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AN APPROXIMATION THEOREM FOR THE ALGEBRAIC RICCATI EQUATION*

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Abstract. For an infinite-dimensional linear quadratic control problem in Hilbert space, approximation of the solution of the algebraic Riccati operator equation in the strong operator topology is considered under conditions weaker than uniform exponential stability of the approximating systems. As an application, strong convergence of the approximating Riccati operators in case of a previously developed spline approximation scheme for delay systems is established. Finally, convergence of the transfer-functions of the approximating systems is investigated.

Key words. linear quadratic control problem in Hilbert space, algebraic Riccati equation, hereditary control systems, spline approximation

AMS(MOS) subject classifications. 34K35, 41A15, 93D15

1. Introduction and hypotheses.

Let H, U, and Y be Hilbert spaces, and consider the linear system

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = \varphi \in H, \qquad y(t) = Cz(t), \tag{1.1}$$

where $A : \operatorname{dom} A \to H$ is the infinitesimal generator of a strongly continuous semigroup $S(t) \in \mathscr{L}(H)$, and $B \in \mathscr{L}(U, H)$, $C \in \mathscr{L}(H, Y)$ are bounded linear operators. Associated with (1.1) we consider the *algebraic Riccati equation*

$$\langle A\psi, P\varphi\rangle + \langle P\psi, A\varphi\rangle - \langle B^*P\psi, B^*P\varphi\rangle + \langle C\psi, C\varphi\rangle = 0 \tag{1.2}$$

for $\varphi, \psi \in \operatorname{dom} A$. This equation has a nonnegative operator solution $P = P^* \in \mathscr{L}(H)$ if and only if for every $\varphi \in H$ there exists a control function $u \in L^2(0, \infty; U)$ such that the integral

$$J(u) = J(u, \varphi) = \int_0^\infty (\|u(t)\|^2 + \|y(t)\|^2)\, dt \tag{1.3}$$

is finite. Under this assumption for every $\varphi \in H$ there exists a unique optimal control that is given by the feedback law

$$u(t) = -B^*Pz(t),$$

where P is the minimal nonnegative solution of (1.2). A nonnegative solution of (1.2) exists under the assumption that system (1.1) is *stabilizable*, meaning that there exists an operator $K \in \mathscr{L}(H, U)$ such that $A + BK$ generates an exponentially stable semigroup. If (1.1) is also *detectable* in the sense that for some operator $L \in \mathscr{L}(Y, H)$ the operator $A + LC$ generates an exponentially stable semigroup, then the solution P of (1.2) is unique in the class of nonnegative operators on H and the closed-loop semigroup generated by $A - BB^*P$ is exponentially stable [1], [6].

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Together with (1.1) we also consider a sequence of approximating control systems

$$\dot{z}^N(t) = A^N z^N(t) + B^N u^N(t), \qquad z^N(0) = \pi^N \varphi, \\ y^N(t) = C^N z^N(t), \tag{1.4}$$

where $z^N \in \mathbf{R}^{k(N)}$, $u^N \in \mathbf{R}^{m(N)}$, $y^N \in \mathbf{R}^{p(N)}$, and A^N, B^N, C^N are matrices of suitable dimensions. We assume that there exist injective linear maps

$$\iota^N : \mathbf{R}^{k(N)} \to H, \quad j^N : \mathbf{R}^{m(N)} \to U, \quad k^N : \mathbf{R}^{p(N)} \to Y$$

and surjective linear maps

$$\pi^N : H \to \mathbf{R}^{k(N)}, \quad \rho^N : U \to \mathbf{R}^{m(N)}, \quad \sigma^N : Y \to \mathbf{R}^{p(N)}$$

such that $\pi^N \iota^N$, $\rho^N j^N$, $\sigma^N k^N$ are identity maps and $\iota^N \pi^N$, $j^N \rho^N$, $k^N \sigma^N$ are orthogonal projections. On the spaces $\mathbf{R}^{k(N)}$, $\mathbf{R}^{m(N)}$, and $\mathbf{R}^{p(N)}$ we will always consider the induced inner products $\langle z, w\rangle_N = \langle \iota^N z, \iota^N w\rangle_H$, $z, w \in \mathbf{R}^{k(N)}$, $\langle u, v\rangle_N = \langle j^N u, j^N v\rangle_U$, $u, v \in \mathbf{R}^{m(N)}$, and $\langle x, y\rangle_N = \langle k^N x, k^N y\rangle_Y$, $x, y \in \mathbf{R}^{p(N)}$. $(A^N)^*$, $(B^N)^*$, $(C^N)^*$, $\cdots$ always denote the adjoint matrices with respect to the induced inner products.

The purpose of this paper is to investigate the convergence properties of the solution matrices $P^N = (P^N)^*$ of the approximating algebraic Riccati equations

$$(A^N)^* P^N + P^N A^N - P^N B^N (B^N)^* P^N + (C^N)^* C^N = 0. \tag{1.5}$$

To formulate the results we introduce the following concepts. The approximating systems (1.4) are called *strongly convergent* to (1.1) if

$$S(t)\varphi = \lim_{N\to\infty} \iota^N e^{A^N t} \pi^N \varphi, \qquad S(t)^*\varphi = \lim_{N\to\infty} \iota^N e^{(A^N)^* t} \pi^N \varphi \tag{1.6}$$

uniformly on compact time intervals for all $\varphi \in H$,

$$\iota^N B^N \rho^N \to B, \quad j^N (B^N)^* \pi^N \to B^*, \quad \text{and} \\ j^N \rho^N \to \mathrm{id}_U \quad \text{strongly} \tag{1.7}$$

and

$$k^N C^N \pi^N \to C, \quad \iota^N (C^N)^* \sigma^N \to C^*, \quad \text{and} \\ k^N \sigma^N \to \mathrm{id}_Y \quad \text{strongly}. \tag{1.8}$$

We will call systems (1.4) *uniformly output stable* if there exists a constant $c > 0$ such

$$\int_0^\infty \| k^N C^N e^{A^N t} \pi^N \varphi \|^2 \, dt \leqq c \|\varphi\|^2$$

for all $\varphi \in H$ and $N = 1, 2, \cdots$. Systems (1.4) are said to be *uniformly input-output stable* if the functions $C^N e^{A^N t} B^N$, $N = 1, 2, \cdots$, are integrable on $0 \leqq t < \infty$ and there exists a constant $c_1 > 0$ such that

$$\| k^N C^N (i\omega I - A^N)^{-1} B^N \rho^N \| \leqq c_1$$

for all $\omega \in \mathbf{R}$ and $N = 1, 2, \cdots$.

