Lagrangian intersections, 3-manifolds with boundary, and the Atiyah-Floer conjecture

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1 Introduction

It has been observed by physicists for a long time that symplectic structures arise naturally from boundary value problems. For example, the **Robbin quotient**

$$V = \operatorname{dom} D^* / \operatorname{dom} D,$$

associated to a symmetric (but not self-adjoint) operator $D : \operatorname{dom} D \to H$ on a Hilbert space H, carries a symplectic structure

$$\omega(v, w) = \langle D^* v, w \rangle - \langle v, D^* w \rangle.$$

Self-adjoint extensions of D correspond to Lagrangian subspaces of V and, moreover, the kernel of D^* determines such a Lagrangian subspace whenever D has a closed range. If D is a symmetric differential operator on a manifold with boundary then, by partial integration, the form ω is given by an integral over the boundary. For example, if D is the Hessian of the symplectic action functional on paths in $\mathbb{R}^{2n}$, then the space $V = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ corresponds to the two boundary values of the path and the symplectic structure is $(-\omega_0) \oplus \omega_0$ where $\omega_0 = \sum_j dx_j \wedge dy_j$ is the standard symplectic structure on $\mathbb{R}^{2n}$. A more interesting example is given by the Chern-Simons functional on 3-manifolds with boundary and we shall discuss this in the Section 2.

In another closely related direction there is a formal analogy between symplectic manifolds with symplectomorphisms and Lagrangian submanifolds on the one hand and oriented Riemann surfaces with orientation preserving diffeomorphisms and 3-dimensional bordisms on the other. If Σ is a compact oriented Riemann surface we denote by $\bar{\Sigma}$ the surface with the opposite orientation. Likewise we denote by M a symplectic manifold without mentioning the symplectic form ω explicitly and by $\bar{M}$ the manifold with reversed symplectic form (i.e. ω is replaced by $-\omega$). The following diagram summarizes the correspondence.

oriented Riemann surface Σ	symplectic manifold M
$\bar{\Sigma}$	$\bar{M}$
or. pres. diffeomorphism	symplectomorphism
$\Sigma = \Sigma_1 \cup \Sigma_2$	$M = M_1 \times M_2$
3-mfld Y with $\partial Y = \Sigma$	Lagrangian submfld $L \subset M$
$Y = Y_1 \cup Y_2$	$L = L_1 \times L_2 \subset M_1 \times M_2$
gluing	symplectic reduction
$\partial Y = \Sigma \cup \bar{\Sigma} \cup \Sigma'$	$L \subset M \times \bar{M} \times M'$,
$\partial Y' = \Sigma'$	$L' = \{x' \in M' \mid \exists\, x : (x, x, x') \in L\}$.

This last correspondence between the gluing operation for 3-manifolds with boundary and symplectic reduction of Lagrangian submanifolds is the most important one. The manifold Y' is obtained from Y by identifying the boundary components Σ and $\bar{\Sigma}$ via the identity map. Similarly, $N = \Delta \times M'$ is a coisotropic submanifold of $M \times \bar{M} \times M'$ with symplectic quotient M', and L' is obtained from L via symplectic reduction. Of course, this analogy can be extended to dimensions other than 2. In the 0-dimensional case where Riemann surfaces are replaced by points and 3-manifolds with boundary by intervals the correspondence is given by the symplectic action. In dimension 2 it is given by the Chern-Simons functional and this will be discussed in the Section 2. In section 3 we shall see that this leads to a natural extension of Floer homology in the form

$$HF^*(Y, L), \qquad L \subset M_\Sigma,$$

where Y is a 3-manifold with boundary $\partial Y = \Sigma$, M_Σ denotes the moduli space of flat SU(2) (or SO(3)) connections over Σ and $L \subset M_\Sigma$ is a Lagrangian submanifold. Such groups were also considered by Fukaya [11] but his definition differs slightly from the one discussed here. Our goal in this note is to explain the definition of these Floer homology groups and to show how they can be used to prove the Atiyah-Floer conjecture for Heegard splittings of homology-3-spheres. We shall only outline the main ideas of the proofs. Details will be published elsewhere.

