J-holomorphic Curves and Symplectic Topology

Second Edition

erratum

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27 November 2019

page 33: Here is a cleaner argument. The map $w' : U_0 \to \mathbb{C}$ defined on page 32 by

$$w'(z) := \prod_{\zeta \in U_0,\, \zeta \sim z} w(z) \tag{1}$$

is holomorphic and nonconstant and satisfies $w'(z_0) = 0$. Hence a theorem in complex analysis asserts that there exists a positive integer $\ell \in \mathbb{N}$, a neighbourhood $U_1 \subset U_0$ of z_0, and a biholomorphic map $\phi : U_1 \to V$ onto an open neighbourhood $V \subset \mathbb{C}$ of zero such that $\phi(z_0) = 0$ and

$$w'(z) = \phi(z)^\ell \qquad \text{for all } z \in U_1$$

(see [1, pp131–133, Thm 11] or [2, Satz 3.61]). Define

$$U := w(U_1), \qquad f := \phi \circ w^{-1} : U \to V.$$

Then $U \subset \mathbb{C}$ is an open neighbourhood of zero, $f : U \to V$ is a biholomorphic map, $f(0) = 0$, and

$$w'(z) = \big(f(w(z))\big)^\ell \qquad \text{for all } z \in U_1. \tag{2}$$

Choose $\delta > 0$ such that

(a) $\delta|w| \le |f(w)| \le \delta^{-1}|w|$ for all $w \in U$ (shrinking U if necessary).

Then the following holds.

(b) $\delta^\ell |w(z)|^\ell \le |w'(z)| \le \delta^{-\ell}|w(z)|^\ell$ for all $z \in U_1$, by (2) and (a).

(c) If $z, \zeta \in U_1$ and $z \sim \zeta$ then $w'(z) = w'(\zeta)$ and hence, by (b),

$$\delta^2 |w(z)| \le |w(\zeta)| \le \delta^{-2}|w(z)|.$$

(d) $\delta^{2m_0}|w(z)|^{m_0} \le |w'(z)| \le \delta^{-2m_0}|w(z)|^{m_0}$ for all $z \in U_1$ by (1) and (c). Here $m_0 := m(z_0)$ is as on page 32.

It follows from (b) and (d) that

$$m_0 = \ell$$

and thus each sufficiently small nonzero complex number has precisely m_0 preimages under w' (again [1, Thm 11, p131] or [2, Satz 3.61]). Hence, for all $z, \zeta \in U_1$ sufficiently close to z_0, we have

$$z \sim \zeta \qquad \Longleftrightarrow \qquad w'(z) = w'(\zeta).$$

This shows that the map

$$U_0' := U_0/\sim \to \mathbb{C} : [z] \mapsto w'(z)$$

is injective and hence is a holomorphic coordinate chart on $\Sigma/\sim$.

page 37: Exercise 2.6.6 is wrong. For example, every branched double cover of $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2$ with positive genus has positive self-intersection number and violates the adjunction inequality.

page 59, line 3: The construction of a diffeomorphism from $\mathbf{P}(L \oplus \mathbb{C})$ to the product $S^2 \times S^2$ that sends the section C_L to the anti-diagonal requires the condition $2k = c_1(L) = 2$.

page 143, last line: The number $\rho_\varepsilon(\mathbf{x}, \mathbf{x}')$ should be defined by

$$\rho_\varepsilon(\mathbf{x}, \mathbf{x}') := |E(\mathbf{u}) - E(\mathbf{u}')| + \inf_{f:T\to T'} \inf_{\{\phi_\alpha\}} \rho_\varepsilon\big(\mathbf{x}, \mathbf{x}'; f, \{\phi_\alpha\}\big).$$

page 144, line 2: The number $\rho_\varepsilon(\mathbf{x}, \mathbf{x}'; f, \{\phi_\alpha\})$ should be defined by

$$
\begin{aligned}
\rho_\varepsilon\big(\mathbf{x}, \mathbf{x}'; f, \{\phi_\alpha\}\big) := &\sum_{\alpha E\beta} \big|E_\alpha\big(\mathbf{u}; B_{2\varepsilon}(z_{\alpha\beta})\big) - E_{f(\alpha)}\big(\mathbf{u}'; \phi_\alpha(B_{2\varepsilon}(z_{\alpha\beta}))\big)\big| \\
&+ \sum_{\alpha\in T} \sup_{z\in S^2\setminus B_\varepsilon(Z_\alpha)} d\big(u'_{f(\alpha)} \circ \phi_\alpha(z), u_\alpha(z)\big) \\
&+ \sum_{\substack{\alpha\neq\beta \\ f(\alpha)=f(\beta)}} \sup_{z\in S^2\setminus B_\varepsilon(z_{\alpha\beta})} d\big(\phi_\beta^{-1} \circ \phi_\alpha(z), z_{\beta\alpha}\big) \\
&+ \sum_{f(\alpha)\neq f(\beta)} d\big(\phi_\beta^{-1}(z'_{f(\beta)f(\alpha)}), z_{\beta\alpha}\big) \\
&+ \sum_{\substack{\alpha\in T \\ 1\leq i\leq n}} d\big(\phi_\alpha^{-1}(z'_{f(\alpha)i}), z_{\alpha i}\big).
\end{aligned} \tag{3}
$$

(This definition is needed in the proof of equation (5.5.7) on page 146 in the proof of Lemma 5.5.9. Thanks to Nate Bottman for pointing this out.)

page 145: There is a gap in the proof of Lemma 5.5.8 on page 145, line 21. The energy limit equation on this line does not follow directly from (5.5.4). The proof can be corrected as follows. (Thanks to Aleksey Zinger for suggesting this argument.)

To prove the *(Map)* and *(Energy)* axioms, note that, since ρ^ν converges to zero, we have

$$
\begin{aligned}
m_{\alpha\beta}(\mathbf{u}) + E\big(u_\alpha; B_{2\varepsilon}(z_{\alpha\beta})\big) &= \lim_{\nu\to\infty} E_{\alpha'}\big(\mathbf{u}^\nu; \phi_\alpha^\nu(B_{2\varepsilon}(z_{\alpha\beta}))\big), \\
m_{\beta\alpha}(\mathbf{u}) + E\big(u_\beta; B_{2\varepsilon}(z_{\beta\alpha})\big) &= \lim_{\nu\to\infty} E_{\beta'}\big(\mathbf{u}^\nu; \phi_\beta^\nu(B_{2\varepsilon}(z_{\beta\alpha}))\big)
\end{aligned} \tag{5.5.4}
$$

whenever $\alpha E\beta$, and the sequence $u_\alpha^\nu := u_{\alpha'}^\nu \circ \phi_\alpha^\nu$ converges to u_α uniformly on $S^2 \setminus \bigcup_{\alpha E\beta} B_\varepsilon(z_{\alpha\beta})$ for all α. Hence, by Lemma 4.6.6, u_α^ν converges uniformly *with all derivatives* on compact subsets of $S^2 \setminus \bigcup_{\alpha E\beta} B_\varepsilon(z_{\alpha\beta})$.