Remarks. (1) Uniform output stability of systems (1.4) in connection with strong convergence to (1.1) implies that system (1.1) is output stable in the sense that

$$\int_0^\infty \| CS(t)\varphi\|^2 \, dt \leqq \text{const.} \|\varphi\|^2 \quad \text{for all } \varphi \in H.$$

(2) If the approximating systems (1.4) are strongly convergent to system (1.1) and the matrices $K^N \in \mathbf{R}^{m(N)\times k(N)}$, $L^N \in \mathbf{R}^{k(N)\times p(N)}$ are chosen such that the operator sequences $j^N K^N \pi^N \in \mathscr{L}(H, U)$, $\iota^N L^N \sigma^N \in \mathscr{L}(Y, H)$ and their adjoints $\iota^N (K^N)^* \rho^N$, $k^N (L^N)^* \pi^N$ converge strongly to K, L and K^*, L^*, respectively, then the feedback

systems

$$\begin{aligned}\dot{z}^N(t)&=(A^N+B^NK^N)z^N(t)+B^Nv^N(t), \qquad z^N(0)=\pi^N\varphi,\\ y^N(t)&=C^Nz^N(t), \qquad w^N(t)=K^Nz^N(t),\end{aligned} \tag{1.9}$$

and the dynamic observers

$$\begin{aligned}\dot{z}^N(t)&=(A^N+L^NC^N)z^N(t)-L^Ny^N(t)+B^Nu^N(t),\\ z^N(0)&=\pi^N\varphi, \qquad w^N(t)=K^Nz^N(t),\end{aligned} \tag{1.10}$$

are also strongly convergent. This can be seen by using the variation of parameters formula, Gronwall's inequality and Lebesgue's dominated convergence theorem.

(3) Let systems (1.4) converge strongly to system (1.1). By (1.6) and the uniform boundedness principle we see that there exists a constant $M_1 \geqq 1$ such that

$$\|\iota^N e^{A^N t}\pi^N\| \leqq M_1$$

for $t \in [0, 1]$ and $N = 1, 2, \cdots$. By standard considerations this implies

$$\|\iota^N e^{A^N t}\pi^N\| \leqq M_1 e^{\alpha t}, \quad t \geqq 0, \quad N=1,2,\cdots, \tag{1.11}$$

where α is some real constant. It follows that $\|S(t)\| \leqq M_1 e^{\alpha t}$, $t \geqq 0$. However, the exponential growth rate of $S(t)$ may be strictly less than $\alpha_0 = \inf \alpha$, where the infimum is over all α for which (1.11) holds with some constant $M_1 \geqq 1$. By the Trotter-Kato theorem we see that

$$\lim_{N\to\infty} \|(\lambda I - A)^{-1}z - \iota^N(\lambda I - A^N)^{-1}\pi^N z\| = 0$$

for all $z \in H$ uniformly for $\operatorname{Re}\lambda \geqq \gamma$ for any $\gamma > \alpha_0$.

(4) From the definition of the norms on $\mathbf{R}^{k(N)}$, $\mathbf{R}^{m(N)}$, and $\mathbf{R}^{p(N)}$ it is obvious that $\|\iota^N\| = \|j^N\| = \|k^N\| = 1$ and also $\|\pi^N\| = \|\rho^N\| = \|\sigma^N\| = 1$. Let $H^N = \text{range } \iota^N\pi^N$, $U^N = \text{range } j^N\rho^N$ and $Y^N = \text{range } k^N\sigma^N$. Then, for instance,

$$\begin{aligned}\|\iota^N B^N \rho^N\|_{\mathscr{L}(U^N, H^N)} &= \|B^N\|_{\mathscr{L}(\mathbf{R}^{m(N)}, \mathbf{R}^{k(N)})},\\ \|k^N C^N \pi^N\|_{\mathscr{L}(H^N, Y^N)} &= \|C^N\|_{\mathscr{L}(\mathbf{R}^{k(N)}, \mathbf{R}^{p(N)})}.\end{aligned}$$

In § 2 we will make use of these observations repeatedly.

2. The convergence result. The following theorem is the main result of this paper.

THEOREM 1. *Let systems* (1.4), $K^N \in \mathbf{R}^{m(N)\times k(N)}$, $L^N \in \mathbf{R}^{k(N)\times p(N)}$, $K \in \mathscr{L}(H, U)$, *and* $L \in \mathscr{L}(Y, H)$ *be given. Assume that*

(i) *Systems* (1.4) *are strongly convergent to* (1.1);

(ii) $j^N K^N \pi^N \to K$, $\iota^N (K^N)^* \rho^N \to K^*$, $\iota^N L^N \sigma^N \to L$, $k^N (L^N)^* \pi^N \to L^*$ *strongly*;

(iii) $A + BK$ *and* $A + LC$ *generate exponentially stable semigroups*;

(iv) *Systems* (1.9) *are uniformly output stable and uniformly input-output stable*; *and*