2 Chern-Simons functional

Let Y be a compact 3-manifold with boundary $\partial Y = \Sigma$ and consider the trivial bundle $Y \times \mathrm{G}$ with structure group $\mathrm{G} = \mathrm{SU}(2)$ and Lie algebra $\mathfrak{g} = \mathfrak{su}(2) = \mathrm{Lie}(\mathrm{G})$. The space $\mathcal{A} = \mathcal{A}(Y) = \Omega^1(Y, \mathfrak{g})$ of SU(2)-connections on Y carries a natural 1-form defined by

$$\alpha \mapsto \mathcal{F}_A(\alpha) = \int_Y \langle F_A \wedge \alpha \rangle \tag{1}$$

for $\alpha \in T_A\mathcal{A} = \Omega^1(Y, \mathfrak{g})$. Here $\langle \xi, \eta \rangle = \mathrm{trace}(\xi^* \eta)$ for $\xi, \eta \in \mathfrak{g}$ and $F_A \in \Omega^2(Y, \mathfrak{g})$ denotes the curvature of A. The 1-form (1) is invariant and horizontal with respect to the action of the gauge group $\mathcal{G}(Y) = \mathrm{Map}(Y, \mathrm{G})$. But it is not closed since

$$d\mathcal{F}_A(\alpha, \beta) = \int_Y \langle d_A\alpha \wedge \beta \rangle - \int_Y \langle \alpha \wedge d_A\beta \rangle = \int_{\partial Y} \langle \alpha \wedge \beta \rangle.$$

This is the standard symplectic structure on the space $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ of connections on Σ. It reflects the failure of the operator $*d_A : \Omega^1(Y, \mathfrak{g}) \to \Omega^1(Y, \mathfrak{g})$ to be self-adjoint. This operator can be interpreted as the differential of the vector field $A \mapsto *F_A$ on $\mathcal{A}(Y)$ associated to the 1-form $\mathcal{F}$.

In order to obtain a closed 1-form we pick some Lagrangian submanifold $\mathcal{L} \subset \mathcal{A}(\Sigma)$ and consider the subspace $\mathcal{A}(Y, \mathcal{L}) \subset \mathcal{A}(Y)$ of those connections on Y which have boundary values in $\mathcal{L}$. The restriction of $\mathcal{F}$ to this space is closed precisely when $\mathcal{L}$ is Lagrangian. Moreover, in order to preserve the invariance of $\mathcal{F}$ under the gauge group we should also assume that $\mathcal{L}$ is invariant under the action of $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, \mathrm{G})$. But this is equivalent to the condition

$$\mathcal{L} \subset \mathcal{A}_{\mathrm{flat}}(\Sigma) = \{A \in \mathcal{A}(\Sigma) \,|\, F_A = 0\}$$

and thus $\mathcal{L}$ determines a Lagrangian submanifold

$$L = \frac{\mathcal{L}}{\mathcal{G}(\Sigma)} \subset \frac{\mathcal{A}_{\mathrm{flat}}(\Sigma)}{\mathcal{G}(\Sigma)} = M_\Sigma$$

of the moduli space M_Σ of flat SU(2)-connections on Σ. This is a $(6g - 6)$-dimensional symplectic manifold (with singularities). We shall assume that L is simply connected and contains the equivalence class of the zero connection. Note that in this case the space $\mathcal{L}$ is not simply connected, but the fundamental group of $\mathcal{L}$ cancels with that of $\mathcal{G}(\Sigma)$. Now the 1-form $\mathcal{F} : T\mathcal{A}(Y, \mathcal{L}) \to \mathbb{R}$ is closed. But since $\mathcal{L}$ is not simply connected $\mathcal{F}$ is not exact. However, it is the differential of the multi-valued **Chern-Simons functional** $\mathcal{CS} : \mathcal{A}(Y, \mathcal{L}) \to \mathbb{R}/4\pi^2\mathbb{Z}$ defined by

$$\mathcal{CS}(A) = \tfrac{1}{2} \int_Y \left(\langle A \wedge dA \rangle + \tfrac{1}{3} \langle [A \wedge A] \wedge A \rangle \right) + \int_0^1 \int_\Sigma \langle A_0(t) \wedge \dot{A}_0(t) \rangle \, dt.$$

Here $A_0(t) \in \mathcal{L}$ is a path with $A_0(0) = 0$ and $A_0(1) = A|_\Sigma$. The homotopy class of this path is not unique and hence the right hand side is only well defined up to an integer multiple of $4\pi^2$.

