $\mathscr{E}_0(M) \to \mathscr{B}_0(M)$ is a submersion and the 2-form (1.7) defines a symplectic connection on $\mathscr{E}_0(M)$. The connection 1-form $\mathscr{A}$ assigns to each Ricci-flat Kähler structure (ω, J) and each closed 2-form $\widehat{\omega}$ the unique element $\widehat{J} = \mathscr{A}_{\omega,J}(\widehat{\omega}) \in \Omega^{0,1}_J(M, TM)$ that satisfies $\bar{\partial}_J \widehat{J} = 0$ and $\Lambda_\rho(J, \widehat{J}) = -d\langle \widehat{\omega}, \omega \rangle \circ J$ and $\widehat{\omega} - J^*\widehat{\omega} = \langle (\widehat{J} - \widehat{J}^*)\cdot, \cdot \rangle$ and $\Omega_J(\widehat{J}, \widehat{J}') = 0$ for all $\widehat{J}' \in \Omega^{0,1}_J(M, TM)$ with $\bar{\partial}_J \widehat{J}' = 0$ and $\widehat{J}' = (\widehat{J}')^*$. The connection is $\mathrm{Diff}^+(M)$-equivariant and is given by*

$$\mathscr{A}_{\omega,J}(\widehat{\omega}) = \mathcal{L}_v J + \widehat{J}_0, \qquad \langle \widehat{J}_0 \cdot, \cdot \rangle = \tfrac{1}{2}\big((\widehat{\omega} - d\widehat{\lambda}) - J^*(\widehat{\omega} - d\widehat{\lambda}) \big), \tag{1.9}$$

where $v \in \mathrm{Vect}(M)$ and $\widehat{\lambda} = \iota(v)\omega \in \Omega^1(M)$ satisfy $d^*(\widehat{\omega} - d\widehat{\lambda}) = 0$, $d^*\widehat{\lambda} = 0$.

Proof. See Lemma 4.5 and Theorem 4.6. $\square$

The Weil–Petersson metric on the fiber $\mathscr{T}_0(M, \omega)$ in Theorem E is Kähler and has been studied by many authors (see e.g. [6, 17, 18, 26, 29], [31]-[38], [41, 43] and the references therein). An important special case arises when $H^{2,0}_J(M) = 0$ for all $J \in \mathscr{J}_{\mathrm{int},0}(M)$. In this case $\mathscr{T}_0(M)$ is Kähler, each polarized fiber $\mathscr{T}_0(M, \omega)$ is an open subset of $\mathscr{T}_0(M)$, the symplectic forms on the fibers agree on the overlaps (as noted by Todorov [37, p 328]), and the connection is trivial.

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2 The Ricci form

Linear complex structures

The standard orientation of $\mathbb{R}^{2\mathsf{n}}$ with the coordinates $x_1, \dots, x_\mathsf{n}, y_1, \dots, y_\mathsf{n}$ is determined by the volume form $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_\mathsf{n} \wedge dy_\mathsf{n}$. The space of linear complex structures on $\mathbb{R}^{2\mathsf{n}}$ compatible with the orientation is given by

$$\mathscr{J}_\mathsf{n} = \left\{ gJ_0g^{-1} \,\middle|\, g \in \mathrm{SL}(2\mathsf{n}, \mathbb{R}) \right\}, \qquad J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{2.1}$$

This is a co-adjoint orbit equipped with a Hamiltonian $\mathrm{SL}(2\mathsf{n}, \mathbb{R})$-action. Abbreviate $\mathrm{G} := \mathrm{SL}(2\mathsf{n}, \mathbb{R})$ and $\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = \mathfrak{sl}(2\mathsf{n}, \mathbb{R})$ and note that $\mathscr{J}_\mathsf{n} \subset \mathfrak{g}$.

Lemma 2.1. *The set $\mathscr{J}_\mathsf{n} \subset \mathbb{R}^{2\mathsf{n}\times 2\mathsf{n}}$ is a connected $2\mathsf{n}^2$-dimensional submanifold and its tangent space at $J \in \mathscr{J}_\mathsf{n}$ is given by*

$$T_J\mathscr{J}_\mathsf{n} = \left\{ \widehat{J} \in \mathbb{R}^{2\mathsf{n}\times 2\mathsf{n}} \,\middle|\, \widehat{J}J + J\widehat{J} = 0 \right\} = \left\{ [\xi, J] \,\middle|\, \xi \in \mathfrak{g} \right\}. \tag{2.2}$$

The formula $\widehat{J} \mapsto -J\widehat{J}$ defines a complex structure on $\mathscr{J}_\mathsf{n}$ and the formula

$$\tau_J\big(\widehat{J}_1, \widehat{J}_2\big) := \tfrac{1}{2}\mathrm{trace}\big(\widehat{J}_1 J \widehat{J}_2\big) = -\mathrm{trace}\big([\xi_1, \xi_2]J\big) \tag{2.3}$$

for $\xi_i \in \mathfrak{g}$ and $\widehat{J}_i := [\xi_i, J]$ defines a symplectic form $\tau \in \Omega^2(\mathscr{J}_\mathsf{n})$. The G-action $\mathrm{G} \times \mathscr{J}_\mathsf{n} \to \mathscr{J}_\mathsf{n} : (g, J) \mapsto gJg^{-1}$ is Hamiltonian and is generated by the G-equivariant moment map $\mu : \mathscr{J}_n \to \mathfrak{g}$ given by $\mu(J) = -J$ for $J \in \mathscr{J}_\mathsf{n}$.

Proof. The set $\mathrm{H} := \{h \in \mathrm{SL}(2n, \mathbb{R}) \,|\, hJ_0 = J_0h\}$ is a Lie subgroup of G and is isomorphic to the group of complex $\mathsf{n} \times \mathsf{n}$-matrices with determinant in the unit circle. So $\dim \mathrm{H} = 2\mathsf{n}^2 - 1$ and $\dim \mathrm{G} = 4\mathsf{n}^2 - 1$ and thus the homogeneous space G/H is a manifold of dimension $2\mathsf{n}^2$. Since G is connected, so is G/H. Next we claim that the map $\mathrm{G} \to \mathbb{R}^{2\mathsf{n}\times 2\mathsf{n}} : g \mapsto gJ_0g^{-1}$ descends to a proper injective immersion $\iota : \mathrm{G}/\mathrm{H} \to \mathbb{R}^{2\mathsf{n}\times 2\mathsf{n}}$. It is injective by definition. To see that ι is an immersion, observe that $T_{[g]}\mathrm{G}/\mathrm{H} \cong g\mathfrak{g}/g\mathfrak{h}$ and $d\iota([g])[g\xi] = g[\xi, J_0]g^{-1}$ for $g \in \mathrm{G}$ and $\xi \in \mathfrak{g}$. To prove that ι is proper, choose $g_k \in \mathrm{G}$ such that the sequence $J_k := g_kJ_0g_k^{-1}$ converges to J_0, and define $h_k := g_k^{-1}[e_1 \cdots e_\mathsf{n}\; J_ke_1 \cdots J_ke_\mathsf{n}]$, where the vectors $e_1, \dots, e_\mathsf{n} \in \mathbb{R}^{2\mathsf{n}}$ form the standard basis of $\mathbb{R}^\mathsf{n} \times \{0\}$. Then $h_k \in \mathrm{H}$ for k sufficiently large and $\lim_{k\to\infty} g_kh_k = \mathbb{1}$. This shows that the map $\iota : \mathrm{G}/\mathrm{H} \to \mathbb{R}^{2\mathsf{n}\times 2\mathsf{n}}$ is a proper injective immersion. Hence its image $\mathscr{J}_\mathsf{n} = \iota(\mathrm{G}/\mathrm{H})$ is a connected $2\mathsf{n}^2$-dimensional submanifold of $\mathbb{R}^{2\mathsf{n}\times 2\mathsf{n}}$.

Now let $J \in \mathscr{J}_\mathsf{n}$. Then $gJg^{-1} \in \mathscr{J}_\mathsf{n}$ for all $g \in \mathrm{G}$ and so $[\xi, J] \in T_J\mathscr{J}_\mathsf{n}$ for all $\xi \in \mathfrak{g}$. Thus $\{[\xi, J] \,|\, \xi \in \mathfrak{g}\} \subset T_J\mathscr{J}_\mathsf{n} \subset \{\widehat{J} \in \mathbb{R}^{2\mathsf{n}\times 2\mathsf{n}} \,|\, \widehat{J}J + J\widehat{J} = 0\}$. Since all three spaces have dimension $2\mathsf{n}^2$, equality holds and this proves (2.2). The formula (2.3) follows by direct calculation. To show that the 2-form τ in (2.3) is nondegenerate, let $\widehat{J} = [\xi, J] \in T_J\mathscr{J}_\mathsf{n} \setminus \{0\}$ and define $\eta := [\xi, J]^T$ and $\widehat{J}' := [\eta, J]$. Then $\tau_J(\widehat{J}, \widehat{J}') = \mathrm{trace}(\eta[\xi, J]) = \mathrm{trace}([\xi, J]^T[\xi, J]) > 0$. The 2-form τ is closed and the complex structure $\widehat{J} \mapsto -J\widehat{J}$ is integrable by Lemma A.1, as both structures are preserved by the torsion-free connection

$$\nabla_t\widehat{J} := \tfrac{d}{dt}\widehat{J} + \tfrac{1}{2}\widehat{J}J\dot{J} + \tfrac{1}{2}\dot{J}J\widehat{J}.$$

The map $\mathscr{J}_\mathsf{n} \to \mathfrak{g} : J \mapsto \mu(J) := -J$ is a moment map for the G-action because $\tau_J([\xi, J], \widehat{J}) = -\mathrm{trace}(\xi\widehat{J}) = \mathrm{trace}((d\mu(J)\widehat{J})\xi)$ for $J \in \mathscr{J}_\mathsf{n}$, $\widehat{J} \in T_J\mathscr{J}_\mathsf{n}$, and $\xi \in \mathfrak{g}$. This proves Lemma 2.1. □

Remark 2.2. The symplectic form τ in (2.3) is a $(1,1)$-form with respect to the complex structure $\widehat{J} \mapsto -J\widehat{J}$. For $\mathsf{n} > 1$ it is not a Kähler form, because the bilinear form $\langle \widehat{J}_1, \widehat{J}_2 \rangle = \frac{1}{2}\mathrm{trace}(\widehat{J}_1\widehat{J}_2)$ is indefinite on each tangent space.

Remark 2.3. Let $\omega_0 := \sum_{i=1}^{\mathsf{n}} dx_i \wedge dy_i$ denote the standard symplectic form on $\mathbb{R}^{2\mathsf{n}}$ and consider the space of ω_0-compatible linear complex structures

$$\mathscr{J}_{\mathsf{n},0} := \left\{ J \in \mathscr{J}_\mathsf{n} \;\middle|\; \begin{array}{l} J^*\omega_0 = \omega_0 \text{ and } \omega_0(\zeta, J\zeta) > 0 \\ \text{for all } \zeta \in \mathbb{R}^{2n} \setminus \{0\} \end{array} \right\}. \tag{2.4}$$

This is a complex submanifold of $\mathscr{J}_\mathsf{n}$ of real dimension $\mathsf{n}^2 + \mathsf{n}$ and the symplectic form (2.3) restricts to a Kähler form on $\mathscr{J}_{\mathsf{n},0}$. The symplectic linear group $\mathrm{Sp}(2\mathsf{n})$ acts on $\mathscr{J}_{\mathsf{n},0}$ by Kähler isometries and a moment map $\mu : \mathscr{J}_{\mathsf{n},0} \to \mathfrak{sp}(2\mathsf{n})$ for this action is again given by $\mu(J) = -J$.

Remark 2.4. The group $\mathrm{Sp}(2\mathsf{n})$ acts on Siegel upper half space $\mathscr{S}_\mathsf{n} \subset \mathbb{C}^{\mathsf{n}\times\mathsf{n}}$ of symmetric matrices with positive definite imaginary part via

$$g_*Z := (AZ + B)(CZ + D)^{-1}, \qquad g =: \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for $g \in \mathrm{Sp}(2\mathsf{n})$ and $Z \in \mathscr{S}_\mathsf{n}$. There is a unique $\mathrm{Sp}(2\mathsf{n})$-equivariant diffeomorphism from $\mathscr{S}_\mathsf{n}$ to $\mathscr{J}_{\mathsf{n},0}$ that sends $\mathbf{i}\mathbb{1} \in \mathscr{S}_\mathsf{n}$ to $J_0 \in \mathscr{J}_{\mathsf{n},0}$. It is given by

$$J(Z) = \begin{pmatrix} XY^{-1} & -Y - XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix} \in \mathscr{J}_{\mathsf{n},0}, \qquad Z = X + \mathbf{i}Y \in \mathscr{S}_\mathsf{n}.$$

This diffeomorphism is a Kähler isometry with respect to the Kähler metric on $\mathscr{S}_\mathsf{n}$ given by $|\widehat{Z}|^2 = \mathrm{trace}((Y^{-1}\widehat{X})^2 + (Y^{-1}\widehat{Y})^2)$ for $\widehat{Z} = \widehat{X} + \mathbf{i}\widehat{Y} \in T_Z\mathscr{S}_\mathsf{n}$.

Definition of the Ricci form

By Lemma 2.1 the space $\mathscr{J}_{\mathsf{n}}$ fits as a fiber into the general framework developed by Donaldson [13]. Starting from this observation we will show that the action of the group of exact volume preserving diffeomorphisms on the space of almost complex structures is a Hamiltonian group action with twice the Ricci form as a moment map. Let M be a closed connected oriented $2\mathsf{n}$-manifold. Assume M admits an almost complex structure compatible with the orientation and denote the space of such almost complex structures by

$$\mathscr{J}(M) := \left\{ J \in \Omega^0(M, \mathrm{End}(TM)) \,\middle|\, \begin{array}{l} J^2 = -\mathbb{1} \text{ and} \\ J \text{ is compatible with} \\ \text{the orientation of } M \end{array} \right\}. \tag{2.5}$$

Thus $\mathscr{J}(M)$ is the space of sections of a bundle each of whose fibers is equipped with a natural symplectic form by Lemma 2.1. It can be viewed formally as an infinite-dimensional manifold whose tangent space at J is the space $T_J\mathscr{J}(M) = \{\widehat{J} \in \Omega^0(M, \mathrm{End}(TM)) \,|\, \widehat{J}J + J\widehat{J} = 0\} = \Omega^{0,1}_J(M, TM)$ of complex anti-linear 1-forms on M with values in TM. Every positive volume form $\rho \in \Omega^{2\mathsf{n}}(M)$ determines a symplectic form Ω_ρ on $\mathscr{J}(M)$ defined by

$$\Omega_{\rho,J}(\widehat{J}_1, \widehat{J}_2) := \tfrac{1}{2}\int_M \mathrm{trace}\left(\widehat{J}_1 J \widehat{J}_2\right)\rho \tag{2.6}$$

for $J \in \mathscr{J}(M)$ and $\widehat{J}_1, \widehat{J}_2 \in T_J\mathscr{J}(M)$. The group $\mathscr{G} = \mathrm{Diff}(M,\rho)$ of volume preserving diffeomorphisms acts on $\mathscr{J}(M)$ contravariantly by $J \mapsto \phi^*J$ for $\phi \in \mathscr{G}$ and $J \in \mathscr{J}(M)$. This action preserves the symplectic form Ω_ρ.

Definition 2.5 (**Ricci Form**). *Fix a positive volume form $\rho \in \Omega^{2\mathsf{n}}(M)$, an almost complex structure $J \in \mathscr{J}(M)$, and a torsion-free ρ-connection ∇ on TM. The* **Ricci form** *of the pair (ρ, J) is the 2-form*

$$\mathrm{Ric}_{\rho,J} := \tfrac{1}{2}\left(\tau^\nabla_J + d\lambda^\nabla_J\right), \tag{2.7}$$

where $\tau^\nabla_J \in \Omega^2(M)$ and $\lambda^\nabla_J \in \Omega^1(M)$ are defined by

$$\begin{aligned} \tau^\nabla_J(u,v) &:= \tfrac{1}{2}\mathrm{trace}\big((\nabla_u J)J(\nabla_v J)\big) + \mathrm{trace}\big(JR^\nabla(u,v)\big), \\ \lambda^\nabla_J(u) &:= \mathrm{trace}\big((\nabla J)u\big) \end{aligned} \tag{2.8}$$

for $u, v \in \mathrm{Vect}(M)$. For $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ define $\Lambda_\rho(J,\widehat{J}) \in \Omega^1(M)$ by

$$\big(\Lambda_\rho(J,\widehat{J})\big)(u) := \mathrm{trace}\big((\nabla\widehat{J})u + \tfrac{1}{2}\widehat{J}J\nabla_u J\big) \qquad \textit{for } u \in \mathrm{Vect}(M). \tag{2.9}$$

The Ricci form as a moment map

The next theorem is the main result of this section. It asserts that the action of the subgroup

$$\mathscr{G}^{\mathrm{ex}} := \left\{ \phi \in \mathrm{Diff}(M) \;\middle|\; \begin{array}{l} \text{there exists a smooth isotopy} \\ [0,1] \times \mathrm{Diff}(M) : t \mapsto \phi_t \\ \text{and a smooth family of vector fields} \\ [0,1] \to \mathrm{Vect}(M) : t \mapsto v_t \\ \text{such that } \iota(v_t)\rho \text{ is exact for all } t \\ \text{and } \partial_t \phi_t = v_t \circ \phi_t \text{ for all } t \\ \text{and } \phi_0 = \mathrm{id} \text{ and } \phi_1 = \phi \end{array} \right\} \tag{2.10}$$

of exact volume preserving diffeomorphisms on $\mathscr{J}(M)$ is a Hamiltonian group action and is generated by the $\mathscr{G}$-equivariant moment map which assigns to each $J \in \mathscr{J}(M)$ twice the Ricci form $\mathrm{Ric}_{\rho,J}$. The moment map must take values in the *dual space* of the *Lie algebra*

$$\mathrm{Lie}(\mathscr{G}^{\mathrm{ex}}) = \mathrm{Vect}^{\mathrm{ex}}(M,\rho) = \{ v \in \mathrm{Vect}(M) \,|\, \iota(v)\rho \text{ is exact} \}.$$

Every $(2\mathsf{n}-2)$-form $\alpha \in \Omega^{2\mathsf{n}-2}(M)$ determines an exact divergence-free vector field $v_\alpha \in \mathrm{Vect}^{\mathrm{ex}}(M,\rho)$ via

$$\iota(v_\alpha)\rho = d\alpha.$$

Thus $\mathrm{Vect}^{\mathrm{ex}}(M,\rho)$ can be identified with the quotient of the space $\Omega^{2\mathsf{n}-2}(M)$ by the space of closed $(2\mathsf{n}-2)$-forms on M. Its dual space can be viewed formally as the space of exact 2-forms on M, in that every exact 2-form τ on M determines a linear functional

$$\mathrm{Vect}^{\mathrm{ex}}(M,\rho) \to \mathbb{R} : v_\alpha \mapsto \int_M \tau \wedge \alpha.$$

With this understood, equation (2.15) in the following theorem is the assertion that the map $J \mapsto 2\mathrm{Ric}_{\rho,J}$ is a moment map for the action of $\mathscr{G}^{\mathrm{ex}}$ on $\mathscr{J}(M)$. In general, however, the Ricci form is only closed and not exact; only its differential in the direction of an infinitesimal almost complex structure is always exact. Thus the map $J \mapsto 2\mathrm{Ric}_{\rho,J}$ is only a moment in the strict sense of the word when restricted to the space of almost complex structures with real first Chern class zero. One could attempt to rectify this situation by subtracting a closed 2-form in the appropriate cohomology class from the Ricci form, however such a modification would destroy the $\mathscr{G}^{\mathrm{ex}}$-equivariance of the moment map unless M has real dimension two.

Theorem 2.6. *Let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form, let $J \in \mathscr{J}(M)$, and let $\widehat{J} \in \Omega^{0,1}_J(M,TM)$. Then the following holds.*

(i) *The Ricci form $\mathrm{Ric}_{\rho,J}$ and the 1-form $\Lambda_\rho(J,\widehat{J})$ are independent of the choice of the torsion-free ρ-connection ∇ used to define them. Moreover,*

$$\mathrm{Ric}_{e^f\rho,J} = \mathrm{Ric}_{\rho,J} + \tfrac{1}{2}d(df\circ J), \quad \Lambda_{e^f\rho}(J,\widehat{J}) = \Lambda_\rho(J,\widehat{J}) + df\circ\widehat{J} \tag{2.11}$$

for all $f\in\Omega^0(M)$ and the Ricci form and Λ_ρ satisfy the naturality condition

$$\phi^*\mathrm{Ric}_{\rho,J} = \mathrm{Ric}_{\phi^*\rho,\phi^*J}, \qquad \phi^*\Lambda_\rho(J,\widehat{J}) = \Lambda_{\phi^*\rho}(\phi^*J,\phi^*\widehat{J}) \tag{2.12}$$

for all $\phi\in\mathrm{Diff}(M)$.

(ii) *Every vector field $v\in\mathrm{Vect}(M)$ satisfies*

$$\int_M \Lambda_\rho(J,\widehat{J})\wedge\iota(v)\rho = \tfrac{1}{2}\int_M \mathrm{trace}\big(\widehat{J}J\mathcal{L}_vJ\big)\rho. \tag{2.13}$$

Moreover, every smooth path $\mathbb{R}\to\mathscr{J}(M): t\mapsto J_t$ of almost complex structures with $J_0 = J$ and $\frac{d}{dt}\big|_{t=0}J_t = \widehat{J}$ satisfies the equations

$$\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) := \frac{d}{dt}\bigg|_{t=0}\mathrm{Ric}_{\rho,J_t} = \tfrac{1}{2}d\big(\Lambda_\rho(J,\widehat{J})\big) \tag{2.14}$$

and

$$\int_M 2\widehat{\mathrm{Ric}}_\rho(J,\widehat{J})\wedge\alpha = \tfrac{1}{2}\int_M \mathrm{trace}\big(\widehat{J}J\mathcal{L}_{v_\alpha}J\big)\rho \tag{2.15}$$

for $\alpha\in\Omega^{2\mathsf{n}-2}(M)$, where $v_\alpha\in\mathrm{Vect}(M)$ is defined by $\iota(v_\alpha)\rho = d\alpha$.

(iii) *Let $\omega\in\Omega^2(M)$ be a nondegenerate 2-form compatible with J such that $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$, let ∇ be the Levi-Civita connection of the Riemannian metric $\langle\cdot,\cdot\rangle = \omega(\cdot,J\cdot)$, and define*

$$\widetilde{\nabla} := \nabla - \tfrac{1}{2}J\nabla J. \tag{2.16}$$

Then $\widetilde{\nabla}$ is a Hermitian connection and

$$\mathrm{Ric}_{\rho,J} = \tfrac{1}{2}\big(\mathrm{trace}(JR^{\widetilde{\nabla}}) + d\lambda_J^{\nabla}\big). \tag{2.17}$$

Thus $\mathrm{Ric}_{\rho,J}$ is closed and represents the class $2\pi c_1(TM,J)$. Moreover,

$$d\omega = 0 \qquad\Longrightarrow\qquad \lambda_J^{\nabla} = 0, \qquad \mathrm{Ric}_{\rho,J} = \tfrac{1}{2}\mathrm{trace}(JR^{\widetilde{\nabla}}). \tag{2.18}$$

Proof. We prove part (i). Choose a smooth function

$$[0,1]\times M\to\mathbb{R}:(t,p)\mapsto f_t(p)$$

with $f_0=0$ and $f_1=f$, define $\rho_t:=e^{f_t}\rho$ for $0\le t\le 1$, and choose a smooth path of torsion-free connections ∇_t on TM such that $\nabla_t\rho_t=0$ for all t. For $0\le t\le 1$ define the 1-forms $A_t\in\Omega^1(M,\mathrm{End}(TM))$ and $\alpha_t\in\Omega^1(M)$ by

$$A_t:=\tfrac{d}{dt}\nabla_t,\qquad \alpha_t(u):=\mathrm{trace}\big(JA_t(u)\big) \tag{2.19}$$

for $u\in\mathrm{Vect}(M)$. Then, for all t and all $u,v\in\mathrm{Vect}(M)$, we have

$$A_t(u)v=A_t(v)u,\qquad \mathrm{trace}(A_t(u))=d(\partial_tf_t)(u), \tag{2.20}$$

$$\tfrac{d}{dt}\nabla_{t,u}J=[A_t(u),J],\qquad \tfrac{d}{dt}R^{\nabla_t}=d^{\nabla_t}A_t. \tag{2.21}$$

It follows from (2.8), (2.19), (2.20), and (2.21) that

$$\begin{aligned}\tfrac{d}{dt}\tau_J^{\nabla_t}(u,v)&=\mathrm{trace}\big((\nabla_{t,u}J)A_t(v)-(\nabla_{t,v}J)A_t(u)\big)+\mathrm{trace}\big(Jd^{\nabla_t}A_t(u,v)\big)\\&=\mathrm{trace}\big(d^{\nabla_t}(JA_t)(u,v)\big)\\&=d\alpha_t(u,v)\end{aligned}$$

and

$$\begin{aligned}\tfrac{d}{dt}\Big|_{t=0}\lambda_J^{\nabla_t}(u)&=\mathrm{trace}\big([A_t,J]u\big)\\&=\mathrm{trace}\big(A_t(Ju)\big)-\mathrm{trace}(JA_t(u))\\&=d(\partial_tf_t)(Ju)-\alpha_t(u)\end{aligned}$$

for all t and all $u,v\in\mathrm{Vect}(M)$. Hence

$$\tfrac{d}{dt}\Big|_{t=0}(\tau_J^{\nabla_t}+d\lambda_J^{\nabla_t})=d\big(d(\partial_tf_t)\circ J\big).$$

Integrate this formula to obtain the first equation in (2.11) and consider the case where $\rho_t=\rho$ is independent of t to deduce that the 2-form $\mathrm{Ric}_{\rho,J}$ is independent of the choice of the torsion-free ρ-connection ∇ used to define it. Moreover, it follows from (2.19), (2.20), and (2.21) that

$$\begin{aligned}\tfrac{d}{dt}\mathrm{trace}\big((\nabla_t\widehat{J})u+\tfrac12\widehat{J}J\nabla_{t,u}J\big)&=\mathrm{trace}\big([A_t,\widehat{J}]u+\tfrac12\widehat{J}J[A_t(u),J]\big)\\&=\mathrm{trace}\big(A_t(\widehat{J}u)\big)\\&=d(\partial_tf_t)(\widehat{J}u).\end{aligned}$$

for all t and all $u\in\mathrm{Vect}(M)$. Integrate this formula to obtain the second equation in (2.11) and consider the case where $\rho_t=\rho$ is independent of t to deduce that the 1-form $\Lambda_\rho(J,\widehat{J})$ is independent of the choice of the torsion-free ρ-connection ∇ used to define it. The naturality condition (2.12) follows directly ftom the definitions and this proves (i).

To prove part (ii) we use the formulas

$$\mathrm{trace}(\nabla u)\rho = d\iota(u)\rho, \tag{2.22}$$

$$(\mathcal{L}_v J)u = J\nabla_u v - \nabla_{Ju} v + (\nabla_v J)u \tag{2.23}$$

for $u, v \in \mathrm{Vect}(M)$. By (2.23), we have

$$\begin{aligned}\mathrm{trace}\big(\widehat{J}J\mathcal{L}_v J\big) &= \mathrm{trace}\big(-\widehat{J}\nabla v - \widehat{J}J\nabla_{J\cdot} v + \widehat{J}J\nabla_v J\big)\\ &= \mathrm{trace}\big(-2\widehat{J}\nabla v + \widehat{J}J\nabla_v J\big)\end{aligned}$$

for all $u, v \in \mathrm{Vect}(M)$. Here the second equality holds because two endomorphisms Φ and $-J\Phi J$ are conjugate and so have the same trace. Thus

$$\begin{aligned}\Lambda_\rho(J,\widehat{J})(v) &= \mathrm{trace}\big((\nabla\widehat{J})v + \tfrac{1}{2}\widehat{J}J\nabla_v J\big)\\ &= \mathrm{trace}\big(\nabla(\widehat{J}v) - \widehat{J}\nabla v + \tfrac{1}{2}\widehat{J}J\nabla_v J\big)\\ &= \mathrm{trace}\big(\nabla(\widehat{J}v)\big) + \tfrac{1}{2}\mathrm{trace}\big(\widehat{J}J\mathcal{L}_v J\big)\end{aligned}$$

for all $v \in \mathrm{Vect}(M)$. Hence it follows from (2.22) with $u = \widehat{J}v$ that

$$\int_M \Lambda_\rho(J,\widehat{J}) \wedge \iota(v)\rho = \int_M \Lambda_\rho(J,\widehat{J})(v)\rho = \tfrac{1}{2}\int_M \mathrm{trace}\big(\widehat{J}J\mathcal{L}_v J\big)\rho$$

for all $v \in \mathrm{Vect}(M)$. This proves (2.13).