(v) *Systems* (1.10) *are uniformly input-output stable.*

Then

$$P\varphi = \lim_{N\to\infty} \iota^N P^N \pi^N \varphi$$

for every $\varphi \in H$, *where* $P \in \mathscr{L}(H)$ *and* $P^N \in \mathbf{R}^{k(N)\times k(N)}$ *are the minimal nonnegative solutions of* (1.2) *and* (1.5), *respectively.*

An earlier version of this convergence theorem was proved in [3] under stronger assumptions. In particular the following property of the approximation scheme is assumed (see [3, Conjecture 7.1]): If the semigroup $S(t)$ is exponentially stable, then the approximating semigroups satisfy an estimate $\|\exp(A^N t)\| \leqq Me^{-\beta t}$, $t \geqq 0$, $N =$

$1, 2, \cdots$, with constants $M \geqq 1$, $\beta > 0$ independent of N. This assumption is not met by the spline approximation scheme for delay systems developed in [4] and [5]. On the other hand, in this case convergence of the P^N's has been observed numerically [4]. In § 3 we will show that the spline scheme indeed satisfies the requirements of Theorem 1.

The proof of Theorem 1 rests on the relationship between the algebraic Riccati equation (1.2) and the optimal control problem (1.3). We first establish two lemmas. For system (1.1), respectively systems (1.4), we define the operators $\mathscr{C}$, $\mathscr{C}^N : H \to L^2(0, \infty; Y)$ by

$$(\mathscr{C}\varphi)(t) = CS(t)\varphi, \qquad t \geqq 0,$$
$$(\mathscr{C}^N\varphi)(t) = k^N C^N e^{A^N t}\pi^N \varphi, \qquad t \geqq 0,$$

respectively. Then the adjoint operators $\mathscr{C}^*$, $(\mathscr{C}^N)^* : L^2(0, \infty; Y) \to H$ are given by

$$\mathscr{C}^* y = \int_0^\infty S(t)^* C^* y(t)\, dt,$$

and

$$(\mathscr{C}^N)^* y = \int_0^\infty \iota^N e^{(A^N)^* t}(C^N)^* \sigma^N y(t)\, dt.$$

LEMMA 1. *Assume that $S(t)$ is exponentially stable and systems* (1.4) *are uniformly output stable and converge strongly to system* (1.1). *Then*

$$\mathscr{C}^N \to \mathscr{C} \quad \text{and} \quad (\mathscr{C}^N)^* \to \mathscr{C}^*$$

strongly as $N \to \infty$.

Proof. For any $T > 0$ we get

$$\begin{aligned} \|\mathscr{C}\varphi - \mathscr{C}^N\varphi\|^2_{L^2(0,\infty;Y)} \leqq \int_0^T & \|CS(t)\varphi - k^N C^N e^{A^N t}\pi^N\varphi\|^2\, dt + 3\int_0^\infty \|CS(T+t)\varphi\|^2\, dt \\ & + 3\int_0^\infty \|k^N C^N e^{A^N t}(e^{A^N T}\pi^N\varphi - \pi^N S(T)\varphi)\|^2\, dt \\ & + 3\int_0^\infty \|k^N C^N e^{A^N t}\pi^N S(T)\varphi\|^2\, dt =: \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4. \end{aligned}$$

The estimate for α_1 is

$$\begin{aligned} \alpha_1 \leqq 2 & \int_0^T \|(C - k^N C^N \pi^N) S(t)\varphi\|^2\, dt \\ & + 2 \sup_N \|k^N C^N \pi^N\|^2 \int_0^T \|S(t)\varphi - \iota^N e^{A^N t}\pi^N\varphi\|^2\, dt. \end{aligned}$$

For any $T > 0$ the right-hand side tends to zero as $N \to \infty$, because systems (1.4) are strongly convergent to (1.1).

For α_2 we get from the exponential stability of $S(t)$ (i.e., $\|S(t)\| \leqq Me^{-\beta t}$, $t \geqq 0$, for some $\beta > 0$)

$$\alpha_2 \leqq 3\|C\|^2 e^{-2\beta T} \frac{M^2}{2\beta} \|\varphi\|^2.$$

Using uniform output stability of systems (1.4) we obtain

$$\alpha_3 \leqq 3c\|\iota^N e^{A^N T}\pi^N\varphi - S(T)\varphi\|^2,$$
$$\alpha_4 \leqq 3c\|S(T)\varphi\|^2 \leqq 3cM^2 e^{-2\beta T}\|\varphi\|^2.$$

These estimates together show that

$$\mathscr{C}^N\varphi \to \mathscr{C}\varphi \quad \text{as } N \to \infty$$

for any $\varphi \in H$.

For the proof of $(\mathscr{C}^N)^*y \to \mathscr{C}^*y$ it is enough to consider y with compact support. Let $\operatorname{supp} y \subset [0, T]$, $T > 0$. Then

$$\|\mathscr{C}^*y - (\mathscr{C}^N)^*y\| \leqq \int_0^T \|S(t)^*C^*y(t) - \iota^N e^{(A^N)^*t}\pi^N C^*y(t)\|\, dt$$

$$+ \int_0^T \|\iota^N e^{(A^N)^*t}\pi^N\|\, \|(C^* - \iota^N(C^N)^*\sigma^N)y(t)\|\, dt.$$

The right-hand side tends to zero by the Lebesgue dominated convergence theorem using strong convergence of systems (1.4) to (1.1) and (1.11). □

Remark. If $\dim Y < \infty$ and systems (1.4) are uniformly exponentially stable (i.e., $\|\iota^N e^{A^N t}\pi^N\| \leqq Me^{-\alpha t}$, $t \geqq 0$, $N = 1, 2, \cdots$, for some constants $M \geqq 1$, $\alpha > 0$), then

$$\|\mathscr{C} - \mathscr{C}^N\| = \|\mathscr{C}^* - (\mathscr{C}^N)^*\| \to 0 \quad \text{as } N \to \infty.$$

We only have to observe that $\|CS(t) - k^N C^N e^{A^N t}\pi^N\|$ is exponentially decaying as $t \to \infty$ uniformly with respect to N.