Now the 3-manifold Y itself also determines a Lagrangian submanifold

$$L_Y = \frac{\mathcal{L}_Y}{\mathcal{G}(\Sigma)}, \qquad \mathcal{L}_Y = \{A|_\Sigma \,|\, A \in \mathcal{A}_{\mathrm{flat}}(Y)\}.$$

Note that under the correspondence $Y \mapsto L_Y$ (from bordisms to Lagrangian submanifolds) the summing of 3-mannifolds along common boundaries translates into symplectic reduction. Note also that the flat connections on Y are in fact the zeros of the 1-form $\mathcal{F} = d\mathcal{CS}$. Hence there is a map

$$\mathrm{Crit}(\mathcal{CS}) \to L_Y \cap L$$

which assigns to every critical point $A \in \mathcal{A}_{\mathrm{flat}}(Y, \mathcal{L})$ of $\mathcal{CS}$ the equivalence class $[A|_\Sigma]$ in $M_\Sigma = \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma)$. In some cases, e.g. when Y is a handle body, the connection $A \in \mathcal{A}_{\mathrm{flat}}(Y)$ is uniquely determined (up to gauge equivalence) by $A|_\Sigma$ and in this case the above map is a bijection.

3 Floer homology

Let Y be a 3-manifold with boundary $\partial Y = \Sigma$ and $\mathcal{L} \subset \mathcal{A}_{\rm flat}(\Sigma)$ be a Lagrangian submanifold with simply connected quotient $L = \mathcal{L}/\mathcal{G}(\Sigma)$. Then the gradient flow lines of the Chern-Simons functional $\mathcal{CS} : \mathcal{A}(Y,\mathcal{L}) \to \mathbb{R}/4\pi^2\mathbb{Z}$ are smooth maps $\mathbb{R} \to \Omega^1(Y,\mathfrak{g}) \times \Omega^0(Y,\mathfrak{g}) : t \mapsto (A(t),\Psi(t))$ which satisfy the boundary value problem

$$\dot{A} - d_A\Psi + *F_A = 0, \qquad A|_\Sigma \in \mathcal{L}, \qquad *A|_\Sigma = 0. \tag{2}$$

For any such gradient line the connection $A + \Psi\, dt$ on the 4-manifold $X = Y \times \mathbb{R}$ is a self-dual Yang-Mills instantons with Lagrangian boundary condition on $\partial X = \Sigma \times \mathbb{R}$. Under suitable conditions on Y and $\mathcal{L}$ the Yang-Mills energy of such an instanton is finite if and only if (in a suitable gauge) there are limits

$$\lim_{t\to\pm\infty} A(t) = A^\pm, \qquad \lim_{t\to\pm\infty} \Psi(t) = 0 \tag{3}$$

where $A^\pm \in \mathcal{A}_{\rm flat}(Y,\mathcal{L})$ are flat connections and hence critical points of $\mathcal{CS}$. If these limits are regular and nondegenerate (i.e. the extended Hessian is bijective) then one can prove that the equations (2) and (3) form a well-posed nonlinear elliptic boundary value problem and so, for a generic metric, the space $\mathcal{M}(A^-,A^+)$ of solutions modulo gauge equivalence is a finite dimensional manifold of dimension

$$\dim \mathcal{M}(A^-,A^+) = \mu(A^-) - \mu(A^+) \pmod 8$$

for some function $\mu : \mathrm{Crit}^*(Y,\mathcal{L}) \to \mathbb{Z}/8\mathbb{Z}$ on the set of irreducible flat connections in $\mathcal{A}(Y,\mathcal{L})$. Here the dimension depends on the component in the space of paths in $\mathcal{A}(Y,\mathcal{L})$ running from A^- to A^+, in contrast to the closed case where any two paths are homotopic and the index ambiguity only comes in after dividing by the gauge group.

Remark 3.1 The well posedness of (2) and (3) extends to general 4-manifold X with boundary $\partial X = \Sigma \times \mathbb{R}$ and cylindrical ends. The proof involves an estimate for the operator $D = d_A^- \oplus d_A^* : \Omega^1_{\mathcal{L}}(X,\mathfrak{g}) \to \Omega^{2,-}(X,\mathfrak{g}) \oplus \Omega^0(X,\mathfrak{g})$ where $\Omega^1_{\mathcal{L}}(X,\mathfrak{g})$ denotes the subset of all $\alpha \in \Omega^1(X,\mathfrak{g})$ which satisfy

$$\alpha|_{\Sigma\times t} \in \Lambda(t) = T_{A|_{\Sigma\times t}}\mathcal{L}, \qquad \alpha \circ \nu_{\partial X} = 0.$$

There is an inequality

$$\|\alpha\|_{W^{1,2}}^2 \le c\left(\|D\alpha\|_{L^2}^2 + \|\alpha\|_{L^2}^2\right) + \int_{\partial X} \langle d_A\alpha \wedge \alpha\rangle$$

and, in view of the Lagrangian boundary conditions, the boundary term can be estimated by

$$\left|\int_{\partial X} \langle d_A\alpha\wedge\alpha\rangle\right| \le c\,\|\alpha\|_{L^2(\partial X)}^2 \le c'\,\|\alpha\|_{W^{1,2}(X)}\,\|\alpha\|_{L^2(X)}\,.$$

Now the elliptic estimate $\|\alpha\|_{W^{1,2}} \le c\,(\|D\alpha\|_{L^2} + \|\alpha\|_{L^2})$ easily follows. This has to be combined with elliptic regularity at the boundary to obtain the required Fredholm theory.