Now fix a torsion-free ρ-connection ∇ and abbreviate

$$\widehat{\lambda}(u) := \mathrm{trace}((\nabla\widehat{J})u) = \tfrac{d}{dt}\Big|_{t=0}\lambda^\nabla_{J_t}(u), \qquad \widehat{\beta}(u) := \tfrac{1}{2}\mathrm{trace}\big(\widehat{J}J\nabla_u J\big)$$

for $u \in \mathrm{Vect}(M)$. Then $\Lambda_\rho(J,\widehat{J}) = \widehat{\lambda} + \widehat{\beta}$ and

$$\begin{aligned}&d\widehat{\beta}(u,v)\\ &= \tfrac{1}{2}\mathcal{L}_u\mathrm{trace}\big(\widehat{J}J\nabla_v J\big) - \tfrac{1}{2}\mathcal{L}_v\mathrm{trace}\big(\widehat{J}J\nabla_u J\big) + \tfrac{1}{2}\mathrm{trace}\big(\widehat{J}J\nabla_{[u,v]}J\big)\\ &= \tfrac{1}{2}\mathrm{trace}\big((\nabla_u(\widehat{J}J))\nabla_v J\big) - \tfrac{1}{2}\mathrm{trace}\big((\nabla_v(\widehat{J}J))\nabla_u J\big)\\ &\quad + \tfrac{1}{2}\mathrm{trace}\big(\widehat{J}J\big(\nabla_u\nabla_v J - \nabla_v\nabla_u J + \nabla_{[u,v]}J\big)\big)\\ &= \tfrac{1}{2}\mathrm{trace}\big((\nabla_u\widehat{J})J(\nabla_v J)\big) - \tfrac{1}{2}\mathrm{trace}\big((\nabla_v\widehat{J})J(\nabla_u J)\big) + \tfrac{1}{2}\mathrm{trace}\big(\widehat{J}J[R^\nabla(u,v),J]\big)\\ &= \tfrac{1}{2}\mathrm{trace}\big((\nabla_u\widehat{J})J(\nabla_v J)\big) + \tfrac{1}{2}\mathrm{trace}\big((\nabla_u J)J(\nabla_v\widehat{J})\big) + \mathrm{trace}\big(\widehat{J}R^\nabla(u,v)\big)\\ &= \tfrac{d}{dt}\Big|_{t=0}\tau^\nabla_{J_t}(u,v)\end{aligned}$$

for all $u, v \in \mathrm{Vect}(M)$. Since $\mathrm{Ric}_{\rho,J_t} = \frac{1}{2}(\tau^\nabla_{J_t} + d\lambda^\nabla_{J_t})$ this proves (2.14). Equation (2.15) follows directly from (2.13), (2.14), and Stokes' theorem and this proves part (ii).

We prove part (iii). The connection $\widetilde{\nabla}$ in (2.16) will in general no longer be torsion-free. However, since the endomorphism $J\nabla_u J$ is skew-adjoint for all $u \in \mathrm{Vect}(M)$, it preserves the Riemannian metric on M and the volume form ρ. In addition it preserves the almost complex structure J because

$$\widetilde{\nabla}_u J = \nabla_u J - \tfrac{1}{2}[J\nabla_u J, J] = \nabla_u J - \tfrac{1}{2}J(\nabla_u J)J + \tfrac{1}{2}JJ\nabla_u J = 0$$

for all $u \in \mathrm{Vect}(M)$. Next we compute the curvature tensor of $\widetilde{\nabla}$. Fix three vector fields $u, v, w \in \mathrm{Vect}(M)$. Then $\widetilde{\nabla}_v w = \nabla_v w - \tfrac{1}{2}J(\nabla_v J)w$ and so

$$\begin{aligned}
\widetilde{\nabla}_u\widetilde{\nabla}_v w &= \widetilde{\nabla}_u\bigl(\nabla_v w - \tfrac{1}{2}J(\nabla_v J)w\bigr) = \widetilde{\nabla}_u\nabla_v w - \tfrac{1}{2}J\widetilde{\nabla}_u\bigl((\nabla_v J)w\bigr) \\
&= \nabla_u\nabla_v w - \tfrac{1}{2}J(\nabla_u J)\nabla_v w - \tfrac{1}{2}J\nabla_u\bigl((\nabla_v J)w\bigr) - \tfrac{1}{4}(\nabla_u J)(\nabla_v J)w \\
&= \nabla_u\nabla_v w - \tfrac{1}{2}J\bigl(\nabla_u\nabla_v J\bigr)w - \tfrac{1}{4}(\nabla_u J)(\nabla_v J)w \\
&\quad - \tfrac{1}{2}J(\nabla_u J)\nabla_v w - \tfrac{1}{2}J(\nabla_v J)\nabla_u w.
\end{aligned}$$

Hence

$$\begin{aligned}
R^{\widetilde{\nabla}}(u,v)w &= \widetilde{\nabla}_u\widetilde{\nabla}_v w - \widetilde{\nabla}_u\widetilde{\nabla}_v w + \widetilde{\nabla}_{[u,v]}w \\
&= \nabla_u\nabla_v w - \tfrac{1}{2}J\bigl(\nabla_u\nabla_v J\bigr)w - \tfrac{1}{4}(\nabla_u J)(\nabla_v J)w \\
&\quad - \nabla_v\nabla_u w + \tfrac{1}{2}J\bigl(\nabla_v\nabla_u J\bigr)w + \tfrac{1}{4}(\nabla_v J)(\nabla_u J)w \\
&\quad + \nabla_{[u,v]}w - \tfrac{1}{2}J(\nabla_{[u,v]}J)w \\
&= R^{\nabla}(u,v)w - \tfrac{1}{2}J[R^{\nabla}(u,v), J]w - \tfrac{1}{4}[\nabla_u J, \nabla_v J]w \\
&= \tfrac{1}{2}R^{\nabla}(u,v)w - \tfrac{1}{2}JR^{\nabla}(u,v)Jw - \tfrac{1}{4}[\nabla_u J, \nabla_v J]w.
\end{aligned}$$

This implies

$$JR^{\widetilde{\nabla}}(u,v) = \tfrac{1}{2}JR^{\nabla}(u,v) + \tfrac{1}{2}R^{\nabla}(u,v)J - \tfrac{1}{4}J[\nabla_u J, \nabla_v J] \tag{2.24}$$

and hence

$$\mathrm{trace}\bigl(JR^{\widetilde{\nabla}}(u,v)\bigr) = \mathrm{trace}\bigl(JR^{\nabla}(u,v)\bigr) + \tfrac{1}{2}\mathrm{trace}\bigl((\nabla_u J)J(\nabla_v J)\bigr). \tag{2.25}$$

Thus $\mathrm{trace}(JR^{\widetilde{\nabla}}) = \tau_J^{\nabla}$ and this proves (2.17). Since $\widetilde{\nabla}$ is a Hermitian connection, the 2-form $\mathrm{trace}(\frac{1}{4\pi}JR^{\widetilde{\nabla}}) = \mathrm{trace}^c(\frac{1}{2\pi}JR^{\widetilde{\nabla}}) \in \Omega^2(M)$ is closed and represents the first Chern class of (TM, J).

If ω is closed, then $\nabla_{Jv}J = -J(\nabla_v J)$ for every vector field $v \in \mathrm{Vect}(M)$ by [30, Lemma 4.1.14], so the endomorphism $v \mapsto (\nabla_v J)u$ anti-commutes with J and therefore has trace zero. Hence $\lambda_J^{\nabla} = 0$. This proves part (iii) and Theorem 2.6. □

For $u \in \mathrm{Vect}(M)$ define $f_u := f_{\rho,u} := \mathrm{div}_\rho(u) \in \Omega^0(M)$, so that

$$f_u \rho = d\iota(u)\rho. \tag{2.26}$$

Theorem 2.7. *Let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form, let $J \in \mathscr{J}(M)$, and let $u \in \mathrm{Vect}(M)$. Then*

$$\Lambda_\rho(J, \mathcal{L}_u J) = 2\iota(u)\mathrm{Ric}_{\rho,J} - df_u \circ J + df_{Ju}. \tag{2.27}$$

Moreover, every smooth map $\mathbb{R}^2 \to \mathscr{J}(M) : (s,t) \mapsto J(s,t)$ satisfies

$$\partial_s \Lambda_\rho(J, \partial_t J) - \partial_t \Lambda_\rho(J, \partial_s J) + \tfrac{1}{2} d\mathrm{trace}\big((\partial_s J)J(\partial_t J)\big) = 0. \tag{2.28}$$

Proof. The proof has six steps.

Step 1. *We prove* (2.28).

Let $v \in \mathrm{Vect}(M)$. Then it follows from equation (2.13) that

$$\begin{aligned}
&\int_M \big(\partial_s \Lambda_\rho(J, \partial_t J) - \partial_t \Lambda_\rho(J, \partial_s J)\big) \wedge \iota(v)\rho \\
&= \tfrac{1}{2}\partial_s \int_M \mathrm{trace}\big((\partial_t J)J(\mathcal{L}_v J)\big)\rho - \tfrac{1}{2}\partial_t \int_M \mathrm{trace}\big((\partial_s J)J(\mathcal{L}_v J)\big)\rho \\
&= \tfrac{1}{2} \int_M \mathrm{trace}\big((\mathcal{L}_v \partial_t J)J(\partial_s J) + (\partial_t J)(\mathcal{L}_v J)(\partial_s J) + (\partial_t J)J(\mathcal{L}_v \partial_s J)\big)\rho \\
&= \tfrac{1}{2} \int_M \big(\mathcal{L}_v \mathrm{trace}\big((\partial_t J)J(\partial_s J)\big)\big)\rho = \tfrac{1}{2} \int_M d\mathrm{trace}\big((\partial_t J)J(\partial_s J)\big) \wedge \iota(v)\rho.
\end{aligned}$$

This proves Step 1.

Step 2. $d\Lambda_\rho(J, \mathcal{L}_u J) = 2d\iota(u)\mathrm{Ric}_{\rho,J} - d(df_u \circ J)$.

Let ϕ_t be the flow of u. Then $\phi_t^* \mathrm{Ric}_{\rho,J} = \mathrm{Ric}_{\phi_t^*\rho, \phi_t^* J}$ by part (i) of Theorem 2.6. Differentiate this equation and use parts (i) and (ii) of Theorem 2.6 to get $d\iota(u)\mathrm{Ric}_{\rho,J} = \widehat{\mathrm{Ric}}_\rho(J, \mathcal{L}_u J) + \tfrac{1}{2} d(df_u \circ J) = \tfrac{1}{2}\big(d\Lambda_\rho(J, \mathcal{L}_u J) + d(df_u \circ J)\big)$.

Step 3. *Suppose $\iota(u)\rho$ is exact. Then u satisfies* (2.27).

Choose $\alpha \in \Omega^{2\mathsf{n}-2}(M)$ such that $\iota(u)\rho = d\alpha$. Then, for all $v \in \mathrm{Vect}(M)$,

$$\begin{aligned}
&\int_M 2\iota(u)\mathrm{Ric}_{\rho,J} \wedge \iota(v)\rho = \int_M 2\mathrm{Ric}_{\rho,J} \wedge \iota(v)d\alpha = -\int_M 2d\iota(v)\mathrm{Ric}_{\rho,J} \wedge \alpha \\
&= -\int_M d\big(\Lambda_\rho(J, \mathcal{L}_v J) + df_v \circ J\big) \wedge \alpha = -\int_M \big(\Lambda_\rho(J, \mathcal{L}_v J) + df_v \circ J\big) \wedge \iota(u)\rho \\
&= \int_M \big(\Lambda_\rho(J, \mathcal{L}_u J) - df_{Ju}\big) \wedge \iota(v)\rho.
\end{aligned}$$

Here the third equality follows from Step 2. This proves Step 3.

Step 4. *Let $\lambda \in \Omega^0(M)$ and $\widehat{J} \in \Omega^{0,1}_J(M,TM)$. Then*

$$f_{\lambda u} = \lambda f_u + d\lambda(u), \tag{2.29}$$

$$\Lambda_\rho(J, \lambda\widehat{J}) = \lambda\Lambda_\rho(J,\widehat{J}) + d\lambda \circ \widehat{J}, \tag{2.30}$$

$$\mathcal{L}_{\lambda u} J = \lambda \mathcal{L}_u J + \widehat{J}_u, \qquad \widehat{J}_u := Ju \otimes d\lambda - u \otimes d\lambda \circ J. \tag{2.31}$$

(2.29) and (2.30) follow from the definitions and (2.31) follows from (2.23).

Step 5. *If u satisfies* (2.27) *and $\lambda \in \Omega^0(M)$, then λu satisfies* (2.27).

Let $\widehat{J}_u \in \Omega^{0,1}_J(M,TM)$ be as in (2.31). We prove the identity

$$\Lambda_\rho(J,\widehat{J}_u) + d\lambda \circ \mathcal{L}_u J = f_{Ju} d\lambda - f_u d\lambda \circ J + d\mathcal{L}_{Ju}\lambda - d\mathcal{L}_u\lambda \circ J. \tag{2.32}$$

To see this, let $v, w \in \mathrm{Vect}(M)$. Then

$$\begin{aligned}
(\nabla_w \widehat{J}_u)v &= \nabla_w\big(d\lambda(v)Ju - d\lambda(Jv)u\big) - d\lambda(\nabla_w v)Ju + d\lambda(J\nabla_w v)u \\
&= d\lambda(v)\nabla_w(Ju) + (\mathcal{L}_w\mathcal{L}_v\lambda)Ju - (\mathcal{L}_{\nabla_w v}\lambda)Ju \\
&\quad - d\lambda(Jv)\nabla_w u - (\mathcal{L}_w\mathcal{L}_{Jv}\lambda)u + (\mathcal{L}_{J\nabla_w v}\lambda)u.
\end{aligned}$$

Hence it follows from (2.22) and (2.23) that

$$\begin{aligned}
\mathrm{trace}\big((\nabla\widehat{J}_u)v\big) &= d\lambda(v)\mathrm{trace}\big(\nabla(Ju)\big) + \mathcal{L}_{Ju}\mathcal{L}_v\lambda - \mathcal{L}_{\nabla_{Ju}v}\lambda \\
&\quad - d\lambda(Jv)\mathrm{trace}\big(\nabla u\big) - \mathcal{L}_u\mathcal{L}_{Jv}\lambda + \mathcal{L}_{J\nabla_u v}\lambda \\
&= d\lambda(v)f_{Ju} - d\lambda(Jv)f_u + \mathcal{L}_v\mathcal{L}_{Ju}\lambda - \mathcal{L}_{Jv}\mathcal{L}_u\lambda \\
&\quad - d\lambda((\nabla_v J)u + (\mathcal{L}_u J)v).
\end{aligned}$$

Since $(\Lambda_\rho(J,\widehat{J}_u))(v) = \mathrm{trace}((\nabla\widehat{J}_u)v) + d\lambda((\nabla_v J)u)$, this proves (2.32). Now suppose u satisfies (2.27) and let $v \in \mathrm{Vect}(M)$. Then, by Step 4, we have

$$\begin{aligned}
\big(\Lambda_\rho(J,\mathcal{L}_{\lambda u}J)\big)(v) &= \big(\Lambda_\rho(J,\lambda\mathcal{L}_u J + \widehat{J}_u)\big)(v) \\
&= \lambda\big(\Lambda_\rho(J,\mathcal{L}_u J)\big)(v) + \big(\Lambda_\rho(J,\widehat{J}_u)\big)(v) + d\lambda((\mathcal{L}_u J)v) \\
&= \lambda\big(2\mathrm{Ric}_{\rho,J}(u,v) - df_u(Jv) + df_{Ju}(v)\big) \\
&\quad + d\lambda(v)f_{Ju} - d\lambda(Jv)f_u + \mathcal{L}_v\mathcal{L}_{Ju}\lambda - \mathcal{L}_{Jv}\mathcal{L}_u\lambda \\
&= 2\mathrm{Ric}_{\rho,J}(\lambda u, v) - df_{\lambda u}(Jv) + df_{\lambda Ju}(v).
\end{aligned}$$

Here the third equality uses (2.32). This proves Step 5.

Step 6. *We prove* (2.27).

There exist finitely many exact divergence-free vector fields u_i and smooth functions λ_i such that $u = \sum_i \lambda_i u_i$. For each i the vector field $\lambda_i u_i$ satisfies (2.27) by Steps 3 and 5. Hence so does u and this proves Theorem 2.7. □

Equation (2.27) is equivalent to the formula

$$\Omega_{\rho,J}(\mathcal{L}_u J, \mathcal{L}_v J) = \int_M \bigl(2\mathrm{Ric}_{\rho,J}(u,v) + f_u f_{Jv} - f_{Ju} f_v\bigr)\rho \tag{2.33}$$

for $u, v \in \mathrm{Vect}(M)$. For exact divergence-free vector fields u, v this is the analogue of the identity $\omega(L_x\xi, L_x\eta) = \langle \mu(x), [\xi,\eta]\rangle$ for Hamiltonian group actions on finite-dimensional symplectic manifolds. The analogue in the scalar curvature setting is discussed in Remark 2.10 below.

Scalar curvature

Let (M,ω) be a $2\mathsf{n}$-dimensional closed connected symplectic manifold and denote by

$$\mathscr{J}(M,\omega) := \left\{ J \in \Omega^0(M, \mathrm{End}(TM)) \;\middle|\; \begin{array}{l} J^2 = -\mathbb{1} \text{ and } J^*\omega = \omega \\ \text{and } \omega(\widehat{x}, J\widehat{x}) > 0 \\ \text{for all } \widehat{x} \in T_xM \setminus \{0\} \end{array} \right\}$$

the space of all almost complex structures that are compatible with ω. This is an infinite-dimensional Kähler submanifold of $\mathscr{J}(M)$ with the tangent spaces $T_J\mathscr{J}(M,\omega) = \{\widehat{J} \in \Omega^{0,1}_J(M,TM) \,|\, \omega(\widehat{J}\cdot,\cdot) + \omega(\cdot,\widehat{J}\cdot) = 0\}$, the symplectic form Ω_ρ in (2.6), and the complex structure $\widehat{J} \mapsto -J\widehat{J}$.

Definition 2.8 (**Scalar Curvature**). *Let ω be a symplectic form on M, let J be an ω-compatible almost complex structure on M, let ∇ be the Levi-Civita connection of the metric $\langle\cdot,\cdot\rangle = \omega(\cdot, J\cdot)$, and let $\widetilde{\nabla} := \nabla - \frac{1}{2}J(\nabla J)$. Define the* **Ricci form** *of (ω, J) by $\mathrm{Ric}_{\omega,J} := \mathrm{Ric}_{\omega^\mathsf{n}/\mathsf{n}!,J} = \frac{1}{2}\mathrm{trace}(JR^{\widetilde{\nabla}})$ and define the* **scalar curvature** *by*

$$S_{\omega,J} := 2\langle \mathrm{Ric}_{\omega,J}, \omega\rangle := \frac{2\mathrm{Ric}_{\omega,J} \wedge \omega^{\mathsf{n}-1}/(\mathsf{n}-1)!}{\omega^\mathsf{n}/\mathsf{n}!} \in \Omega^0(M). \tag{2.34}$$

By Theorem 2.6 the scalar curvature $S_{\omega,J}$ in (2.34) satisfies

$$\int_M S_{\omega,J} \frac{\omega^\mathsf{n}}{\mathsf{n}!} = 4\pi \left\langle c_1(TM,J) \smile \frac{[\omega]^{\mathsf{n}-1}}{(\mathsf{n}-1)!}, [M] \right\rangle \tag{2.35}$$

and $\phi^* S_{\omega,J} = S_{\phi^*\omega,\phi^*J}$ for every diffeomorphism $\phi : M \to M$. The following result was proved by Donaldson [12], and independently by Fujiki [17] (in the integrable case) and Quillen (for Riemann surfaces).

Corollary 2.9 (Fujiki–Quillen–Donaldson). *The map $J \mapsto S_{\omega,J}$ is an equivariant moment map for the action of* $\mathrm{Ham}(M,\omega)$ *on* $\mathscr{J}(M,\omega)$*, i.e. if $H \in \Omega^0(M)$ and $v_H \in \mathrm{Vect}(M)$ is the Hamiltonian vector field defined by $\iota(v_H)\omega = dH$, then every smooth path $\mathbb{R} \to \mathscr{J}(M,\omega) : t \mapsto J_t$ satisfies*

$$\frac{d}{dt}\int_M S_{\omega,J_t} H \frac{\omega^{\mathsf{n}}}{\mathsf{n}!} = \tfrac{1}{2}\int_M \mathrm{trace}\Big((\partial_t J_t)J_t(\mathcal{L}_{v_H}J_t)\Big)\frac{\omega^{\mathsf{n}}}{\mathsf{n}!}. \tag{2.36}$$

Proof. Define $J := J_0$, $\widehat{J} := \frac{d}{dt}\big|_{t=0} J_t$, and $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$. Then

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\int_M S_{\omega,J_t} H \frac{\omega^{\mathsf{n}}}{\mathsf{n}!} &= \int_M 2H\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) \wedge \frac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!} \\
&= \int_M H d\Lambda_\rho(J,\widehat{J}) \wedge \frac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!} \\
&= \int_M \Lambda_\rho(J,\widehat{J}) \wedge \iota(v_H)\rho.
\end{aligned}$$

Hence the assertion follows from Theorem 2.6. □

Remark 2.10. For a closed connected symplectic $2\mathsf{n}$-manifold (M,ω) with volume form $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$, an almost complex structure $J \in \mathscr{J}(M,\omega)$, and two Hamiltonian functions $F, G : M \to \mathbb{R}$ equation (2.33) takes the form

$$\begin{aligned}
\Omega_{\rho,J}(\mathcal{L}_{v_F}J, \mathcal{L}_{v_G}J) &= \int_M 2\mathrm{Ric}_{\omega,J}(v_F, v_G)\rho \\
&= \int_M 2\mathrm{Ric}_{\omega,J} \wedge \iota(v_G)\iota(v_F)\rho \\
&= \int_M 2\mathrm{Ric}_{\omega,J} \wedge \iota(v_G)\left(dF \wedge \frac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!}\right) \\
&= \int_M 2\mathrm{Ric}_{\omega,J} \wedge \{F,G\}\frac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!} \\
&= \int_M S_{\omega,J}\{F,G\}\rho.
\end{aligned} \tag{2.37}$$

Here $\{F,G\} := \omega(v_F, v_G)$ denotes the Poisson bracket. If the scalar curvature is constant, equation (2.37) implies that $\mathcal{L}_{v_F}J$ and $J\mathcal{L}_{v_G}J$ are L^2 orthogonal and hence $\|\mathcal{L}_{v_F}J + J\mathcal{L}_{v_G}J\|^2 = \|\mathcal{L}_{v_F}\|^2 + \|\mathcal{L}_{v_G}J\|^2$ for all $F, G \in \Omega^0(M)$. If J is integrable, the scalar curvature is constant, and $H^1(M;\mathbb{R}) = 0$, this in turn implies that the Lie algebra of holomorphic vector fields is the complexification of the Lie algebra of Killing vector fields and is therefore reductive (Matsushima's Theorem).

Symplectic complements

The next theorem examines symplectic complements in $T_J\mathscr{J}(M)$. It shows that the regular part of the Marsden–Weinstein quotient

$$\mathscr{W}_0(M,\rho) := \{J \in \mathscr{J}(M) \,|\, \mathrm{Ric}_{\rho,J} = 0\}/\mathrm{Diff}^{\mathrm{ex}}(M,\rho) \tag{2.38}$$

is an infinite-dimensional symplectic manifold.

Theorem 2.11 (Complements). *Let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form and $J \in \mathscr{J}(M)$, $\widehat{J} \in \Omega^{0,1}_J(M,TM)$, $\widehat{\lambda} \in \Omega^1(M)$. Then the following holds.*

(i) *There exists a $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ such that $\Lambda_\rho(J,\widehat{J}') = \widehat{\lambda}$ if and only if $\int_M \widehat{\lambda} \wedge \iota(v)\rho = 0$ for all $v \in \mathrm{Vect}(M)$ with $\mathcal{L}_vJ = 0$.*

(ii) *There exists a $v \in \mathrm{Vect}(M)$ with $\mathcal{L}_vJ = \widehat{J}$ if and only if $\Omega_{\rho,J}(\widehat{J},\widehat{J}') = 0$ for all $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ with $\Lambda_\rho(J,\widehat{J}') = 0$.*

(iii) *There exists a $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ such that $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}') = d\widehat{\lambda}$ if and only if $\int_M d\widehat{\lambda} \wedge \alpha = 0$ for all $\alpha \in \Omega^{2\mathsf{n}-2}(M)$ with $\mathcal{L}_{v_\alpha}J = 0$.*

(iv) *There exists a $v \in \mathrm{Vect}(M)$ such that $\mathcal{L}_vJ = \widehat{J}$ and $\iota(v)\rho$ is exact if and only if $\Omega_{\rho,J}(\widehat{J},\widehat{J}') = 0$ for all $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ such that $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}') = 0$.*

Proof. See page 19. □

To prove Theorem 2.11 it is convenient to choose a nondegenerate 2-form $\omega \in \Omega^2(M)$ that is compatible with J and satisfies $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$. Let ∇ be the Levi-Civita connection of the Riemannian metric $\langle\cdot,\cdot\rangle = \omega(\cdot,J\cdot)$ and define the linear operator $\bar\partial_J : \Omega^0(M,TM) \to \Omega^{0,1}_J(M,TM)$ by

$$(\bar\partial_J v)u := -\tfrac{1}{2}J(\mathcal{L}_vJ)u = \tfrac{1}{2}\big(\nabla_u v + J\nabla_{Ju}v - J(\nabla_vJ)u\big) \tag{2.39}$$

for $u,v \in \mathrm{Vect}(M)$. Let $\bar\partial_J^*$ be the formal adjoint operator of $\bar\partial_J$ with respect to the standard L^2-inner products. Then both $\bar\partial_J$ and $\bar\partial_J^*$ are bounded linear operators with closed images between appropriate Sobolev completions.

Lemma 2.12. *Let $\widehat{J} \in \Omega^{0,1}_J(M,TM)$. Then $\Lambda_\rho(J,\widehat{J}) = \iota(J\bar\partial_J^*\widehat{J}^*)\omega$.*

Proof. Let $v \in \mathrm{Vect}(M)$. Then part (ii) of Theorem 2.6 yields

$$\begin{aligned}\int_M \Lambda_\rho(J,\widehat{J}) \wedge \iota(v)\rho &= \tfrac{1}{2}\int_M \mathrm{trace}\big(\widehat{J}J\mathcal{L}_vJ\big)\rho = -\langle \widehat{J}^*, \bar\partial_J v\rangle_{L^2} \\ &= -\langle \bar\partial_J^*\widehat{J}^*, v\rangle_{L^2} = \int_M \omega(J\bar\partial_J^*\widehat{J}^*, v)\rho = \int_M \iota(J\bar\partial_J^*\widehat{J}^*)\omega \wedge \iota(v)\rho.\end{aligned} \tag{2.40}$$

This proves Lemma 2.12. □

Proof of Theorem 2.11. Choose ω as in Lemma 2.12. We prove part (i). The condition is necessary by (2.13). Conversely, assume $\int_M \widehat{\lambda}\wedge\iota(v)\rho = 0$ for all $v\in\mathrm{Vect}(M)$ with $\mathcal{L}_vJ=0$. Define the vector field u by $\iota(Ju)\omega := \widehat{\lambda}$. Then $\langle u,v\rangle_{L^2} = \int_M \omega(u,Jv)\rho = -\int_M \widehat{\lambda}\wedge\iota(v)\rho = 0$ for all $v\in\ker\bar\partial_J$. Hence there exists a $\widehat{J}'\in\Omega^{0,1}_J(M,TM)$ such that $\bar\partial_J^*(\widehat{J}')^* = u$ and so by Lemma 2.12 we have $\widehat{\lambda} = \iota(Ju)\omega = \iota(J\bar\partial_J^*(\widehat{J}')^*)\omega = \Lambda_\rho(J,\widehat{J}')$. This proves (i).