It remains to show that the sequence u_α^ν does not exhibit bubbling in $B_{2\varepsilon}(z_{\alpha\beta}) \setminus B_\delta(z_{\alpha\beta})$ for any $\delta > 0$. Assume first that $\alpha' \neq \beta'$. Then $\alpha' E' \beta'$ and $z_{\alpha\gamma} = \lim_{\nu\to\infty} (\phi_\alpha^\nu)^{-1}(z_{\alpha'\gamma'}^\nu)$ for all $\gamma \in T$ with $\alpha' \neq \gamma'$ by part (ii) of Lemma 5.5.6, and $z_{\beta\gamma} = \lim_{\nu\to\infty} (\phi_\beta^\nu)^{-1}(z_{\beta'\gamma'}^\nu)$ for all $\gamma \in T$ with $\beta' \neq \gamma'$. Thus

$$
\begin{aligned}
E_{\alpha'}\big(\mathbf{u}^\nu; \phi_\alpha^\nu(B_{2\varepsilon}(z_{\alpha\beta}))\big) &= m_{\alpha'\beta'}(\mathbf{u}^\nu) + E\big(u_\alpha^\nu; B_{2\varepsilon}(z_{\alpha\beta})\big), \\
E_{\beta'}\big(\mathbf{u}^\nu; \phi_\beta^\nu(B_{2\varepsilon}(z_{\beta\alpha}))\big) &= m_{\beta'\alpha'}(\mathbf{u}^\nu) + E\big(u_\beta^\nu; B_{2\varepsilon}(z_{\beta\alpha})\big)
\end{aligned}
$$

for ν sufficiently large and hence, by adding the two equations in (5.5.4) and using

$$\lim_{\nu\to\infty}\left(m_{\alpha'\beta'}(\mathbf{u}^\nu)+m_{\beta'\alpha'}(\mathbf{u}^\nu)\right)=\lim_{\nu\to\infty}E(\mathbf{u}^\nu)=E(\mathbf{u})=m_{\alpha\beta}(\mathbf{u})+m_{\beta\alpha}(\mathbf{u}),$$

we obtain

$$\lim_{\nu\to\infty}\big(E\big(u_\alpha^\nu;B_{2\varepsilon}(z_{\alpha\beta})\big)+E\big(u_\beta^\nu;B_{2\varepsilon}(z_{\beta\alpha})\big)\big)<\hbar.$$

Thus $E\big(u_\alpha^\nu;B_{2\varepsilon}(z_{\alpha\beta})\big)<\hbar$ for ν sufficiently large, and so the sequence u_α^ν does not exhibit any bubbling in $B_{2\varepsilon}(z_{\alpha\beta})$. In the case $\alpha'=\beta'$ we use the fact that the sequence $\phi_{\alpha\beta}^\nu:=(\phi_\alpha^\nu)^{-1}\circ\phi_\beta^\nu$ converges to $z_{\alpha\beta}$ u.c.s. on $S^2\setminus\{z_{\beta\alpha}\}$ as well as $z_{\alpha\gamma}=\lim_{\nu\to\infty}(\phi_\alpha^\nu)^{-1}(z_{\alpha'\gamma'}^\nu)$ for all $\gamma\in T$ with $\alpha'\neq\gamma'$ by part (ii) of Lemma 5.5.6. Thus

$$\begin{aligned}&E_{\alpha'}\big(\mathbf{u}^\nu;\phi_\alpha^\nu(B_{2\varepsilon}(z_{\alpha\beta}))\big)+E_{\alpha'}\big(\mathbf{u}^\nu;\phi_\beta^\nu(B_{2\varepsilon}(z_{\beta\alpha}))\big)\\&=E(\mathbf{u}^\nu)+E\big(u_\alpha^\nu;B_{2\varepsilon}(z_{\alpha\beta})\cap\phi_{\alpha\beta}^\nu(B_{2\varepsilon}(z_{\beta\alpha}))\big)\end{aligned}$$

for ν sufficiently large and hence, by adding the two equations in (5.5.4) and using $\lim_{\nu\to\infty}E(\mathbf{u}^\nu)=E(\mathbf{u})=m_{\alpha\beta}(\mathbf{u})+m_{\beta\alpha}(\mathbf{u})$, we obtain

$$\lim_{\nu\to\infty}E\big(u_\alpha^\nu;B_{2\varepsilon}(z_{\alpha\beta})\cap\phi_{\alpha\beta}^\nu(B_{2\varepsilon}(z_{\beta\alpha}))\big)<\hbar.$$

This implies that the sequence u_α^ν does not exhibit bubbling in the spherical shell $B_{2\varepsilon}(z_{\beta\alpha})\setminus B_\delta(z_{\beta\alpha})$ for any $\delta>0$ as claimed.

page 148, line 19: The assertion of Lemma 5.6.4 should be

$$\mathcal{C}(\mathcal{U}(\mathcal{C}))=\mathcal{C}.$$

The uniqueness statement is wrong (and not addressed in the proof). A counterexample is the space ℓ^1 of summable sequences, because a sequence in ℓ^1 converges strongly if and only if it converges weakly.

page 150, Theorem 5.6.6: There is a gap in the proof of part (ii) of Theorem 5.6.6. The proof only shows that the moduli space $\overline{\mathcal{M}}_{0,n}(M,A;J)$ is first countable and separable, but this does not imply that it is second countable. An example is the *Sorgenfrey line* with the nonstandard topology on the real axis in which the open sets are unions of half-open intervals $[a,b)$. This space is separable and first countable. However, it is not second countable. Namely, if $\mathcal{B}\subset 2^{\mathbb{R}}$ is a basis of the Sorgenfrey topology, then for every $a\in\mathbb{R}$ there exists an open set $B_a\in\mathcal{B}$ with $a\in B_a\subset[a,\infty)$; thus $a=\inf B_a$, hence $B_a\neq B_b$ for $a\neq b$, and so the collection $\mathcal{B}$ is uncountable.

Here is a proof of second countablility for the moduli space of genus zero stable maps with n marked points. The proof will take up ten pages. Assume throughout that (M,ω) is a closed symplectic manifold, let J be an ω-tame almost complex structure on M, equip M with the Riemannian metric $\langle\cdot,\cdot\rangle = \frac{1}{2}(\omega(\cdot,J\cdot)-\omega(J\cdot,\cdot))$, and let $A\in H_2(M;\mathbb{Z})$.

Theorem A. *The moduli space $\overline{\mathcal{M}}_{0,n}(M,A;J)$ is second countable.*

The proof requires some preparation. For two stable maps

$$\begin{aligned}\mathbf{x} &= (\mathbf{u},\mathbf{z}) = \bigl(\{u_\alpha\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le n}\bigr),\\ \mathbf{x}' &= (\mathbf{u}',\mathbf{z}') = \bigl(\{u'_{\alpha'}\}_{\alpha'\in T'}, \{z'_{\alpha'\beta'}\}_{\alpha' E'\beta'}, \{\alpha'_i, z'_i\}_{1\le i\le n}\bigr)\end{aligned}$$

of genus zero with n marked points, a sufficiently small constant $\varepsilon>0$, a surjective tree homomorphism $f:T\to T'$ satisfying $f(\alpha_i)=\alpha'_i$ for $i=1,\dots,n$, and a tuple $\phi=\{\phi_\alpha\}_{\alpha\in T}\in \mathrm{G}_T$ of Möbius transformations ϕ_α, abbreviate

$$\begin{aligned}\rho^1_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}') &:= \sum_{\alpha E\beta}\Bigl|E_\alpha\bigl(\mathbf{u};B_{2\varepsilon}(z_{\alpha\beta})\bigr) - E_{f(\alpha)}\bigl(\mathbf{u}';\phi_\alpha(B_{2\varepsilon}(z_{\alpha\beta}))\bigr)\Bigr|,\\ \rho^2_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}') &:= \sum_{\alpha\in T}\sup_{z\in S^2\setminus B_\varepsilon(Z_\alpha)} d_M\bigl(u'_{f(\alpha)}\circ\phi_\alpha(z),u_\alpha(z)\bigr),\\ \rho^3_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}') &:= \sum_{\substack{\alpha\ne\beta\\ f(\alpha)=f(\beta)}}\sup_{z\in S^2\setminus B_\varepsilon(z_{\alpha\beta})} d_{S^2}\bigl(\phi_\beta^{-1}\circ\phi_\alpha(z),z_{\beta\alpha}\bigr),\\ \rho^4_{f,\phi}(\mathbf{x},\mathbf{x}') &:= \sum_{f(\alpha)\ne f(\beta)} d_{S^2}\bigl(\phi_\beta^{-1}(z'_{f(\beta)f(\alpha)}),z_{\beta\alpha}\bigr),\\ \rho^5_{f,\phi}(\mathbf{x},\mathbf{x}') &:= \sum_{\substack{\alpha\in T\\ 1\le i\le n}} d_{S^2}\bigl(\phi_\alpha^{-1}(z'_{f(\alpha)i}),z_{\alpha i}\bigr).\end{aligned}\tag{4}$$