To obtain finitess in the case where the index difference is 1 we must employ Uhlenbeck's compactness theorem in the case of bounded curvature and combine this with the usual bubbling argument if there is only an L^2-bound on the curvature. Such a bound is always guaranteed since

$$YM(A + \Psi\, dt) = \int_{-\infty}^{\infty} \left\|F_{A(t)}\right\|_{L^2(Y)}^2 dt = \mathcal{CS}(A^-) - \mathcal{CS}(A^+)$$

for every solution of (2) and (3). However, bubbling near the boundary produces nontrivial finite-energy instantons on a half-space

$$\mathbb{H}^4 = \left\{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \,|\, x_0 \ge 0\right\}$$

which on the boundary $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ are flat on each $\mathbb{R}^2$-slice. Such instantons should have Yang-Mills energy equal to an integer multiple of $8\pi^2$.

Conjecture 3.1 *Let $A = \sum_{j=0}^{3} A_j dx_j \in \Omega^1(\mathbb{H}^4, \mathfrak{g})$ be a connection such that*

$$F_{01} = F_{23}, \quad F_{02} = F_{31}, \quad F_{03} = F_{12}, \qquad A_0|_{\partial\mathbb{H}^4} = 0, \quad F_{23}|_{\partial\mathbb{H}^4} = 0,$$

where $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. Then either $F_{ij} = 0$ for all i, j or

$$\mathcal{YM}(A) = \frac{1}{2}\int_{\mathbb{H}^4} \sum_{i<j} |F_{ij}|^2 \ge 8\pi^2.$$

The proof will go along the lines of Uhlenbeck's removable singularity theorem. At the time of writing I have not carried out the details. If this holds then in the case of index difference 1 the space $\mathcal{M}(A^-, A^+)$ will consist of only finitely many connecting orbits (moduli time shift) and, as in the case of closed 3-manifolds [9, 6], counting these gives rise to a Floer chain complex

$$CF_*(Y, \mathcal{L}) = \bigoplus_{[A] \in \mathcal{A}^*_{\mathrm{flat}}(Y,\mathcal{L})/\mathcal{G}(Y)} \mathbb{Z}\langle A\rangle$$

generated by the gauge equivalence classes of irreducible flat connections. The boundary operator is defined by

$$\partial \left\langle A^- \right\rangle = \sum_{A^+} \#\{\mathcal{M}_1(A^-, A^+)/\mathbb{R}\} \left\langle A^+ \right\rangle$$

where the sum runs over all equivalence classes $[A^+] \in \mathcal{A}^*_{\mathrm{flat}}(Y, \mathcal{L})/\mathcal{G}(Y)$ with $\mu(A^+) = \mu(A^-) - 1(\mathrm{mod}\ 8)$ and $\mathcal{M}_1$ denotes the 1-dimensional components of the moduli space. As in Floer's original work [9] one uses a gluing theorem to prove that $\partial^2 = 0$ The resulting Floer cohomology groups are denoted by $HF^*(Y, L) = H^*(CF, \partial)$. They are graded modulo 8.

Remark 3.2 **(i)** To make these ideas work we must impose certain conditions on Y and L which guarantee that there are no reducible flat connections in $\mathcal{A}(Y, \mathcal{L})$ other than the equivalence class of the zero connection. Here **reducible** means that the kernel of $d_A : \Omega^0(Y, \mathfrak{g}) \to \Omega^1(Y, \mathfrak{g})$ is zero. For example, we may assume that $L = L_{Y'}$ where Y' is a handlebody with $\partial Y' = \bar{\Sigma}$ and $Y \cup_\Sigma Y'$ is a homology-3-sphere.

(ii) A connection $A \in \mathcal{A}_{\text{flat}}(Y, \mathcal{L})$ is called **nondegenerate** if every $\alpha \in \Omega^1(Y, \mathfrak{g})$ with $d_A\alpha = d_A^*\alpha = 0$ and $*\alpha|_\Sigma = 0$, $\alpha|_\Sigma \in T_A\mathcal{L}$ is equal to zero. If there are degenerate flat connections then we must choose a perturbation of the Chern-Simons functional (as in the case of closed 3-manifolds) to obtain well-defined Floer homology groups.