We prove part (ii). The condition is necessary by (2.13). Conversely, assume $\Omega_{\rho,J}(\widehat{J},\widehat{J}')=0$ for all $\widehat{J}'\in\Omega^{0,1}_J(M,TM)$ that satisfy $\Lambda_\rho(J,\widehat{J}')=0$. Let $v\in\mathrm{Vect}(M)$ with $\bar\partial_J^*\big(\bar\partial_Jv+\frac{1}{2}J\widehat{J}\big)=0$ and define $\widehat{J}' := (\bar\partial_Jv+\frac{1}{2}J\widehat{J})^*$. Then $\bar\partial_J^*(\widehat{J}')^*=0$, hence $\Lambda_\rho(J,\widehat{J}')=0$ by (2.40), and hence $\Omega_{\rho,J}(\widehat{J},\widehat{J}')=0$. This implies $\int_M|\widehat{J}'|^2\rho = \int_M \mathrm{trace}\big(\widehat{J}'(\bar\partial_Jv+\frac{1}{2}J\widehat{J})\big)\rho = \langle(\widehat{J}')^*,\bar\partial_Jv\rangle_{L^2} = 0$. Thus $\widehat{J}'=0$ and so $\widehat{J}=2J\bar\partial_Jv=\mathcal{L}_vJ$ by (2.39). This proves (ii).

We prove part (iii). The condition is necessary by (2.15). Conversely, assume $\int_M d\widehat{\lambda}\wedge\alpha=0$ for all $\alpha\in\Omega^{2\mathsf{n}-2}(M)$ with $\mathcal{L}_{v_\alpha}J=0$. Choose a basis $u_1,\dots,u_\ell$ of $V:=\{u\in\mathrm{Vect}(M)\,|\,\mathcal{L}_uJ=0\}$ such that $u_{k+1},\dots,u_\ell$ form a basis of $\{u\in V\,|\,\iota(u)\rho\in\mathrm{im}d\}$. Then $\iota(u_1)\rho,\dots,\iota(u_k)\rho$ are linearly independent in the quotient $\Omega^{2\mathsf{n}-1}(M)/\mathrm{im}d$. Hence, by Poincaré duality, there exist closed 1-forms $\lambda_1,\dots,\lambda_k\in\Omega^1(M)$ such that $\int_M\lambda_i\wedge\iota(u_j)\rho=\delta_{ij}$ for $i,j\le k$. Define $\widehat{\lambda}' := \widehat{\lambda}-\sum_{i=1}^k(\int_M\widehat{\lambda}\wedge\iota(u_i)\rho)\lambda_i$. Then we have $\int_M\widehat{\lambda}'\wedge\iota(u_j)\rho=0$ for $j=1,\dots,\ell$. Hence by (i) there exists a 1-form $\widehat{J}'\in\Omega^{0,1}_J(M,TM)$ such that $\Lambda_\rho(J,\widehat{J}')=2\widehat{\lambda}'$. Thus $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}')=d\widehat{\lambda}'=d\widehat{\lambda}$ and this proves (iii).

We prove part (iv). The condition is necessary by (2.15). Conversely, assume $\Omega_{\rho,J}(\widehat{J},\widehat{J}')=0$ for all $\widehat{J}'\in\Omega^{0,1}_J(M,TM)$ such that $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}')=0$. Then by (ii) there is a $v\in\mathrm{Vect}(M)$ with $\mathcal{L}_vJ=\widehat{J}$. Choose u_i,λ_i as in the proof of part (iii) and define $v_0:=v-\sum_{i=1}^k x_iu_i$, $x_i:=\int_M\lambda_i\wedge\iota(v)\rho$. Then $\mathcal{L}_{v_0}J=\widehat{J}$. We prove that $\iota(v_0)\rho$ is exact. To see this, let $\widehat{\lambda}\in\Omega^1(M)$ be any closed 1-form and define $\widehat{\lambda}' := \widehat{\lambda}-\sum_{i=1}^k y_i\lambda_i$, $y_i:=\int_M\widehat{\lambda}\wedge\iota(u_i)\rho$. Then $\int_M\widehat{\lambda}'\wedge\iota(u_j)\rho=0$ for $j=1,\dots,\ell$. Hence by (i) there exists a 1-form $\widehat{J}'\in\Omega^{0,1}_J(M,TM)$ such that $\Lambda_\rho(J,\widehat{J}')=\widehat{\lambda}'$. Thus $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}')=0$, hence $\Omega_{\rho,J}(\widehat{J},\widehat{J}')=0$, and therefore

$$\int_M\widehat{\lambda}\wedge\iota(v_0)\rho=\int_M\widehat{\lambda}\wedge\iota(v)\rho-\sum_{i=1}^k x_iy_i=\int_M\widehat{\lambda}'\wedge\iota(v)\rho=\Omega_{\rho,J}\big(\widehat{J}',\widehat{J}\big)=0.$$

This shows that $\iota(v_0)\rho$ is exact and completes the proof of Theorem 2.11. □

3 The integrable case

Let M be a closed connected oriented $2\mathsf{n}$-manifold. In this section we restrict attention to (integrable) complex structures that are compatible with the orientation. Denote the space of such complex structures by $\mathscr{J}_{\mathrm{int}}(M)$.

The Ricci form in the integrable case

Let $J \in \mathscr{J}_{\mathrm{int}}(M)$. Then (TM, J) is a holomorphic vector bundle with the Cauchy–Riemann operator $\bar{\partial}_J : \Omega_J^{0,q}(M,TM) \to \Omega_J^{0,q+1}(M,TM)$. It satisfies

$$2J\bar{\partial}_{J,u}v = J\nabla_u v - \nabla_{Ju}v = (\mathcal{L}_v J)u \tag{3.1}$$

and

$$\begin{aligned} 2J\bar{\partial}_J\widehat{J}(u,v) &= J(\nabla_u\widehat{J})v - J(\nabla_v\widehat{J})u - J(\nabla_{Ju}\widehat{J})Jv + J(\nabla_{Jv}\widehat{J})Ju \\ &= -\tfrac{d}{dt}\Big|_{t=0} N_{J_t}(u,v) \end{aligned} \tag{3.2}$$

for all $u, v \in \mathrm{Vect}(M)$, all $\widehat{J} \in \Omega_J^{0,1}(M,TM)$, and every smooth path of almost complex structures $\mathbb{R} \to \mathscr{J}(M) : t \mapsto J_t$ with $J_0 = J$ and $\frac{d}{dt}|_{t=0}J_t = \widehat{J}$. Here ∇ is a torsion-free connection on TM with $\nabla J = 0$, equation (3.1) follows from (2.23), and (3.2) follows by differentiating (A.2).

Next observe that

$$d(df \circ J)(u,v) - d(df \circ J)(Ju, Jv) = df(JN_J(u,v)). \tag{3.3}$$

for all $f \in \Omega^0(M)$ and all $u, v \in \mathrm{Vect}(M)$. Hence an almost complex structure J is integrable if and only if the 2-form $d(df \circ J)$ is of type $(1,1)$ for all $f \in \Omega^0(M)$. Theorem 3.1 below uses the Bott–Chern cohomology group $H^{1,1}_{\mathrm{BC}}(M,J) := (\ker d \cap \Omega_J^{1,1}(M))/\{d(df \circ J) \mid f \in \Omega^0(M)\}$ [1, 2, 3, 5]. It shows that $\mathrm{Ric}_{\rho,J}$ is the standard Ricci form in the integrable case.

Theorem 3.1. *Let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form, let $J \in \mathscr{J}_{\mathrm{int}}(M)$, and let ∇ be a torsion-free ρ-connection with $\nabla J = 0$. The following holds.*

(i) $\mathrm{Ric}_{\rho,J} = \frac{1}{2}\mathrm{trace}(JR^\nabla)$ *is a closed $(1,1)$-form and $\frac{1}{2\pi}\mathrm{Ric}_{\rho,J}$ represents the first Bott–Chern class of the holomorphic tangent bundle (TM, J).*

(ii) *There exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ such that $\mathrm{Ric}_{\rho,\phi^*J} = 0$ if and only if the first Bott–Chern class of (TM, J) vanishes.*

(iii) *Let $\phi : M \to M$ be an orientation preserving diffeomorphism and suppose that $\mathrm{Ric}_{\rho,J} = \mathrm{Ric}_{\rho,\phi^*J} = 0$. Then $\phi^*\rho = \rho$. If in addition ϕ is isotopic to the identity, then $\phi \in \mathrm{Diff}_0(M,\rho)$.*

Proof. The formula $\mathrm{Ric}_{\rho,J} = \frac{1}{2}\mathrm{trace}(JR^\nabla)$ follows from Definition 2.5. Moreover, $\mathrm{Ric}_{\rho,J}$ is independent of the choice of ∇ by part (i) of Theorem 2.6, is closed and represents the cohomology class $2\pi c_1(TM,J) \in H^2(M;\mathbb{R})$ by part (iii) of Theorem 2.6, and is a $(1,1)$-form by Lemma A.2.

Now choose a nondegenerate 2-form $\omega \in \Omega^2(M)$, compatible with J, such that ρ is the volume form of the metric $\langle\cdot,\cdot\rangle = \omega(\cdot, J\cdot)$. Let ∇ be the Levi–Civita connection of this metric and define

$$\widetilde{\nabla} := \nabla - \tfrac{1}{2}J\nabla J, \qquad \widehat{\widehat{\nabla}} := \widetilde{\nabla} - \tfrac{1}{4}(A - A^*), \tag{3.4}$$

where $A \in \Omega^1(M, \mathrm{End}(TM))$ is the endomorphism valued 1-form defined by

$$A(u)v := J(\nabla_v J)u + (\nabla_{Jv}J)u \tag{3.5}$$

for $u, v \in \mathrm{Vect}(M)$. Then, for all $u \in \mathrm{Vect}(M)$,

$$A(u)J = JA(u) = -A(Ju), \qquad A(u)^*J = JA(u)^* = A(Ju)^*. \tag{3.6}$$

This shows that $\widehat{\widehat{\nabla}}$ is a Hermitian connection on TM and induces the same Cauchy–Riemann operator on TM as the connection $\widetilde{\nabla} - \frac{1}{4}A$. The latter preserves J by (3.6) and is torsion-free by (A.2) (but it need not preserve ρ). Hence, for all $u, v \in \mathrm{Vect}(M)$, we have

$$\bar\partial^{\widehat{\widehat{\nabla}}}_{J,u} v = \bar\partial^{\widetilde{\nabla} - \frac{1}{4}A}_{J,u} v = \bar\partial_{J,u} v = \tfrac{1}{2}\Big(\nabla_u v + J\nabla_{Ju} v - J(\nabla_v J)u\Big).$$

Here the last equality holds because ∇ is torsion-free and J is integrable. Thus $\widehat{\widehat{\nabla}}$ is the unique Hermitian connection on TM with $\bar\partial^{\widehat{\widehat{\nabla}}}_J = \bar\partial_J$.

The curvature tensor of $\widehat{\widehat{\nabla}}$ is given by

$$R^{\widehat{\widehat{\nabla}}} = R^{\widetilde{\nabla}} + \tfrac{1}{4}d^{\widetilde{\nabla}}(A^* - A) + \tfrac{1}{32}[(A^* - A) \wedge (A^* - A)].$$

Since J commutes with $A^* - A$ by (3.6), we obtain

$$\begin{aligned}
\mathrm{trace}(JR^{\widehat{\widehat{\nabla}}}) &= \mathrm{trace}(JR^{\widetilde{\nabla}}) + \tfrac{1}{4}\mathrm{trace}\big(Jd^{\widetilde{\nabla}}(A^* - A)\big) \\
&= \mathrm{trace}(JR^{\widetilde{\nabla}}) + \tfrac{1}{4}\mathrm{trace}\big(d^{\widetilde{\nabla}}(JA^* - JA)\big) \\
&= \mathrm{trace}(JR^{\widetilde{\nabla}}) + \tfrac{1}{2}d\big(\mathrm{trace}(A) \circ J\big) \\
&= \mathrm{trace}(JR^{\widetilde{\nabla}}) + d\lambda^\nabla_J = 2\mathrm{Ric}_{\rho,J}.
\end{aligned}$$

Here the third equality follows from (3.6) and the fact that the endomorphisms $A(Ju)$ and $A(Ju)^*$ have the same trace, the fourth equality uses the fact that the two summands in $v \mapsto A(Ju)v = (\nabla_v J)u + (\nabla_{Jv}J)Ju$ have the same trace, both equal to $\lambda^\nabla_J(u)$ (see equation (2.8)), and the last equality follows from part (iii) of Theorem 2.6. This proves (i).

We prove part (ii). Let $\phi \in \mathrm{Diff}_0(M)$ such that $\mathrm{Ric}_{\rho,\phi^*J} = 0$. Then we have $\mathrm{Ric}_{\phi_*\rho,J} = \phi_*\mathrm{Ric}_{\rho,\phi^*J} = 0$ by part (i) of Theorem 2.6. Define the function $f \in \Omega^0(M)$ by $e^{-f}\rho := \phi_*\rho$. Then

$$\mathrm{Ric}_{\rho,J} = \mathrm{Ric}_{\rho,J} - \mathrm{Ric}_{\phi_*\rho,J} = \mathrm{Ric}_{\rho,J} - \mathrm{Ric}_{e^{-f}\rho,J} = \tfrac{1}{2}d(df \circ J).$$

Here the last equality uses (2.11). Since $\mathrm{Ric}_{\rho,J}$ represents 2π times the first Bott–Chern class of (TM, J) by (i), this shows that $c_{1,\mathrm{BC}}(TM, J) = 0$.

Conversely, assume $c_{1,\mathrm{BC}}(TM, J) = 0$. Then, by part (i), there exists a smooth function $f : M \to \mathbb{R}$ such that $\mathrm{Ric}_{\rho,J} = \frac{1}{2}d(df \circ J)$. Choose $c \in \mathbb{R}$ such that $e^c \int_M \rho = \int_M e^{-f}\rho$ and replace f by $f + c$ to obtain $\int_M e^{-f}\rho = \int_M \rho$. Then by Moser isotopy there exists a smooth isotopy $\{\phi_t\}_{0\le t\le 1}$ of M such that $\phi_0 = \mathrm{id}$ and $\phi_t^*\big((1-t)\rho + te^{-f}\rho\big) = \rho$ for $0 \le t \le 1$. Thus the diffeomorphism $\phi := \phi_1$ is isotopic to the identity and satisfies $\phi^*(e^{-f}\rho) = \rho$. Hence

$$\mathrm{Ric}_{\rho,\phi^*J} = \mathrm{Ric}_{\phi^*(e^{-f}\rho),\phi^*J} = \phi^*\mathrm{Ric}_{e^{-f}\rho,J} = \phi^*\Big(\mathrm{Ric}_{\rho,J} - \tfrac{1}{2}d(df \circ J)\Big) = 0.$$

This proves (ii).

We prove part (iii). Let $\phi \in \mathrm{Diff}(M)$ be orientation preserving, assume that $\mathrm{Ric}_{\rho,\phi^*J} = \mathrm{Ric}_{\rho,J} = 0$, and define $f \in \Omega^0(M)$ by $e^{-f}\rho := \phi_*\rho$. Then

$$\tfrac{1}{2}d(df \circ J) = \mathrm{Ric}_{\rho,J} - \mathrm{Ric}_{e^{-f}\rho,J} = -\mathrm{Ric}_{\phi_*\rho,J} = -\phi_*\mathrm{Ric}_{\rho,\phi^*J} = 0.$$

Thus f is constant. Since $\int_M e^{-f}\rho = \int_M \phi_*\rho = \int_M \rho$, it follows that $f = 0$ and so $\phi_*\rho = \rho$. Moreover, $\mathrm{Diff}_0(M,\rho) = \mathrm{Diff}(M,\rho) \cap \mathrm{Diff}_0(M)$ by Moser isotopy. This proves part (iii) and Theorem 3.1. □

Example 3.2. Assume $\mathsf{n} = 1$, suppose M has genus $g \ge 1$, define $V := \int_M \rho$ and $c := 2\pi(2-2g)V^{-1} \le 0$, and let $K_{\rho,J} := \mathrm{Ric}_{\rho,J}/\rho$ be the Gaußian curvature. Then the moment map

$$\mathscr{J}(M) \to \Omega^2(M) : J \mapsto 2(\mathrm{Ric}_{\rho,J} - c\rho) = 2(K_{\rho,J} - c)\rho$$

is $\mathscr{G}$-equivariant and takes values in the space of exact 2-forms. The uniformization theorem for Riemann surfaces asserts that for every $J \in \mathscr{J}(M)$ there exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ such that $K_{\phi_*\rho,J} = c$ and therefore $\mathrm{Ric}_{\rho,\phi^*J} = c\rho$. Moreover, if $\mathrm{Ric}_{\rho,J} = \mathrm{Ric}_{\rho,\phi^*J} = c\rho$ for some orientation preserving diffeomorphism ϕ and $\phi_*\rho =: e^f\rho$, then $\frac{1}{2}d(df \circ J) = c(e^f - 1)\rho$. Hence $d^*df = 2c(e^f - 1)$ and this implies $\int_M |df|^2\rho = 2c\int_M f(e^f - 1)\rho \le 0$. Thus f is constant and $\int_M e^f\rho = \int_M \phi_*\rho = \int_M \rho$, so $f \equiv 0$ and $\phi^*\rho = \rho$.

Let (M, ω, J) be a closed connected Kähler manifold. For a Kähler potential $h : M \to \mathbb{R}$ (with mean value zero) let $\omega_h := \omega + \mathbf{i}\bar\partial\partial h = \omega + \frac{1}{2}d(dh \circ J)$ be the associated symplectic form and let $\rho_h := \omega_h^{\mathsf{n}}/\mathsf{n}!$. The Calabi conjecture asserts that the map $h \mapsto \mathrm{Ric}_{\rho_h,J}$ is a bijection onto the space of closed $(1,1)$-forms representing the cohomology class $2\pi c_1(TM, J)$. Injectivity was proved by Calabi [8, 9] and surjectivity by Yau [44, 45].

Corollary 3.3 (Calabi–Yau). *Let (M, ω, J) be a closed connected Kähler manifold and let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form with $\int_M \rho = \int_M \omega^{\mathsf{n}}/\mathsf{n}!$. Then the following holds.*

(i) *There exists a unique Kähler potential $h : M \to \mathbb{R}$ such that $\rho_h = \rho$.*

(ii) *Assume $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$ and $c_1(TM, J) = 0 \in H^2(M; \mathbb{R})$. Then there exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ such that*

$$\mathrm{Ric}_{\rho,\phi^*J} = 0 \text{ and } \phi^*J \text{ is compatible with } \omega. \tag{3.7}$$

(iii) *Assume $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$ and $\mathrm{Ric}_{\rho,J} = 0$. Suppose $\phi \in \mathrm{Diff}(M)$ satisfies (3.7) and the 2-form $\phi^*\omega - \omega$ is exact. Then $\phi^*\omega = \omega$.*

Proof. We prove part (i). By part (i) of Theorem 3.1, $\mathrm{Ric}_{\rho,J}$ is a closed $(1,1)$-form representing the cohomology class $2\pi c_1(TM, J)$. Hence, by Yau's existence theorem [44, 45] and Calabi's uniqueness theorem [8, 9], there exists a unique Kähler potential h such that $\mathrm{Ric}_{\rho_h,J} = \mathrm{Ric}_{\rho,J}$. Since $\int_M \rho_h = \int_M \rho$ by assumption, this implies $\rho_h = \rho$ by equation (2.11) in part (i) of Theorem 2.6.

We prove part (ii). By assumption and part (i) of Theorem 3.1 $\mathrm{Ric}_{\rho,J}$ is an exact $(1,1)$-form. Since J admits a compatible Kähler form, this implies that there exists a function $f \in \Omega^0(M)$ such that

$$\mathrm{Ric}_{\rho,J} = \tfrac{1}{2}d(df \circ J), \qquad \int_M e^{-f}\rho = \int_M \rho.$$

Hence $\mathrm{Ric}_{e^{-f}\rho,J} = 0$ by part (i) of Theorem 2.6. Now it follows from (i) that there exists a Kähler potential h such that $\rho_h = e^{-f}\rho$. Since ω_h and ω are compatible with J, Moser isotopy yields a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ with $\phi^*\omega_h = \omega$. Thus ϕ^*J is compatible with ω and $\phi^*\rho_h = \rho$. This implies $\mathrm{Ric}_{\rho,\phi^*J} = \phi^*\mathrm{Ric}_{\rho_h,J} = 0$ by part (i) of Theorem 2.6.

To prove (iii), note that $(\phi^{-1})^*\omega$ is compatible with J and represents the cohomology class of ω. Thus there is a Kähler potential h with $\omega_h = (\phi^{-1})^*\omega$. Hence $\phi^*\rho_h = \rho$ and $\phi^*\mathrm{Ric}_{\rho_h,J} = \mathrm{Ric}_{\rho,\phi^*J} = 0$ by part (i) of Theorem 2.6. Thus $h = 0$ by Calabi uniqueness, so $\phi^*\omega = \omega$. This proves Corollary 3.3. $\square$

Ricci-flat Kähler manifolds

Let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form. Then the symplectic form Ω_ρ on $\mathscr{J}(M)$ is a $(1,1)$-form for the complex structure $\widehat{J} \mapsto -J\widehat{J}$. However, the resulting symmetric bilinear form $\langle \widehat{J}_1, \widehat{J}_2 \rangle_{\rho,J} = \frac{1}{2}\int_M \mathrm{trace}(\widehat{J}_1\widehat{J}_2)\rho$ is indefinite, so $\mathscr{J}(M)$ is not Kähler and complex submanifolds need not be symplectic. An example is the space of (integrable) complex structures with real first Chern class zero and nonempty Kähler cone. It is denoted by

$$\mathscr{J}_{\mathrm{int},0}(M) := \left\{ J \in \mathscr{J}_{\mathrm{int}}(M) \;\middle|\; \begin{array}{l} c_1(TM,J) = 0 \in H^2(M;\mathbb{R}) \\ \text{and } J \text{ admits a Kähler form} \end{array} \right\}.$$

Its tangent space at J is the kernel of $\bar{\partial}_J : \Omega^{0,1}_J(M,TM) \to \Omega^{0,2}_J(M,TM)$.

Theorem 3.4. *Let $J \in \mathscr{J}_{\mathrm{int},0}(M)$ with $\mathrm{Ric}_{\rho,J} = 0$ and let $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ such that $\bar{\partial}_J\widehat{J} = 0$. Then the following holds.*

(i) *$\Omega_{\rho,J}(\widehat{J},\widehat{J}') = 0$ for all $\widehat{J}'$ with $\bar{\partial}_J\widehat{J}' = 0$ if and only if there exists a vector field v such that $f_v = f_{Jv} = 0$ and $\mathcal{L}_v J = \widehat{J}$.*

(ii) *Assume $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) = 0$. Then $\Omega_{\rho,J}(\widehat{J},\widehat{J}') = 0$ for all $\widehat{J}'$ with $\bar{\partial}_J\widehat{J}' = 0$ and $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}') = 0$ if and only if there exists a vector field v such that $f_v = 0$ and $\mathcal{L}_v J = \widehat{J}$ or, equivalently, there exists an $\alpha \in \Omega^{2\mathsf{n}-2}(M)$ with $\mathcal{L}_{v_\alpha} J = \widehat{J}$.*

Proof. See page 28. □

Define $\mathscr{J}_{\mathrm{int},0}(M,\rho) := \{J \in \mathscr{J}_{\mathrm{int},0}(M) \,|\, \mathrm{Ric}_{\rho,J} = 0\}$. Part (ii) of Theorem 3.4 (compare with part (iv) of Theorem 2.11) implies that the Teichmüller space $\mathscr{T}_0(M,\rho) := \mathscr{J}_{\mathrm{int},0}(M,\rho)/\mathrm{Diff}_0(M,\rho)$ is a symplectic submanifold of the infinite-dimensional symplectic quotient $\mathscr{W}_0(M,\rho)$ in (2.38). The Teichmüller space will be discussed further in Section 4.

The proof of Theorem 3.4 relies on three lemmas about Ricci-flat Kähler manifolds, which examine Λ_ρ (Lemma 3.8), show that holomorphic vector fields correspond to harmonic 1-forms (Lemma 3.9), and show that the space of harmonic infinitesimal complex structures is invariant under $\widehat{J} \mapsto \widehat{J}^*$ (Lemma 3.10). These in turn require three preparatory lemmas about Hamiltonian and gradient vector fields (Lemma 3.5), about infinitesimal compatibility (Lemma 3.6), and about vector fields v such that $\mathcal{L}_v J$ is self-adjoint (Lemma 3.7). While some of this material is well-known, we include full proofs for completeness of the exposition. For a symplectic manifold (M,ω) and $J \in \mathscr{J}(M,\omega)$ the Hamiltonian and gradient vector fields of $H \in \Omega^0(M)$ are given by $\iota(v_H)\omega = dH$ and $\nabla H = Jv_H$.

Lemma 3.5 (**Hamiltonian and Gradient Vector Fields**). *Let (M,ω) be a symplectic $2\mathsf{n}$-manifold, let $J \in \mathscr{J}(M,\omega)$, let $H \in \Omega^0(M)$, and define $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$. Then $f_{v_H} = 0$ and $f_{\nabla H} = -d^*dH$. Moreover, if $\mathrm{Ric}_{\rho,J} = 0$, then $\Lambda_\rho(J, \mathcal{L}_{\nabla H}J) = dd^*dH \circ J$ and $\Lambda_\rho(J, \mathcal{L}_{v_H}J) = -dd^*dH$.*

Proof. Since $*\lambda = -(\lambda \circ J) \wedge \omega^{\mathsf{n}-1}/(\mathsf{n}-1)!$ for all $\lambda \in \Omega^1(M)$, we have

$$*\iota(v)\omega = -(\iota(v)\omega \circ J) \wedge \tfrac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!} = \iota(Jv)\omega \wedge \tfrac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!} = \iota(Jv)\rho \qquad (3.8)$$

for all $v \in \mathrm{Vect}(M)$. Hence $f_{\nabla H} = *d\iota(Jv_H)\rho = *d*dH = -d^*dH$ and the remaining assertions follow from (2.27). This proves Lemma 3.5. □

Lemma 3.6 (**Infinitesimal Compatibility**). *Let M be an oriented $2\mathsf{n}$-manifold and let $J \in \mathscr{J}(M)$. If $\tau \in \Omega^{1,1}_J(M)$ and $v \in \mathrm{Vect}(M)$, then the Lie derivatives $\widehat{\tau} := \mathcal{L}_v\tau$ and $\widehat{J} := \mathcal{L}_vJ$ satisfy the equation*

$$\widehat{\tau}(u,u') - \widehat{\tau}(Ju,Ju') = \tau(Ju,\widehat{J}u') + \tau(\widehat{J}u,Ju') \qquad (3.9)$$

for all $u,u' \in \mathrm{Vect}(M)$. If J is integrable and $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ satisfies the equation $\bar{\partial}_J\widehat{J} = 0$, then $\tau := \mathrm{Ric}_{\rho,J}$ and $\widehat{\tau} := \widehat{\mathrm{Ric}}_\rho(J,\widehat{J})$ satisfy equation (3.9) for every positive volume form $\rho \in \Omega^{2\mathsf{n}}(M)$.

Proof. Let $\tau \in \Omega^{1,1}_J(M)$, let ϕ_t be the flow of $v \in \mathrm{Vect}(M)$, and let $J_t := \phi_t^*J$. If $\widehat{\tau} := \mathcal{L}_v\tau$ and $\widehat{J} := \mathcal{L}_vJ$, differentiate the identity $\tau_t(u,u') = \tau_t(J_tu,J_tu')$ with $\tau_t := \phi_t^*\tau$ to obtain (3.9). Now assume J is integrable and $\tau = \mathrm{Ric}_{\rho,J}$. If $\widehat{\tau} := \widehat{\mathrm{Ric}}_\rho(J,\mathcal{L}_vJ)$, then $\mathcal{L}_v\tau - \widehat{\tau} = \frac{1}{2}d(df_v \circ J) \in \Omega^{1,1}_J(M)$ by Theorem 2.7, so (3.9) holds with $\widehat{J} = \mathcal{L}_vJ$. Now use the holomorphic Poincaré Lemma. □

Lemma 3.7 (**Self-Adjoint Lie Derivative $\mathcal{L}_vJ$**). *Let (M,ω) be a closed connected symplectic $2\mathsf{n}$-manifold, let $J \in \mathscr{J}(M,\omega)$, and let $v \in \mathrm{Vect}(M)$. Then the following holds (with $\rho = \omega^{\mathsf{n}}/\mathsf{n}!$ in part (iii)).*

(i) *$\mathcal{L}_vJ$ is self-adjoint if and only if $d\iota(v)\omega \in \Omega^{1,1}_J(M)$.*

(ii) *If J is integrable, then $\mathcal{L}_vJ$ is self-adjoint if and only if there exists a function $F \in \Omega^0(M)$ such that $d\iota(v + \nabla F)\omega = 0$.*

(iii) *$\iota(v)\omega$ is harmonic if and only if $f_v = f_{Jv} = 0$ and $\mathcal{L}_vJ = (\mathcal{L}_vJ)^*$.*

Proof. Part (i) follows from Lemma 3.6 with $\tau = \omega$.