Here we use the notation $Z_\alpha := \{z_{\alpha\beta}\,|\,\beta\in T,\ \alpha E\beta\}$. Thus

$$\rho_\varepsilon(\mathbf{x},\mathbf{x}';f,\{\phi_\alpha\}) = \sum_{i=1,2,3}\rho^i_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}') + \sum_{i=4,5}\rho^i_{f,\phi}(\mathbf{x},\mathbf{x}')$$

(see equation (3) on page 3 of this erratum) and

$$\rho_\varepsilon(\mathbf{x},[\mathbf{x}']) := \rho_\varepsilon(\mathbf{x},\mathbf{x}') = \inf_{f,\phi}\rho_\varepsilon(\mathbf{x},\mathbf{x}';f,\{\phi_\alpha\}).$$

Note that $\rho_\varepsilon(\mathbf{x},[\mathbf{x}'])$ depends only on the isomorphism class of $\mathbf{x}'$, while it depends on the parametrization of $\mathbf{x}$. In particular, even if $\mathbf{x},\mathbf{x}'$ are both modelled on the same tree T, it is not symmetric in $\mathbf{x}$ and $\mathbf{x}'$. It also does not satisfy the triangle inequality, which is the essential ingredient of the proof that a first countable metric space is second countable. The crucial step in our proof of second countability is part (iii) of Lemma B that establishes a substitute for the triangle inequality that is sufficient for our purposes.

Notice that $\rho^i_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}')$ is nonincreasing in ε for $i=2$ and $i=3$. It is therefore useful to choose different values of ε for $i=1$ and $i=2,3$ before taking the infimum over all f and $\phi=\{\phi_\alpha\}$. Thus, for $0<\varepsilon'\le\varepsilon$ we define

$$\rho_{\varepsilon,\varepsilon'}(\mathbf{x},\mathbf{x}';f,\{\phi_\alpha\}) := \rho^1_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}') + \sum_{i=2,3}\rho^i_{f,\phi,\varepsilon'}(\mathbf{x},\mathbf{x}') + \sum_{i=4,5}\rho^i_{f,\phi}(\mathbf{x},\mathbf{x}'),$$

$$\rho_{\varepsilon,\varepsilon'}(\mathbf{x},[\mathbf{x}']) := \rho_{\varepsilon,\varepsilon'}(\mathbf{x},\mathbf{x}') = \inf_{f,\phi}\rho_{\varepsilon,\varepsilon'}(\mathbf{x},\mathbf{x}';f,\{\phi_\alpha\}).$$

Just as with ρ_ε, the function $\rho_{\varepsilon,\varepsilon'}$ has the property that, for fixed $\varepsilon,\varepsilon'>0$ and $\mathbf{x}\in\mathcal{SC}_{0,n}(M,A;J)$, a sequence $[\mathbf{x}^\nu]$ Gromov converges to $[\mathbf{x}]$ if and only if $\rho_{\varepsilon,\varepsilon'}(\mathbf{x},[\mathbf{x}^\nu])\to 0$.

Further, slightly abusing notation, we will write $\mathbf{y}\in\mathcal{SC}_{0,n}(M,A;J)$ to mean that $\mathbf{y}$ is a genus zero stable map with n-marked points representing the homology class A, even though the collection of such stable maps is a proper class and not a set. The isomorphism classes $[\mathbf{y}]$ of such stable maps, however, do form a set $\overline{\mathcal{M}}_{0,n}(M,A;J)$ equipped with the Gromov topology. For $\mathbf{y}\in\mathcal{SC}_{0,n}(M,A;J)$ and real numbers $\varepsilon\ge\varepsilon'>0$ and $\varepsilon''>0$ the set

$$\mathcal{N}_{\varepsilon,\varepsilon',\varepsilon''}(\mathbf{y}) := \left\{[\mathbf{y}']\in\overline{\mathcal{M}}_{0,n}(M,A;J)\,|\,\rho_{\varepsilon,\varepsilon'}(\mathbf{y},[\mathbf{y}'])<\varepsilon''\right\}$$

is open in $\overline{\mathcal{M}}_{0,n}(M,A;J)$. Our goal is to prove that a countable collection of such open sets is a basis of the topology of $\overline{\mathcal{M}}_{0,n}(M,A;J)$.

Fix an n-labelled tree (T,Λ) (as on p 115) and a collection of homology classes $A_\alpha\in H_2(M;\mathbb{Z})$, one for each $\alpha\in T$, with $\sum_{\alpha\in T}A_\alpha=A$. The stability condition asserts that $\#Z_\alpha+\#\Lambda_\alpha\ge 3$ for each $\alpha\in T$ with $A_\alpha=0$. Given these data, denote by $\mathcal{SC}_{0,n}(M,A;J,T)$ the set[1] of all genus zero stable maps $\mathbf{x}=(\mathbf{u},\mathbf{z})=(\{u_\alpha\}_{\alpha\in T},\{z_{\alpha\beta}\}_{\alpha E\beta},\{\alpha_i,z_i\}_{1\le i\le n})$ with n marked points that satisfy $[u_\alpha]=A_\alpha$ for all $\alpha\in T$. For $\mathbf{x},\mathbf{x}'\in\mathcal{SC}_{0,n}(M,A;J,T)$ define

$$\begin{aligned} d_{C^0}(\mathbf{x},\mathbf{x}') &:= \sup_{\alpha\in T}\sup_{z\in S^2} d_M(u_\alpha(z),u'_\alpha(z)) \\ &\quad + \sum_{\alpha\ne\beta} d_{S^2}(z_{\alpha\beta},z'_{\alpha\beta}) + \sum_{\alpha,i} d_{S^2}(z_{\alpha i},z'_{\alpha i}). \end{aligned} \tag{5}$$

[1]This is a set because the domain of each of these stable maps is a fixed union of spheres.

(See p 121 for the notation $z_{\alpha i}$ and p 136 for the notation $z_{\alpha\beta}$.) With this distance function $\mathcal{SC}_{0,n}(M,A;J,T)$ is a separable, and hence second countable, metric space. Let $\hbar > 0$ be the smallest energy of a nonconstant holomorphic sphere.