To state the next lemma we introduce the operators $\mathscr{G}$, $\mathscr{G}^N : L^2(0, \infty; U) \to L^2(0, \infty; Y)$ by

$$(\mathscr{G}u)(t) = \int_0^t CS(t-s)Bu(s)\, ds, \qquad t \geqq 0,$$

$$(\mathscr{G}^N u)(t) = \int_0^t k^N C^N e^{A^N(t-s)} B^N \rho^N u(s)\, ds, \qquad t \geqq 0,$$

for $u \in L^2(0, \infty; U)$. The adjoint operators are given by

$$(\mathscr{G}^*y)(t) = \int_t^\infty B^*S(s-t)^*C^*y(s)\, ds, \qquad t \geqq 0,$$

$$((\mathscr{G}^N)^*y)(t) = \int_t^\infty j^N(B^N)^* e^{(A^N)^*(s-t)}(C^N)^*\sigma^N y(s)\, ds, \qquad t \geqq 0,$$

for $y \in L^2(0, \infty; Y)$.

LEMMA 2. *Assume that $S(t)$ is exponentially stable, systems (1.4) are strongly convergent to system (1.1) and, furthermore, systems (1.4) are uniformly input-output stable and uniformly output stable. Then*

$$\mathscr{G}^N \to \mathscr{G} \quad \textit{and} \quad (\mathscr{G}^N)^* \to \mathscr{G}^*$$

strongly as $N \to \infty$.

Proof. Using Parseval's equality and uniform input-output stability we obtain the estimate

$$\|\mathscr{G}^N u\|^2 = \int_0^\infty \left\| \int_0^t k^N C^N e^{A^N(t-s)} B^N \rho^N u(s)\, ds \right\|^2 dt$$

$$\leqq \int_{-\infty}^\infty \|k^N C^N (i\omega I - A^N)^{-1} B^N \rho^N\|^2 \|\hat{u}(\omega)\|^2\, d\omega$$

$$\leqq c_1^2 \int_{-\infty}^\infty \|\hat{u}(\omega)\|^2\, d\omega = c_1^2 \|u\|^2,$$

which implies uniform boundedness of the operators $\mathcal{G}^N$. Therefore it is enough to consider input functions u with compact support, $\operatorname{supp} u \subset [0, T]$. Let $y = \mathcal{G}u$ and $y^N = \mathcal{G}^N u$. Then the estimate

$$\|y(t) - y^N(t)\| \leqq \int_0^t \|(C - k^N C^N \pi^N) S(s) B u(t-s)\| \, ds$$
$$+ \sup_N \|k^N C^N \pi^N\| \int_0^t \|(S(s) - \iota^N e^{A^N s} \pi^N) B u(t-s)\| \, ds$$
$$+ M_1 \, e^{\alpha T} \sup_N \|k^N C^N \pi^N\| \int_0^t \|(B - \iota^N B^N \rho^N) u(t-s)\| \, ds$$

for $0 \leqq t \leqq T$, shows that $\|y(\cdot) - y^N(\cdot)\|_{L^2(0,T;Y)} \to 0$ as $N \to \infty$ (using strong convergence of systems (1.4) to system (1.1) and the Lebesgue dominated convergence theorem). Moreover, we have

$$y(t+T) = (\mathcal{E}\varphi)(t) \quad \text{with } \varphi = \int_0^T S(T-s) B u(s) \, ds,$$

$$y^N(t+T) = (\mathcal{E}^N \varphi^N)(t) \quad \text{with } \varphi^N = \iota^N \int_0^T e^{A^N(T-s)} B^N \rho^N u(s) \, ds$$

and hence it follows from Lemma 1 that $y^N \to y$ in $L^2(T, \infty; Y)$ as $N \to \infty$.

For the adjoint operators we again need to consider y with compact support only, say $\operatorname{supp} y \subset [0, T]$, $T > 0$. Then

$$\|(\mathcal{G}^N)^* y - \mathcal{G}^* y\|^2$$
$$= \int_0^T \left\| \int_t^T (B^* S(s-t)^* C^* - j^N (B^N)^* \, e^{(A^N)^*(s-t)} (C^N)^* \sigma^N) y(s) \, ds \right\|^2 dt.$$

Using strong convergence of systems (1.4) to (1.1) (together with the estimate (1.11)) we see that we can apply the Lebesgue dominated convergence theorem twice. □

Proof of Theorem 1. Let $S_K(t)$ and $S_L(t)$ denote the semigroups generated by $A + BK$ and $A + LC$, respectively. We first observe that $J(u) < \infty$ for $u \in L^2(0, \infty; U)$ if and only if $v = u - Kz \in L^2(0, \infty; U)$, where $z(t)$ is the mild solution of $\dot{z}(t) = Az(t) + Bu(t)$, $z(0) = \varphi$, i.e., $z(t) = S(t)\varphi + \int_0^t S(t-s) B u(s) \, ds$. Indeed, since

$$z(t) = S_K(t)\varphi + \int_0^t S_K(t-s) B v(s) \, ds$$

(this is rather obvious for $\varphi \in \operatorname{dom} A$ and u being differentiable and follows by a density argument in the general case), $z(t)$ is square integrable if v is. But then $u = v + Kz \in L^2(0, \infty; U)$ and $y = Cz \in L^2(0, \infty; Y)$, i.e., $J(u) < \infty$. Conversely, if $J(u) < \infty$ then the formula

$$z(t) = S_L(t)\varphi + \int_0^t S_L(t-s)(Bu(s) - Ly(s)) \, ds$$

shows that z and $v = u - Kz$ are square integrable.