(iii) The Floer cohomology groups $HF^*(Y, L)$ are independent of the choice of the metric and the perturbation used to define them. They depend on the Lagrangian submanifold L only up to Hamiltonian isotopy. More precisely, for different choices there are natural isomorphisms of Floer homology.

(iv) In [11] Fukaya proposed an alternative construction of Floer homology groups on 3-manifolds with boundary.

Conjecture 3.2 *If $Y = Y_0 \cup_\Sigma Y_1$ is a homology-3-sphere, and Y_0 is a handlebody, then there is a natural isomorphism $HF^*(Y) \cong HF^*(Y_1, L_{Y_0})$. If Y_1 is also a handlebody (i.e. in the case of a Heegard splitting) there is a natural isomorphism*

$$HF^*(Y) \cong HF^*(\Sigma \times [0,1], L_{Y_0} \times L_{Y_1}).$$

At the time of writing the details of the proof have not been carried out. However, here is the main idea. Choose a map $f : \Sigma \times [0,1] \to Y_0$ which identifies $\Sigma \times \{1\}$ with ∂Y_0 and shrinks $\Sigma \times \{0\}$ onto the 1-skeleton of Y_0. Then the pullback of any connection on Y_0 onto $\Sigma \times \{0\}$ is in $\mathcal{L}_{Y_0}$ (A connection on a 1-manifold is just given by the holonomy.) The ASD equation on $\Sigma \times [0,1] \times \mathbb{R}$ with the pullback metric then takes the form

$$\partial_s A - d_A\Phi + *_s(\partial_t A - d_A\Psi) = 0, \quad \partial_s\Psi - \partial_t\Phi + [\Phi, \Psi] + *_s F_A = 0 \qquad (4)$$

where the change of the metric is not in the same conformal class and degenerates at $s = 0$. Note that (4) is just the ASD equation on (half of) the closed manifold $Y \times \mathbb{R}$. If, however, we consider the equation on the interval $s \geq \varepsilon$ for some $\varepsilon > 0$ then we obtain a genuine boundary value problem. The solutions of these should converge to those on the closed manifold as $\varepsilon \to 0$ and this will prove Conjecture 3.2. Note that the degeneration of the metric at $s = 0$ is related to the choice of the Lagrangian boundary condition in L_{Y_0}. The case $\Sigma = S^2$ is slightly simpler. In this case $Y_1 = B^3$ and we can take the map $S^2 \times [0,1] \to B^3 : (x, s) \mapsto sx$. Then the change in the metric is conformal and so the Hodge-$*$-operator on 1-forms is independent of s while $*_s F_A = s^{-2} * F_A$. Similar equations were recently studied by Fukaya [12].

4 Atiyah-Floer conjecture

In [8] Floer introduced what is now called Floer homology for Lagrangian intersections. Assume for simplicity that (M, ω) is a compact simply connected symplectic manifold which is **positive** in the sense that the first Chern class $c_1 = c_1(TM)$ (with respect to an ω-compatible almost complex structure) is a positive multiple of the cohomology class $[\omega]$. We also assume that the minimal Chern number N, defined by $\langle c_1, \pi_2(M)\rangle = N\mathbb{Z}$, is at least 2. Then for any two Lagrangian submanifolds $L_0, L_1 \subset M$ with torsion fundamental group there are Floer homology groups

$$HF^*_{\mathrm{symp}}(L_0, L_1)$$

which are graded modulo $2N$. In this theory the critical points are the intersection points $x \in L_0 \cap L_1$ and the connecting orbits are J-holomorphic strips $u : [0,1] \times \mathbb{R} \to M$ which satisfy

$$\partial_s u + J(u)\partial_t u = 0, \quad u(0,t) \in L_0, \quad u(1,t) \in L_1, \quad \lim_{t\to\pm\infty} u(s,t) = x^{\pm}, \tag{5}$$

where $x^{\pm} \in L_0 \cap L_1$ and J is an almost complex structure on M which is compatible with ω in the sense that $\langle \xi, \eta\rangle = \omega(J\xi, \eta)$ is a Riemannian metric. This construction requires transversal intersections of the Lagrangian submanifolds and surjectivity of the linearized Cauchy-Riemann operators. These conditions can be achieved by a suitable Hamiltonian perturbation and, as before, the resulting Floer homology groups are indepedent of the almost complex structure and the Hamiltonian perturbation used to define them [8, 15].