Now suppose J is integrable. Then $\mathcal{L}_vJ$ and $\mathcal{L}_{Jv}J = J\mathcal{L}_vJ$ are self-adjoint for every symplectic vector field v by (i). Conversely, assume $\mathcal{L}_vJ = (\mathcal{L}_vJ)^*$. Then $d\iota(v)\omega \in \Omega^{1,1}_J(M)$ by (i), and so there exists a function $F \in \Omega^0(M)$ such that $d\iota(v)\omega = d(dF \circ J) = -d\iota(\nabla F)\omega$. This proves (ii).

To prove (iii) define $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$. Then (3.8) shows that $d^*\iota(v)\omega = 0$ if and only if $f_{Jv} = 0$. If $d\iota(v)\omega = 0$, then $f_v = 0$ and $\mathcal{L}_v J$ is self-adjoint by (i). Conversely, assume $f_v = 0$ and $\mathcal{L}_v J = (\mathcal{L}_v J)^*$. Then $d\iota(v)\omega \in \Omega_J^{1,1}(M)$ by (i) and $\langle d\iota(v)\omega, \omega\rangle = f_v = 0$. Thus

$$*d\iota(v)\omega = -d\iota(v)\omega \wedge \tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!},$$

so $d{*}d\iota(v)\omega = 0$, and hence $d\iota(v)\omega = 0$. This proves Lemma 3.7. □

Lemma 3.8 (**Λ_ρ in the Ricci-flat Case).** *Let (M, J, ω) be a closed connected $2\mathsf{n}$-dimensional Ricci-flat Kähler manifold with volume form $\rho = \omega^{\mathsf{n}}/\mathsf{n}!$ and let $\widehat{J} \in \Omega_J^{0,1}(M, TM)$ such that $\bar\partial_J \widehat{J} = 0$. Then $\widehat{\mathrm{Ric}}_\rho(J, \widehat{J}) \in \Omega_J^{1,1}(M)$ and there exists a unique pair of functions $f = f_{\widehat{J}}, g = f_{J\widehat{J}} \in \Omega^0(M)$ such that*

$$\Lambda_\rho(J, \widehat{J}) = -df \circ J + dg, \qquad \int_M f\rho = \int_M g\rho = 0. \tag{3.10}$$

Moreover, if $\bar\partial_J^* \widehat{J} = 0$ then $\Lambda_\rho(J, \widehat{J}) = 0$.

Proof. It follows directly from Lemma 3.6 that $\widehat{\mathrm{Ric}}_\rho(J, \widehat{J}) \in \Omega_J^{1,1}(M)$. Now assume $\bar\partial_J^* \widehat{J} = 0$ and let $v := \bar\partial_J^* \widehat{J}^* \in \mathrm{Vect}(M)$. Then $\iota(v)\omega = -\Lambda_\rho(J, J\widehat{J})$ by Lemma 2.12, hence $d\iota(v)\omega = -2\widehat{\mathrm{Ric}}_\rho(J, J\widehat{J})$ is an exact $(1,1)$-form, thus $\mathcal{L}_v J$ is self-adjoint by Lemma 3.7, and so is $J\mathcal{L}_v J = -2\bar\partial_J v = 2\bar\partial_J \bar\partial_J^*(\widehat{J} - \widehat{J}^*)$. Thus $\bar\partial_J \bar\partial_J^*(\widehat{J}^* - \widehat{J})$ is L^2 orthogonal to $\widehat{J}^* - \widehat{J}$ and so $\bar\partial_J^* \widehat{J}^* = \bar\partial_J^*(\widehat{J}^* - \widehat{J}) = 0$. Hence $\Lambda_\rho(J, \widehat{J}) = 0$ by Lemma 2.12. To prove (3.10), choose $v \in \mathrm{Vect}(M)$ such that $\bar\partial_J^*(\widehat{J} - \mathcal{L}_v J) = 0$. Then $\Lambda_\rho(J, \widehat{J}) = \Lambda_\rho(J, \mathcal{L}_v J)$ and hence $f := f_v$ and $g := f_{Jv}$ satisfy (3.10) by Theorem 2.7. This proves Lemma 3.8. □

Lemma 3.9 (**Holomorphic Vector Fields).** *Let M be a closed connected oriented $2\mathsf{n}$-manifold, fix a positive volume form $\rho \in \Omega^{2\mathsf{n}}(M)$ and an almost complex structure $J \in \mathscr{J}(M)$ such that $\mathrm{Ric}_{\rho,J} = 0$, and let $v \in \mathrm{Vect}(M)$. Then the following holds.*

(i) *$\Lambda_\rho(J, \mathcal{L}_v J) = 0$ if and only if $d\iota(v)\rho = d\iota(Jv)\rho = 0$.*

(ii) *Assume J is compatible with a symplectic form ω such that $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$ and that $\mathcal{L}_v J = 0$. Then $\iota(v)\omega$ is a harmonic 1-form.*

(iii) *Assume J is integrable and compatible with a symplectic form ω such that $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$. Then $\mathcal{L}_v J = 0$ if and only if $\iota(v)\omega$ is harmonic. If $d\iota(v)\rho = 0$, then there exists a $v_0 \in \mathrm{Vect}(M)$ such that $\iota(v_0)\rho$ is exact and $\mathcal{L}_{v_0} J = \mathcal{L}_v J$.*

Proof. To prove (i), observe that $\Lambda_\rho(J,\mathcal{L}_vJ) = -df_v\circ J + df_{Jv}$ by (2.27). Assume $\Lambda_\rho(J,\mathcal{L}_vJ)=0$ and choose a nondegenerate 2-form $\omega\in\Omega^2(M)$ that is compatible with J and satisfies $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$. Then equation (3.8) yields

$$0 = *\Big(\big(d(df_v\circ J)\big)\wedge\tfrac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!}\Big) = d^*df_v - *\Big((df_v\circ J)\wedge d\omega\wedge\tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!}\Big).$$

By the Hopf maximum principle, this implies that f_v is locally constant. Thus f_{Jv} is also locally constant. Since f_v and f_{Jv} have mean value zero on each connected component of M, it follows that $f_v = f_{Jv} = 0$. This proves (i).

Part (ii) follows directly from (i) and Lemma 3.7. To prove (iii), assume J is integrable and ω is closed. If $\iota(v)\omega$ is harmonic, then $(\mathcal{L}_vJ)^* = \mathcal{L}_vJ$ and $f_v = f_{Jv} = 0$ by Lemma 3.7, thus it follows from (i) and Lemma 2.12 that

$$0 = \Lambda_\rho(J,\mathcal{L}_vJ) = \iota(J\bar\partial_J^*(\mathcal{L}_vJ)^*)\omega = \iota(J\bar\partial_J^*\mathcal{L}_vJ)\omega = -2\iota(\bar\partial_J^*\bar\partial_Jv)\omega,$$

and so $\mathcal{L}_vJ = 2J\bar\partial_Jv = 0$. Now assume $d\iota(v)\rho = 0$, choose $\alpha_0\in\Omega^{2\mathsf{n}-2}(M)$ such that $d{*}d\alpha_0 = d\iota(Jv)\omega$, and define $v_0\in\mathrm{Vect}(M)$ by $\iota(v_0)\rho := d\alpha_0$. Then $\iota(Jv_0)\omega = *d\alpha_0$ by (3.8). Hence $d\iota(Jv_0)\omega = d{*}d\alpha_0 = d\iota(Jv)\omega$. We also have $d{*}\iota(J(v-v_0))\omega = -d\iota(v-v_0)\rho = 0$. Thus $\iota(J(v-v_0))\omega$ is harmonic and so $\mathcal{L}_{v-v_0}J = -J\mathcal{L}_{J(v-v_0)}J = 0$. This proves Lemma 3.9. □

Lemma 3.10 (Harmonic Complex Anti-Linear Endomorphisms $\widehat{J}$). *Let (M,J,ω) be a closed connected $2\mathsf{n}$-dimensional Ricci-flat Kähler manifold with volume form $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$ and let $\widehat{J}\in\Omega^{0,1}_J(M,TM)$ such that $\bar\partial_J\widehat{J} = 0$ and $\bar\partial_J^*\widehat{J} = 0$. Then $\bar\partial_J\widehat{J}^* = 0$ and $\bar\partial_J^*\widehat{J}^* = 0$.*

Proof. Let ∇ be the Levi-Civita connection of a Kähler metric $\omega(\cdot,J\cdot)$. Then the Bochner–Kodaira–Nakano identity for $\widehat{J}\in\Omega^{0,1}_J(M,TM)$ takes the form

$$\bar\partial_J^*\bar\partial_J\widehat{J} + \bar\partial_J\bar\partial_J^*\widehat{J} = \tfrac12\nabla^*\nabla\widehat{J} + \tfrac12[JQ,\widehat{J}] + \mathcal{T}(\widehat{J}), \tag{3.11}$$

where

$$\nabla^*\nabla\widehat{J} = -\sum_i(\nabla_{e_i}\nabla_{e_i}\widehat{J} + \mathrm{div}(e_i)\nabla_{e_i}\widehat{J}), \qquad \mathcal{T}(\widehat{J})u = \sum_i R^\nabla(e_i,u)\widehat{J}e_i$$

for a local orthonormal frame $e_1,\dots,e_{2\mathsf{n}}$, and the skew-adjoint endomorphism Q is defined by $\langle Q\cdot,\cdot\rangle = \mathrm{Ric}_{\rho,J}$. (See [11].) Since $\mathcal{T}(\widehat{J})^* = \mathcal{T}(\widehat{J}^*)$, it follows that the operator $\bar\partial_J^*\bar\partial_J + \bar\partial_J\bar\partial_J^*$ commutes with the operator $\widehat{J}\mapsto\widehat{J}^*$ in the Kähler–Einstein case $Q = \kappa J$. This proves Lemma 3.10. □

Proof of Theorem 3.4. Let $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ with $\bar\partial_J \widehat{J} = 0$. Then part (iii) of Lemma 3.9 shows that the last two assertions in (ii) are equivalent. Next observe the following.

Claim 1. $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) = 0$ *if and only if* $f_{\widehat{J}} = 0$.

Claim 2. $\Omega_{\rho,J}(\widehat{J}, \mathcal{L}_v J) = \int_M \bigl(f_{\widehat{J}} f_{Jv} - f_{J\widehat{J}} f_v\bigr)\rho$ *for all* $v \in \mathrm{Vect}(M)$.

By (2.14) and Lemma 3.8 we have $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) = \frac{1}{2} d\Lambda_\rho(J,\widehat{J}) = -\frac{1}{2} d(df_{\widehat{J}} \circ J)$ and this proves Claim 1. Claim 2 follows from (2.13) and Lemma 3.8.

Sufficiency in (i) and (ii) follows directly from Claim 1 and Claim 2. To prove necessity, choose a symplectic form ω such that J is compatible with ω and $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$. Then (M,J,ω) is a Ricci-flat Kähler manifold.

We prove part (i). Assume $\Omega_{\rho,J}(\widehat{J},\widehat{J}') = 0$ for all $\widehat{J}'$ with $\bar\partial_J \widehat{J}' = 0$ and choose $v \in \mathrm{Vect}(M)$ such that $\bar\partial_J^*(\widehat{J} - \mathcal{L}_v J) = 0$. Then $\Lambda_\rho(J, \widehat{J} - \mathcal{L}_v J) = 0$ by Lemma 3.8 and $\bar\partial_J(\widehat{J} - \mathcal{L}_v J)^* = 0$ by Lemma 3.10. Thus it follows from the assumption and equations (2.13) and (2.33) that

$$0 = \Omega_{\rho,J}(\widehat{J} - \mathcal{L}_v J, \mathcal{L}_{Jv} J) = -\Omega_{\rho,J}(\mathcal{L}_v J, \mathcal{L}_{Jv} J) = \int_M \bigl(f_v^2 + f_{Jv}^2\bigr)\rho.$$

Hence $f_v = f_{Jv} = 0$. Now fix an element $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ with $\bar\partial_J \widehat{J}' = 0$. Then $\Omega_{\rho,J}(\widehat{J} - \mathcal{L}_v J, J\widehat{J}') = 0$ by assumption and Claim 2. Thus

$$\langle (\widehat{J} - \mathcal{L}_v J)^*, \widehat{J}' \rangle_{L^2} = \int_M \mathrm{trace}((\widehat{J} - \mathcal{L}_v J)\widehat{J}')\rho = -2\Omega_{\rho,J}(\widehat{J} - \mathcal{L}_v J, J\widehat{J}') = 0.$$

This implies that there exists a $\tau \in \Omega^{0,2}_J(M,TM)$ with $\bar\partial_J^* \tau = (\widehat{J} - \mathcal{L}_v J)^*$. Hence $\bar\partial_J \bar\partial_J^* \tau = \bar\partial_J(\widehat{J} - \mathcal{L}_v J)^* = 0$ and so $\widehat{J} = \mathcal{L}_v J$. This proves (i).

We prove part (ii). Assume $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) = 0$ and $\Omega_{\rho,J}(\widehat{J},\widehat{J}') = 0$ for all $\widehat{J}'$ that satisfy $\bar\partial_J \widehat{J}' = 0$ and $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}') = 0$. Then $f_{\widehat{J}} = 0$ by Claim 1 and we choose $H \in \Omega^0(M)$ such that $d^*dH = -f_{J\widehat{J}}$. Then $\Lambda_\rho(J, \widehat{J} - \mathcal{L}_{v_H} J) = 0$ by Lemma 3.5 and Lemma 3.8. Now let $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ with $\bar\partial_J \widehat{J}' = 0$ and choose $F \in \Omega^0(M)$ such that $d^*dF = -f_{\widehat{J}'}$. Then $\widehat{\mathrm{Ric}}_\rho(J, \widehat{J}' - \mathcal{L}_{\nabla F} J) = 0$ by (2.14), Lemma 3.5, and Lemma 3.8. Hence

$$\Omega_{\rho,J}(\widehat{J} - \mathcal{L}_{v_H} J, \widehat{J}') = \Omega_{\rho,J}(\widehat{J} - \mathcal{L}_{v_H} J, \widehat{J}' - \mathcal{L}_{\nabla F} J) = \Omega_{\rho,J}(\widehat{J}, \widehat{J}' - \mathcal{L}_{\nabla F} J) = 0$$

by assumption and Theorem 2.6. So part (i) asserts that there exists a vector field u with $f_u = f_{Ju} = 0$ and $\mathcal{L}_u J = \widehat{J} - \mathcal{L}_{v_H} J$. Thus $v := u + v_H$ is a divergence-free vector field with $\mathcal{L}_v J = \widehat{J}$. This proves Theorem 3.4. $\square$

We close this section with a well known lemma (see [22]) that is used in Theorem 4.6. We include a proof for completeness of the exposition.

Lemma 3.11 (**Harmonic** $(0,2)$**-Forms).** *Let* (M,J,ω) *be a closed connected* $2\mathsf{n}$*-dimensional Ricci-flat Kähler manifold, let* ∇ *be the Levi-Civita connection of the Kähler metric, let* $\widehat{J}\in\Omega^{0,1}_J(M,TM)$ *with* $\widehat{J}+\widehat{J}^*=0$*, and define* $\widehat{\omega}:=\langle\widehat{J}\cdot,\cdot\rangle\in\Omega^2(M)$*. Then* $\widehat{\omega}^{1,1}_J=0$ *and the following are equivalent.*

(i) $\bar\partial_J\widehat{J}=0$ *and* $\bar\partial_J^*\widehat{J}=0$.

(ii) $\nabla\widehat{J}=0$.

(iii) $\nabla\widehat{\omega}=0$.

(iv) $\widehat{\omega}$ *is a harmonic* 2*-form.*

(v) $d\widehat{\omega}=0$.

Proof. It follows directly from the definition that $\widehat{\omega}^{1,1}_J=0$. To prove that (i) is equivalent to (ii), let $\mathcal{T}$ be the operator in the proof of Lemma 3.10 and assume that $\widehat{J}$ is skew-adjoint. Then, by the first Bianchi identity, we have

$$\begin{aligned}\mathcal{T}(\widehat{J})u&=\sum_i R^\nabla(e_i,u)\widehat{J}e_i=\sum_{i,j}\langle e_j,\widehat{J}e_i\rangle R^\nabla(e_i,u)e_j\\&=\sum_{i,j}\langle\widehat{J}e_j,e_i\rangle\big(R^\nabla(e_j,e_i)u+R^\nabla(u,e_j)e_i\big)=\sum_j R^\nabla(e_j,\widehat{J}e_j)u-\mathcal{T}(\widehat{J})u.\end{aligned}$$

Hence, for a local orthonormal frame with $e_{\mathsf{n}+i}=Je_i$, we obtain

$$\mathcal{T}(\widehat{J})=\tfrac12\sum_i R^\nabla(e_i,\widehat{J}e_i)=\tfrac14\sum_i\Big(R^\nabla(e_i,\widehat{J}e_i)+R^\nabla(Je_i,\widehat{J}Je_i)\Big)=0.$$

Thus $\|\bar\partial_J\widehat{J}\|^2+\|\bar\partial_J^*\widehat{J}\|^2=\|\nabla\widehat{J}\|^2$ by (3.11) with $Q=0$ and so (i) is equivalent to (ii). The equivalence of (ii) and (iii) follows directly from the definition of $\widehat{\omega}$. That (iii) implies (v) follows from Lemma A.1, and (iv) is equivalent to (v) because $\widehat{\omega}^{1,1}_J=0$ and so $*\widehat{\omega}=\widehat{\omega}\wedge\omega^{\mathsf{n}-2}/(\mathsf{n}-2)!$. That (iv) implies (iii) is a consequence of the Weitzenböck formula

$$\begin{aligned}&\Big(d^*d\widehat{\omega}-dd^*\widehat{\omega}-\nabla^*\nabla\widehat{\omega}\Big)(u,v)\\&=\sum_i\Big(\widehat{\omega}\big(e_i,R^\nabla(u,v)e_i\big)-\widehat{\omega}\big(u,R^\nabla(v,e_i)e_i\big)+\widehat{\omega}\big(v,R^\nabla(u,e_i)e_i\big)\Big)\end{aligned}\tag{3.12}$$

for $\widehat{\omega}\in\Omega^2(M)$, $u,v\in\mathrm{Vect}(M)$, and a local orthonormal frame $e_1,\dots,e_{2\mathsf{n}}$. In the Ricci-flat Kähler case with $\widehat{\omega}^{1,1}_J=0$ the right hand side in (3.12) vanishes and hence $\|d\widehat{\omega}\|^2+\|d^*\widehat{\omega}\|^2=\|\nabla\widehat{\omega}\|^2$. This proves Lemma 3.11. □

4 Teichmüller space

The Calabi–Yau Teichmüller space

Fix a closed connected oriented $2\mathsf{n}$-manifold M. The **Calabi–Yau Teichmüller space** is the space of isotopy classes of complex structures J with real first Chern class zero and nonempty Kähler cone $\mathcal{K}_J$. It is denoted by

$$\begin{aligned}\mathscr{T}_0(M) &:= \mathscr{J}_{\mathrm{int},0}(M)/\mathrm{Diff}_0(M),\\ \mathscr{J}_{\mathrm{int},0}(M) &:= \left\{ J \in \mathscr{J}_{\mathrm{int}}(M) \;\middle|\; \begin{array}{l} c_1(TM,J) = 0 \in H^2(M;\mathbb{R}) \\ \text{and } J \text{ admits a Kähler form} \end{array} \right\}.\end{aligned} \tag{4.1}$$

For every $J \in \mathscr{J}_{\mathrm{int},0}(M)$ the space of holomorphic vector fields is isomorphic to $H^1(M;\mathbb{R})$ by part (iii) of Lemma 3.9 and the Calabi–Yau Theorem. Moreover, the Bogomolov–Tian–Todorov theorem asserts that the obstruction class vanishes [4, 36, 37]. Hence the cohomology of the complex

$$\Omega^0(M,TM) \xrightarrow{\bar\partial_J} \Omega^{0,1}_J(M,TM) \xrightarrow{\bar\partial_J} \Omega^{0,2}_J(M,TM) \tag{4.2}$$

has constant dimension. (This assertion can also be derived from [42, Proposition 9.30] and Lemma D.4.) It follows that the Teichmüller space $\mathscr{T}_0(M)$ is a smooth manifold [10, 23, 24, 25, 27, 28] whose tangent space at $J \in \mathscr{J}_{\mathrm{int},0}(M)$ is the cohomology of the complex (4.2), i.e.

$$T_{[J]}\mathscr{T}_0(M) = \frac{\ker(\bar\partial_J : \Omega^{0,1}_J(M,TM) \to \Omega^{0,2}_J(M,TM))}{\mathrm{im}(\bar\partial_J : \Omega^0(M,TM) \to \Omega^{0,1}_J(M,TM))}. \tag{4.3}$$

Remark 4.1. The Teichmüller space is in general not Hausdorff, even for the K3 surface [19, 39]. Let (M,J) be a K3-surface that admits an embedded holomorphic sphere $C \subset M$ with self-intersection number $C \cdot C = -2$, and let $\tau : M \to M$ be a Dehn twist about C. Then there exists a smooth family of complex structures $\{J_t \in \mathscr{J}_{\mathrm{int},0}(M)\}_{t\in\mathbb{C}}$ and a smooth family of diffeomorphisms $\{\phi_t \in \mathrm{Diff}_0(M)\}_{t\in\mathbb{C}\setminus\{0\}}$ such that $J_0 = J$ and $\phi_t^* J_t = \tau^* J_{-t}$ for all $t \in \mathbb{C}\setminus\{0\}$. Thus J_t and $\tau^* J_{-t}$ represent the same class in Teichmüller space, however, their limits $\lim_{t\to0} J_t = J_0$ and $\lim_{t\to0} \tau^* J_{-t} = \tau^* J_0$ do not represent the same class in Teichmüller space because their effective cones differ. Namely, the class $[C] \in H_2(M;\mathbb{Z})$ belongs to the effective cone of J_0 while the class $-[C] \in H_2(M;\mathbb{Z})$ belongs to the effective cone of $\tau^* J_0$.

For general hyperKähler manifolds the Teichmüller space becomes Hausdorff after identifying inseparable complex structures (see Verbitsky [39, 40]), which are biholomorphic by a theorem of Huybrechts [20].

Teichmüller space as a symplectic quotient

Fix a positive volume form ρ on M and define

$$\begin{aligned}\mathscr{T}_0(M,\rho) &:= \mathscr{J}_{\mathrm{int},0}(M,\rho)/\mathrm{Diff}_0(M,\rho),\\ \mathscr{J}_{\mathrm{int},0}(M,\rho) &:= \big\{J\in\mathscr{J}_{\mathrm{int},0}(M)\,|\,\mathrm{Ric}_{\rho,J}=0\big\}.\end{aligned}\tag{4.4}$$

The tangent space of $\mathscr{T}_0(M,\rho)$ at $J\in\mathscr{J}_{\mathrm{int},0}(M,\rho)$ is the quotient

$$T_{[J]}\mathscr{T}_0(M,\rho)=\frac{\big\{\widehat{J}\in\Omega^{0,1}_J(M,TM)\,|\,\bar\partial_J\widehat{J}=0,\ \widehat{\mathrm{Ric}}_\rho(J,\widehat{J})=0\big\}}{\big\{\mathcal{L}_vJ\,|\,v\in\mathrm{Vect}(M),\ d\iota(v)\rho=0\big\}}.\tag{4.5}$$

Lemma 4.2. *The inclusion $\iota_\rho:\mathscr{T}_0(M,\rho)\to\mathscr{T}_0(M)$ is a diffeomorphism.*

Proof. The map ι_ρ is bijective by Theorem 3.1. The derivative of ι_ρ at an element $J\in\mathscr{J}_{\mathrm{int},0}(M,\rho)$ is the inclusion of the quotient (4.5) into (4.3). It is injective because $\widehat{\mathrm{Ric}}_\rho(J,\mathcal{L}_vJ)=-\frac{1}{2}d(df_v\circ J)$, and so $\widehat{\mathrm{Ric}}_\rho(J,\mathcal{L}_vJ)=0$ implies $f_v=0$ and thus $d\iota(v)\rho=0$. It is surjective because, if $\widehat{J}\in\Omega^{0,1}_J(M,TM)$ satisfies $\bar\partial_J\widehat{J}=0$ and $F\in\Omega^0(M)$ is the unique solution of $d^*dF=f_{\widehat{J}}$ with mean value zero, then $\Lambda_\rho(J,\mathcal{L}_{\nabla F}J)=dd^*dF\circ J=df_{\widehat{J}}\circ J$ by Lemma 3.5 and so $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}+\mathcal{L}_{\nabla F}J)=0$ by Lemma 3.8. This proves Lemma 4.2. □

By Lemma 3.9 the quotient group $\mathrm{Diff}_0(M,\rho)/\mathrm{Diff}^{\mathrm{ex}}(M,\rho)$ acts trivially on $\mathscr{J}_{\mathrm{int},0}(M,\rho)/\mathrm{Diff}^{\mathrm{ex}}(M,\rho)$. Hence $\mathscr{T}_0(M,\rho)$ is a submanifold of the infinite-dimensional symplectic quotient $\mathscr{W}_0(M,\rho)=\mathscr{J}_0(M,\rho)/\mathrm{Diff}^{\mathrm{ex}}(M,\rho)$ in (2.38), which is regular near $\mathscr{T}_0(M,\rho)$ by Lemma 3.9 and Theorem 2.11. Moreover, Ω_ρ descends to a symplectic form on $\mathscr{T}_0(M,\rho)$ by Theorem 3.4. Here is a formula for the pushforward of this symplectic form under the diffeomorphism ι_ρ in Lemma 4.2. Let $V>0$. By Theorem 3.1 every $J\in\mathscr{J}_{\mathrm{int},0}(M)$ admits a unique positive volume form $\rho=\rho_J\in\Omega^{2\mathsf{n}}(M)$ such that

$$\mathrm{Ric}_{\rho,J}=0,\qquad \int_M\rho=V.\tag{4.6}$$

Definition 4.3 (Weil–Petersson Symplectic Form). *For $J\in\mathscr{J}_{\mathrm{int},0}(M)$, for the volume form ρ_J in* (4.6)*, and for $\widehat{J}_1,\widehat{J}_2\in\Omega^{0,1}_J(M,TM)$ with $\bar\partial_J\widehat{J}_i=0$ and f_i,g_i as in Lemma 3.8, define*

$$\Omega_J(\widehat{J}_1,\widehat{J}_2):=\int_M\Big(\tfrac{1}{2}\mathrm{trace}\big(\widehat{J}_1J\widehat{J}_2\big)-f_1g_2+f_2g_1\Big)\rho_J,\tag{4.7}$$

$$\langle\widehat{J}_1,\widehat{J}_2\rangle_J:=\Omega_J(\widehat{J}_1,-J\widehat{J}_2)=\int_M\Big(\tfrac{1}{2}\mathrm{trace}\big(\widehat{J}_1\widehat{J}_2\big)-f_1f_2-g_1g_2\Big)\rho_J.\tag{4.8}$$

Theorem 4.4. **(i)** *The 2-form Ω_J in (4.7) descends to a nondegenerate 2-form on the quotient space (4.3) and defines a symplectic form on $\mathscr{T}_0(M)$. Its pullback to $\mathscr{T}_0(M,\rho)$ under the diffeomorphism ι_ρ of Lemma 4.2 is the symplectic form induced by Ω_ρ.*
(ii) *If $\phi : M \to M$ is an orientation preserving diffeomorphism then*

$$\Omega_{\phi^*J}(\phi^*\widehat{J}_1, \phi^*\widehat{J}_2) = \Omega_J(\widehat{J}_1, \widehat{J}_2) \tag{4.9}$$

for all $\widehat{J}_1, \widehat{J}_2 \in \Omega^{0,1}_J(M,TM)$ such that $\bar{\partial}_J\widehat{J}_i = 0$. Thus the mapping class group $\Gamma := \mathrm{Diff}^+(M)/\mathrm{Diff}_0(M)$ acts on $\mathscr{T}_0(M)$ by symplectomorphisms.
(iii) *For a symplectic form $\omega \in \Omega^2(M)$ with real first Chern class zero define*

$$\mathscr{T}(M,\omega) := \mathscr{J}_{\mathrm{int}}(M,\omega)/\!\sim, \quad \mathscr{J}_{\mathrm{int}}(M,\omega) := \mathscr{J}_{\mathrm{int}}(M) \cap \mathscr{J}(M,\omega), \tag{4.10}$$

where $J_0 \sim J_1$ iff there is a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ such that $\phi^*J_0 = J_1$. This space is a complex submanifold of $\mathscr{T}_0(M)$ and the symplectic form (4.7) restricts to the standard Kähler form on $\mathscr{T}(M,\omega)$. The symmetric bilinear form (4.8) is positive on $T_{[J]}\mathscr{T}(M,\omega)$ and is negative on its symplectic complement.