Therefore the control problem of minimizing (1.3) subject to (1.1) is equivalent to the problem of minimizing

$$J_K(v) = J_K(v, \varphi) = \int_0^\infty (\|v(t) + Kz(t)\|^2 + \|y(t)\|^2) \, dt \tag{2.1}$$

subject to

(2.2) $$\dot z=(A+BK)z+Bv,\quad z(0)=\varphi,\quad y=Cz.$$

The functional (2.1) is bounded for all $v\in L^2(0,\infty;U)$ and can be written in the form

$$J_K(v,\varphi)=\|\mathscr{C}\varphi+\mathscr{T}v\|^2,$$

where the operators $\mathscr{C}:H\to L^2(0,\infty;U\times Y)$ and $\mathscr{T}:L^2(0,\infty;U)\to L^2(0,\infty;U\times Y)$ are defined by

$$(\mathscr{C}\varphi)(t)=(KS_K(t)\varphi,\ CS_K(t)\varphi),$$
$$(\mathscr{T}v)(t)=\left(v(t)+K\int_0^t S_K(t-s)Bv(s)\,ds,\ C\int_0^t S_K(t-s)Bv(s)\,ds\right).$$

Hence the optimal control $\hat v$ satisfies

(2.3) $$\mathscr{T}^*\mathscr{T}\hat v+\mathscr{T}^*\mathscr{C}\varphi=0.$$

We define the operator $\mathscr{F}:L^2(0,\infty;U\times Y)\to L^2(0,\infty;U)$ by

$$\mathscr{F}(u,y)(t)=u(t)-K\int_0^t S_L(t-s)(Bu(s)-Ly(s))\,ds.$$

Then straightforward computations show that, for $t\geqq 0$,

$$(\mathscr{F}\mathscr{T}v)(t)=v(t)+Kz(t)-K\int_0^t S_L(t-s)(Bv(s)+BKz(s)-LCz(s))\,ds,$$

where $z(t)=\int_0^t S_K(t-s)Bu(s)\,ds$. Let $w(t)$ denote the integral term in the above equation. Then $w(t)$ is the unique mild solution of $\dot w=(A+LC)w+Bv(t)+BKz(t)-LCz(t)$, $w(0)=0$. Obviously, $z(t)$ is also a mild solution of this problem, i.e., $w(t)\equiv z(t)$. Thus we have

$$(\mathscr{F}\mathscr{T}v)(t)=v(t)+Kz(t)-Kz(t)=v(t),\qquad t\geqq 0,$$

i.e.,

$$\mathscr{F}\mathscr{T}v=v\quad\text{for all } v\in L^2(0,\infty;U).$$

This implies $\|v\|^2\leqq\|\mathscr{F}\|^2\|\mathscr{T}v\|^2=\|\mathscr{F}\|^2\langle v,\mathscr{T}^*\mathscr{T}v\rangle\leqq\|\mathscr{F}\|^2\|v\|\,\|\mathscr{T}^*\mathscr{T}v\|$, i.e.,

$$\|\mathscr{T}^*\mathscr{T}v\|\geqq\|\mathscr{F}\|^{-2}\|v\|\quad\text{for all } v\in L^2(0,\infty;U).$$

Hence the operator $\mathscr{T}^*\mathscr{T}$ is boundedly invertible, and from (2.3) we get

$$\hat v=-(\mathscr{T}^*\mathscr{T})^{-1}\mathscr{T}^*\mathscr{C}\varphi.$$

The identity $J_K(\hat v,\varphi)=J(\hat u,\varphi)=\langle\varphi,P\varphi\rangle$ shows that

$$\langle\varphi,P\varphi\rangle=\langle\mathscr{C}\varphi,\mathscr{C}\varphi+\mathscr{T}\hat v\rangle=\langle\varphi,\mathscr{C}^*\mathscr{C}\varphi-\mathscr{C}^*\mathscr{T}(\mathscr{T}^*\mathscr{T})^{-1}\mathscr{T}^*\mathscr{C}\varphi\rangle$$

and hence

(2.4) $$P=\mathscr{C}^*(I-\mathscr{T}(\mathscr{T}^*\mathscr{T})^{-1}\mathscr{T}^*)\mathscr{C}.$$

Defining the approximating operators

$$\mathscr{C}^N:\mathbf{R}^{k(N)}\to L^2(0,\infty;\mathbf{R}^{k(N)}\times\mathbf{R}^{p(N)}),$$
$$\mathscr{T}^N:L^2(0,\infty;\mathbf{R}^{m(N)})\to L^2(0,\infty;\mathbf{R}^{m(N)}\times\mathbf{R}^{p(N)}),$$
$$\mathscr{F}^N:L^2(0,\infty;\mathbf{R}^{m(N)}\times\mathbf{R}^{p(N)})\to L^2(0,\infty;\mathbf{R}^{m(N)})$$

in the obvious way we get analogously as above

$$P^N = (\mathscr{C}^N)^*(I - \mathscr{T}^N((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}(\mathscr{T}^N)^*)\mathscr{C}^N. \tag{2.5}$$

Lemma 1 applied to systems (1.9) shows that

$$(j^N \oplus k^N)\mathscr{C}^N\pi^N \to \mathscr{C} \quad \text{and} \quad \iota^N(\mathscr{C}^N)^*(\rho^N \oplus \sigma^N) \to \mathscr{C}^* \quad \text{strongly.} \tag{2.6}$$

Here $j^N \oplus k^N$ denotes the direct sum of j^N and k^N defined by $(j^N \oplus k^N)(u^N, y^N) = (j^N u^N, k^N y^N)$, $u^N \in \mathbf{R}^{m(N)}$, $y^N \in \mathbf{R}^{p(N)}$ etc. Moreover, in abuse of notation we define $j^N u$ for $u \in L^2(0, \infty; \mathbf{R}^{m(N)})$ by $(j^N u)(t) = j^N u(t)$, $t \geqq 0$, etc.