Now consider the case where $M = M_\Sigma$ is the moduli space of flat SU(2)-connections on a compact oriented Riemann surface Σ and $L_i = L_{Y_i}$ for $i = 0, 1$ where Y_0 and Y_1 are handlebodies with $\partial Y_0 = \Sigma$ and $\partial Y_1 = \bar{\Sigma}$. Then the manifold M_Σ is simply connected and positive in the above sense with minimal Chern number 4. Moreover, the Lagrangian manifolds L_0 and L_1 are diffeomorphic to the quotient of $\mathrm{SU}(2)^g$ by simultaneous conjugacy and hence are obviously simply connected. However, some care must be taken when extending symplectic Floer homology to M_Σ in view of the singularities. To obtain a well-defined theory we must assume that $Y = Y_0 \cup_\Sigma Y_1$ is a homology-3-sphere so that 0 is the only singular intersection point of L_0 and L_1. Another point is to give the right definition of holomorphic curves when they pass through the singular part of M_Σ. The correct definition should be that they can be represented locally by a smooth map $\mathbb{C} \to \mathcal{A}(\Sigma) \oplus \Omega^0(\Sigma) \oplus \Omega^0(\Sigma) : s + it \mapsto (A(s,t), \Phi(s,t), \Psi(s,t))$ such that

$$\partial_s A - d_A\Phi + *(\partial_t A - d_A\Psi) = 0, \qquad F_A = 0. \tag{6}$$

Using local Coulomb gauge in $\mathcal{A}(\Sigma)$, in a neighbourhood of a (possibly singular) connection $A_0 = A(0,0)$, one can show that every $W^{1,p}$-solution of (6) is gauge equivalent to a smooth solution. One should then be able to use a transversality argument in the moduli space $\mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}_0(\Sigma)$ of flat connections modulo pointed gauge transformations to prove that generic holomorphic curves avoid the singular set, because it is of codimension larger than 2 if the genus of Σ is at least 3.

As a result there are symplectic Floer cohomology groups for $(M_\Sigma, L_{Y_0}, L_{Y_1})$ whenever $Y = Y_0 \cup_\Sigma Y_1$ is a Heegard splitting of a homology-3-sphere and it was conjectured by Atiyah and Floer that there should be a natural isomorphism

$$HF^*(Y) = HF^*_{\mathrm{symp}}(M_\Sigma, L_{Y_0}, L_{Y_1}).$$

In view of Conjecture 3.2 this reduces to the following.

Conjecture 4.1 *For every Heegard splitting $Y = Y_0 \cup_\Sigma Y_1$ of a homology-3-sphere there is a natural isomorphism of Floer cohomologies*

$$HF^*(\Sigma \times [0,1], L_{Y_0} \times L_{Y_1}) \cong HF^*_{\mathrm{symp}}(M_\Sigma, L_{Y_0}, L_{Y_1}).$$

The proof of Conjecture 4.1 follows the line of argument in [7] for mapping cylinders. The key idea is to conformally rescale the metric on Σ by a factor $\varepsilon^2 > 0$ and prove that in the limit $\varepsilon \to 0$ the ASD instantons on $\Sigma \times [0,1] \times \mathbb{R}$ degenerate into holomorphic curves. More precisely, the ASD equation on $\Sigma \times [0,1] \times \mathbb{R}$ with respect to the rescaled metric takes the form

$$\partial_s A - d_A \Phi + *(\partial_t A - d_A \Psi) = 0, \quad \partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \frac{1}{\varepsilon^2} * F_A = 0, \tag{7}$$

with boundary conditions

$$A(0,t) \in \mathcal{L}_{Y_0}, \qquad A(1,t) \in \mathcal{L}_{Y_1}, \qquad \Phi(0,t) = \Phi(1,t) = 0. \tag{8}$$

The proof that for $\varepsilon \to 0$ the solutions of (7) and (8) converge to those of (6) is almost word by word the same as in [7]. An important ingredient in the proof is the observation that the Yang-Mills energy (with respect to the ε-dependent metric) of a connection $\Xi = A + \Phi\, ds + \Psi\, dt$ which satisfies (7) is given by

$$\mathcal{YM}_\varepsilon(\Xi) = \int_{-\infty}^{\infty} \int_0^1 \left(\|\partial_s A - d_A \Phi\|^2_{L^2(\Sigma)} + \frac{1}{\varepsilon^2} \|F_A\|^2_{L^2(\Sigma)} \right) dsdt.$$

The main differences in the proof are that, firstly, the estimates on the curvature in [7], Section 7, must be established near the boundary, secondly, the bubbling argument requires Conjecture 3.1, and thirdly, care must be taken near the singularities of the moduli space. Details will be carried out elsewhere.