Definition. *For $\mathbf{x} \in \mathcal{SC}_{0,n}(M,A;J,T)$, let $\varepsilon_0 = \varepsilon_0(\mathbf{x}) \in (0,\pi/2]$ be the largest number that satisfies the following conditions for each vertex $\alpha \in T$:*

$$\alpha E\beta,\ \alpha E\gamma,\ \beta \neq \gamma \qquad \Longrightarrow \qquad B_{2\varepsilon_0}(z_{\alpha\beta}) \cap B_{2\varepsilon_0}(z_{\alpha\gamma}) = \emptyset, \tag{6}$$

$$0 < \varepsilon < \varepsilon_0 \qquad \Longrightarrow \qquad \sum_{\beta \in T,\, \alpha E\beta} E(u_\alpha, B_{2\varepsilon}(z_{\alpha\beta})) < \frac{\hbar}{2}. \tag{7}$$

Lemma B. *Let $\mathbf{x} \in \mathcal{SC}_{0,n}(M,A;J,T)$ and $0 < \varepsilon < \varepsilon_0(\mathbf{x})/2$. Then there exist constants $0 < \kappa < \varepsilon/2$ and $K \geq 1$ such that the following holds for every stable map $\mathbf{y} \in \mathcal{SC}_{0,n}(M,A;J,T)$ modelled over the same tree.*

(i) *If $d_{C^0}(\mathbf{x},\mathbf{y}) < \kappa$, then $\varepsilon < \varepsilon_0(\mathbf{y})$.*

(ii) *If $d_{C^0}(\mathbf{x},\mathbf{y}) < \kappa$ and $0 < \varepsilon' \leq \varepsilon$, then*

$$\rho_{\varepsilon,\varepsilon'}(\mathbf{x},[\mathbf{y}]) \leq K d_{C^0}(\mathbf{x},\mathbf{y}), \qquad \rho_{\varepsilon,\varepsilon'}(\mathbf{y},[\mathbf{x}]) \leq K d_{C^0}(\mathbf{x},\mathbf{y}).$$

(iii) *If $d_{C^0}(\mathbf{x},\mathbf{y}) < \kappa$ and $\mathbf{x}' \in \mathcal{SC}_{0,n}(M,A;J)$ is a stable map satisfying the inequality $\rho_{\varepsilon,\varepsilon'}(\mathbf{y},[\mathbf{x}']) < \kappa$ with $\varepsilon' := \varepsilon/2$, then*

$$\rho_\varepsilon(\mathbf{x},[\mathbf{x}']) \leq K d_{C^0}(\mathbf{x},\mathbf{y}) + \rho_{\varepsilon,\varepsilon'}(\mathbf{y},[\mathbf{x}']). \tag{8}$$

Proof of Theorem A, assuming Lemma B. Pick a countable collection $\mathcal{T}$ of labelled trees $T = (T,\Lambda,\{A_\alpha\})$, one for each isomorphism class, and for each $T \in \mathcal{T}$ choose a dense sequence $\{\mathbf{x}_i^T\}_{i\in\mathbb{N}}$ in the space $\mathcal{SC}_{0,n}(M,A;J,T)$. We prove that the sets

$$\mathcal{N}_{\varepsilon,\varepsilon',\varepsilon''}(\mathbf{x}_i^T) := \left\{[\mathbf{x}'] \in \overline{\mathcal{M}}_{0,n}(M,A;J) \,|\, \rho_{\varepsilon,\varepsilon'}(\mathbf{x}_i^T,[\mathbf{x}']) < \varepsilon''\right\},$$

with ε,ε' positive rational numbers satisfying $\varepsilon' < \varepsilon < \varepsilon_0(\mathbf{x}_i^T)/2$, form a basis of the topology of $\overline{\mathcal{M}}_{0,n}(M,A;J)$. To see this, choose $\mathbf{x} \in \mathcal{SC}_{0,n}(M,A;J)$ and let $\mathcal{U} \subset \overline{\mathcal{M}}_{0,n}(M,A;J)$ be an open set containing the class $[\mathbf{x}]$. Assume that $\mathbf{x}$ is modelled over a labelled tree $T \in \mathcal{T}$ and fix a rational number $0 < \varepsilon < \varepsilon_0(\mathbf{x})/2$. Then, by Lemma 5.5.8 and Theorem 5.6.6, there exists a constant $\delta > 0$ such that

$$\mathcal{V} := \left\{[\mathbf{x}'] \in \overline{\mathcal{M}}_{0,n}(M,A;J) \,|\, \rho_\varepsilon(\mathbf{x},[\mathbf{x}']) < \delta\right\} \subset \mathcal{U}.$$

Let $0 < \kappa < \varepsilon/2$ and $K \geq 1$ be as in Lemma B. Choose a rational number ε'' such that $0 < \varepsilon'' < \min\{\delta/2, \kappa\}$ and choose i with $d_{C^0}(\mathbf{x}, \mathbf{x}_i^T) < \varepsilon''/K < \kappa$. Then $\varepsilon < \varepsilon_0(\mathbf{x}_i^T)$ by part (i) of Lemma B. Now define $\varepsilon' := \varepsilon/2$ so that part (iii) of Lemma B holds with $\mathbf{y} = \mathbf{x}_i^T$. We claim that

$$[\mathbf{x}] \in \mathcal{N}_{\varepsilon,\varepsilon',\varepsilon''}(\mathbf{x}_i^T) \subset \mathcal{U}. \tag{9}$$

To see this, note first that $\rho_{\varepsilon,\varepsilon'}(\mathbf{x}_i^T, [\mathbf{x}]) \leq K d_{C^0}(\mathbf{x}, \mathbf{x}_i^T) < \varepsilon''$ by part (ii) of Lemma B, and so $[\mathbf{x}] \in \mathcal{N}_{\varepsilon,\varepsilon',\varepsilon''}(\mathbf{x}_i^T)$. Now let $[\mathbf{x}'] \in \mathcal{N}_{\varepsilon,\varepsilon',\varepsilon''}(\mathbf{x}_i^T)$. Then we have $\rho_{\varepsilon,\varepsilon'}(\mathbf{x}_i^T, [\mathbf{x}']) < \varepsilon'' < \kappa$ and hence, by part (iii) of Lemma B,

$$\rho_\varepsilon(\mathbf{x}, [\mathbf{x}']) \leq K d_{C^0}(\mathbf{x}, \mathbf{x}_i^T) + \rho_{\varepsilon,\varepsilon'}(\mathbf{x}_i^T, [\mathbf{x}']) < \varepsilon'' + \varepsilon'' < \delta.$$

Thus $[\mathbf{x}'] \in \mathcal{V} \subset \mathcal{U}$. This proves (9) and Theorem A. □

It remains to prove Lemma B, and this proof in turn will be based on the following energy estimate for pairs of J-holomorphic curves that are sufficiently close in C^0-distance.

Lemma C. *Let (Σ, j) be a closed Riemann surface and let $u : \Sigma \to M$ be a J-holomorphic curve. Then there exist constants $\gamma > 0$ and $c > 0$ with the following significance. If $v : \Sigma \to M$ is a J-holomorphic curve satisfying*

$$\sup_{z\in\Sigma} d_M(u(z), v(z)) < \gamma,$$

then

$$\big|E(u;U) - E(v;U)\big| \leq c \sup_{z\in\Sigma} d_M(u(z), v(z)) \tag{10}$$

for every open set $U \subset \Sigma$.

Proof. By combining the elliptic estimate in Lemma C.2.1 (p 586) with the estimate in Proposition 3.5.3 (p 70), we find that the C^0-distance of u and v controls their $W^{1,p}$-distance. More precisely, fix a volume form $\mathrm{dvol}_\Sigma \in \Omega^2(\Sigma)$ and a constant $p > 2$, assume that $\gamma > 0$ is smaller than the injectivity radius of M, write $v = \exp_u(\xi)$ for $\xi \in \Omega^0(\Sigma, u^*TM)$ with $\|\xi\|_{L^\infty} < \gamma$, and denote by $\mathcal{F}_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$ the map in (3.1.3) on page 41. Then $\mathcal{F}_u(0) = \mathcal{F}_u(\xi) = 0$ and so, by Lemma C.2.1 and Proposition 3.5.3,

$$\begin{aligned}
\|\xi\|_{W^{1,p}} &\leq c_1 \left(\|D_u\xi\|_{L^p} + \|\xi\|_{L^p}\right) \\
&\leq c_1 \left(\sup_{0\leq t\leq 1} \|D_u\xi - d\mathcal{F}_u(t\xi)\xi\|_{L^p} + \|\xi^\nu_\alpha\|_{L^p}\right) \\
&\leq c_0 c_1 \|\xi\|_{L^\infty} \|\xi\|_{W^{1,p}} + c_1 \|\xi\|_{L^p}.
\end{aligned}$$