Proof. The proof has three steps.

Step 1. Let $J \in \mathscr{J}_{\mathrm{int},0}(M)$, let $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ such that $\bar{\partial}_J\widehat{J} = 0$, and let $v \in \mathrm{Vect}(M)$. Then $\Omega_J(\widehat{J}, \mathcal{L}_vJ) = 0$.

Let $\rho := \rho_J$ and let $f = f_{\widehat{J}}$ and $g = f_{J\widehat{J}}$ be as in Lemma 3.8. Then

$$\begin{aligned}\Omega_J(\widehat{J}, \mathcal{L}_vJ) &= \tfrac{1}{2}\int_M \mathrm{trace}\big(\widehat{J}J\mathcal{L}_vJ\big) - \int_M \big(ff_{Jv} - gf_v\big)\rho \\ &= \int_M \Lambda_\rho(J,\widehat{J}) \wedge \iota(v)\rho - \int_M f d\iota(Jv)\rho + \int_M g d\iota(v)\rho = 0\end{aligned}$$

because $\Lambda_\rho(J,\widehat{J}) = -df \circ J + dg$. This proves Step 1.

Step 2. *Let $J \in \mathscr{J}_{\mathrm{int},0}(M)$ and let $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ such that $\bar{\partial}_J\widehat{J} = 0$ and $\Omega_J(\widehat{J}, \widehat{J}') = 0$ for all $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ with $\bar{\partial}_J\widehat{J}' = 0$. Then $\widehat{J} \in \mathrm{im}\bar{\partial}_J$.*

Choose a symplectic form ω such that $J \in \mathscr{J}_{\mathrm{int}}(M,\omega)$ and $\omega^{\mathsf{n}}/\mathsf{n}! = \rho_J$, and choose functions $F, G \in \Omega^0(M)$ such that $d^*dF = -f_{J\widehat{J}}$ and $d^*dG = -f_{\widehat{J}}$. Then $\Lambda_\rho(J, \mathcal{L}_{v_F+\nabla G}J) = \Lambda_\rho(J,\widehat{J})$ by Lemma 3.5 and hence, by Step 1,

$$\Omega_{\rho,J}(\widehat{J} - \mathcal{L}_{v_F+\nabla G}J, \widehat{J}') = \Omega_J(\widehat{J} - \mathcal{L}_{v_F+\nabla G}J, \widehat{J}') = \Omega_J(\widehat{J}, \widehat{J}') = 0$$

for all $\widehat{J}' \in \Omega^{0,1}_J(M,TM)$ with $\bar{\partial}_J\widehat{J}' = 0$. Hence, by part (i) of Theorem 3.4, there exists a vector field v with $\mathcal{L}_vJ = \widehat{J} - \mathcal{L}_{v_F+\nabla G}J$. This proves Step 2.

Step 3. *We prove Theorem 4.4.*

By Step 1 the 2-form (4.7) on $\mathscr{J}_{\mathrm{int},0}(M)$ descends to $\mathscr{T}_0(M)$ and by Step 2 the induced 2-form on $\mathscr{T}_0(M)$ is nondegenerate. Its pullback under the diffeomorphism ι_ρ in Lemma 4.2 is the restricton of the symplectic form Ω_ρ on $\mathscr{J}(M)$ to the subquotient $\mathscr{T}_0(M,\rho)$. Hence it is closed and this proves part (i). Part (ii) follows directly from the definitions and part (iii) holds because the tangent space $T_{[J]}\mathscr{T}(M,\omega)$ is the quotient of the space of self-adjoint endomorphisms $\widehat{J}=\widehat{J}^*\in\Omega^{0,1}_J(M,TM)$ with $\bar\partial_J\widehat{J}=0$ by those generated by Hamiltonian and gradient vector fields. This proves Theorem 4.4. □

A symplectic connection

Consider the fibration $\mathscr{T}_0(M,\omega)\hookrightarrow\mathscr{E}_0(M)\to\mathscr{B}_0(M)$, where $\mathscr{B}_0(M)$ denotes the space of isotopy classes of symplectic forms with real first Chern class zero which admit compatible complex structures and $\mathscr{E}_0(M)$ denotes the space of isotopy classes of Ricci-flat Kähler structures (ω,J) on M. Thus

$$\begin{aligned}
\mathscr{T}_0(M,\omega) &:= \mathscr{J}_{\mathrm{int},0}(M,\omega)/\mathrm{Symp}(M,\omega)\cap\mathrm{Diff}_0(M),\\
\mathscr{J}_{\mathrm{int},0}(M,\omega) &:= \left\{J\in\mathscr{J}_{\mathrm{int}}(M,\omega)\,|\,\mathrm{Ric}_{\omega,J}=0\right\},\\
\mathscr{B}_0(M) &:= \mathscr{S}_0(M)/\mathrm{Diff}_0(M),\\
\mathscr{S}_0(M) &:= \left\{\omega\in\Omega^2(M)\;\middle|\;\begin{array}{l} d\omega=0,\omega^{\mathsf{n}}>0,\ c_1^{\mathbb{R}}(\omega)=0,\\ \mathscr{J}_{\mathrm{int}}(M,\omega)\neq\emptyset\end{array}\right\},\\
\mathscr{E}_0(M) &:= \mathscr{K}_0(M)/\mathrm{Diff}_0(M),\\
\mathscr{K}_0(M) &:= \left\{(\omega,J)\,|\,\omega\in\mathscr{S}_0(M),\ J\in\mathscr{J}_{\mathrm{int}}(M,\omega),\ \mathrm{Ric}_{\omega,J}=0\right\}.
\end{aligned}\tag{4.11}$$

These quotient spaces are finite-dimensional manifolds. Here is a list of the real dimensions for the cases where M is the $2\mathsf{n}$-torus, the K3 surface, the Enriques surface, the quintic in $\mathbb{CP}^4$, the banana manifold B in [7], and a rigid Calabi–Yau 3-fold JB introduced recently by Jim Bryan (as yet unpublished).

M	$\mathscr{T}_0(M)$ $2h^{\mathsf{n}-1,1}_{M,L}$	$\mathcal{K}_J$ $h^{1,1}$	$\mathscr{E}_0(M)$ $h^{1,1}+2h^{\mathsf{n}-1,1}_{M,L}$	$\mathscr{B}_0(M)$ $h^{1,1}+2h^{2,0}$	$\mathscr{T}_0(M,\omega)$ $2h^{\mathsf{n}-1,1}_{M,L}-2h^{2,0}$
$\mathbb{T}^{2\mathsf{n}}$	$2\mathsf{n}^2$	n^2	$3\mathsf{n}^2$	$2\mathsf{n}^2-\mathsf{n}$	$\mathsf{n}^2+\mathsf{n}$
K3	40	20	60	22	38
Enriques	20	10	30	10	20
Quintic	202	1	203	1	202
B	16	20	36	20	16
JB	0	4	4	4	0

The symplectic form (4.7) in Theorem 4.4 gives rise to a closed 2-form on $\mathscr{E}_0(M)$ which restricts to the canonical Kähler form on each fiber and whose kernel at (ω, J) is the space $H^{1,1}_J(M)\times\{0\}$. It gives rise to a symplectic connection on $\mathscr{E}_0(M)$ as in [30, Chapter 6]. To describe this connection, it will be convenient to use the notation

$$\begin{aligned}(J^*\widehat{\omega})(u,u') &:= \widehat{\omega}(Ju,Ju'),\\ (\iota(J)\widehat{\omega})(u,u') &:= \widehat{\omega}(Ju,u')+\widehat{\omega}(u,Ju').\end{aligned} \tag{4.12}$$

for $\widehat{\omega}\in\Omega^2(M)$ and $u,u'\in\mathrm{Vect}(M)$. The 2-forms $J^*\widehat{\omega}$ and $\iota(J)\widehat{\omega}$ are closed whenever $\widehat{\omega}$ is harmonic, because $\widehat{\omega}^{2,0}_J=\frac{1}{4}(\widehat{\omega}-J^*\widehat{\omega})-\frac{\mathbf{i}}{4}\iota(J)\widehat{\omega}$.

The tangent space of $\mathscr{K}_0(M)$ at (ω, J) with $\rho:=\omega^{\mathsf{n}}/\mathsf{n}!$ is the space of all pairs $(\widehat{\omega},\widehat{J})\in\Omega^2(M)\times\Omega^{0,1}_J(M,TM)$ that satisfy the conditions

$$\widehat{\omega}(u,u')-\widehat{\omega}(Ju,Ju')=\omega(\widehat{J}u,Ju')+\omega(Ju,\widehat{J}u') \tag{4.13}$$

for all $u,u'\in\mathrm{Vect}(M)$ and

$$d\widehat{\omega}=0,\qquad \bar{\partial}_J\widehat{J}=0,\qquad \widehat{\mathrm{Ric}}_\rho(J,\widehat{J})+\tfrac{1}{2}d(d\langle\widehat{\omega},\omega\rangle\circ J)=0. \tag{4.14}$$

We will strengthen the last condition in (4.14) and require

$$\Lambda_\rho(J,\widehat{J})=-d\langle\widehat{\omega},\omega\rangle\circ J. \tag{4.15}$$

The definition of the connection is based on the next lemma.

Lemma 4.5. *Let $(\omega,J)\in\mathscr{K}_0(M)$ be a Ricci-flat Kähler structure, denote by $\langle\cdot,\cdot\rangle:=\omega(\cdot,J\cdot)$ the Kähler metric, define $\rho:=\omega^{\mathsf{n}}/\mathsf{n}!$, and let $\widehat{\omega}\in\Omega^2(M)$ be a closed 2-form. Then, for $\widehat{J}\in\Omega^{0,1}_J(M,TM)$, the following are equivalent.*

(a) *$\widehat{J}$ satisfies (4.13), (4.14), (4.15), and, for all $\widehat{J}'\in\Omega^{0,1}_J(M,TM)$,*

$$\widehat{J}'=(\widehat{J}')^*,\quad \bar{\partial}_J\widehat{J}'=0\qquad\Longrightarrow\qquad \Omega_J(\widehat{J},\widehat{J}')=0. \tag{4.16}$$

(b) *If $v\in\mathrm{Vect}(M)$ satisfies*

$$d^*(\widehat{\omega}-d\iota(v)\omega)=0,\qquad d^*\iota(v)\omega=0, \tag{4.17}$$

and $\widehat{\omega}_0\in\Omega^2(M)$ and $\widehat{J}_0\in\Omega^{0,1}_J(M,TM)$ are defined by

$$\widehat{\omega}_0:=\widehat{\omega}-d\iota(v)\omega,\qquad \langle\widehat{J}_0\cdot,\cdot\rangle=\tfrac{1}{2}\big(\widehat{\omega}_0-J^*\widehat{\omega}_0\big), \tag{4.18}$$

then $\widehat{J}=\mathcal{L}_vJ+\widehat{J}_0$.

Moreover, for every closed 2-form $\widehat{\omega}$ there exists a unique $\widehat{J}\in\Omega^{0,1}_J(M,TM)$ that satisfies the equivalent conditions (a) and (b).

Proof. We prove in three steps that (a) is equivalent to (b).

Step 1. *Suppose $\widehat{J}_1$ and $\widehat{J}_2$ satisfy (a). Then $\widehat{J}_1 = \widehat{J}_2$.*

The difference $\widehat{J} := \widehat{J}_1 - \widehat{J}_2$ satisfies (a) with $\widehat{\omega} = 0$. Hence $\widehat{J} = \widehat{J}^*$ by (4.13), and $\bar{\partial}_J \widehat{J} = 0$ by (4.14), and $\Lambda_\rho(J, \widehat{J}) = 0$ by (4.15). Let $\widehat{J}' \in \Omega^{0,1}_J(M, TM)$ with $\bar{\partial}_J \widehat{J}' = 0$, choose a vector field v' such that $\bar{\partial}_J^*(\widehat{J}' - \mathcal{L}_{v'}J) = 0$, and define $\big(\widehat{J}' - \mathcal{L}_{v'}J\big)^+ := \frac{1}{2}(\widehat{J}' - \mathcal{L}_{v'}J + (\widehat{J}' - \mathcal{L}_{v'}J)^*)$. Then $\bar{\partial}_J(\widehat{J}' - \mathcal{L}_{v'}J)^+ = 0$ by Lemma 3.10, hence $\Omega_J(\widehat{J}, (\widehat{J}' - \mathcal{L}_{v'}J)^+) = 0$ by (4.16), and this implies that $\Omega_J(\widehat{J}, \widehat{J}') = \Omega_J(\widehat{J}, \widehat{J}' - \mathcal{L}_{v'}J) = \Omega_J(\widehat{J}, (\widehat{J}' - \mathcal{L}_{v'}J)^+) = 0$. Thus by Theorem 4.4 there exists a vector field v with $\mathcal{L}_v J = \widehat{J}$. Since $\widehat{J} = \widehat{J}^*$, Lemma 3.7 asserts that there exist functions $F, H \in \Omega^0(M)$ and a vector field v_0 such that $v := v_0 + v_H + \nabla F$ and $\iota(v_0)\omega$ is a harmonic 1-form. Thus $\mathcal{L}_{v_0}J = 0$ by Lemma 3.9 and $dd^*dF \circ J - dd^*dH = \Lambda_\rho(J, \mathcal{L}_v J) = 0$ by Lemma 3.5. Hence F and H are constant, so $\widehat{J} = \mathcal{L}_{v_0}J = 0$ and $\widehat{J}_1 = \widehat{J}_2$. This proves Step 1.

Step 2. *Suppose $v \in \mathrm{Vect}(M)$ satisfies (4.17), define $\widehat{\omega}_0$ and $\widehat{J}_0$ by (4.18), and define $\widehat{J} := \mathcal{L}_v J + \widehat{J}_0$. Then $\widehat{J}$ satisfies (a).*

By (4.18) and (3.8), we have

$$\begin{aligned} f_v\rho &= d\iota(v)\rho = d\iota(v)\omega \wedge \tfrac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!} = \langle \widehat{\omega} - \widehat{\omega}_0, \omega\rangle\rho, \\ f_{Jv}\rho &= d\iota(Jv)\rho = d{*}\iota(v)\omega = 0. \end{aligned} \tag{4.19}$$

Moreover, $\widehat{\omega}_0$ is a harmonic 2-form and so is $\widehat{\omega}_0 - J^*\widehat{\omega}_0$. Thus $\bar{\partial}_J \widehat{J}_0 = 0$ and $\bar{\partial}_J^* \widehat{J}_0 = 0$ by Lemma 3.11, hence $\Lambda_\rho(J, \widehat{J}_0) = 0$ by Lemma 3.8, and therefore $\Lambda_\rho(J, \widehat{J}) = \Lambda_\rho(J, \mathcal{L}_v J) = -df_v \circ J = -d\langle \widehat{\omega}, \omega\rangle \circ J$ by (4.19). Since $\widehat{J}_0$ is skew-adjoint by (4.18), we have $\Omega_J(\widehat{J}, \widehat{J}') = \Omega_J(\mathcal{L}_v J, \widehat{J}') = 0$ for all $\widehat{J}'$ with $\widehat{J}' = (\widehat{J}')^*$ and $\bar{\partial}_J \widehat{J}' = 0$ and, by Lemma 3.6,

$$\langle (\widehat{J} - \widehat{J}^*)\cdot, \cdot\rangle = \langle (\mathcal{L}_v J - (\mathcal{L}_v J)^*)\cdot, \cdot\rangle + 2\langle \widehat{J}_0\cdot, \cdot\rangle = \widehat{\omega} - J^*\widehat{\omega}.$$

Hence $\widehat{J}$ satisfies (a) and this proves Step 2.

Step 3. *(a) is equivalent to (b).*

By Step 2, (b) implies (a). Now assume $\widehat{J}$ satisfies (a) and v satisfies (4.17). Define $\widehat{\omega}_0$ and $\widehat{J}_0$ by (4.18). Then $\mathcal{L}_v J + \widehat{J}_0$ satisfies (a) by Step 2 and so $\widehat{J} = \mathcal{L}_v J + \widehat{J}_0$ by Step 1. Thus $\widehat{J}$ satisfies (b) and this proves Step 3.

Thus we have established the equivalence of (a) and (b). By Step 2 and Hodge theory there exists an element $\widehat{J} \in \Omega^{0,1}_J(M, TM)$ that satisfies (a). Uniqueness was established in Step 1 and this proves Lemma 4.5. □

Theorem 4.6 (Symplectic Connection). (i) *Let* $(\omega, J) \in \mathscr{K}_0(M)$ *and let* $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$. *Then there exists a unique linear map*

$$\mathscr{A}_{\omega,J} : \Omega^2(M) \supset \ker d \to \Omega^{0,1}_J(M, TM)$$

which assigns to every closed real valued 2*-form* $\widehat{\omega} \in \Omega^2(M)$ *the unique infinitesimal complex structure* $\widehat{J} = \mathscr{A}_{\omega,J}(\widehat{\omega}) \in \Omega^{0,1}_J(M, TM)$ *that satisfies the equivalent conditions (a) and (b) in Lemma 4.5.*

(ii) *The* 1*-form* $\mathscr{A}$ *is* $\mathrm{Diff}(M)$*-equivariant, i.e.*

$$\mathscr{A}_{\phi^*\omega,\phi^*J}(\phi^*\widehat{\omega}) = \phi^*\mathscr{A}_{\omega,J}(\widehat{\omega}) \tag{4.20}$$

for every $(\omega, J) \in \mathscr{K}_0(M)$, *every closed* 2*-form* $\widehat{\omega}$, *and every orientation preserving diffeomorphism* $\phi : M \to M$. *Moreover,*

$$\mathscr{A}_{\omega,J}(d\iota(v)\omega) = \mathcal{L}_v J \tag{4.21}$$

for all $(\omega, J) \in \mathscr{K}_0(M)$ *and all* $v \in \mathrm{Vect}(M)$ *with* $d\iota(Jv)\omega^{\mathsf{n}} = 0$.

(iii) *The curvature of the connection* $\mathscr{A}$ *is a* $\mathrm{Diff}_0(M)$*-equivariant* 2*-form on* $\mathscr{S}_0(M)$ *with values in the space of smooth functions on the fiber* $\mathscr{T}_0(M, \omega)$. *It assigns to every* $\omega \in \mathscr{S}_0(M)$ *and every pair* $\widehat{\omega}_1, \widehat{\omega}_2$ *of closed* 2*-forms on* M *the Hamiltonian function* $\mathscr{H}_{\omega;\widehat{\omega}_1,\widehat{\omega}_2} : \mathscr{T}_0(M, \omega) \to \mathbb{R}$ *given by*

$$\begin{aligned} \mathscr{H}_{\omega;\widehat{\omega}_1,\widehat{\omega}_2}(J) &:= -\Omega_J\big(\mathscr{A}_{\omega,J}(\widehat{\omega}_1), \mathscr{A}_{\omega,J}(\widehat{\omega}_2)\big) \\ &= \tfrac{1}{2}\int_M \big(\iota(J)(\widehat{\omega}_1 - d\widehat{\lambda}_1)\big) \wedge \widehat{\omega}_2 \wedge \tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!} \end{aligned} \tag{4.22}$$

for $J \in \mathscr{J}_{\mathrm{int}}(M, \omega)$ *with* $\mathrm{Ric}_{\omega,J} = 0$, *where the* 1*-form* $\widehat{\lambda}_1 \in \Omega^1(M)$ *is chosen such that* $d^*(\widehat{\omega}_1 - d\widehat{\lambda}_1) = 0$ *with respect to the Kähler metric* $\langle\cdot,\cdot\rangle := \omega(\cdot, J\cdot)$. *The Hamiltonian vector field on* $\mathscr{T}_0(M, \omega)$ *generated by this function is the vertical part of the Lie bracket of the horizontal lifts of two vector fields on* $\mathscr{B}_0(M)$ *that take the values* $\widehat{\omega}_i$ *at* ω *(see [30, Lemma 6.4.8]).*

Proof. Part (i) and (4.21) follow directly from Lemma 4.5, while (4.20) follows by combining uniqueness in part (i) with the naturality conditions in Theorem 2.6 and Theorem 4.4. This proves (i) and (ii).

For part (iii) we must verify the second equality in (4.22). Fix a symplectic form $\omega \in \Omega^2(M)$ with real first Chern class zero, define $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$, and let $\widehat{\omega}_1, \widehat{\omega}_2 \in \Omega^2(M)$ be closed. Let $J \in \mathscr{J}_{\mathrm{int}}(M, \omega)$ such that $\mathrm{Ric}_{\rho,J} = 0$, choose $\widehat{\lambda}_i \in \Omega^1(M)$ such that $d^*(\widehat{\omega}_i - d\widehat{\lambda}_i) = 0$ and $d^*\widehat{\lambda}_i = 0$ with respect to the Kähler metric $\langle\cdot,\cdot\rangle = \omega(\cdot, J\cdot)$, and define $v_i \in \mathrm{Vect}(M)$ by $\iota(v_i)\omega := \widehat{\lambda}_i$.

By (i) and Lemma 4.5, we have

$$\mathscr{A}_{\omega,J}(\widehat{\omega}_i) = \widehat{J}_i + \mathcal{L}_{v_i}J, \quad \langle \widehat{J}_i\cdot,\cdot\rangle = \tfrac{1}{2}\big((\widehat{\omega}_i - d\widehat{\lambda}_i) - J^*(\widehat{\omega}_i - d\widehat{\lambda}_i)\big). \tag{4.23}$$

Since $\Lambda_\rho(J,\widehat{J}_i) = 0$ and $f_{Jv_i} = 0$ by (4.19), equation (4.23) yields

$$\mathscr{H}_{\omega;\widehat{\omega}_1,\widehat{\omega}_2}(J) = -\Omega_J\big(\widehat{J}_1 + \mathcal{L}_{v_1}J, \widehat{J}_2 + \mathcal{L}_{v_2}J\big) = -\tfrac{1}{2}\int_M \operatorname{trace}\big(\widehat{J}_1 J \widehat{J}_2\big)\rho.$$

Now choose a local orthonormal frame $e_1,\dots,e_{2\mathsf{n}}$. Then

$$\begin{aligned}
-\tfrac{1}{2}\operatorname{trace}\big(\widehat{J}_1 J\widehat{J}_2\big) &= -\tfrac{1}{2}\sum_i \langle J\widehat{J}_1 e_i, \widehat{J}_2 e_i\rangle = -\tfrac{1}{2}\sum_i \langle J\widehat{J}_1 e_i, e_j\rangle\langle e_j, \widehat{J}_2 e_i\rangle \\
&= \sum_{i<j}\langle \widehat{J}_1 e_i, J e_j\rangle\langle e_j,\widehat{J}_2 e_i\rangle = \tfrac{1}{2}\langle \iota(J)\tau_1,\tau_2\rangle,
\end{aligned}$$

where $\tau_i := \langle \widehat{J}_i\cdot,\cdot\rangle = \frac{1}{2}(\widehat{\omega}_i - d\widehat{\lambda}_i - J^*(\widehat{\omega}_i - d\widehat{\lambda}_i))$. This 2-form satisfies $(\tau_i)_J^{1,1} = 0$ and hence $*\tau_i = \tau_i\wedge\omega^{\mathsf{n}-2}/(\mathsf{n}-2)!$. Moreover, $\iota(J)\tau_i = \iota(J)(\widehat{\omega}_i - d\widehat{\lambda}_i)$. Thus

$$\begin{aligned}
\mathscr{H}_{\omega;\widehat{\omega}_1,\widehat{\omega}_2}(J) &= -\tfrac{1}{2}\int_M \operatorname{trace}\big(\widehat{J}_1 J\widehat{J}_2\big)\rho = \tfrac{1}{2}\int_M \big(\iota(J)\tau_1\big)\wedge *\tau_2 \\
&= \tfrac{1}{4}\int_M \big(\iota(J)(\widehat{\omega}_1 - d\widehat{\lambda}_1)\big)\wedge\big(\widehat{\omega}_2 - d\widehat{\lambda}_2 - J^*(\widehat{\omega}_2 - d\widehat{\lambda}_2)\big)\wedge\tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!} \\
&= \tfrac{1}{2}\int_M \big(\iota(J)(\widehat{\omega}_1 - d\widehat{\lambda}_1)\big)\wedge\big(\widehat{\omega}_2 - d\widehat{\lambda}_2\big)\wedge\tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!} \\
&= \tfrac{1}{2}\int_M \big(\iota(J)(\widehat{\omega}_1 - d\widehat{\lambda}_1)\big)\wedge\widehat{\omega}_2\wedge\tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!}.
\end{aligned}$$

This proves (4.22). The right hand side of (4.22) depends only on the cohomology classes of $\widehat{\omega}_1$ and $\widehat{\omega}_2$. Hence it is invariant under the action of $\mathrm{Diff}_0(M)\cap\mathrm{Symp}(M,\omega)$ on J, because $\phi^*\widehat{\omega}_i - \widehat{\omega}_i$ is exact for $\phi\in\mathrm{Diff}_0(M)$. Thus it descends to a function on $\mathscr{T}_0(M,\omega)$. This proves Theorem 4.6. □

The quotient of the Calabi–Yau Teichmüller space by the mapping class group is the Calabi–Yau moduli space $\mathscr{M}_0(M) := \mathscr{J}_{\mathrm{int},0}(M)/\mathrm{Diff}^+(M)$. For each Kählerable symplectic form ω with real first Chern class zero there is a polarized Calabi–Yau moduli space $\mathscr{M}_0(M,\omega) := \mathscr{J}_{\mathrm{int},0}(M,\omega)/\mathrm{Symp}(M,\omega)$, the quotient of the polarized Teichmüller space $\mathscr{T}_0(M,\omega)$ by the symplectic mapping class group. The study of the geometry of these moduli spaces is a vast and extremely active area of research in both mathematics and physics. An overview of the subject and many references can be found in the article [46] by Shing-Tung Yau.

A Torsion-free connections

Let M be an oriented $2\mathsf{n}$-manifold. We prove that a nondegenerate 2-form on M is preserved by a torsion-free connection if and only if it is closed, and that an almost complex structure on M is preserved by a torsion-free connection if and only if it is integrable. We use the sign conventions

$$[\mathcal{L}_u, \mathcal{L}_v] + \mathcal{L}_{[u,v]} = 0$$

for the Lie bracket and

$$N_J(u,v) = [u,v] + J[Ju,v] + J[u,Jv] - [Ju,Jv] \tag{A.1}$$

for the Nijenhuis tensor. If ∇ is a torsion-free connection on TM then

$$N_J(u,v) = (\nabla_u J)Jv + (\nabla_{Ju}J)v - (\nabla_v J)Ju - (\nabla_{Jv}J)u. \tag{A.2}$$

Lemma A.1. *Let M be a $2\mathsf{n}$-manifold.*

(i) *An almost complex structure J is integrable if and only if there exists a torsion-free connection ∇ on TM such that $\nabla J = 0$. If J is integrable and $\rho \in \Omega^{2\mathsf{n}}(M)$ is a volume form inducing the same orientation as J, then there exists a torsion-free connection ∇ on TM such that $\nabla\rho = 0$ and $\nabla J = 0$.*

(ii) *A nondegenerate 2-form $\omega \in \Omega^2(M)$ is closed if and only if there exists a torsion-free connection ∇ on TM such that $\nabla\omega = 0$.*

Proof. We prove part (i). If ∇ is a torsion-free connection with $\nabla J = 0$ it follows directly from (A.2) that $N_J = 0$. Conversely suppose J is integrable and let ρ be a volume form on M inducing the same orientation as J. Fix a background metric g on M. Then $g_J := g + J^*g$ is a metric with respect to which J is skew-adjoint and, if $\mathrm{dvol}_J \in \Omega^{2\mathsf{n}}(M)$ is the volume form of this metric, then the metric $g_{\rho,J} := (\rho/\mathrm{dvol}_J)^{1/\mathsf{n}} g_J$ has the volume form ρ. Let ∇ be the Levi-Civita connection of the metric $g_{\rho,J}$. Then ∇ is torsion-free and $\nabla\rho = 0$. Let $\alpha(u) := \frac{1}{2}\mathrm{trace}(J(\nabla J)u)$ and define

$$\begin{aligned}\widehat{\nabla}_u v := \nabla_u v - \tfrac{1}{2}J(\nabla_u J)v - \tfrac{1}{4}J(\nabla_v J)u - \tfrac{1}{4}(\nabla_{Jv}J)u \\ + \frac{\alpha(u)v + \alpha(v)u - \alpha(Ju)Jv - \alpha(Jv)Ju}{2\mathsf{n}+2}.\end{aligned} \tag{A.3}$$

Then $\widehat{\nabla}\rho = 0$, $\widehat{\nabla}J = 0$, and a calculation shows that $\mathrm{Tor}^{\widehat{\nabla}} = -\frac{1}{4}N_J$, so $\widehat{\nabla}$ is torsion-free if and only if J is integrable. This proves (i).