By Lemma 2 applied to systems (1.9) we obtain

$$(j^N \oplus k^N)\mathscr{T}^N\rho^N \to \mathscr{T} \quad \text{and} \quad j^N(\mathscr{T}^N)^*(\rho^N \oplus \sigma^N) \to \mathscr{T}^* \quad \text{strongly.} \tag{2.7}$$

By assumption (v) of Theorem 1 we have

$$\sup_N \|\mathscr{F}^N\| < \infty.$$

This and the estimate

$$\|(\mathscr{T}^N)^*\mathscr{T}^N v^N\| \geqq \|\mathscr{F}^N\|^{-2}\|v^N\|$$

for all $v^N \in L^2(0, \infty; \mathbf{R}^{m(N)})$ show that $\|((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}\|$ are uniformly bounded. By Remark (4) of § 1 also $\|j^N((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}\rho^N\|$ are uniformly bounded. Then for $v \in L^2(0, \infty; U)$

$$j^N((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}\rho^N v - (\mathscr{T}^*\mathscr{T})^{-1}v = j^N((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}\rho^N v - j^N\rho^N(\mathscr{T}^*\mathscr{T})^{-1}v$$
$$- ((\mathscr{T}^*\mathscr{T})^{-1}v - j^N\rho^N(\mathscr{T}^*\mathscr{T})^{-1}v).$$

The second term on the right-hand side converges to zero as $N \to \infty$. For the first term we get

$$j^N((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}\rho^N v - j^N\rho^N(\mathscr{T}^*\mathscr{T})^{-1}v$$
$$= j^N((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}\rho^N(\mathscr{T}^*\mathscr{T} - j^N(\mathscr{T}^N)^*\mathscr{T}^N\rho^N)(\mathscr{T}^*\mathscr{T})^{-1}v,$$

which proves

$$j^N((\mathscr{T}^N)^*\mathscr{T}^N)^{-1}\rho^N \to (\mathscr{T}^*\mathscr{T})^{-1} \quad \text{strongly.} \tag{2.8}$$

The representations (2.4) and (2.5) together with (2.6)-(2.8) prove that

$$\iota^N P^N \pi^N \to P \quad \text{strongly.}$$ □

Remark. If the matrices $e^{(A^N + B^N K^N)t}$, $t \geqq 0$, are uniformly exponentially stable, then the operators $\mathscr{C}^N\pi^N$ converge to $\mathscr{C}$ in the uniform operator topology (see the remark following Lemma 1). Then it follows that the operators $\iota^N P^N \pi^N$ also converge in the uniform operator topology provided $\dim U < \infty$ and $\dim Y < \infty$. It remains an open question whether convergence of the P^N in the uniform operator topology can be established under weaker assumptions.

3. Spline approximation for delay equations. The system

$$\begin{aligned}\dot{x}(t)&=A_0x(t)+A_1x(t-h)+B_0u(t), \qquad y(t)=C_0x(t),\\ x(0)&=\varphi^0, \quad x(\tau)=\varphi^1(\tau) \quad \text{for } -h\leqq\tau<0,\end{aligned} \tag{3.1}$$

with $x(t)\in\mathbf{R}^n$, $u(t)\in\mathbf{R}^m$, $y(t)\in\mathbf{R}^p$ and $\varphi=(\varphi^0,\varphi^1)\in\mathbf{R}^n\times L^2(-h,0;\mathbf{R}^n)$ is equivalent to system (1.1) in the Hilbert space

$$H=M^2=\mathbf{R}^n\times L^2(-h,0;\mathbf{R}^n).$$

Operators A, B, and C are given by

$$\begin{aligned}&\operatorname{dom} A=\{\varphi=(\varphi^0,\varphi^1)\in M^2 \mid \varphi^1\in W^{1,2}(-h,0;\mathbf{R}^n),\ \varphi^0=\varphi^1(0)\},\\ &A=(A_0\varphi^0+A_1\varphi^1(-h),\ \dot{\varphi}^1) \quad \text{for } \varphi\in\operatorname{dom} A,\\ &Bu=(B_0u,0) \quad \text{for } u\in\mathbf{R}^m,\\ &C\varphi=C_0\varphi^0 \quad \text{for } \varphi\in M^2.\end{aligned}$$

In [4] and [5] we have considered a sequence of approximating systems (1.4) where $k(N)=n(N+2)$, $m(N)=m$, $p(N)=p$ and the matrices A^N, B^N, C^N are given by $A^N=(Q^N)^{-1}H^N$ with

$$Q^N=\begin{bmatrix} I & 0 & \cdot & \cdot\ \ \cdot & \cdot & 0\\ 0 & \frac{h}{3N}I & \frac{h}{6N}I & 0\ \ \cdot & \cdot & 0\\ 0 & \frac{h}{6N}I & \frac{2h}{3N}I & \cdot\ \ \cdot & & \cdot\\ \cdot & 0 & \cdot & \cdot\ \ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\ \ \cdot & \cdot & 0\\ \cdot & \cdot & & \cdot\ \ \cdot & \frac{2h}{3N}I & \frac{h}{6N}I\\ 0 & 0 & \cdot & \cdot\ \ 0 & \frac{h}{6N}I & \frac{h}{3N}I \end{bmatrix},$$

$$H^N=\begin{bmatrix} A_0 & 0 & \cdot & \cdot & \cdot & \cdot & A_1\\ I & -\frac{1}{2}I & -\frac{1}{2}I & 0 & \cdot & \cdot & 0\\ 0 & \frac{1}{2}I & 0 & \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & & \cdot & \cdot & \cdot & \cdot & 0\\ \cdot & & & \cdot & \cdot & 0 & -\frac{1}{2}I\\ 0 & \cdot & \cdot & \cdot & 0 & \frac{1}{2}I & -\frac{1}{2}I \end{bmatrix},$$

$$B^N=\begin{bmatrix} B_0\\ 0\\ \vdots\\ 0\end{bmatrix}, \qquad C^N=(C_0\cdots 0).$$