With $\|\xi\|_{L^\infty} \le 1/2c_0c_1$ this gives $\|\xi\|_{W^{1,p}} \le 2c_1 \|\xi\|_{L^p}$. Hence

$$\begin{aligned}\Big|E(v;U) - E(u;U)\Big| &= \left|\tfrac{1}{2}\int_U \Big(|d\exp_u(\xi)|^2 - |du|^2\Big)\mathrm{dvol}_\Sigma\right| \\ &\le c_2 \int_U \Big(\big(|\nabla\xi| + |\xi|\big)^2 + |\nabla\xi| + |\xi|\Big)\mathrm{dvol}_\Sigma \\ &\le c_2c_3\left(\|\xi\|_{W^{1,p}}^2 + \|\xi\|_{W^{1,p}}\right) \\ &\le c_2c_3\left(4c_1^2\|\xi\|_{L^p}^2 + 2c_1\|\xi\|_{L^p}\right) \\ &\le c_4\|\xi\|_{L^\infty} \\ &= c_4 \sup_{z\in\Sigma} d_M(u(z), v(z))\end{aligned}$$

for every open set $U \subset \Sigma$. This proves Lemma C. □

Proof of Lemma B. Fix a stable map $\mathbf{x} \in \mathcal{SC}_{0,n}(M, A; J, T)$, modelled over a tree $T = (T, \Lambda, \{A_\alpha\})$, and a constant $0 < \varepsilon < \varepsilon_0(\mathbf{x})/2$.

We prove part (i). Assume, by contradiction, that (i) does not hold. Then there exists a sequence of stable maps $\mathbf{x}^\nu \in \mathcal{SC}_{0,n}(M, A; J, T)$, modelled over the same tree $T = (T, \Lambda, \{A_\alpha\})$, such that

$$\lim_{\nu\to\infty} d_{C^0}(\mathbf{x}, \mathbf{x}^\nu) = 0, \qquad \sup_{\nu\in\mathbb{N}} \varepsilon_0(\mathbf{x}^\nu) \le \varepsilon. \tag{11}$$

By definition of d_{C^0} in (5) this implies that the sequence $(u_\alpha^\nu)_{\nu\in\mathbb{N}}$ of J-holomorphic spheres converges to u_α in the C^0 topology, and hence in the C^∞ topology, for every $\alpha \in T$. It follows also from the definition of d_{C^0} that

$$\lim_{\nu\to\infty} d_{S^2}(z_{\alpha\beta}, z_{\alpha\beta}^\nu) = 0$$

for all $\alpha, \beta \in T$ with $\alpha E\beta$. Choose ν_0 so large that $d_{S^2}(z_{\alpha\beta}, z_{\alpha\beta}^\nu) < \varepsilon$ for all $\alpha, \beta \in T$ with $\alpha E\beta$ and all $\nu \ge \nu_0$. Then, for all $\nu \ge \nu_0$ and all $\alpha, \beta, \gamma \in T$ with $\alpha E\beta$, $\alpha E\gamma$, and $\beta \ne \gamma$, it follows from the inequality $0 < \varepsilon < \varepsilon_0(\mathbf{x})/2$ and (6) that

$$\begin{aligned}d(z_{\alpha\beta}^\nu, z_{\alpha\gamma}^\nu) &\ge d(z_{\alpha\beta}, z_{\alpha\gamma}) - d(z_{\alpha\beta}, z_{\alpha\beta}^\nu) - d(z_{\alpha\gamma}, z_{\alpha\gamma}^\nu) \\ &> 4\varepsilon_0(\mathbf{x}) - 2\varepsilon \\ &> 6\varepsilon,\end{aligned}$$

and therefore

$$B_{3\varepsilon}(z_{\alpha\beta}^\nu) \cap B_{3\varepsilon}(z_{\alpha\gamma}^\nu) = \emptyset.$$

Moreover, $B_{3\varepsilon}(z^\nu_{\alpha\beta}) \subset B_{4\varepsilon}(z_{\alpha\beta})$ for all $\alpha, \beta \in T$ with $\alpha E \beta$ and all $\nu \geq \nu_0$. Hence, by (7) with $2\varepsilon < \varepsilon_0(\mathbf{x})$, we have

$$\begin{aligned}
\lim_{\nu\to\infty} \sum_{\beta\in T,\, \alpha E\beta} E(u^\nu_\alpha, B_{3\varepsilon}(z^\nu_{\alpha\beta})) &\leq \lim_{\nu\to\infty} \sum_{\beta\in T,\, \alpha E\beta} E(u^\nu_\alpha, B_{4\varepsilon}(z_{\alpha\beta})) \\
&= \sum_{\beta\in T,\, \alpha E\beta} E(u_\alpha, B_{4\varepsilon}(z_{\alpha\beta})) \\
&< \frac{\hbar}{2}.
\end{aligned}$$

for all $\alpha \in T$. Thus $\varepsilon_0(\mathbf{x}^\nu) \geq 3\varepsilon/2$ for ν sufficiently large, in contradiction to (11). This proves part (i).

We prove part (ii). Assume, by contradiction, that (ii) does not hold. Then there exists a sequence of stable maps $\mathbf{x}^\nu \in \mathcal{SC}_{0,n}(M, A; J, T)$, modelled over the same tree $T = (T, \Lambda, \{A_\alpha\})$, and a sequence of real numbers $0 < \varepsilon^\nu < \varepsilon$ such that

$$\begin{aligned}
&\lim_{\nu\to\infty} d_{C^0}(\mathbf{x}, \mathbf{x}^\nu) = 0, \\
&\max\{\rho_{\varepsilon,\varepsilon^\nu}(\mathbf{x}, [\mathbf{x}^\nu])\}, \rho_{\varepsilon,\varepsilon^\nu}(\mathbf{x}^\nu, [\mathbf{x}])\} > \nu\, d_{C^0}(\mathbf{x}, \mathbf{x}^\nu)
\end{aligned} \tag{12}$$

for all $\nu \in \mathbb{N}$. With $f = \mathrm{id} : T \to T$ and $\phi_\alpha = \mathrm{id} \in \mathrm{G}$ for all $\alpha \in T$, it follows from the definitions in (4) and (5) that

$$\begin{aligned}
&\rho^3_{f,\phi,\varepsilon^\nu}(\mathbf{x}, \mathbf{x}^\nu) = \rho^3_{f,\phi,\varepsilon^\nu}(\mathbf{x}^\nu, \mathbf{x}) = 0, \\
&\rho^2_{f,\phi,\varepsilon^\nu}(\mathbf{x}, \mathbf{x}^\nu) + \sum_{i=4,5} \rho^i_{f,\phi}(\mathbf{x}, \mathbf{x}^\nu) \leq d_{C^0}(\mathbf{x}, \mathbf{x}^\nu), \\
&\rho^2_{f,\phi,\varepsilon^\nu}(\mathbf{x}^\nu, \mathbf{x}) + \sum_{i=4,5} \rho^i_{f,\phi}(\mathbf{x}^\nu, \mathbf{x}) \leq d_{C^0}(\mathbf{x}, \mathbf{x}^\nu).
\end{aligned}$$

Next we observe that, again for $f = \mathrm{id}$ and $\phi = \mathrm{id}$, we have

$$\begin{aligned}
\rho^1_{f,\phi,\varepsilon}(\mathbf{x}^\nu, \mathbf{x}) &= \sum_{\alpha E\beta} \bigl| E(u_\alpha, B_{2\varepsilon}(z^\nu_{\alpha\beta})) - E(u^\nu_\alpha, B_{2\varepsilon}(z^\nu_{\alpha\beta})) \bigr|, \\
\rho^1_{f,\phi,\varepsilon}(\mathbf{x}, \mathbf{x}^\nu) &= \sum_{\alpha E\beta} \bigl| E(u_\alpha, B_{2\varepsilon}(z_{\alpha\beta})) - E(u^\nu_\alpha, B_{2\varepsilon}(z_{\alpha\beta})) \bigr|.
\end{aligned}$$

Hence it follows from Lemma C that there exists a constant $c > 0$ such that

$$\rho^1_{f,\phi,\varepsilon}(\mathbf{x}^\nu, \mathbf{x}) \leq c\, d_{C^0}(\mathbf{x}, \mathbf{x}^\nu), \qquad \rho^1_{f,\phi,\varepsilon}(\mathbf{x}, \mathbf{x}^\nu) \leq c\, d_{C^0}(\mathbf{x}, \mathbf{x}^\nu)$$

for ν sufficiently large. This contradicts (12) for ν sufficiently large and completes the proof of part (ii).