We prove part (ii). If ∇ is torsion-free and $\nabla\omega = 0$ then

$$\begin{aligned} d\omega(u,v,w) &= \mathcal{L}_u(\omega(v,w)) + \mathcal{L}_v(\omega(w,u)) + \mathcal{L}_w(\omega(u,v)) \\ &\quad + \omega([v,w],u) + \omega([w,u],v) + \omega([u,v],w) \\ &= \omega([v,w] - \nabla_w v + \nabla_v w, u) + \omega([w,u] - \nabla_u w + \nabla_w u, v) \\ &\quad + \omega([u,v] - \nabla_v u + \nabla_u v, w) = 0. \end{aligned}$$

Conversely, suppose ω is a symplectic form and choose an almost complex structure J on M that is compatible with ω, so $\langle\cdot,\cdot\rangle := \omega(\cdot, J\cdot)$ is a Riemannian metric. Let ∇ be its Levi-Civita connection. Then

$$\langle(\nabla_u J)v, w\rangle + \langle(\nabla_v J)w, u\rangle + \langle(\nabla_w J)u, v\rangle = d\omega(u,v,w) = 0 \tag{A.4}$$

for all $u, v, w \in \mathrm{Vect}(M)$ by [30, Lemma 4.1.14]. Define

$$\widetilde{\widetilde{\nabla}}_u v := \nabla_u v + A(u)v, \qquad A(u)v := -\tfrac{1}{3}J\big((\nabla_u J)v + (\nabla_v J)u\big). \tag{A.5}$$

This connection is torsion-free and satisfies $JA(u) + A(u)^* J = \nabla_u J$ for every vector field $u \in \mathrm{Vect}(M)$ by a straight forward calculation. Hence

$$\begin{aligned} \omega(\widetilde{\widetilde{\nabla}}_u v, w) + \omega(v, \widetilde{\widetilde{\nabla}}_u w) &= \langle J\nabla_u v + JA(u)v, w\rangle + \langle Jv, \nabla_u w + A(u)w\rangle \\ &= \langle (JA(u) + A(u)^*J)v, w\rangle + \langle J\nabla_u v, w\rangle + \langle Jv, \nabla_u w\rangle \\ &= \langle (\nabla_u J)v, w\rangle + \langle J\nabla_u v, w\rangle + \langle Jv, \nabla_u w\rangle \\ &= \mathcal{L}_u\langle Jv, w\rangle = \mathcal{L}_u\big(\omega(v,w)\big) \end{aligned}$$

for all $u, v, w \in \mathrm{Vect}(M)$. This proves Lemma A.1. □

Lemma A.2. *Let M be an oriented $2\mathsf{n}$-manifold, let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form, let $J \in \mathscr{J}_{\mathrm{int}}(M)$ be a complex structure compatible with the orientation, and let ∇ be a torsion-free ρ-connection such that $\nabla J = 0$. Then* $\mathrm{trace}(JR^\nabla)$ *is a $(1,1)$-form.*

Proof. Since ∇ is torsion-free, R^∇ satisfies the first Bianchi identity. Thus

$$\begin{aligned} &R(u,v)w + JR(Ju,v)w + JR(u,Jv)w - R(Ju,Jv)w \\ &= R(u,v)w + JR(Ju,v)w + JR(u,Jv)w + R(Jv,w)Ju + R(w,Ju)Jv \\ &= R(u,v)w + JR(Ju,v)w + JR(w,Ju)v + JR(u,Jv)w + JR(Jv,w)u \\ &= R(u,v)w - JR(v,w)Ju - JR(w,u)Jv \\ &= R(u,v)w + R(v,w)u + R(w,u)v = 0 \end{aligned}$$

and so $JR(u,v) - R(Ju,v) - R(u,Jv) - JR(Ju,Jv) = 0$. Take the trace to obtain $\mathrm{trace}(JR(u,v)) = \mathrm{trace}(JR(Ju,Jv))$. This proves Lemma A.2. □

B The Bochner–Kodaira–Nakano identity

This appendix gives a self-contained proof of the Bochner–Kodaira–Nakano identity (3.11) for complex anti-linear 1-forms with values in the tangent bundle. This formula plays a central role in the proofs of Lemma 3.10 and Lemma 3.11. Assume throughout that (M, ω, J) is a $2\mathsf{n}$-dimensional Kähler manifold with the volume form $\rho := \omega^{\mathsf{n}}/\mathsf{n}!$, denote by ∇ the Levi-Civita connection of the Kähler metric, by $R^\nabla \in \Omega^2(M, \mathrm{End}(TM))$ the Riemann curvature tensor, by $\mathrm{Ric}_{\rho,J} := \frac{1}{2}\mathrm{trace}(JR^\nabla) \in \Omega^2(M)$ the Ricci-form, and define the complex linear skew-adjoint endomorphism $Q \in \Omega^0(M, \mathrm{End}(TM))$ by

$$\langle Qu, v\rangle = \mathrm{Ric}_{\rho,J}(u, v) \tag{B.1}$$

for $u, v \in \mathrm{Vect}(M)$. Define the map $\mathcal{T} : \Omega^{0,1}_J(M, TM) \to \Omega^{0,1}_J(M, TM)$ by

$$\mathcal{T}(\widehat{J})u := \sum_{i=1}^{2\mathsf{n}} R^\nabla(e_i, u)\widehat{J}e_i \tag{B.2}$$

for $\widehat{J} \in \Omega^{0,1}_J(M, TM)$ and $u \in \mathrm{Vect}(M)$, where $e_1, \dots, e_{2\mathsf{n}}$ is a local orthonormal frame of the tangent bundle. With this notation the operator $\nabla^*\nabla$ on the space of sections of the endomorphism bundle is given by

$$\nabla^*\nabla\widehat{J} = -\sum_i \left(\nabla_{e_i}\nabla_{e_i}\widehat{J} + \mathrm{div}(e_i)\nabla_{e_i}\widehat{J}\right) \tag{B.3}$$

for $\widehat{J} \in \Omega^{0,1}_J(M, TM)$.

Theorem B.1 (Bochner–Kodaira–Nakano). *Every $\widehat{J} \in \Omega^{0,1}_J(M, TM)$ satisfies the equation*

$$\bar\partial_J^*\bar\partial_J\widehat{J} + \bar\partial_J\bar\partial_J^*\widehat{J} = \tfrac{1}{2}\nabla^*\nabla\widehat{J} + \tfrac{1}{2}[JQ, \widehat{J}] + \mathcal{T}(\widehat{J}). \tag{B.4}$$

Proof. See page 44. □

The proof relies on the following three lemmas. Throughout we use the notation $e_1, \dots, e_{2\mathsf{n}}$ for a local orthonormal frame of the tangent bundle, and it will sometimes be convenient to choose the frame such that $e_{\mathsf{n}+i} = Je_i$ for $i = 1, \dots, \mathsf{n}$. We will use the notation $\mathrm{div}(u)$ for the divergence of a vector field $u \in \mathrm{Vect}(M)$; thus $\mathrm{div}(u) = \mathrm{trace}(\nabla u)$ and $\mathrm{div}(u)\rho = d\iota(u)\rho$. For $u \in \mathrm{Vect}(M)$ we denote by $\mathcal{L}_u$ the Lie derivative (of any object on M) in the direction of u; thus $\mathcal{L}_u f = df \circ u$ for $f \in \Omega^0(M)$ and $\mathcal{L}_u v = [v, u]$ for $v \in \mathrm{Vect}(M)$. (Note the sign convention for the Lie bracket.)

Lemma B.2. *Let $B : TM \otimes TM \to E$ be any bilinear form on the tangent bundle with values in a vector bundle $E \to M$. Then*

$$\sum_{i=1}^{2n}\Big(B(e_i, \nabla_u e_i) + B(\nabla_u e_i, e_i)\Big) = 0 \tag{B.5}$$

for all $u \in \mathrm{Vect}(M)$. Moreover,

$$\sum_{i=1}^{2n}(\nabla_{e_i} e_i + \mathrm{div}(e_i)e_i) = 0. \tag{B.6}$$

Proof. Define the coefficients c_{ij} by

$$c_{ij} := \langle \nabla_u e_i, e_j\rangle$$

so that $\nabla_u e_i = \sum_j c_{ij} e_j$. Then $c_{ij} + c_{ji} = \mathcal{L}_u\langle e_i, e_j\rangle = 0$, because the frame is orthonormal and ∇ is a Riemannian connection. Hence

$$\begin{aligned}\sum_{i=1}^{2n} B(e_i, \nabla_u e_i) &= \sum_{i=1}^{2n}\sum_{j=1}^{2n} c_{ij} B(e_i, e_j) \\ &= -\sum_{j=1}^{2n}\sum_{i=1}^{2n} c_{ji} B(e_i, e_j) \\ &= -\sum_{j=1}^{2n} B(\nabla_u e_j, e_j)\end{aligned}$$

and this proves (B.5). Since

$$0 = \mathcal{L}_{e_i}\langle e_j, e_i\rangle = \langle \nabla_{e_i} e_j, e_i\rangle + \langle e_j, \nabla_{e_i} e_i\rangle$$

for all i and j, we also have

$$\begin{aligned}\sum_{i=1}^{2n}\langle e_j, \mathrm{div}(e_i)e_i + \nabla_{e_i} e_i\rangle &= \mathrm{div}(e_j) - \sum_{i=1}^{2n}\langle \nabla_{e_i} e_j, e_i\rangle \\ &= \mathrm{div}(e_j) - \mathrm{trace}(\nabla e_j) \\ &= 0\end{aligned}$$

for all j. This proves (B.6) and Lemma B.2. □

Lemma B.3. *The endomorphism Q in* (B.1) *is given by*

$$Qu = -\tfrac{1}{2}\sum_{i=1}^{2\mathsf{n}} R^\nabla(e_i, Je_i)u = \sum_{i=1}^{2\mathsf{n}} JR^\nabla(u, e_i)e_i \tag{B.7}$$

for $u \in \mathrm{Vect}(M)$.

Proof. Let $u, v \in \mathrm{Vect}(M)$. Then by (B.1) we have

$$\begin{aligned}
\langle Qu, v\rangle &= \mathrm{Ric}_{\rho,J}(u, v) \\
&= \tfrac{1}{2}\mathrm{trace}(JR^\nabla(u, v)) \\
&= \tfrac{1}{2}\sum_{i=1}^{2\mathsf{n}} \langle e_i, JR^\nabla(u, v)e_i\rangle \\
&= -\tfrac{1}{2}\sum_{i=1}^{2\mathsf{n}} \langle R^\nabla(u, v)e_i, Je_i\rangle \\
&= -\tfrac{1}{2}\sum_{i=1}^{2\mathsf{n}} \langle R^\nabla(e_i, Je_i)u, v\rangle.
\end{aligned}$$

This proves the first equality in (B.7). Moreover, it follows from the first Bianchi identity that

$$\begin{aligned}
Qu &= -\tfrac{1}{2}\sum_{i=1}^{2\mathsf{n}} R^\nabla(e_i, Je_i)u \\
&= \tfrac{1}{2}\sum_{i=1}^{2\mathsf{n}} \Big(R^\nabla(u, e_i)Je_i + R^\nabla(Je_i, u)e_i\Big) \\
&= \tfrac{1}{2}\sum_{i=1}^{2\mathsf{n}} \Big(R^\nabla(u, e_i)Je_i - R^\nabla(e_i, u)Je_i\Big) \\
&= \sum_{i=1}^{2\mathsf{n}} R^\nabla(u, e_i)Je_i \\
&= \sum_{i=1}^{2\mathsf{n}} JR^\nabla(u, e_i)e_i.
\end{aligned}$$

Here the third step uses a frame that satisfies $e_{\mathsf{n}+i} = Je_i$ for $i = 1, \dots, \mathsf{n}$. This proves (B.7) and Lemma B.3. □

Lemma B.4. *Let* $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ *and* $\tau \in \Omega^{0,2}_J(M,TM)$. *Then*

$$\bar\partial_J^* \widehat{J} = -\sum_{i=1}^{2n} (\nabla_{e_i}\widehat{J})e_i \in \mathrm{Vect}(M), \tag{B.8}$$

$$\bar\partial_J^* \tau = -\sum_{i=1}^{2n} (\nabla_{e_i}\tau)(e_i,\cdot) \in \Omega^{0,1}_J(M,TM). \tag{B.9}$$

Proof. For $u \in \mathrm{Vect}(M)$ the formal adjoint operator of the covariant derivative ∇_u is given by ${\nabla_u}^* = -\nabla_u - \mathrm{div}(u)$. Fix a vector field $X \in \mathrm{Vect}(M)$. Since the $(1,0)$-form $\partial_J X = \frac{1}{2}(\nabla X - J(\nabla X)J)$ is orthogonal to $\widehat{J}$, we have

$$\begin{aligned}
\langle \bar\partial_J^*\widehat{J}, X\rangle_{L^2} &= \langle \widehat{J}, \bar\partial_J X\rangle_{L^2} = \langle \widehat{J}, \nabla X\rangle_{L^2} = \int_M \mathrm{trace}\left(\widehat{J}^*\nabla X\right)\rho \\
&= \sum_{i=1}^{2n}\int_M \langle \widehat{J}e_i, \nabla_{e_i}X\rangle\rho = -\sum_{i=1}^{2n}\int_M \langle \nabla_{e_i}(\widehat{J}e_i) + \mathrm{div}(e_i)\widehat{J}e_i, X\rangle\rho \\
&= -\sum_{i=1}^{2n}\int_M \langle \nabla_{e_i}(\widehat{J}e_i) - \widehat{J}\nabla_{e_i}e_i, X\rangle\rho = -\sum_{i=1}^{2n}\int_M \langle (\nabla_{e_i}\widehat{J})e_i, X\rangle\rho.
\end{aligned}$$

Here the penultimate step uses (B.6) in Lemma B.2. This proves (B.8). Now

$$\begin{aligned}
\langle \bar\partial_J^*\tau, \widehat{J}\rangle_{L^2} &= \langle \tau, d^\nabla \widehat{J}\rangle_{L^2} = \frac{1}{2}\sum_{i,j=1}^{2n}\int_M \langle \tau(e_i,e_j), (\nabla_{e_i}\widehat{J})e_j - (\nabla_{e_j}\widehat{J})e_i\rangle\rho \\
&= \sum_{i,j=1}^{2n}\int_M \langle \tau(e_i,e_j), (\nabla_{e_i}\widehat{J})e_j\rangle\rho \\
&= \sum_{i,j=1}^{2n}\int_M \langle \tau(e_i,e_j), \nabla_{e_i}(\widehat{J}e_j) - \widehat{J}\nabla_{e_i}e_j\rangle\rho \\
&= -\sum_{i,j=1}^{2n}\int_M \langle \nabla_{e_i}(\tau(e_i,e_j)) + \mathrm{div}(e_i)\tau(e_i,e_j) - \tau(e_i,\nabla_{e_i}e_j), \widehat{J}e_j\rangle\rho \\
&= -\sum_{i,j=1}^{2n}\int_M \langle (\nabla_{e_i}\tau)(e_i,e_j), \widehat{J}e_j\rangle\rho.
\end{aligned}$$

Here the last but one step uses equation (B.5) and the last step uses equation (B.6) in Lemma B.2. This proves equation (B.9) and Lemma B.4. □

Proof of Theorem B.1. Let $\widehat{J} \in \Omega^{0,1}_J(M,TM)$ and let $X := \bar{\partial}_J^* \widehat{J} \in \mathrm{Vect}(M)$ and $\tau := \bar{\partial}_J \widehat{J} \in \Omega^{0,2}_J(M,TM)$. Then, by Lemma B.4, we have

$$\begin{aligned} X &= -\sum_i (\nabla_{e_i}\widehat{J})e_i, \\ (\bar{\partial}_J X)(u) &= \tfrac{1}{2}\big(\nabla_u X + J\nabla_{Ju}X\big), \\ \tau(u,v) &= \tfrac{1}{2}\big((\nabla_u\widehat{J})v - (\nabla_v\widehat{J})u - (\nabla_{Ju}\widehat{J})Jv + (\nabla_{Jv}\widehat{J})Ju\big), \\ (\bar{\partial}_J^*\tau)(u) &= -\sum_i (\nabla_{e_i}\tau)(e_i,u) \end{aligned}$$

for all $u, v \in \mathrm{Vect}(M)$. Hence, by Lemma B.2,

$$\begin{aligned} &(\bar{\partial}_J^*\bar{\partial}_J\widehat{J} + \bar{\partial}_J\bar{\partial}_J^*\widehat{J})(u) = (\bar{\partial}_J^*\tau)(u) + (\bar{\partial}_J X)(u) \\ &= -\sum_i \nabla_{e_i}\big(\tau(e_i,u)\big) + \sum_i \tau(\nabla_{e_i}e_i,u) + \sum_i \tau(e_i,\nabla_{e_i}u) \\ &\quad - \tfrac{1}{2}\sum_i \Big(\nabla_u((\nabla_{e_i}\widehat{J})e_i) + J\nabla_{Ju}((\nabla_{e_i}\widehat{J})e_i)\Big) \\ &= -\tfrac{1}{2}\sum_i \nabla_{e_i}\Big((\nabla_{e_i}\widehat{J})u - (\nabla_u\widehat{J})e_i - (\nabla_{Je_i}\widehat{J})Ju + (\nabla_{Ju}\widehat{J})Je_i\Big) \\ &\quad - \tfrac{1}{2}\sum_i \mathrm{div}(e_i)\Big((\nabla_{e_i}\widehat{J})u - (\nabla_u\widehat{J})e_i - (\nabla_{Je_i}\widehat{J})Ju - (\nabla_{Ju}\widehat{J})Je_i\Big) \\ &\quad + \tfrac{1}{2}\sum_i \Big((\nabla_{e_i}\widehat{J})\nabla_{e_i}u - (\nabla_{\nabla_{e_i}u}\widehat{J})e_i - (\nabla_{Je_i}\widehat{J})J\nabla_{e_i}u + (\nabla_{J\nabla_{e_i}u}\widehat{J})Je_i\Big) \\ &\quad - \tfrac{1}{2}\sum_i \Big((\nabla_u\nabla_{e_i}\widehat{J})e_i + (\nabla_{e_i}\widehat{J})\nabla_u e_i + J(\nabla_{Ju}\nabla_{e_i}\widehat{J})e_i + J(\nabla_{e_i}\widehat{J})\nabla_{Ju}e_i\Big) \\ &= -\tfrac{1}{2}\sum_i \Big(\nabla_{e_i}\nabla_{e_i}\widehat{J} + \mathrm{div}(e_i)\nabla_{e_i}\widehat{J}\Big)u + \tfrac{1}{2}\sum_i \Big(\nabla_{e_i}\nabla_{Je_i}\widehat{J} + \mathrm{div}(e_i)\nabla_{Je_i}\widehat{J}\Big)Ju \\ &\quad + \tfrac{1}{2}\sum_i \Big(\nabla_{e_i}\nabla_u\widehat{J} - \nabla_u\nabla_{e_i}\widehat{J} + \nabla_{[e_i,u]}\widehat{J}\Big)e_i \\ &\quad - \tfrac{1}{2}\sum_i \Big(\nabla_{e_i}\nabla_{Ju}\widehat{J} - \nabla_{Ju}\nabla_{e_i}\widehat{J} + \nabla_{[e_i,Ju]}\widehat{J}\Big)Je_i \\ &= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u + \tfrac{1}{2}\sum_i \Big(\nabla_{e_i}\nabla_{Je_i}\widehat{J} + \mathrm{div}(e_i)\nabla_{Je_i}\widehat{J}\Big)Ju \\ &\quad + \tfrac{1}{2}\sum_i [R^\nabla(e_i,u),\widehat{J}]e_i - \tfrac{1}{2}\sum_i [R^\nabla(e_i,Ju),\widehat{J}]Je_i. \end{aligned}$$

Here we have used equation (B.3) and the formula

$$[R^\nabla(u,v),\widehat{J}] = \nabla_u\nabla_v\widehat{J} - \nabla_v\nabla_u\widehat{J} + \nabla_{[u,v]}\widehat{J}$$

for the Riemann curvature tensor. Now use the identities (B.6) in Lemma B.2 and (B.7) in Lemma B.3 to obtain

$$\begin{aligned}
-[Q,\widehat{J}] &= \tfrac{1}{2}\sum_i [R^\nabla(e_i,Je_i),\widehat{J}] \\
&= \tfrac{1}{2}\sum_i \Big(\nabla_{e_i}\nabla_{Je_i}\widehat{J} - \nabla_{Je_i}\nabla_{e_i}\widehat{J} + \nabla_{[e_i,Je_i]}\widehat{J}\Big) \\
&= \tfrac{1}{2}\sum_i \Big(\nabla_{e_i}\nabla_{Je_i}\widehat{J} - \nabla_{\nabla_{e_i}(Je_i)}\widehat{J}\Big) - \tfrac{1}{2}\sum_i \Big(\nabla_{Je_i}\nabla_{e_i}\widehat{J} - \nabla_{\nabla_{Je_i}e_i}\widehat{J}\Big) \\
&= \sum_i \Big(\nabla_{e_i}\nabla_{Je_i}\widehat{J} - \nabla_{\nabla_{e_i}(Je_i)}\widehat{J}\Big) \\
&= \sum_i \Big(\nabla_{e_i}\nabla_{Je_i}\widehat{J} + \mathrm{div}(e_i)\nabla_{Je_i}\widehat{J}\Big).
\end{aligned}$$

This yields the formula

$$\begin{aligned}
&(\bar{\partial}_J^*\bar{\partial}_J\widehat{J} + \bar{\partial}_J\bar{\partial}_J^*\widehat{J})u \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u - \tfrac{1}{2}[Q,\widehat{J}]Ju + \tfrac{1}{2}\sum_i\Big([R^\nabla(e_i,u),\widehat{J}]e_i - [R^\nabla(e_i,Ju),\widehat{J}]Je_i\Big) \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u - \tfrac{1}{2}[Q,\widehat{J}]Ju + \tfrac{1}{2}\sum_i\Big([R^\nabla(e_i,u),\widehat{J}]e_i + [R^\nabla(Je_i,Ju),\widehat{J}]e_i\Big) \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u - \tfrac{1}{2}[Q,\widehat{J}]Ju + \sum_i [R^\nabla(e_i,u),\widehat{J}]e_i \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u - \tfrac{1}{2}[Q,\widehat{J}]Ju - \sum_i \widehat{J}R^\nabla(e_i,u)e_i + \sum_i R^\nabla(e_i,u)\widehat{J}e_i \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u - \tfrac{1}{2}[Q,\widehat{J}]Ju - \sum_i \widehat{J}JJR^\nabla(u,e_i)e_i + \mathcal{T}(\widehat{J})u \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u - \tfrac{1}{2}Q\widehat{J}Ju + \tfrac{1}{2}\widehat{J}QJu - \widehat{J}JQu + \mathcal{T}(\widehat{J})u \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u + \tfrac{1}{2}JQ\widehat{J}u - \tfrac{1}{2}\widehat{J}JQu + \mathcal{T}(\widehat{J})u \\
&= \tfrac{1}{2}(\nabla^*\nabla\widehat{J})u + \tfrac{1}{2}[JQ,\widehat{J}]u + \mathcal{T}(\widehat{J})u.
\end{aligned}$$

Here we have used (B.2) and Lemma B.3. This proves Theorem B.1. □

C Bott–Chern cohomology

Let M be a closed connected $2\mathsf{n}$-manifold and let J be an almost complex structure on M.

Lemma C.1. *The almost complex structure J is integrable if and only if 2-form $d(df \circ J)$ is of type $(1,1)$ for every smooth function $f : M \to \mathbb{R}$.*

Proof. The assertion follows directly from the identity

$$d(df \circ J)(u,v) - d(df \circ J)(Ju, Jv) = df(JN_J(u,v)). \tag{C.1}$$

for all $f \in \Omega^0(M)$ and all $u, v \in \mathrm{Vect}(M)$. To prove this equation, choose a Riemannian metric on M with respect to which the almost complex structure is skew-adjoint, let ∇ be the Levi-Civita connection of this metric, denote by ∇f the gradient of f, and abbreviate $\tau_f := d(df \circ J)$. Then

$$\begin{aligned}
\tau_f(u,v) &= \mathcal{L}_u(df(Jv)) - \mathcal{L}_v(df(Ju)) + df(J[u,v]) \\
&= \mathcal{L}_u\langle \nabla f, Jv\rangle - \mathcal{L}_v\langle \nabla f, Ju\rangle + \langle \nabla f, J[u,v]\rangle \\
&= \langle \nabla_u \nabla f, Jv\rangle - \langle \nabla_v \nabla f, Ju\rangle + \langle \nabla f, (\nabla_u J)v - (\nabla_v J)u\rangle
\end{aligned}$$

and hence

$$\tau_f(Ju, Jv) = -\langle \nabla_{Ju}\nabla f, v\rangle + \langle \nabla_{Jv}\nabla f, u\rangle + \langle \nabla f, (\nabla_{Ju} J)Jv - (\nabla_{Jv} J)Ju\rangle.$$

Take the difference of these equations and use the fact that the covariant Hessian of f is symmetric to obtain

$$\begin{aligned}
\tau_f(u,v) - \tau_f(Ju,Jv) &= \langle \nabla f, (\nabla_u J)v - (\nabla_v J)u - (\nabla_{Ju} J)Jv + (\nabla_{Jv} J)Ju\rangle \\
&= \langle \nabla f, JN_J(u,v)\rangle \\
&= df(JN_J(u,v)).
\end{aligned}$$

Here the second equality follows from (A.2). This proves Lemma C.1. □

Throughout the remainder of this appendix we assume that J is integrable. Then $\bar\partial \circ \bar\partial = 0$ and so

$$\mathbf{i}\partial\bar\partial f = \mathbf{i}d\bar\partial f = -\tfrac{1}{2}d(df \circ J) \tag{C.2}$$

for every smooth function $f : M \to \mathbb{R}$.

The Bott–Chern cohomology groups of M are defined by

$$H^{p,q}_{\mathrm{BC}}(M;\mathbb{C}) := \frac{\left\{\tau\in\Omega^{p,q}(M;\mathbb{C})\,|\,\partial\tau=0,\,\bar\partial\tau=0\right\}}{\left\{\partial\bar\partial\sigma\,|\,\sigma\in\Omega^{p-1,q-1}(M;\mathbb{C})\right\}}.$$

These are finite-dimensional complex vector spaces [5]. In the Kähler case the $\partial\bar\partial$-lemma asserts that every exact (p,q)-form on M belongs to the image of the operator $\partial\bar\partial:\Omega^{p-1,q-1}(M;\mathbb{C})\to\Omega^{p,q}(M;\mathbb{C})$. This implies that the Bott–Chern cohomology group agrees with the deRham cohomology group

$$H^{p,q}_{\mathrm{dR}}(M;\mathbb{C}) := \frac{\{\tau\in\Omega^{p,q}(M;\mathbb{C})\,|\,d\tau=0\}}{\{d\alpha\,|\,\alpha\in\Omega^{p+q-1}(M;\mathbb{C}),\,d\alpha\in\Omega^{p,q}(M)\}}$$

and the direct sum of these groups is $H^{p+q}_{\mathrm{dR}}(M;\mathbb{C})$. In general, there is a surjective map $H^{p,q}_{\mathrm{BC}}(M;\mathbb{C})\to H^{p,q}_{\mathrm{dR}}(M;\mathbb{C})$ which may have a nontrivial kernel.