The injections ι^N are given by

$$\iota^N z=\left(z_0,\ \sum_{j=0}^{N} z_{ij}s_j^N\right),$$

where $z=\operatorname{col}(z_0, z_{10},\cdots,z_{1N})\in\mathbf{R}^{n(N+2)}$ and the functions s_j^N are the basis splines

$$s_0^N(\tau)=\max\left(0,\ N\frac{\tau}{h}+1\right), \qquad s_N^N(\tau)=\max\left(0,1-N-N\frac{\tau}{h}\right),$$

and, for $j=1,\cdots,N-1$,

$$s_j^N(\tau)=\begin{cases} N\frac{\tau}{h}+j+1 & \text{for } -(j+1)\frac{h}{N}\leqq\tau\leqq -j\frac{h}{N},\\ -N\frac{\tau}{h}-j+1 & \text{for } -j\frac{h}{N}\leqq\tau\leqq-(j-1)\frac{h}{N},\\ 0 & \text{elsewhere.}\end{cases}$$

The induced inner product on $\mathbf{R}^{n(N+2)}$ is given by $\langle z, w\rangle_N=z^TQ^Nw$. Of course, $U=U^N=\mathbf{R}^m$ and $Y=Y^N=\mathbf{R}^p$ for all N.

The approximating systems (1.4) with these matrices are strongly convergent [4] and if the delay system (3.1) is stable in the sense that $\operatorname{Re}\lambda<0$ for all roots of $\det(\lambda I-A_0-e^{-\lambda h}A_1)=0$, then the approximating systems (1.4) are uniformly output stable [5]. Moreover, the approximating transfer functions are in this case given by

$$C^N(i\omega I-A^N)^{-1}B^N=C_0(i\omega I-A_0-\alpha^N(i\omega)A_1)^{-1}B_0, \tag{3.2}$$

where $\alpha^N(\lambda)$ is a sequence of rational functions converging to $e^{-\lambda h}$ uniformly on compact sets and satisfying $|\alpha^N(\lambda)|\leqq 2$ on $\operatorname{Re}\lambda\geqq 0$ for all $N=1,2,\cdots$ [5]. This shows that the approximating systems (1.4) are uniformly input-output stable for N sufficiently large provided that the delay system (3.1) is stable (which is equivalent to exponential stability of the corresponding system (1.1)).

THEOREM 2. *Suppose that there exist matrices $K_0\in\mathbf{R}^{m\times n}$ and $L_0\in\mathbf{R}^{n\times p}$ such that the delay systems*

$$\dot{x}(t)=(A_0+B_0K_0)x(t)+A_1x(t-h),$$
$$\dot{x}(t)=(A_0+L_0C_0)x(t)+A_1x(t-h)$$

are stable and let the matrices A^N, B^N, C^N be defined as above. Then there exist unique nonnegative solutions

$$P\in\mathscr{L}(M^2)\quad \text{and}\quad P^N\in\mathbf{R}^{n(N+2)\times n(N+2)}$$

of (1.2) *and* (1.5), *respectively, and for every* $\varphi\in M^2$

$$P\varphi=\lim_{N\to\infty}\iota^NP^N\pi^N\varphi.$$

Proof. Define the matrices

$$K^N=(K_0\quad 0\quad\cdots\quad 0),\qquad L^N=\begin{pmatrix}L_0\\0\\ \vdots\\0\end{pmatrix}$$

and apply Theorem 1. □

Numerical examples for this convergence result are reported in [4].

Remark. The conditions of Theorem 2 are stronger than stabilizability and detectability of the delay system (3.1). However, we are not aware of a stabilizable delay system that cannot be stabilized by a feedback law of the form $u(t)=K_0x(t)$.

4. Convergence of transfer functions. In this section we give a short discussion of the connection between strong convergence of systems (1.4) and convergence of the corresponding transfer functions on the imaginary axis.

If the semigroup $S(t)$ is exponentially stable and systems (1.4) are strongly convergent to system (1.1) and are uniformly input-output stable, then we can show that

$$\lim_{N\to\infty} k^N C^N(\lambda I - A^N)^{-1} B^N \rho^N = C(\lambda I - A)^{-1} B$$

uniformly on compact subsets of $\operatorname{Re}\lambda > 0$. The proof involves Vitali's theorem on sequences of holomorphic functions (see, for instance, [2, p. 309]). Despite the fact that under the assumption of uniform input-output stability the functions $k^N C^N(\lambda I - A^N)^{-1} B^N \rho^N$ are uniformly bounded on $\operatorname{Re}\lambda \geqq 0$ (and not only on compact subsets of $\operatorname{Re}\lambda > 0$ as required in Vitali's theorem) we cannot conclude uniform convergence of these functions on compact subsets of the imaginary axis. This is demonstrated by the following example.

Example. Let $H = l^2$ and $U = Y = \mathbf{R}$. For an element $b = (b_1, b_2, \cdots) \in l^2$ with $b_j > 0$ for all j we consider

$$\dot z(t) = -z(t) + bu(t), \quad t \geqq 0, \qquad y(t) = \langle b, z(t)\rangle_{l^2}. \tag{4.1}$$

The solution semigroup of the homogeneous problem is $S(t) = e^{-t}I$, which obviously is exponentially stable. We consider the approximating systems

$$\dot z^N(t) = A^N z^N(t) + b^N u(t), \quad t \geqq 0, \qquad y(t) = (b^N)^T z^N(t), \tag{4.2}$$

where

$$A^N = \operatorname{diag}(-1, \cdots, -1, -b_{N+1}^2) \in \mathbf{R}^{(N+1)\times(N+1)}$$
$$b^N = \operatorname{col}(b_1, \cdots, b_{N+1}) \in \mathbf{R}^{N+1}.$$

The embedding $\iota^N : \mathbf{R}^{N+1} \to l^2$ is given by $\iota^N z^N = (z_1, \cdots, z_{N+1}, 0, \cdots)$ for $z^N = \operatorname{col}(z_1, \cdots, z_{N+1}) \in \mathbf{R}^{N+1}$ and the "projections" π^N by $\pi^N z = \operatorname{col}(z_1, \cdots, z_{N+1})$ for $z = (z_1, z_2, \cdots) \in l^2$.