We prove part (iii) in two steps. We show first that a stable map $\mathbf{x}'$ (modelled on T') that is $\rho_{\varepsilon,\varepsilon'}$ close to $\mathbf{y}$ (modelled on T), which in turn is d_0-close to $\mathbf{x}$ (modelled on T), has bounded derivative away from the bubble points of $\mathbf{x}$.

Step 1. *There exist constants $\kappa > 0$ and $K \geq 1$ such that the following holds. Assume $d_{C^0}(\mathbf{x}, \mathbf{y}) < \kappa$, $\mathbf{x}' = (\mathbf{u}', \mathbf{z}') \in \mathcal{SC}_{0,n}(M, A; J, T')$ is a stable map modelled over a labelled tree T', $f : T \to T'$ is a surjective tree homomorphism, and $\phi = \{\phi_\alpha\} \in \mathrm{G}_T$ such that $\rho_{\varepsilon,\varepsilon'}(\mathbf{y}, \mathbf{x}'; f, \phi) < \kappa$ with $\varepsilon' := \varepsilon/2$. Then*

$$\sup_{\alpha \in T} \sup_{S^2 \setminus B_\varepsilon(Z_\alpha(\mathbf{x}))} \left| d(u'_{f(\alpha)} \circ \phi_\alpha) \right| \leq K,$$

where $Z_\alpha(\mathbf{x}) := \{z_{\alpha\beta}(\mathbf{x}) \,|\, \beta \in T, \, \alpha E \beta\}$.

Suppose, by contradiction, that this does not hold. Then there exists a sequence of stable maps $\mathbf{y}^\nu = (\mathbf{v}^\nu, \mathbf{z}(\mathbf{y}^\nu))$ modelled over T, a sequence of stable maps $\mathbf{x}^\nu = (\mathbf{u}^\nu, \mathbf{z}^\nu)$ modelled over a sequence of trees T^ν, a sequence of surjective tree homomorphisms $f^\nu : T \to T^\nu$, and a sequence $\phi^\nu = \{\phi^\nu_\alpha\} \in \mathrm{G}_T$ such that

$$\lim_{\nu \to \infty} d_{C^0}(\mathbf{x}, \mathbf{y}^\nu) = 0, \qquad \lim_{\nu \to \infty} \rho_\varepsilon(\mathbf{y}^\nu, \mathbf{x}^\nu; f^\nu, \phi^\nu) = 0, \tag{13}$$

$$\sup_{S^2 \setminus B_\varepsilon(Z_\alpha(\mathbf{x}))} \left| d(u^\nu_{f^\nu(\alpha)} \circ \phi^\nu_\alpha) \right| > \nu. \tag{14}$$

By (13) we have

$$\lim_{\nu \to \infty} \left(\sup_{\alpha \in T} \sup_{z \in S^2} d_M\big(v^\nu_\alpha(z), u_\alpha(z)\big) + \sum_{\alpha E \beta} d_{S^2}\big(z_{\alpha\beta}(\mathbf{y}^\nu), z_{\alpha\beta}(\mathbf{x})\big) \right) = 0,$$

$$\lim_{\nu \to \infty} \sum_{\alpha \in T} \sup_{z \in S^2 \setminus B_{\varepsilon/2}(Z_\alpha(\mathbf{y}^\nu))} d_M\big(u^\nu_{f^\nu(\alpha)} \circ \phi^\nu_\alpha(z), v^\nu_\alpha(z)\big) = 0.$$

This implies $B_{\varepsilon/2}(Z_\alpha(\mathbf{y}^\nu)) \subset B_{3\varepsilon/4}(Z_\alpha(\mathbf{x}))$ for ν sufficiently large and hence

$$\lim_{\nu \to \infty} \sum_{\alpha \in T} \sup_{z \in S^2 \setminus B_{3\varepsilon/4}(Z_\alpha(\mathbf{x}))} d_M\big(u^\nu_{f^\nu(\alpha)} \circ \phi^\nu_\alpha(z), u_\alpha(z)\big) = 0.$$

Thus, for every $\alpha \in T$, the sequence $u^\nu_{f^\nu(\alpha)} \circ \phi^\nu_\alpha$ converges to u_α uniformly on the open set $S^2 \setminus \overline{\mathrm{B}}_{3\varepsilon/4}(Z_\alpha(\mathbf{x}))$ and therefore, by the standard bubbling argument, its first derivatives are uniformly bounded on the compact subset $S^2 \setminus B_\varepsilon(Z_\alpha(\mathbf{x}))$. This contradicts (14) and proves Step 1.

Step 2. *We prove part (iii).*

Denote by $e(T)$ the number of edges of T and by $v(T) := \#T$ the number of vertices. Choose $\gamma > 0$ and $c > 0$ such that the assertion of Lemma C holds with $u = u_\alpha$ for each vertex $\alpha \in T$ and choose constants $\kappa > 0$ and $K \geq 1$ such that parts (i) and (ii) of Lemma B are satisfied and the assertion of Step 1 holds. Shrinking κ if necessary, we may also assume that

$$\kappa < \varepsilon/2, \qquad \max_{\alpha \in T}\big(\|du_\alpha\|_{L^\infty} + 1\big)\kappa < \gamma. \tag{15}$$

Define $\varepsilon' := \varepsilon/2$ and let $\mathbf{y} = (\mathbf{v}, \mathbf{z}(\mathbf{y})) \in \mathcal{SC}_{0,n}(M, A; J, T)$ with $d_{C^0}(\mathbf{x}, \mathbf{y}) < \kappa$.

We will prove that there exist constants $K_i > 0$ for $i = 1, 2, 3, 4, 5$ such that the following holds. *If $\mathbf{x}' = (\mathbf{u}', \mathbf{z}') \in \mathcal{SC}_{0,n}(M, A; J, T')$ is a stable map modelled over a labelled tree T', $f : T \to T'$ is a surjective tree homomorphism, and $\phi \in \mathrm{G}_T$ such that*

$$\rho_{\varepsilon,\varepsilon'}(\mathbf{y}, \mathbf{x}'; f, \phi) < \kappa,$$

then

$$\begin{aligned}
\rho^1_{f,\phi,\varepsilon}(\mathbf{x}, \mathbf{x}') &\leq K_1\, d_{C_0}(\mathbf{x}, \mathbf{y}) + \rho^1_{f,\phi,\varepsilon}(\mathbf{y}, \mathbf{x}'), \\
\rho^i_{f,\phi,\varepsilon}(\mathbf{x}, \mathbf{x}') &\leq K_i\, d_{C_0}(\mathbf{x}, \mathbf{y}) + \rho^i_{f,\phi,\varepsilon'}(\mathbf{y}, \mathbf{x}'), \qquad \textit{for } i = 2, 3, \\
\rho^i_{f,\phi}(\mathbf{x}, \mathbf{x}') &\leq K_i\, d_{C_0}(\mathbf{x}, \mathbf{y}) + \rho^i_{f,\phi}(\mathbf{y}, \mathbf{x}'), \qquad \textit{for } i = 4, 5.
\end{aligned} \tag{16}$$

Once this has been established, take the sum of the inequalities in (16) and then take the infimum over all f and ϕ satisfying $\rho_{\varepsilon,\varepsilon'}(\mathbf{y}, \mathbf{x}'; f, \phi) < \kappa$ to obtain the estimate (8) for every stable map $\mathbf{x}' \in \mathcal{SC}_{0,n}(M, A; J, T')$ that satisfies $\rho_{\varepsilon,\varepsilon'}(\mathbf{y}, [\mathbf{x}']) < \kappa$. Thus it remains to prove (16).