5 Products

There are interesting product structures on Floer cohomology due to Donaldson. Let (M, ω) be a compact simply connected symplectic manifold which is positive in the above sense with minimal Chern number $N \geq 2$. Then there is a **quantum category** $\mathcal{C}_M$ whose objects are the Lagrangian submanifolds $L \subset M$ with torsion fundamental group and whose morphisms are Floer cohomology classes. Thus $\mathrm{Mor}(L_0, L_1) = HF^*_{\mathrm{symp}}(L_0, L_1)$. The composition rule appears as a product structure

$$HF^*_{\mathrm{symp}}(L_0, L_1) \otimes HF^*_{\mathrm{symp}}(L_1, L_2) \to HF^*_{\mathrm{symp}}(L_0, L_2).$$

On the chain level this homomorphism is given by *counting J-holomorphic triangles*. More precisely, one considers J-holomorphic maps $u : \Omega \to M$ defined on a domain $\Omega \subset \mathbb{C}$ with three smooth boundary components and three cylindrical ends which map the boundary components to L_0, L_1, and L_2, respectively, and in the cylindrical ends converge to intersection points. To obtain a well-defined

Fredholm theory one can choose Hamiltonian perturbations in the cylindrical ends. The resulting product is associative in homology but not on the chain level. The proof of associativity involves domains with four cylindrical ends and leads to the A_∞-category of Fukaya [11].

Now there are similar product structures for homology-3-spheres. If Y_0, Y_1, Y_2 are three handle bodies with boundary $\partial Y_i = \Sigma$ such that the manifolds $Y_i \cup \bar{Y}_j$ are homology-3-spheres for $i \neq j$ then there is a product

$$HF^*(Y_0 \cup \bar{Y}_1) \otimes HF^*(Y_1 \cup \bar{Y}_2) \to HF^*(Y_0 \cup \bar{Y}_2).$$

This can be defined in terms of ASD instantons on a cobordism X which is obtained from $\Omega \times \Sigma$ by gluing $Y_0 \times \mathbb{R}$, $Y_1 \times \mathbb{R}$, $Y_2 \times \mathbb{R}$ to the three boundary components (which are all diffeomorphic to $\Sigma \times \mathbb{R}$). The natural extension of the Atiyah-Floer conjecture asserts that these two product structures should correspond under the isomorphisms of Conjectures 3.2 and 4.1 if in the symplectic case we choose $M = M_\Sigma$ and $L_i = L_{Y_i}$. This can be proved with the same techniques as above.

An interesting special case occurs when the symplectic manifold M is replaced by $\bar{M} \times M$ and $L_0 = \Delta$, $L_1 = \mathrm{graph}\,(\phi)$, $L_2 = \mathrm{graph}\,(\psi\phi)$. This gives rise to Floer cohomology groups

$$HF^*_{\mathrm{symp}}(\phi) = HF^*_{\mathrm{symp}}(\bar{M} \times M, \Delta, \mathrm{graph}\,(\phi)).$$

Intuitively, the Floer cohomology of ϕ can be interpreted as the *middle-dimensional* cohomology of the space Ω_ϕ of paths $\gamma : [0,1] \to M$ with $\gamma(1) = \phi(\gamma(0))$

$$HF^*_{\mathrm{symp}}(\phi) = H^{\frac{1}{2}\infty}(\Omega_\phi).$$

These groups are invariant under conjugacy, i.e. $HF^*_{\mathrm{symp}}(\phi) = HF^*_{\mathrm{symp}}(\psi^{-1}\phi\psi)$, and the Donaldson product structure takes the form

$$HF^*_{\mathrm{symp}}(\phi) \otimes HF^*_{\mathrm{symp}}(\psi) \to HF^*_{\mathrm{symp}}(\psi\phi). \qquad (9)$$

In the case $\phi = \psi = \mathrm{id}$ there is a natural isomorphism $HF^*(\mathrm{id}) = H^*(M)$ (with the grading made periodic with period $2N$) and the above product reduces to quantum cohomology [17]. (See [13] for an exposition of quantum cohomology.)

Let us now specialize further to the case where $M = M_\Sigma$ is the moduli space of flat connections on the **nontrivial** $\mathrm{SO}(3)$**-bundle** $P \to \Sigma$. The mapping class group of Σ acts on this space by symplectomorphisms $\phi_f : M_\Sigma \to M_\Sigma$ (modulo some finite ambiguity in the choice of a lift). An automorphism $f : P \to P$ also determines a mapping cylinder Y_f and there are corresponding Floer cohomology groups $HF^*(Y_f)$, defined in terms of ASD instantons on $Y_f \times \mathbb{R}$. In [7] it was shown that there are natural isomorphisms

$$HF^*(Y_f) \cong HF^*_{\mathrm{symp}}(\phi_f).$$

Now there is again a product structure

$$HF^*(Y_f) \otimes HF^*(Y_g) \to HF^*(Y_{gf}) \qquad (10)$$

defined in terms of ASD instantons on suitable 4-dimensional cobordisms. In [16] it is shown that these agree with the products in (i) under the above isomorphisms.