In the present paper the relevant case is $p=q=1$. Denote by $\Omega^{1,1}(M)$ the space of real valued 2-forms $\tau\in\Omega^2(M)$ that satisfy $\tau(u,v)=\tau(Ju,Jv)$ for all $u,v\in\mathrm{Vect}(M)$ and define

$$H^{1,1}_{\mathrm{BC}}(M) := \frac{\{\tau\in\Omega^{1,1}(M)\,|\,d\tau=0\}}{\{d(df\circ J)\,|\,f\in\Omega^0(M;\mathbb{R})\}}. \tag{C.3}$$

The kernel of the homomorphism $H^{1,1}_{\mathrm{BC}}(M)\to H^{1,1}_{\mathrm{dR}}(M)$ has the dimension

$$\kappa(J) := \dim\frac{\{d\lambda\,|\,\lambda\in\Omega^1(M),\,d\lambda\in\Omega^{1,1}(M)\}}{\{d(df\circ J)\,|\,f\in\Omega^0(M)\}}. \tag{C.4}$$

To examine this invariant, choose a nondegenerate 2-form ω that is compatible with J, so $\langle\cdot,\cdot\rangle:=\omega(\cdot,J\cdot)$ is a Riemannian metric on M, and denote by $*:\Omega^k(M)\to\Omega^{2\mathsf{n}-k}(M)$ the Hodge $*$-operator of this metric. Then

$$*\lambda = -(\lambda\circ J)\wedge\frac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!},\qquad *\left(\lambda\wedge\frac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!}\right) = -\lambda\circ J \tag{C.5}$$

for $\lambda\in\Omega^1(M)$. For $\mathsf{n}\ge2$ the operator $\tau\mapsto*(\tau\wedge\omega^{\mathsf{n}-2}/(\mathsf{n}-2)!)$ on $\Omega^2(M)$ has trace zero and eigenvalues ±1 and $\mathsf{n}-1$. Define

$$\Omega^{\pm}(M) := \left\{\tau\in\Omega^2(M)\,\middle|\,*\left(\tau\wedge\frac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!}\right)=\pm\tau\right\}.$$

Then $\Omega^{1,1}(M)=\Omega^0(M)\omega\oplus\Omega^-(M)$ and

$$*\left(\tau\wedge\frac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!}\right) = \langle\tau,\omega\rangle\omega - J^*\tau,\qquad \langle\tau,\omega\rangle\frac{\omega^{\mathsf{n}}}{\mathsf{n}!} := \tau\wedge\frac{\omega^{\mathsf{n}-1}}{(\mathsf{n}-1)!}, \tag{C.6}$$

for all $\tau\in\Omega^2(M)$, where $(J^*\tau)(u,v):=\tau(Ju,Jv)$ for $u,v\in\mathrm{Vect}(M)$.

Lemma C.2. **(i)** *The numbers*

$$\begin{aligned}\kappa_0(J,\omega) &:= \dim \frac{\{\langle d\lambda,\omega\rangle \,|\, \lambda\in\Omega^1(M),\, d\lambda\in\Omega^{1,1}(M)\}}{\{\langle d(df\circ J),\omega\rangle \,|\, f\in\Omega^0(M)\}},\\ \kappa_1(J,\omega) &:= \dim\left\{d\lambda \,|\, \lambda\in\Omega^1(M),\, d\lambda\in\Omega^{1,1}(M),\, \langle d\lambda,\omega\rangle = 0\right\}\end{aligned} \tag{C.7}$$

are finite and $\kappa_0(J,\omega)\in\{0,1\}$. *Moreover, the number* $\kappa_0(J,\omega)$ *vanishes whenever* $\omega^{\mathsf{n}-1}$ *is closed, and* $\kappa_1(J,\omega)$ *vanishes whenever* $\omega^{\mathsf{n}-2}$ *is closed.*
(ii) $\kappa(J) = \kappa_0(J,\omega) + \kappa_1(J,\omega)$.
(iii) $\kappa(J) = 0$ *if and only if for every exact* $(1,1)$*-form* $\tau\in\Omega^{1,1}(M)$ *there exists a function* $f\in\Omega^0(M)$ *such that* $d(df\circ J) = \tau$. *Moreover,* $\kappa(J) = 0$ *whenever* J *admits a Kähler form.*

Proof. We prove part (i). Define the operator $L_0 : \Omega^0(M)\to\Omega^0(M)$ by

$$L_0 f := \langle d(df\circ J),\omega\rangle = d^*df - \frac{(df\circ J)\wedge d(\omega^{\mathsf{n}-1}/(\mathsf{n}-1)!)}{\omega^{\mathsf{n}}/\mathsf{n}!} \tag{C.8}$$

for $f\in\Omega^0(M)$. Here the second equation follows from (C.5). Then L_0 is a second order elliptic operator without zeroth order terms. Thus its kernel consists of the constant functions by the Hopf maximum principle. Moreover, L_0 is an index zero Fredholm operator and so has a one-dimensional cokernel. Thus $\kappa_0(J,\omega)\in\{0,1\}$. If $\omega^{\mathsf{n}-1}$ is closed then $L_0 = d^*d$ and the function $\langle d\lambda,\omega\rangle$ has mean value zero for all $\lambda\in\Omega^1(M)$, so $\kappa_0(J,\omega) = 0$.

Next define the operator $d^+ : \Omega^1(M)\to\Omega^1(M)$ by

$$d^+\lambda := d\lambda + *\left(d\lambda\wedge\tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!}\right) \tag{C.9}$$

for $\lambda\in\Omega^1(M)$. Then it follows from (C.6) that a 1-form $\lambda\in\Omega^1(M)$ satisfies $d^+\lambda = 0$ if and only if $d\lambda\in\Omega^{1,1}(M)$ and $\langle d\lambda,\omega\rangle = 0$. Thus

$$\kappa_1(J,\omega) = \dim(d(\ker d^+)) = \dim(\ker d^+/\ker d). \tag{C.10}$$

Now define the operator $L_1 : \Omega^1(M)\to\Omega^1(M)$ by

$$L_1\lambda := d^*d^+\lambda + dd^*\lambda = (d^*d + dd^*)\lambda - *\left(d\lambda\wedge d\tfrac{\omega^{\mathsf{n}-2}}{(\mathsf{n}-2)!}\right). \tag{C.11}$$

Then L_1 is a second order elliptic operator and so has a finite dimensional kernel. Its kernel contains the space $\ker d^+\cap\ker d^*$, which in turn contains the space $H^1(M) := \ker d\cap\ker d^*$ of harmonic 1 forms. Moreover, the quotient $(\ker d^+\cap\ker d^*)/H^1(M)$ has dimension $\kappa_1(J,\omega)$ by (C.10). Thus

$$\dim H^1(M) + \kappa_1(J,\omega) = \dim(\ker d^+\cap\ker d^*) \le \dim\ker L_1 < \infty.$$

If $d\omega^{\mathsf{n}-2} = 0$ then $L_1 = d^*d + dd^*$ and so $\kappa_1(J,\omega) = 0$. This proves part (i).

To prove part (ii), assume first that $\kappa_0(J,\omega)=1$ and choose $\lambda_0\in\Omega^1(M)$ such that $d\lambda_0\in\Omega^{1,1}(M)$ and $\langle d\lambda_0,\omega\rangle\notin \mathrm{im}L_0$. Then

$$\Omega^0(M)=\mathrm{im}L_0\oplus\mathbb{R}\langle d\lambda_0,\omega\rangle. \tag{C.12}$$

We prove that

$$\Omega^{1,1}(M)\cap\mathrm{im}d=\mathbb{R}d\lambda_0\oplus d(\ker d^+)\oplus\{d(df\circ J)\,|\,f\in\Omega^0(M)\}. \tag{C.13}$$

First note that $d\lambda_0\notin d(\ker d^+)$ by (C.6) and (C.9), and that $d\lambda_0\neq d(df\circ J)$ for all $f\in\Omega^0(M)$ because $\langle d\lambda_0,\omega\rangle\notin\mathrm{im}L_0$. Moreover, if $\lambda\in\Omega^1(M)$ satisfies $d^+\lambda=0$ and $d\lambda=d(df\circ J)$ for some $f\in\Omega^0(M)$, then

$$L_0f=\langle d(df\circ J),\omega\rangle=\langle d\lambda,\omega\rangle=0,$$

and so f is constant. Thus the right hand side of (C.13) is a direct sum decomposition. Now let $\lambda\in\Omega^1(M)$ with $d\lambda\in\Omega^{1,1}(M)$. Then by (C.12) there exist a real number s and a function $f\in\Omega^0(M)$ such that

$$\langle d\lambda,\omega\rangle=\langle d(df\circ J),\omega\rangle+s\langle d\lambda_0,\omega\rangle.$$

Define $\lambda_1:=\lambda-df\circ J-s\lambda_0$. Then $d\lambda_1\in\Omega^{1,1}(M)$ and $\lambda_1\in\ker d^+$ by (C.6) and (C.9), and we have $d\lambda=sd\lambda_0+d\lambda_1+d(df\circ J)$. This proves (C.13). It follows from (C.10) and (C.13) that $\kappa(J)=1+\kappa_1(J,\omega)$. This proves part (ii) in the case $\kappa_0(J,\omega)=1$. The proof in the case $\kappa_0(J,\omega)=0$ is analogous.

Part (iii) follows directly from the definitions and parts (i) and (ii). This proves Lemma C.2. □

Corollary C.3. *Let M be closed connected oriented smooth four-manifold, let J be a complex structure on M that is compatible with the orientation, let $\omega\in\Omega^2(M)$ be a nondegenerate 2-form that is compatible with J, and equip M with the Riemannian metric $\langle\cdot,\cdot\rangle:=\omega(\cdot,J\cdot)$. Then*

$$\kappa_1(J,\omega)=0$$

and $\kappa(J)=\kappa_0(J,\omega)\in\{0,1\}$. Moreover, the following are equivalent.

(i) *$\kappa(J)=1$.*

(ii) *$\Omega^0(M)=\{\langle d\lambda,\omega\rangle\,|\,\lambda\in\Omega^1(M),\,d\lambda\in\Omega^{1,1}(M)\}$.*

(iii) *Every self-dual harmonic 2-form $\tau\in\Omega^2(M)$ satisfies $\langle\tau,\omega\rangle\equiv0$.*

(iv) *$H^{2,+}_{\omega,J}(M)\subset\Omega^{2,0}_J(M)\oplus\Omega^{0,2}_J(M)$.*

Proof. Since M has dimension four, $\omega^{\mathsf{n}-2}$ is the constant function 1 and so $\kappa_1(J,\omega)=0$ by Lemma C.2. This shows that $\kappa(J)=\kappa_0(J,\omega)\in\{0,1\}$. The equivalence of (i) and (ii) follows from the fact that the operator L_0 in (C.8) has a one-dimensional cokernel and that $\kappa_0(J,\omega)=1$ if and only if

$$\mathrm{im}L_0 \subsetneq \left\{\langle d\lambda,\omega\rangle \,\middle|\, \lambda\in\Omega^1(M),\, d\lambda\in\Omega^{1,1}(M)\right\}\subset\Omega^0(M).$$

Next observe that the L^2-orthogonal complement of the image of the operator $d^+:\Omega^1(M)\to\Omega^+_{\omega,J}(M)$ is the space $H^{2,+}_{\omega,J}(M)$ of self-dual harmonic 2-forms, and that (ii) holds if and only if $\Omega^0(M)\omega\subset\mathrm{im}d^+$ (see equation (C.6)). Thus (ii) holds if and only if the spaces $H^{2,+}_{\omega,J}(M)$ and $\Omega^0(M)\omega$ are L^2 orthogonal to each other, and this is equivalent to (iii). The equivalence of (iii) and (iv) follows from the fact that the space $\Omega^{2,0}_J(M;\mathbb{C})\oplus\Omega^{0,2}_J(M;\mathbb{C})$ intersects $\Omega^2(M)$ in the space of all $\tau\in\Omega^2(M)$ that satisfy $\tau(u,v)+\tau(Ju,Jv)=0$ for all $u,v\in\mathrm{Vect}(M)$, the L^2 orthogonal complement of $\Omega^0(M)\omega$ in $\Omega^{2,+}_{\omega,J}(M)$. This proves Corollary C.3. □

On a closed connected oriented smooth four-manifold M Corollary C.3 shows that the set of complex structures J that satisfy $\kappa(J)=0$ is open, and that $\kappa(J)=1$ whenever $b^{2,+}(M)=0$.

Remark C.4. Let E be a holomorphic vector bundle over a complex manifold (M,J). Then, for every Hermitian metric h on E, there exists a unique Hermitian connection ∇ on E such that $\bar\partial^\nabla=\bar\partial:\Omega^0(M,E)\to\Omega^{0,1}(M,E)$. The real valued 2-form $\frac{\mathbf{i}}{2\pi}\mathrm{trace}^c(F^\nabla)$ is a closed $(1,1)$-form which represents the first Chern class of E. If $h'(s_1,s_2)=h(s_1,Qs_2)$ is another Hermitian structure (with Q a section of the bundle of positive definite Hermitian automorphisms with respect to h) then the corresponding Hermitian connection is given by $\nabla'=\nabla+Q^{-1}\partial Q$ and the complex trace of its curvature is

$$\mathrm{trace}^c(F^{\nabla'})=\mathrm{trace}^c(F^\nabla)-\partial\bar\partial f,\qquad f:=\det{}^c(Q):M\to\mathbb{R}.$$

Thus by (C.2) the first Chern class

$$c_1(E)=\left[\frac{\mathbf{i}}{2\pi}\mathrm{trace}^c(F^\nabla)\right]_{\mathrm{dR}}\in H^2_{\mathrm{dR}}(M)$$

in de Rham cohomology lifts to a well defined class

$$c_{1,\mathrm{BC}}(E):=\left[\frac{\mathbf{i}}{2\pi}\mathrm{trace}^c(F^\nabla)\right]_{\mathrm{BC}}\in H^{1,1}_{\mathrm{BC}}(M)$$

in Bott–Chern cohmology, called the **first Bott–Chern class of** E. (For more details see [1, 2, 3, 5]).

D Complex structures and n-forms

Fix a closed connected oriented $2\mathsf{n}$-manifold M and a complex line bundle $L \to M$ with a Hermitian form $\langle s_1, s_2 \rangle$ for $s_1, s_2 \in \Omega^0(M, L)$ (complex anti-linear in the first variable and complex linear in the second variable). Define

$$c_{\mathsf{n}} := (-1)^{\frac{\mathsf{n}(\mathsf{n}+1)}{2}} \mathbf{i}^{\mathsf{n}} = \begin{cases} 1, & \text{if } \mathsf{n} \text{ is even}, \\ -\mathbf{i}, & \text{if } \mathsf{n} \text{ is odd}. \end{cases} \tag{D.1}$$

Lemma D.1. *Let $J \in \mathscr{J}(M)$ be an almost complex structure compatible with the orientation. Then $c_1(TM, J) = c_1(L) \in H^2(M; \mathbb{Z})$ if and only if there exists a nowhere vanishing n-form $\theta \in \Omega_J^{\mathsf{n},0}(M, L)$. If this holds then*

$$\rho := c_{\mathsf{n}} \langle \theta \wedge \theta \rangle \in \Omega^{2\mathsf{n}}(M) \tag{D.2}$$

is a positive volume form on M.

Proof. The first Chern class of (TM, J) agrees with minus the first Chern class of the complex line bundle $\Lambda_J^{\mathsf{n},0} T^*M$. Hence $c_1(TM, J) = c_1(L)$ if and only if $E := \Lambda_J^{\mathsf{n},0} T^*M \otimes L$ admits a a trivialization or, equivalently, a nowhere vanishing section, and such a section is an $(\mathsf{n}, 0)$-form $\theta \in \Omega_J^{\mathsf{n},0}(M, L)$.

To show that, for any nowhere vanishing n-form $\theta \in \Omega_J^{\mathsf{n},0}(M)$, the formula (D.2) defines a positive volume form on M, fix an element $m \in M$ and choose a complex isomorphism $(\mathbb{C}^{\mathsf{n}}, \mathbf{i}) \to (T_m M, J)$. Let $z_i = x_i + \mathbf{i} y_i$ for $i = 1, \dots, \mathsf{n}$ be the coordinates on $\mathbb{C}^{\mathsf{n}}$. Then there is an element $\lambda \in L_m$ (the fiber of L over m) such that

$$\theta_m = \lambda dz_1 \wedge \cdots \wedge dz_{\mathsf{n}}.$$

Hence

$$\begin{aligned} \rho_m &= c_{\mathsf{n}} \langle \theta_m \wedge \theta_m \rangle \\ &= \frac{(-1)^{\frac{\mathsf{n}(\mathsf{n}-1)}{2}}}{\mathbf{i}^{\mathsf{n}}} |\lambda|^2 d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{\mathsf{n}} \wedge dz_1 \wedge \cdots \wedge dz_{\mathsf{n}} \\ &= 2^{\mathsf{n}} |\lambda|^2 \frac{d\bar{z}_1 \wedge dz_1}{2\mathbf{i}} \wedge \cdots \wedge \frac{d\bar{z}_{\mathsf{n}} \wedge dz_{\mathsf{n}}}{2\mathbf{i}} \\ &= 2^{\mathsf{n}} |\lambda|^2 dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{\mathsf{n}} \wedge dy_{\mathsf{n}}. \end{aligned}$$

Thus ρ is a positive volume form on M and this proves Lemma D.1. $\square$

Lemma D.2. *Let $J \in \mathscr{J}(M)$ be an almost complex structure compatible with the orientation, let $\theta \in \Omega_J^{\mathsf{n},0}(M,L)$ be a nowhere vanishing $(\mathsf{n},0)$-form, let $\omega \in \Omega^2(M)$ be a nondegenerate 2-form that is compatible with J such that*

$$\frac{\omega^{\mathsf{n}}}{\mathsf{n}!} = c_{\mathsf{n}}\langle\theta\wedge\theta\rangle =: \rho, \tag{D.3}$$

and let $* : \Omega_J^{p,q}(M,L) \to \Omega_J^{\mathsf{n}-q,\mathsf{n}-p}(M,L)$ be the Hodge $*$-operator of the Riemannian metric $\langle\cdot,\cdot\rangle := \omega(\cdot, J\cdot)$. Then the following holds.

(i) *For every $\widehat{J} \in \Omega_J^{0,1}(M,TM)$ there is a unique $\beta \in \Omega_J^{\mathsf{n}-1,1}(M,L)$ such that*

$$\mathbf{i}\iota(u)\beta - \iota(Ju)\beta = \iota(\widehat{J}u)\theta \tag{D.4}$$

for all $u \in \mathrm{Vect}(M)$.

(ii) *For every $\beta \in \Omega_J^{\mathsf{n}-1,1}(M,L)$ there exists a unique $\widehat{J} \in \Omega_J^{0,1}(M,TM)$ such that* (D.4) *holds for all $u \in \mathrm{Vect}(M)$.*

(iii) *Suppose $\beta \in \Omega_J^{\mathsf{n}-1,1}(M,L)$ and $\widehat{J} \in \Omega_J^{0,1}(M,TM)$ satisfy equation* (D.4)*. Then*

$$\mathbf{i}\iota(u)*\beta - \iota(Ju)*\beta = -c_{\mathsf{n}}\iota(\widehat{J}^*u)\theta \tag{D.5}$$

for all $u \in \mathrm{Vect}(M)$. Moreover,

$$\widehat{J} = \widehat{J}^* \quad\iff\quad *\beta = -c_{\mathsf{n}}\beta \quad\iff\quad \beta\wedge\omega = 0, \tag{D.6}$$

$$\widehat{J} + \widehat{J}^* = 0 \quad\iff\quad *\beta = c_{\mathsf{n}}\beta \quad\iff\quad \beta \in \Omega_J^{\mathsf{n}-2,0}(M,L)\wedge\omega. \tag{D.7}$$

(iv) *Suppose $\beta \in \Omega_J^{\mathsf{n}-1,1}(M,L)$ and $\widehat{J} \in \Omega_J^{0,1}(M,TM)$ satisfy equation* (D.4) *and let $\widehat{\omega} \in \Omega^2(M)$. Then*

$$\omega\wedge\beta + \widehat{\omega}\wedge\theta = 0 \quad\iff\quad \begin{array}{l}\widehat{\omega}(u,v) - \widehat{\omega}(Ju,Jv) = \langle(\widehat{J} - \widehat{J}^*)u, v\rangle \\ \text{for all } u,v \in \mathrm{Vect}(M).\end{array} \tag{D.8}$$

(v) *Let $\beta, \beta' \in \Omega_J^{\mathsf{n}-1,1}(M,L)$ and $\widehat{J}, \widehat{J}' \in \Omega_J^{0,1}(M,TM)$ be given such that the pairs $(\beta, \widehat{J})$ and $(\beta', \widehat{J}')$ satisfy* (D.4)*. Then the pointwise inner product of β and β' is given by*

$$\langle\beta,\beta'\rangle = \mathrm{Re}\left(\frac{\langle\beta\wedge*\beta'\rangle}{\rho}\right) = \tfrac{1}{8}\mathrm{trace}\big(\widehat{J}^*\widehat{J}'\big)\rho. \tag{D.9}$$

Moreover, we have

$$c_{\mathsf{n}}\langle\beta\wedge\beta'\rangle = -\tfrac{1}{8}\mathrm{trace}\big(\widehat{J}\widehat{J}'\big)\rho + \tfrac{\mathbf{i}}{8}\mathrm{trace}\big(\widehat{J}J\widehat{J}'\big)\rho. \tag{D.10}$$

Proof. Define $\beta \in \Omega^{\mathsf{n}}(M, L)$ by

$$\begin{aligned}\beta(v_1, \dots, v_{\mathsf{n}}) :=& \ \theta(-\tfrac{1}{2} J\widehat{J} v_1, v_2, \dots, v_{\mathsf{n}}) \\ &+ \theta(v_1, -\tfrac{1}{2} J\widehat{J} v_2, v_3, \dots, v_{\mathsf{n}}) \\ &+ \cdots + \theta(v_1, \dots, v_{\mathsf{n}-1}, -\tfrac{1}{2} J\widehat{J} v_{\mathsf{n}})\end{aligned}$$

for $v_1, \dots, v_{\mathsf{n}} \in \mathrm{Vect}(M)$. Then

$$\begin{aligned}\beta(Ju, v_2, \dots, v_{\mathsf{n}}) + \theta(\widehat{J}u, v_2, \dots, v_{\mathsf{n}}) &= \theta(\tfrac{1}{2}\widehat{J}u, v_2, \dots, v_{\mathsf{n}}) \\ &\quad + \theta(Ju, -\tfrac{1}{2} J\widehat{J} v_2, v_3, \dots, v_{\mathsf{n}}) \\ &\quad + \cdots + \theta(Ju, v_2, \dots, v_{\mathsf{n}-1}, -\tfrac{1}{2} J\widehat{J} v_{\mathsf{n}}) \\ &= \mathbf{i}\beta(u, v_2, \dots, v_{\mathsf{n}})\end{aligned}$$

for all $u, v_2, \dots, v_{\mathsf{n}} \in \mathrm{Vect}(M)$. Thus β is an $(\mathsf{n}-1, 1)$-form that satisfies equation (D.4). If β' is another $(\mathsf{n}-1, 1)$-form that satisfies equation (D.4), then $\iota(Ju)(\beta' - \beta) = \mathbf{i}\iota(u)(\beta' - \beta)$, thus $\beta' - \beta \in \Omega_J^{\mathsf{n},0}(M, L)$, and so $\beta' = \beta$. This proves (i).

We prove part (ii). Thus let $\beta \in \Omega_J^{\mathsf{n}-1,1}(M, L)$ be given. Then for every vector field $u \in \mathrm{Vect}(M)$ the $(\mathsf{n}-1)$-form $\mathbf{i}\iota(u)\beta - \iota(Ju)\beta$ is of type $(\mathsf{n}-1, 0)$ and hence can be written in the form $\iota(v)\theta$ for some vector field $v \in \mathrm{Vect}(M)$ that is uniquely determined by u. This shows that there exists a unique section $\widehat{J} \in \Omega^0(M, \mathrm{End}(TM))$ of the endomorphism bundle that satisfies (D.4) for all $u \in \mathrm{Vect}(M)$. By (D.4) we have

$$\iota(\widehat{J}Ju)\theta = \mathbf{i}\iota(Ju)\beta + \iota(u)\beta = -\mathbf{i}\iota(\widehat{J}u)\theta = -\iota(J\widehat{J}u)\theta$$

for all $u \in \mathrm{Vect}(M)$ and thus $\widehat{J}J + J\widehat{J} = 0$. This proves (ii).

We prove part (iii). It suffices to consider the trivial line bundle and the standard structures on $\mathbb{R}^{2\mathsf{n}}$ with the coordinates $x_1, \dots, x_{\mathsf{n}}, y_1, \dots, y_{\mathsf{n}}$. They are given by

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \omega = \sum_{i=1}^{\mathsf{n}} dx_i \wedge dy_i, \qquad \theta = \frac{dz_1}{\sqrt{2}} \wedge \cdots \wedge \frac{dz_{\mathsf{n}}}{\sqrt{2}}, \quad \text{(D.11)}$$

where $z_i := x_i + \mathbf{i}y_i$ for $i = 1, \dots, \mathsf{n}$. A complex anti-linear endomorphism has the form

$$\widehat{J} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, \qquad A + \mathbf{i}B = (a_{ij})_{i,j=1,\dots,\mathsf{n}} \in \mathbb{C}^{\mathsf{n}\times\mathsf{n}}. \qquad \text{(D.12)}$$

The corresponding $(\mathsf{n}-1,1)$-form $\beta\in\Omega_J^{\mathsf{n}-1,1}(\mathbb{R}^{2\mathsf{n}})$ is given by

$$\beta=\frac{1}{2\mathbf{i}}\sum_{i,j=1}^{\mathsf{n}}a_{ij}\frac{dz_1}{\sqrt{2}}\wedge\cdots\frac{dz_{i-1}}{\sqrt{2}}\wedge\frac{d\bar{z}_j}{\sqrt{2}}\wedge\frac{dz_{i+1}}{\sqrt{2}}\wedge\cdots\wedge\frac{dz_{\mathsf{n}}}{\sqrt{2}}. \tag{D.13}$$

Now $\widehat{J}^*$ is represented by the transposed matrix $A^T+\mathbf{i}B^T=(a_{ji})_{i,j=1,\dots,\mathsf{n}}$ and

$$*\beta=\frac{-c_{\mathsf{n}}}{2\mathbf{i}}\sum_{i,j=1}^{\mathsf{n}}a_{ji}\frac{dz_1}{\sqrt{2}}\wedge\cdots\frac{dz_{i-1}}{\sqrt{2}}\wedge\frac{d\bar{z}_j}{\sqrt{2}}\wedge\frac{dz_{i+1}}{\sqrt{2}}\wedge\cdots\wedge\frac{dz_{\mathsf{n}}}{\sqrt{2}}.$$

This proves (D.5). Now (D.6) and (D.7) follow from (D.5) and the eigenspace decomposition of the Hodge $*$-operator on $\Omega^{\mathsf{n}-1,1}(M)$. This proves (iii).

We prove part (iv). Continue the notation in the proof of part (iii), so $J,\omega,\theta,\widehat{J},\beta$ are as in (D.11), (D.12), and (D.13). Then a 2-form $\widehat{\omega}\in\Omega^2(M)$ satisfies $\widehat{\omega}(u,v)-\widehat{\omega}(Ju,Jv)=\langle(\widehat{J}-\widehat{J}^*)u,v\rangle$ for all $u,v\in\mathrm{Vect}(M)$ if and only if its $(0,2)$-part is given by $\widehat{\omega}^{0,2}=-\frac{1}{4}\sum_{i,j}a_{ij}d\bar{z}_i\wedge d\bar{z}_j$, and this in turn is equivalent to the equation $\widehat{\omega}\wedge\theta=-\omega\wedge\beta$. This proves (iv).