For each edge $\alpha E \beta$ choose an isometry $\iota_{\alpha\beta} : S^2 \to S^2$ such that

$$\iota_{\alpha\beta}(z^{\mathbf{x}}_{\alpha\beta}) = z^{\mathbf{y}}_{\alpha\beta}, \qquad \sup_{z \in S^2} d_{S^2}(z, \iota_{\alpha\beta}(z)) \leq d(z^{\mathbf{x}}_{\alpha\beta}, z^{\mathbf{y}}_{\alpha\beta}) < \kappa.$$

Here we shorten the notation and denote the tuples $\mathbf{z}(\mathbf{x}), \mathbf{z}(\mathbf{y})$ respectively by $(z^{\mathbf{x}}_{\alpha\beta})$, $(z^{\mathbf{y}}_{\alpha\beta})$. Then $\iota_{\alpha\beta}(B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta})) = B_{2\varepsilon}(z^{\mathbf{y}}_{\alpha\beta})$, and by (15)

$$\sup_{z \in S^2} d_M(u_\alpha(z), v_\alpha \circ \iota_{\alpha\beta}(z)) < \big(\|du_\alpha\|_{L^\infty} + 1\big)\kappa < \gamma$$

for every edge $\alpha E \beta$.

We now deduce from Lemma C that

$$
\begin{aligned}
\rho^1_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}') &= \sum_{\alpha E\beta} \left| E_\alpha\big(\mathbf{u}; B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta})\big) - E_{f(\alpha)}\big(\mathbf{u}'; \phi_\alpha(B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta}))\big) \right| \\
&\le \sum_{\alpha E\beta} \left| E_\alpha\big(\mathbf{u}; B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta})\big) - E_\alpha\big(\mathbf{v}; B_{2\varepsilon}(z^{\mathbf{y}}_{\alpha\beta})\big) \right| \\
&\quad + \sum_{\alpha E\beta} \left| E_\alpha\big(\mathbf{v}; B_{2\varepsilon}(z^{\mathbf{y}}_{\alpha\beta})\big) - E_{f(\alpha)}\big(\mathbf{u}'; \phi_\alpha(B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta}))\big) \right| \\
&\le \sum_{\alpha E\beta} \left| E\big(u_\alpha; B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta})\big) - E\big(v_\alpha \circ \iota_{\alpha\beta}; B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta})\big) \right| + \rho^1_{f,\phi,\varepsilon}(\mathbf{y},\mathbf{x}') \\
&\quad + \sum_{\alpha E\beta} \left| E_{f(\alpha)}\big(\mathbf{u}'; \phi_\alpha(B_{2\varepsilon}(z^{\mathbf{y}}_{\alpha\beta}))\big) - E_{f(\alpha)}\big(\mathbf{u}'; \phi_\alpha(B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta}))\big) \right| \\
&\le e(T)(c + 8\pi\varepsilon K^2)\, d_{C_0}(\mathbf{x},\mathbf{y}) + \rho^1_{f,\phi,\varepsilon}(\mathbf{y},\mathbf{x}').
\end{aligned}
$$

Here the last inequality uses the fact that both the sets $B_{2\varepsilon}(z^{\mathbf{y}}_{\alpha\beta}) \setminus B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta})$ and $B_{2\varepsilon}(z^{\mathbf{x}}_{\alpha\beta}) \setminus B_{2\varepsilon}(z^{\mathbf{y}}_{\alpha\beta})$ are contained in the annulus

$$2\varepsilon - d_{S^2}(z^{\mathbf{x}}_{\alpha\beta}, z^{\mathbf{y}}_{\alpha\beta}) \le d_{S^2}(z, z^{\mathbf{x}}_{\alpha\beta}) \le 2\varepsilon + d_{S^2}(z^{\mathbf{x}}_{\alpha\beta}, z^{\mathbf{y}}_{\alpha\beta})$$

whose area is bounded above by $8\pi\varepsilon\, d_{C^0}(\mathbf{x},\mathbf{y})$ and which is contained in the set $S^2 \setminus B_\varepsilon(Z_\alpha(\mathbf{x}))$, on which the first derivative of $u'_{f(\alpha)} \circ \phi_\alpha$ is bounded above by K (see Step 1). This proves (16) for $i = 1$.

Next we prove (16) for $i = 2, 3$. Observe that

$$d_{S^2}(z^{\mathbf{x}}_{\alpha\beta}, z^{\mathbf{y}}_{\alpha\beta}) \le d_{C_0}(\mathbf{x},\mathbf{y}) < \kappa < \varepsilon/2$$

for all $\alpha, \beta \in T$ with $\alpha \ne \beta$. Hence, because we assume $\varepsilon' = \varepsilon/2$, we have $B_{\varepsilon'}(z^{\mathbf{y}}_{\alpha\beta}) \subset B_\varepsilon(z^{\mathbf{x}}_{\alpha\beta})$ so that $S^2 \setminus B_\varepsilon(Z_\alpha(\mathbf{x})) \subset S^2 \setminus B_{\varepsilon'}(Z_\alpha(\mathbf{y}))$, for all $\alpha \in T$. Thus it follows from (4) that

$$
\begin{aligned}
\rho^2_{f,\phi,\varepsilon}(\mathbf{x},\mathbf{x}') &= \sum_{\alpha\in T} \sup_{z \in S^2 \setminus B_\varepsilon(Z_\alpha(\mathbf{x}))} d_M\big(u_\alpha(z), u'_{f(\alpha)} \circ \phi_\alpha(z)\big) \\
&\le \sum_{\alpha\in T} \sup_{z \in S^2 \setminus B_{\varepsilon'}(Z_\alpha(\mathbf{y}))} d_M\big(u_\alpha(z), u'_{f(\alpha)} \circ \phi_\alpha(z)\big) \\
&\le \sum_{\alpha\in T} \sup_{z \in S^2} d_M(u_\alpha(z), v_\alpha(z)) \\
&\quad + \sum_{\alpha\in T} \sup_{z \in S^2 \setminus B_{\varepsilon'}(Z_\alpha(\mathbf{y}))} d_M\big(v_\alpha(z), u'_{f(\alpha)} \circ \phi_\alpha(z)\big) \\
&\le v(T)\, d_{C_0}(\mathbf{x},\mathbf{y}) + \rho^2_{f,\phi,\varepsilon'}(\mathbf{y},\mathbf{x}').
\end{aligned}
$$