The solutions of $\dot z = -z$, $z(0) = \varphi = (\varphi_1, \varphi_2, \cdots) \in l^2$, and $\dot z^N = A^N z^N$, $z^N(0) = \pi^N \varphi$, are given by

$$z(t) = e^{-t}\varphi, \qquad z^N(t) = e^{-t} \sum_{j=1}^{N} \varphi_j + e^{-b_{N+1}^2 t} \varphi_{N+1},$$

respectively. Therefore

$$\|z(t) - \iota^N z^N(t)\|_{l^2}^2 = (e^{-t} - e^{-b_{N+1}^2 t})^2 |\varphi_{N+1}|^2 + e^{-2t} \sum_{j=N+2}^{\infty} |\varphi_j|^2$$
$$\leqq \sum_{N+1}^{\infty} |\varphi_j|^2 \to 0 \quad \text{as } N \to \infty.$$

It is obvious that $\|b - \iota^N b^N\|_{l^2} \to 0$ as $N \to \infty$. Thus systems (4.2) are strongly convergent to (4.1).

Obviously $(b^N)^T e^{A^N t} b^N$ is integrable on $t \geqq 0$. The transfer functions $G^N(\lambda) = (b^N)^T(\lambda I - A^N)^{-1} b^N$ are given by

$$G^N(\lambda) = \frac{1}{1+\lambda} \sum_{j=1}^{N} b_j^2 + \frac{b_{N+1}^2}{b_{N+1}^2 + \lambda}.$$

Therefore

$$|G^N(i\omega)| \leqq \frac{1}{|1+i\omega|} \|b\|_{l^2}^2 + \frac{b_{N+1}^2}{|b_{N+1}^2 + i\omega|} \leqq \|b\|_{l^2}^2 + 1$$

for all $\omega \in \mathbf{R}$ and $N = 1, 2, \cdots$, i.e., systems (4.2) are uniformly input-output stable.

For $\varphi=(\varphi_1, \varphi_2, \cdots)\in l^2$ we get

$$(b^N)^T e^{A^N t}\pi^N\varphi = e^{-t}\sum_{j=1}^{N} b_j\varphi_j + e^{-b_{N+1}^2 t} b_{N+1}\varphi_{N+1}.$$

Therefore

$$\begin{aligned}|(b^N)^T e^{A^N t}\pi^N\varphi|^2 &\leqq (e^{-t}\|b\|_{l^2}\|\varphi\|_{l^2} + e^{-b_{N+1}^2 t} b_{N+1}|\varphi_{N+1}|)^2\\ &\leqq 2e^{-2t}\|b\|_{l^2}^2\|\varphi\|_{l^2}^2 + 2e^{-2b_{N+1}^2 t} b_{N+1}^2|\varphi_{N+1}|^2\end{aligned}$$

and

$$\int_0^\infty |(b^N)^T e^{A^N t}\pi^N\varphi|^2\,dt \leqq \|b\|_{l^2}^2\|\varphi\|_{l^2}^2 + |\varphi_{N+1}|^2 \leqq (\|b\|_{l^2}^2+1)\|\varphi\|_{l^2}^2,$$

which proves uniform output stability of systems (4.2).

Finally, if we define

$$G(\lambda)=\langle b, (\lambda I - A)^{-1} b\rangle_{l^2} = \frac{1}{1+\lambda}\|b\|_{l^2}^2,$$

then we immediately see that for $\lambda \neq -1$ (note that $b_{N+1}^2 \to 0$ as $N\to\infty$)

$$\lim_{N\to\infty} G^N(\lambda) = \begin{cases} G(\lambda) & \text{for } \lambda \neq 0,\\ G(\lambda)+1 & \text{for } \lambda = 0.\end{cases}$$

This example shows that even under additional assumptions we cannot obtain uniform convergence of the transfer functions of the approximating systems on bounded subsets of $\mathbf{R}$ in general. But we can prove the following proposition.

PROPOSITION 1. *Under the assumptions of Lemma* 1 *we have*

$$\int_{-\infty}^{\infty} \|C(i\omega I - A)^{-1}B\xi - k^N C^N(i\omega I - A^N)^{-1}B^N\rho^N\xi\|_Y^2 \to 0$$

as $N\to\infty$ *for any* $\xi\in U$.

Proof. Using Parseval's identity we get

$$\begin{aligned}&\int_{-\infty}^{\infty} \|C(i\omega I - A)^{-1}\varphi - k^N C^N(i\omega I - A^N)^{-1}\pi^N\varphi^N\|^2\,d\omega\\ &\quad= \int_0^\infty \|CS(t)\varphi - k^N C^N e^{A^N t}\pi^N\varphi^N\|^2\,dt = \|\mathscr{C}\varphi - \mathscr{C}^N\varphi^N\|_{L^2(0,\infty;Y)}^2\end{aligned}$$

for φ, $\varphi^N\in H$. Hence the result follows from Lemma 1 and (1.7) if we choose $\varphi = B\xi$ and $\varphi^N = \iota^N B^N\rho^N\xi$. □

In case of the spline scheme discussed in § 3 we have uniform convergence of the transfer functions (3.2) on compact intervals to the transfer function of the delay system (1.1) [5].

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