We prove part (v). Continue the notation in the proof of part (iii) and use the same notation for $(\beta',\widehat{J}')$ with A,B,a_{ij} replaced by A',B',a'_{ij}. Then

$$\begin{aligned}
\overline{\beta}\wedge *\beta' &= \frac{-c_{\mathsf{n}}}{4}\sum_{i,j=1}^{\mathsf{n}}\sum_{k,\ell=1}^{\mathsf{n}}\bar{a}_{ij}a'_{\ell k}\frac{d\bar{z}_1}{\sqrt{2}}\wedge\cdots\wedge\frac{d\bar{z}_{i-1}}{\sqrt{2}}\wedge\frac{dz_j}{\sqrt{2}}\wedge\frac{d\bar{z}_{i+1}}{\sqrt{2}}\wedge\cdots\wedge\frac{d\bar{z}_{\mathsf{n}}}{\sqrt{2}} \\
&\qquad\wedge\frac{dz_1}{\sqrt{2}}\wedge\cdots\wedge\frac{dz_{k-1}}{\sqrt{2}}\wedge\frac{d\bar{z}_\ell}{\sqrt{2}}\wedge\frac{dz_{k+1}}{\sqrt{2}}\wedge\cdots\wedge\frac{dz_{\mathsf{n}}}{\sqrt{2}} \\
&= \frac{c_{\mathsf{n}}}{4}\sum_{i,j=1}^{\mathsf{n}}\bar{a}_{ij}a'_{ji}\frac{d\bar{z}_1}{\sqrt{2}}\wedge\cdots\wedge\frac{d\bar{z}_{\mathsf{n}}}{\sqrt{2}}\wedge\frac{dz_1}{\sqrt{2}}\wedge\cdots\wedge\frac{dz_{\mathsf{n}}}{\sqrt{2}} \\
&= \frac{1}{4}\sum_{i,j=1}^{\mathsf{n}}\bar{a}_{ij}a'_{ji}c_{\mathsf{n}}\overline{\theta}\wedge\theta \\
&= \frac{1}{4}\mathrm{trace}(A-\mathbf{i}B)^T(A'+\mathbf{i}B')\rho.
\end{aligned}$$

Thus $\mathrm{Re}(\overline{\beta}\wedge *\beta')=\frac{1}{4}\mathrm{trace}(A^TA'+B^TB')\rho=\frac{1}{8}\mathrm{trace}(\widehat{J}^T\widehat{J}')\rho$ and this proves equation (D.9). Moreover, the pair $(\bar{c}_{\mathsf{n}}*\beta,-\widehat{J}^*)$ satisfies (D.4) by part (iii). Thus $\mathrm{Re}(c_{\mathsf{n}}\overline{\beta}\wedge\beta')=\mathrm{Re}(\overline{\bar{c}_{\mathsf{n}}*\beta}\wedge\beta')=-\frac{1}{8}\mathrm{trace}(\widehat{J}\widehat{J}')\rho$. This confirms (D.10) for the real part. The formula for the imaginary part holds because both sides of the equation are complex linear in $\widehat{J}'$ with respect to the complex structure $\widehat{J}'\mapsto J\widehat{J}'$. This proves (v) and Lemma D.2. □

The next lemma adapts an observation by Donaldson in [14, Lemma 1] to the present setting.

Lemma D.3. *Let ρ be a positive volume form and let $J \in \mathscr{J}(M)$ be a positive almost complex structure such that $c_1(TM, J) = c_1(L) \in H^2(M;\mathbb{Z})$. Then the following are equivalent.*

(i) *J is integrable.*

(ii) *There exists a nowhere vanishing n-form $\theta \in \Omega_J^{\mathsf{n},0}(M, L)$ and a Hermitian connection ∇_L on L such that $d^{\nabla_L}\theta = 0$ and $c_{\mathsf{n}}\langle\theta\wedge\theta\rangle = \rho$.*

If (i) holds then the pair (∇_L, θ) in (ii) is uniquely determined by J up to unitary gauge equivalence. If (ii) holds then

$$(F^{\nabla_L})_J^{0,2} = 0, \qquad \mathrm{Ric}_{\rho,J} = \mathbf{i}F^{\nabla_L}. \tag{D.14}$$

If (i) and (ii) hold and ∇ is a torsion-free connection on TM that satisfies $\nabla J = 0$, then $\nabla\rho = 0$ if and only if $\nabla\theta = 0$.

Proof. We prove that (i) implies (ii). By Lemma D.1 there exists a nowhere vanishing $(\mathsf{n}, 0)$-form $\theta \in \Omega_J^{\mathsf{n},0}(M, L)$ such that $c_{\mathsf{n}}\langle\theta\wedge\theta\rangle = \rho$. Choose any Hermitian connection ∇_0 on L. Then $d^{\nabla_0}\theta \in \Omega_J^{n,1}(M, L)$ because J is integrable and hence there exists a unique 1-form $\eta \in \Omega_J^{0,1}(M)$ such that $\eta \wedge \theta = d^{\nabla_0}\theta$. Define the Hermitian connection ∇_L by $\nabla_L := \nabla_0 + \overline{\eta} - \eta$. Then

$$d^{\nabla_L}\theta = d^{\nabla_0}\theta + (\overline{\eta} - \eta) \wedge \theta = \overline{\eta} \wedge \theta = 0$$

because $\overline{\eta} \in \Omega_J^{1,0}(M)$. This shows that (i) implies (ii). Moreover, (ii) implies

$$(F^{\nabla_L})_J^{0,2} \wedge \theta = F^{\nabla_L} \wedge \theta = d^{\nabla_L}d^{\nabla_L}\theta = 0$$

and hence $(F^{\nabla_L})_J^{0,2} = 0$.

We prove uniqueness in (ii). If (θ', ∇_L') is any other pair as in (ii), then there exists a unique unitary gauge transformation $g : M \to S^1$ such that

$$\theta' = g^{-1}\theta$$

Hence the 1-form $\alpha := \nabla_L' - \nabla_L \in \Omega^1(M, \mathbf{i}\mathbb{R})$ satisfies

$$0 = d^{\nabla_L'}\theta' = d^{\nabla_L+\alpha}(g^{-1}\theta) = \alpha \wedge g^{-1}\theta + dg^{-1} \wedge \theta = (\alpha^{0,1} - g^{-1}\bar{\partial}g) \wedge g^{-1}\theta.$$

Hence $\alpha^{0,1} = g^{-1}\bar{\partial}g$ and so $\alpha = g^{-1}\bar{\partial}_J g - \bar{g}^{-1}\partial_J\bar{g} = g^{-1}dg$ because $g^{-1}dg$ is a 1-form on M with values in $\mathbf{i}\mathbb{R}$. Thus

$$\nabla_L' = \nabla_L + g^{-1}dg = g^*\nabla_L$$

and this proves uniqueness up to unitary gauge equivalence.

We prove that (ii) implies (i). If $\theta \in \Omega_J^{\mathsf{n},0}(M,L)$ and ∇_L is any complex connection on L then $(d^{\nabla_L}\theta)^{\mathsf{n}-1,2} = \frac{1}{4}\iota(N_J)\theta$, where

$$\begin{aligned}&(\iota(N_J)\theta)(v_1,\dots,v_{\mathsf{n}+1})\\&:= \sum_{i<j}(-1)^{i+j-1}\theta\left(N_J(v_i,v_j),v_1,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_{\mathsf{n}+1}\right)\end{aligned} \tag{D.15}$$

for $v_1,\dots,v_{\mathsf{n}+1} \in \mathrm{Vect}(M)$. If $d^{\nabla_L}\theta = 0$ it follows that $\iota(N_J)\theta = 0$. If θ vanishes nowhere this implies $N_J = 0$. To see this, fix two vector fields v_1, v_2. Then $\iota(N_J(v_1,v_2))\theta$ is a nonzero $(\mathsf{n}-1,0)$-form while the remaining summands on the right in (D.15) are of type $(\mathsf{n}-2,1)$ or $(\mathsf{n}-3,2)$. This implies that $\iota(N_J(v_1,v_2))\theta = 0$ and hence $N_J(v_1,v_2) = 0$ because θ vanishes nowhere. Thus $N_J = 0$ and therefore J is integrable.

Now assume (ii) and let ∇ be a torsion-free connection on TM that satisfies $\nabla J = 0$ and $\nabla\rho = 0$. Such a connection exists by part (i) of Lemma A.1. For $u \in \mathrm{Vect}(M)$ the n-form $\nabla_u\theta \in \Omega^{\mathsf{n}}(M,L)$ is defined by

$$\begin{aligned}(\nabla_u\theta)(v_1,\dots v_{\mathsf{n}}) := {} & \nabla_{L,u}\big(\theta(v_1,\dots,v_{\mathsf{n}})\big)\\ & - \theta(\nabla_u v_1, v_2,\dots,v_{\mathsf{n}}) - \cdots - \theta(v_2,\dots,v_{\mathsf{n}-1},\nabla_u v_{\mathsf{n}})\end{aligned}$$

Since $\nabla J = 0$, this is an $(\mathsf{n},0)$-form. Hence there exists a unique complex valued 1-form $\alpha \in \Omega^1(M,\mathbb{C})$ such that

$$\nabla_u\theta = \alpha(u)\theta$$

for all $u \in \mathrm{Vect}(M)$. Now the equation $d^{\nabla_L}\theta = 0$ can be expressed in the form

$$(\nabla_u\theta)(v_1,\dots v_{\mathsf{n}}) = \sum_{i=1}^{n}(-1)^{i-1}(\nabla_{v_i}\theta)(u,v_1,\dots,v_{i-1},v_{i+1},\dots,v_{\mathsf{n}})$$

for $u, v_1,\dots,v_{\mathsf{n}} \in \mathrm{Vect}(M)$. The right hand side of this equation is complex linear in u and this implies $\alpha \in \Omega^{1,0}(M)$, i.e.

$$\alpha(Ju) = \mathbf{i}\alpha(u)$$

for all $u \in \mathrm{Vect}(M)$. Since $\rho = c_{\mathsf{n}}\langle\theta\wedge\theta\rangle$ and $\nabla\rho = 0$, we also have $\mathrm{Re}\alpha = 0$, hence $\alpha = 0$, and so $\nabla\theta = 0$. This implies $\mathrm{trace}^c(R^\nabla) = F^{\nabla_L}$ and therefore

$$\mathrm{Ric}_{\rho,J} = \tfrac{1}{2}\mathrm{trace}(JR^\nabla) = \mathbf{i}\mathrm{trace}^c(R^\nabla) = \mathbf{i}F^{\nabla_L}.$$

This proves Lemma D.3. □

Lemma D.4. *Let $\rho \in \Omega^{2\mathsf{n}}(M)$ be a positive volume form, let $J \in \mathscr{J}_{\mathrm{int}}(M)$, let ∇_L be a Hermitian connection on L, and let $\theta \in \Omega_J^{\mathsf{n},0}(M,L)$ be nowhere vanishing such that $d^{\nabla_L}\theta = 0$ and $c_{\mathsf{n}}\langle\theta\wedge\theta\rangle = \rho$. Then the following holds.*

(i) *Let $v \in \mathrm{Vect}(M)$ and define $\widehat{J} := \mathcal{L}_v J$ and $\beta := \bar\partial_J^{\nabla_L}\iota(v)\theta \in \Omega_J^{\mathsf{n}-1,1}(M,L)$ and $h := \frac{1}{2}(f_v - \mathbf{i}f_{Jv})$. Then $d^{\nabla_L}\iota(v)\theta = \beta + h\theta$ and (D.4) holds for all u.*

(ii) *Suppose $\widehat{J} \in \Omega_J^{0,1}(M,TM)$ and $\beta \in \Omega_J^{\mathsf{n}-1,1}(M,L)$ satisfy (D.4). Then*

$$\bar\partial_J^*\widehat{J} = 0 \qquad \Longleftrightarrow \qquad \big(\bar\partial_J^{\nabla_L}\big)^*\beta = 0, \tag{D.16}$$

$$\bar\partial_J\widehat{J} = 0 \qquad \Longleftrightarrow \qquad \bar\partial_J^{\nabla_L}\beta = 0. \tag{D.17}$$

(iii) *Let $\widehat{J}$ and β be as in (ii) and let $\Lambda_\rho(J,\widehat{J})$ be as in (2.9). Then*

$$\mathbf{i}\partial_J^{\nabla_L}\beta + \tfrac{1}{2}\Lambda_\rho(J,\widehat{J}) \wedge \theta = 0. \tag{D.18}$$

(iv) *Let $\widehat{J}$ and β be as in (ii) with $\bar\partial_J\widehat{J} = 0$ and let $\mathrm{Ric}_{\rho,J}$ and $\widehat{\mathrm{Ric}}_\rho(J,\widehat{J})$ be as in Theorem 2.6. Then $\mathrm{Ric}_{\rho,J} \wedge \beta + \widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) \wedge \theta = 0$.*

(v) *Let $\widehat{J}$ and β be as in (ii) with $\bar\partial_J\widehat{J} = 0$ and assume $F^{\nabla_L} = 0$ and J admits a Kähler form. Then there exists a unique function $h \in \Omega^0(M,\mathbb{C})$ such that*

$$d^{\nabla_L}\big(\beta + h\theta\big) = 0, \qquad \int_M h\rho = 0. \tag{D.19}$$

Moreover, $h = \frac{1}{2}(f - \mathbf{i}g)$ in the notation of Lemma 3.8, and $\beta + h\theta \in \mathrm{im} d^{\nabla_L}$ if and only if there exists a vector field v such that $\widehat{J} = \mathcal{L}_v J$.

Proof. Fix a torsion-free connection ∇ such that $\nabla J = 0$ and $\nabla\theta = 0$. Next define $\mathcal{L}_v^{\nabla_L}\alpha := d^{\nabla_L}\iota(v)\alpha + \iota(v)d^{\nabla_L}\alpha$ for $\alpha \in \Omega^k(M,L)$ and $v \in \mathrm{Vect}(M)$ Then

$$(\mathcal{L}_v^{\nabla_L}\alpha)(v_1,\dots,v_k) = \nabla_{L,v}\big(\alpha(v_1,\dots,v_k)\big) - \sum \alpha(\dots,v_{i-1},[v_i,v],v_{i+1},\dots)$$

for $v, v_1, \dots, v_k \in \mathrm{Vect}(M)$ and $\mathcal{L}_v^{\nabla_L}\theta = d^{\nabla_L}\iota(v)\theta$. Hence, by the Leibniz rule,

$$(d^{\nabla_L}\iota(v)\theta)(v_1,\dots,v_{\mathsf{n}}) = \theta(\nabla_{v_1}v, v_2,\dots,v_{\mathsf{n}}) + \cdots + \theta(v_1,\dots,v_{\mathsf{n}-1},\nabla_{v_{\mathsf{n}}}v)$$

for all $v, v_1, \dots, v_{\mathsf{n}} \in \mathrm{Vect}(M)$. Since θ is complex multi-linear this implies

$$\mathbf{i}\iota(u)d^{\nabla_L}\iota(v)\theta - \iota(Ju)d^{\nabla_L}\iota(v)\theta = \iota(J\nabla_u v - \nabla_{Ju}v)\theta = \iota((\mathcal{L}_v J)u)\theta \tag{D.20}$$

for all $u, v \in \mathrm{Vect}(M)$. Hence

$$\iota((\mathcal{L}_v J)u)\theta = \mathbf{i}\iota(u)(d^{\nabla_L}\iota(v)\theta)_J^{\mathsf{n}-1,1} - \iota(Ju)(d^{\nabla_L}\iota(v)\theta)_J^{\mathsf{n}-1,1} = \mathbf{i}\iota(u)\beta - \iota(Ju)\beta$$

for all u, v and this proves (D.4). The equation $\langle\theta\wedge\partial_J^{\nabla_L}\iota(v)\theta\rangle = \langle\theta\wedge h\theta\rangle$ follows directly from the definitions and the formula $\rho = c_{\mathsf{n}}\langle\theta\wedge\theta\rangle$. This proves (i).

We prove part (ii). The equivalence in (D.16) follows from the identity

$$\langle \beta, \bar\partial_J^{\nabla_L}\iota(v)\theta\rangle_{L^2} = \tfrac{1}{8}\int_M \mathrm{trace}\big(\widehat{J}^*\mathcal{L}_vJ\big)\rho = \tfrac{1}{4}\langle \widehat{J}, J\bar\partial_J v\rangle_{L^2} = \tfrac{1}{4}\langle \bar\partial_J^*\widehat{J}, Jv\rangle_{L^2}$$

for $v\in\mathrm{Vect}(M)$. Here we have used part (v) of Lemma D.2 as well as (i). To prove (D.17), define $\alpha_u\in\Omega^{\mathsf{n}}(M,L)$ by

$$\alpha_u := \mathbf{i}\iota(u)d^{\nabla_L}\beta - \iota(Ju)d^{\nabla_L}\beta \tag{D.21}$$

for $u\in\mathrm{Vect}(M)$. We will prove that, for all $u,v\in\mathrm{Vect}(M)$,

$$\mathbf{i}\iota(v)\alpha_u - \iota(Jv)\alpha_u = \iota(J\bar\partial_J\widehat{J}(u,v))\theta. \tag{D.22}$$

Equation (D.22) shows that $\bar\partial_J\widehat{J}=0$ if and only if $\alpha_u\in\Omega_J^{\mathsf{n},0}(M,L)$ for every vector field $u\in\mathrm{Vect}(M)$. By (D.21) this is equivalent to the condition $d^{\nabla_L}\beta\in\Omega_J^{\mathsf{n},1}(M,L)$ or, equivalently, to $\bar\partial_J^{\nabla_L}\beta=(d^{\nabla_L}\beta)_J^{\mathsf{n}-1,2}=0$.

To prove (D.22), fix a torsion-free connection ∇ that satisfies $\nabla J=0$ and $\nabla\theta=0$. Then it follows from (D.20) with v replaced by $\widehat{J}v$ that

$$\begin{aligned}&\mathbf{i}\iota(u)d^{\nabla_L}\iota(\widehat{J}v)\theta - \iota(Ju)d^{\nabla_L}\iota(\widehat{J}v)\theta\\ &= \iota\big(J(\nabla_u\widehat{J})v - (\nabla_{Ju}\widehat{J})v\big)\theta + \iota\big(J\widehat{J}\nabla_u v - \widehat{J}\nabla_{Ju}v\big)\theta\end{aligned} \tag{D.23}$$

for all $u,v\in\mathrm{Vect}(M)$. Moreover,

$$\alpha_u = \mathbf{i}\iota(u)d^{\nabla_L}\beta - \iota(Ju)d^{\nabla_L}\beta = \mathbf{i}\mathcal{L}_u^{\nabla_L}\beta - \mathcal{L}_{Ju}^{\nabla_L}\beta - d^{\nabla_L}\iota(\widehat{J}u)\theta \tag{D.24}$$

for all $u\in\mathrm{Vect}(M)$ by (D.4). With this understood, we obtain

$$\begin{aligned}\mathbf{i}\iota(v)\alpha_u - \iota(Jv)\alpha_u &= -\iota(v)\mathcal{L}_u^{\nabla_L}\beta - \mathbf{i}\iota(v)\mathcal{L}_{Ju}^{\nabla_L}\beta - \mathbf{i}\iota(v)d\iota(\widehat{J}u)\theta\\ &\quad - \mathbf{i}\iota(Jv)\mathcal{L}_u^{\nabla_L}\beta + \iota(Jv)\mathcal{L}_{Ju}^{\nabla_L}\beta + \iota(Jv)d^{\nabla_L}\iota(\widehat{J}u)\theta\\ &= -\mathcal{L}_u^{\nabla_L}\iota(v)\beta - \iota([u,v])\beta - \mathbf{i}\mathcal{L}_{Ju}^{\nabla_L}\iota(v)\beta - \mathbf{i}\iota([Ju,v])\beta\\ &\quad - \mathbf{i}\mathcal{L}_u^{\nabla_L}\iota(Jv)\beta - \mathbf{i}\iota([u,Jv])\beta + \mathcal{L}_{Ju}^{\nabla_L}\iota(Jv)\beta + \iota([Ju,Jv])\beta\\ &\quad - \mathbf{i}\iota(v)d\iota(\widehat{J}u)\theta + \iota(Jv)d\iota(\widehat{J}u)\theta\\ &= \mathbf{i}\iota(u)d^{\nabla_L}\iota(\widehat{J}v)\theta - \iota(Ju)d^{\nabla_L}\iota(\widehat{J}v)\theta - \mathbf{i}\iota(v)d^{\nabla_L}\iota(\widehat{J}u)\theta\\ &\quad + \iota(Jv)d^{\nabla_L}\iota(\widehat{J}u)\theta + \iota(J\widehat{J}[u,v])\theta + \iota(\widehat{J}J[Ju,Jv])\theta\\ &= \iota\big(J(\nabla_u\widehat{J})v - (\nabla_{Ju}\widehat{J})v - J(\nabla_v\widehat{J})u + (\nabla_{Jv}\widehat{J})u\big)\theta\\ &= \iota(J\bar\partial_J\widehat{J}(u,v))\theta.\end{aligned}$$

Here the first equality follows from (D.24), the third from (D.4), and the fourth from (D.23). This proves (D.22) and (ii).

We prove part (iii). Since $\mathbf{i}\partial_J^{\nabla_L}\beta \in \Omega_J^{n,1}(M,L)$, there is an $\eta \in \Omega_J^{0,1}(M)$ such that

$$\mathbf{i}\partial_J^{\nabla_L}\beta + \eta \wedge \theta = 0. \tag{D.25}$$

Now let $v \in \mathrm{Vect}(M)$. Then the pair $(\mathcal{L}_v J, \bar{\partial}_J^{\nabla_L}\iota(v)\theta)$ satisfies (D.4) by (i). Hence, by (2.13) and (D.10), we have

$$\begin{aligned}
\tfrac{1}{4}\int_M \Lambda_\rho(J,\widehat{J}) \wedge \iota(v)\rho &= \tfrac{1}{8}\int_M \mathrm{trace}\big(\widehat{J}J\mathcal{L}_v J\big)\rho \\
&= \mathrm{Im}\left(\int_M c_{\mathsf{n}}\langle \beta \wedge \bar{\partial}_J^{\nabla_L}\iota(v)\theta\rangle\right) \\
&= \mathrm{Re}\left(\int_M c_{\mathsf{n}}\langle (\mathbf{i}\beta) \wedge d^{\nabla_L}\iota(v)\theta\rangle\right) \\
&= (-1)^{\mathsf{n}+1}\mathrm{Re}\left(\int_M c_{\mathsf{n}}\langle (\mathbf{i}d^{\nabla_L}\beta) \wedge \iota(v)\theta\rangle\right) \\
&= -\mathrm{Re}\left(\int_M c_{\mathsf{n}}\langle (\iota(v)\mathbf{i}\partial_J^{\nabla_L}\beta) \wedge \theta\rangle\right) \\
&= \mathrm{Re}\left(\int_M \overline{\eta(v)} c_{\mathsf{n}}\langle \theta \wedge \theta\rangle\right) \\
&= \int_M \mathrm{Re}(\eta) \wedge \iota(v)\rho.
\end{aligned}$$

Here the penultimate equality follows from (D.25). Thus $\mathrm{Re}(\eta) = \frac{1}{4}\Lambda_\rho(J,\widehat{J})$, hence $\eta = \frac{1}{2}\Lambda_\rho(J,\widehat{J})_J^{0,1}$, and so (D.18) follows from (D.25). This proves (iii).

We prove part (iv). Since $\bar{\partial}_J\widehat{J} = 0$ it follows from part (ii) that $\bar{\partial}_J^{\nabla_L}\beta = 0$. Hence $\mathbf{i}d^{\nabla_L}\beta + \frac{1}{2}\Lambda_\rho(J,\widehat{J}) \wedge \theta = 0$ by (D.18) and so, by Lemma D.3, we have

$$\mathrm{Ric}_{\rho,J} \wedge \beta = \mathbf{i}F^{\nabla_L} \wedge \beta = \mathbf{i}d^{\nabla_L}d^{\nabla_L}\beta = -\tfrac{1}{2}d\Lambda_\rho(J,\widehat{J}) \wedge \theta = -\widehat{\mathrm{Ric}}_\rho(J,\widehat{J}) \wedge \theta.$$

If $\mathrm{Ric}_{\rho,J}$ is nondegenerate, the assertion follows directly from Lemma 3.6 and part (iv) of Lemma D.2. This proves (iv).

We prove part (v). Since $F^{\nabla_L} = 0$ we have $\mathrm{Ric}_{\rho,J} = 0$ by Lemma D.3. Hence Lemma 3.8 asserts that there exists a unique pair of smooth functions $f, g \in \Omega^0(M)$ of mean value zero such that $\Lambda_\rho(J,\widehat{J}) = -df \circ J + dg$. Let $h := \frac{1}{2}(f - \mathbf{i}g)$. Then $\bar{\partial}_J h = -\frac{\mathbf{i}}{2}\Lambda_\rho(J,\widehat{J})_J^{0,1}$ and so $d^{\nabla_L}(\beta + h\theta) = 0$ by (D.18). If $\beta + h\theta \in \mathrm{im}\, d^{\nabla_L}$, choose a Kähler form ω such that $\omega^{\mathsf{n}}/\mathsf{n}! = \rho$ and use the identity $d^{\nabla_L}(d^{\nabla_L})^* + (d^{\nabla_L})^* d^{\nabla_L} = 2(\bar{\partial}_J^{\nabla_L}(\bar{\partial}_J^{\nabla_L})^* + (\bar{\partial}_J^{\nabla_L})^*\bar{\partial}_J^{\nabla_L})$ to deduce that $\beta = \bar{\partial}_J^{\nabla_L}\iota(v)\theta$ for some vector field v. This proves Lemma D.4. $\square$

If $\widehat{J}\in\Omega^{0,1}_J(M,TM)$ satisfies $\bar\partial_J\widehat{J}=0$, then the n-form

$$\widehat{\theta}:=\beta+h\theta\in\Omega^{\mathsf{n}}(M,L)$$

in part (v) of Lemma D.4 should be thought of as the tangent vector associated to $\widehat{J}$ in the projective space of closed complex valued n-forms modulo scaling. Namely, if $t\mapsto J_t$ is a smooth path of (integrable) complex structures such that $\partial_t|_{t=0}J_t=\widehat{J}$, and $t\mapsto\theta_t\in\Omega^{\mathsf{n},0}_{J_t}(M,L)$ is a smooth path of nowhere vanishing closed $(\mathsf{n},0)$-forms, then $\partial_t|_{t=0}\theta_t\in\widehat{\theta}+\mathbb{C}\theta$.

Corollary D.5. *Let M be an closed connected oriented $2\mathsf{n}$-manifold, let J be a complex structure on M with real first Chern class zero and nonempty Kähler cone, let $L\to M$ be a Hermitian line bundle equipped with a flat connection ∇_L such that $c_1(L)=c_1(TM,J)\in H^2(M;\mathbb{Z})$, let $\theta\in\Omega^{\mathsf{n},0}_J(M,L)$ be a nowhere vanishing $(\mathsf{n},0)$-form such that $d^{\nabla_L}\theta=0$, and define $\rho:=\mathsf{c}_{\mathsf{n}}\langle\theta\wedge\theta\rangle$. For $i=1,2$ let $\widehat{J}_i\in\Omega^{0,1}_J(M,TM)$ such that $\bar\partial_J\widehat{J}_i=0$, let $\beta_i\in\Omega^{\mathsf{n}-1,1}_J(M,L)$ satisfy* (D.4) *for all $u\in\mathrm{Vect}(M)$ with $\widehat{J}=\widehat{J}_i$, let $h_i\in\Omega^0(M,\mathbb{C})$ be the unique function that satisfies* (D.19) *with $\beta=\beta_i$, and define $\widehat{\theta}_i:=\beta_i+h_i\theta$. Then*

$$\begin{aligned}\mathrm{Re}\left(c_{\mathsf{n}}\int_M\langle\widehat{\theta}_1\wedge\widehat{\theta}_2\rangle\right)&=-\tfrac{1}{8}\int_M\mathrm{trace}(\widehat{J}_1\widehat{J}_2)\rho+\int_M\mathrm{Re}(\bar h_1h_2)\rho,\\ \mathrm{Im}\left(c_{\mathsf{n}}\int_M\langle\widehat{\theta}_1\wedge\widehat{\theta}_2\rangle\right)&=\tfrac{1}{8}\int_M\mathrm{trace}(\widehat{J}_1J\widehat{J}_2)\rho+\int_M\mathrm{Im}(\bar h_1h_2)\rho.\end{aligned}\tag{D.26}$$

Proof. This follows directly from (D.9) and the definition of $\widehat{\theta}_i$. □

The discussion in this appendix is inspired by Donaldson's symplectic form on the space of complex structures on a Fano manifold in [14]. He proved in [14, Theorem 1] in the Fano case that the Hermitian form

$$(\widehat{J}_1,\widehat{J}_2)\mapsto c_{\mathsf{n}}\int_M\langle\widehat{\theta}_1\wedge\widehat{\theta}_2\rangle$$

is negative definite on the space of complex structures compatible with a fixed symplectic form ω. In the Calabi–Yau case (with the symplectic form not fixed) this Hermitian form on the kernel of $\bar\partial_J:\Omega^{0,1}_J(M,TM)\to\Omega^{0,2}_J(M,TM)$ vanishes on the image of the operator $\bar\partial_J:\Omega^0(M,TM)\to\Omega^{0,1}_J(M,TM)$ and descends to a well-defined and nondegenerate, but indefinite, Hermitian form on the quotient space $\ker\bar\partial_J/\mathrm{im}\bar\partial_J=T_{[J]}\mathscr{T}_0(M)$. Its imaginary part is the symplectic form on Teichmüller space in Theorem 4.4.

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