This proves (16) for $i = 2$. Since $B_{\varepsilon'}(z^{\mathbf{y}}_{\alpha\beta}) \subset B_\varepsilon(z^{\mathbf{x}}_{\alpha\beta})$ as noted above, we have $S^2 \setminus B_\varepsilon(z^{\mathbf{x}}_{\alpha\beta}) \subset S^2 \setminus B_{\varepsilon'}(z^{\mathbf{y}}_{\alpha\beta})$ for every edge $\alpha E\beta$, and hence

$$\begin{aligned}
\rho^3_{f,\phi,\varepsilon}(\mathbf{x}, \mathbf{x}') &= \sum_{\substack{\alpha\neq\beta \\ f(\alpha)=f(\beta)}} \sup_{z\in S^2\setminus B_\varepsilon(z^{\mathbf{x}}_{\alpha\beta})} d_{S^2}\big(z^{\mathbf{x}}_{\beta\alpha}, \phi_\beta^{-1}\circ\phi_\alpha(z)\big) \\
&\le \sum_{\substack{\alpha\neq\beta \\ f(\alpha)=f(\beta)}} \sup_{z\in S^2\setminus B_{\varepsilon'}(z^{\mathbf{y}}_{\alpha\beta})} d_{S^2}\big(z^{\mathbf{x}}_{\beta\alpha}, \phi_\beta^{-1}\circ\phi_\alpha(z)\big) \\
&\le \sum_{\substack{\alpha\neq\beta \\ f(\alpha)=f(\beta)}} \left(d_{S^2}(z^{\mathbf{x}}_{\beta\alpha}, z^{\mathbf{y}}_{\beta\alpha}) + \sup_{z\in S^2\setminus B_{\varepsilon'}(z^{\mathbf{y}}_{\alpha\beta})} d_{S^2}\big(z^{\mathbf{y}}_{\beta\alpha}, \phi_\beta^{-1}\circ\phi_\alpha(z)\big)\right) \\
&\le d_{C_0}(\mathbf{x}, \mathbf{y}) + \rho^3_{f,\phi,\varepsilon'}(\mathbf{y}, \mathbf{x}').
\end{aligned}$$

Here the last inequality follows from (4) and (5). Thus we have proved that (16) holds for $i = 1, 2, 3$. For $i = 4, 5$ and $K_i = 1$ the estimate follows directly from the triangle inequality and the definitions in (4) and (5). This proves (16), part (iii), and Lemma B. □

page 161, line 16: Replace π^E by ev^E.

page 162, line 20: The set Δ^E is always a submanifold of M^E.

page 165, line 13: Replace π^E by ev^E.

page 171: In Exercise 6.4.10 a *symplectic embedding* is understood as an embedding whose image is a symplectic submanifold. The resulting J-holomorphic sphere in this example has Chern number $2 - 2m$ (and not $2 - m$).

page 246: The proof of Lemma 7.5.5 contains a mistake. The element $\mathbf{w}_I \in \overline{\mathrm{M}}_{0,I}$ defined by (7.5.1) is not a regular value of the projection

$$\pi_{k,I} : \overline{\mathrm{M}}_{0,k} \to \overline{\mathrm{M}}_{0,I}, \tag{17}$$

and so $Y_{k,I} := \pi_{k,I}^{-1}(\mathbf{w}_I)$ is not a submanifold of $\overline{\mathrm{M}}_{0,k}$. Moreover, even if $\mathbf{w}_I$ is chosen as a regular value of the projection (17), and if $k \in I$ and $\#I \ge 4$, then, while $Y_{k,I}$ and $Y_{k-1,I\setminus\{k\}}$ have the same dimension $2(k - \#I)$, the projection

$$\pi_{0,k} : Y_{k,I} \to Y_{k-1,I\setminus\{k\}} \tag{18}$$

(which forgets the kth marked point) is not necessarily a holomorphic diffeomorphism; it may collapse certain submanifolds to points. Nevertheless, Lemma 7.5.5 is correct and the proof can be fixed as follows.

First, the class $\beta_{k,I}$ can be represented by any fibre of the projection (17), whether or not it is the preimage of a regular value. The fibres are all connected and a point $\mathbf{w}_I \in \overline{\mathrm{M}}_{0,I}$ is a regular value of (17) if and only if it belongs to the top stratum.

Second, if $\mathbf{w}_I$ is a regular value of the projection (17) and $k \in I$, then the point $\mathbf{w}_{I \setminus \{k\}} \in \overline{\mathrm{M}}_{0,I\setminus\{k\}}$ (obtained by deleting all the crossratios involving the index k) is a regular value of the projection $\pi_{k-1,I\setminus\{k\}} : \overline{\mathrm{M}}_{0,k-1} \to \overline{\mathrm{M}}_{0,I\setminus\{k\}}$ and the two preimages $Y_{k,I}$ and $Y_{k-1,I\setminus\{k\}}$ both have have dimension $2(k - \#I)$.

Third, the restricted projecton (18) will typically be a kind of blow-up map, collapsing some submanifolds. However, it has degree one and hence maps the fundamental class of $Y_{k,I}$ to that of $Y_{k-1,I\setminus\{k\}}$. So the forgetful map

$$\pi_{0,k} : \overline{\mathrm{M}}_{0,k} \to \overline{\mathrm{M}}_{0,k-1}$$

sends the homology class $\beta_{k,I}$ represented by the fundamental class of $Y_{k,I}$ to the class $\beta_{k-1,I\setminus\{k\}}$ represented by the fundamental class of $Y_{k-1,I\setminus\{k\}}$. This proves part (ii) of Lemma 7.5.5 in the case $k \in I$.

page 342, line -8: To use Theorem 9.4.7 we must prove that $\widetilde{\mathrm{M}}$ is minimal.

page 343, line -12: To use Theorem 9.4.2 we must prove that $\widetilde{\mathrm{M}}$ is minimal.

page 368, line 15/16: Eliashberg–Mishachev.

page 534–546: The discussion of determinant bundles needs rewriting to correct signs [3].

page 584, line -9: In equation (C.1.8) replace $\Omega^{1,1}(\Sigma, E^*)$ by $\Omega^{1,1}(\Sigma)$.

page 637, line -12: Replace the first displayed equation in the second paragraph by the equation

$$w_{m,m+1,n,i} = \frac{w_{1,m,m+1,n} - 1}{w_{1,m,m+1,n} - w_{1,m,m+1,i}}.$$

This holds for $1 < i < m$ and for all w near w^0 by (D.4.3). To see this, one must verify that $(1, \infty, w_{1,m,m+1,n}, w_{1,m,m+1,i}) \notin \Delta_3$ at the relevant points. Indeed, $w_{1,m,m+1,n}(\mathbf{z}^0) = \infty$ by assumption and $w_{1,m,m+1,i}(\mathbf{z}^0) \neq \infty$, because the points $z_{\alpha 1}, z_{\alpha m}, z_{\alpha m+1}$ are pairwise distinct and $z_{\alpha i} \neq z_{\alpha m+1}$ when $i \leq m$.

page 644, line 5: In Exercise D.6.2 the set $\mathcal{M}_{0,n+1}$ is contained in but is not equal to the set of regular points of the projection $\pi : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ which forgets the $(n+1)$st marked point. The equivalence class of a tuple

$$\mathbf{z} = \left(\{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le n+1}\right) \in \mathcal{SC}_{0,n+1}$$

is a singular point of π if and only if $n_{\alpha_{n+1}} = 3$ and $\Lambda_{\alpha_{n+1}} = \{n+1\}$. **Exercise:** Characterize this condition in terms of the corresponding tuple $\{w_{ijk\ell}\}_{1 \le i,j,k,\ell \le n+1} := \mathbf{w}(\mathbf{z}) \in \overline{\mathrm{M}}_{0,n+1}$ of crossratios.

References

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[2] Dietmar A. Salamon, *Funktionentheorie*. Birkhäuser, 2012.
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[3] Dietmar A. Salamon, Notes on the universal determinant bundle. Preprint, 2013.
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