Remark 5.1 **(i)** In his thesis [4] Callahan examines these product structures in detail and uses them to find examples of symplectomorphisms $\phi_f : M_\Sigma \to M_\Sigma$ which are homotopic to the identity but not symplectically so. In his examples the automorphism f is generated by a Dehn twist on a loop which divides Σ into two components.

(ii) There is an alternative way to interprete these product structures (in the case $g = \mathrm{id}$) by intersecting the spaces of connecting orbits with suitable submanifolds of finite codimension in either $\mathcal{B}_Y = \mathcal{A}(Y)/\mathcal{G}(Y)$ or Ω_ϕ. In the symplectic context this gives rise to an action

$$H^*(\Omega_\phi) \otimes HF^*_{\mathrm{symp}}(\phi) \to HF^*_{\mathrm{symp}}(\phi).$$

Intuitively, $HF^*_{\mathrm{symp}}(\phi) = H^{\frac{1}{2}\infty}(\Omega_\phi)$ and this is the *cup-product* in Ω_ϕ. The map $\Omega_\phi \to M : \gamma \mapsto \gamma(0)$ induces a homomorphism $H^*(M) \to H^*(\Omega_\phi)$ and the resulting product $H^*(M) \otimes HF^*_{\mathrm{symp}}(\phi) \to HF^*_{\mathrm{symp}}(\phi)$ agrees with (9) in the case $\psi = \mathrm{id}$.

(iii) A loop $\gamma : S^1 \to Y$ determines a submanifold $V_\gamma \subset \mathcal{B}_Y$ via Donaldson's map $\mu : H_1(Y) \to H^3(\mathcal{B}_Y)$ and the induced homomorphism of Floer cohomology appears as the second boundary map in the Fukaya-Floer cohomology groups $HFF^*(Y, \gamma)$ [3]. In the symplectic context these operators correspond to the action of $H^*(\Omega_\phi)$ on $HF^*_{\mathrm{symp}}(\phi)$. If $M = M_\Sigma$ and $\phi = \phi_f$ for some automorphism $f : P \to P$ then a loop $\gamma : S^1 \to Y_f$ determines a codimension-2 submanifold $W_\gamma \subset \Omega_{\phi_f}$ and there is a commuting diagram

$$\begin{array}{ccc} HF^*(Y_f) & \stackrel{V_\gamma}{\to} & HF^*(Y_f) \\ \downarrow & & \downarrow \\ HF^*_{\mathrm{symp}}(\phi_f) & \stackrel{W_\gamma}{\to} & HF^*_{\mathrm{symp}}(\phi_f) \end{array}.$$

If the loop γ lies entirely in $\Sigma \times \{0\}$ then these maps can be interpreted in terms of the product structures (9) and (10) with $g = \mathrm{id}$ and $\psi = \psi_g = \mathrm{id}$. In [5] Donaldson has computed the quantum cohomology of M_Σ for a surface of genus 2.

(iv) In the instanton case the maps in **Floer's exact sequence** can be interpreted in terms of the Donaldson product structures [2]. It was proposed by Donaldson and Callahan [4] that there should be a symplectic analogue of this exact sequence. In special cases this should be related to Floer's original sequence by the Atiyah-Floer conjecture.

(v) There is a related question what the effect of symplectic reduction is on Floer homology. This should also be related to the quantum product structures discussed here. An interesting example is provided by surgery on a loop $\gamma \subset Y$ in a three manifold with boundary $\partial Y = \Sigma$. Cut out a neighbourhood U of γ and write $Y = U \cup_T (Y - U)$. Then the disjoint union $U \cup (Y - U)$ has three boundary components $T \cup \bar{T} \cup \partial Y$. Different ways of gluing in U correspond to different symplectic reductions in $M_T \cup \bar{M}_T \cup M_\Sigma$.

(vi) If Y is a three manifold with boundary $\partial Y = \Sigma$ then the quantum category $\mathcal{C}_{M_\Sigma}$ acts on the Floer cohomology groups $HF^*(Y, L)$ via natural product type maps

$$HF^*(Y, L_0) \otimes HF^*_{\mathrm{symp}}(L_0, L_1) \to HF^*(Y, L_1).$$

This was already observed by Fukaya [11]. So far these product structures are little understood.

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