Notes on flat connections and the loop group

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Contents

1 Introduction

The moduli space $M_D = \mathcal{A}_D^{\text{flat}}/\mathcal{G}_D$ of framed flat G-connections over the 2-disc D can be identified with the based loop group ΩG. The symplectic action functional on the space of a paths of flat connections gives rise to a cocycle $\theta : \mathcal{A}_D^{\text{flat}} \times \mathcal{G}_D \to S^1$ and hence an action of $\mathcal{G}_D$ on $\mathcal{A}_D^{\text{flat}} \times \mathbb{C}$ via

$$g^*(A, z) = (g^*A, \theta(A, g)z).$$

The line bundle $\mathcal{L}_D = \mathcal{A}_D^{\text{flat}} \times_{\mathcal{G}_D} \mathbb{C}$ over M_D carries a canonical connection whose curvature form is equal to the symplectic form. Section 3 of his paper describes a canonical group operation

$$(A_{h_0}, z_0) \cdot (A_{h_1}, z_1) = (A_{h_0 h_1}, z_0 z_1 \lambda(h_0, h_1))$$

on $\mathcal{A}_D^{\text{flat}} \times S^1$ which commutes with the action of the gauge group. This identifies the unit circle bundle in $\mathcal{L}_D$ with the central extension $\widetilde{\Omega \mathrm{G}}$ of the loop group. The one-parameter subgroups in $\widetilde{\Omega \mathrm{G}}$ are horizontal lifts of the one-parameter subgroups of ΩG with respect to the canonical connection.

Section 4 reviews the relation between holomorphic curves in the loop group and anti-self-dual Yang-Mills instantons on the four sphere, as described by Jarvis and Norbury [8]. Their work is placed in the context of the correspondence between flat G-connections on the disc and based loops in G. Section 4 explains how the Jarvis-Norbury theorem can be viewed as a special case of Donaldson's theorem about the correspondence between Hermitian Yang-Mills connections and holomorphic bundles over Kähler manifolds Z with boundary. The Kähler manifold in question is the product $Z = S^2 \times D$ of the two-sphere and the two-disc, but the metric is singular near the boundary.

2 Flat connections and line bundles

Fix a compact connected Lie group G and choose an invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g} = \text{Lie(G)}$ such that the 3-form

$$\tau = \frac{1}{24\pi} \langle [g^{-1}dg \wedge g^{-1}dg] \wedge g^{-1}dg \rangle \in \Omega^3(G)$$

determines an integral cohomology class. Let $P \to \Sigma$ be a principal G-bundle over a compact Riemann surface Σ without boundary and denote by

M_Σ the moduli space of flat connections on P. Following [10] we use the Chern-Simons functional to construct a line bundle $\mathcal{L}_\Sigma \to M_\Sigma$ with a natural connection. The first Chern class of this line bundle agrees with the cohomology class of the symplectic form on M_Σ. Sections 2.1 and 2.2 discuss background material about flat connections and the symplectic action, Section 2.3 gives a construction of the line bundle $\mathcal{L}_\Sigma$ in terms of a natural cocycle, and Section 2.4 discusses a canonical connection on this line bundle.

2.1 Flat connections over Riemann surfaces

Let $\mathcal{A}_\Sigma$ denote the space of connections on P and $\mathcal{G}_\Sigma$ the identity component in the group of gauge transformations. Throughout we think of a connection A as an equivariant vertical Lie algebra valued 1-form on P and of a gauge tranformation g as an equivariant function from P to G. Associated to a connection A is the elliptic complex

$$\Omega^0(\Sigma, \mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\Sigma, \mathfrak{g}_P) \xrightarrow{d_A} \Omega^2(\Sigma, \mathfrak{g}_P).$$

Here $\mathfrak{g}_P \to \Sigma$ is the Lie algebra bundle associated to P via the adjoint action of G. The sections of this bundle form the Lie algebra of the gauge group $\mathcal{G}_\Sigma$, the 1-forms are the tangent vectors of $\mathcal{A}_\Sigma$, the covariant derivative $d_A : \Omega^0 \to \Omega^1$ is the infinitesimal action of $\mathcal{G}_\Sigma$, and the covariant derivative $d_A : \Omega^1 \to \Omega^2$ is the differential of the curvature function

$$\mathcal{A}_\Sigma \to \Omega^2(\Sigma, \mathfrak{g}_P) : A \mapsto F_A.$$

The space $\mathcal{A}_\Sigma$ carries a natural symplectic form given by

$$\omega(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle$$

for $\alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g}_P)$. Let us denote by

$$M_\Sigma = \mathcal{A}_\Sigma^{\mathrm{flat}} / \mathcal{G}_\Sigma$$

the moduli space of flat connections on P. In [2] Atiyah and Bott describe this space as a symplectic quotient. They observe that the action of $\mathcal{G}_\Sigma$ on $\mathcal{A}_\Sigma$ preserves the symplectic structure and that the curvature can be interpreted as a moment map. The moduli space M_Σ is a manifolds with singularities at the reducible connections. At a regular point $[A]$ the tangent space of M_Σ is the quotient

$$T_{[A]}M_\Sigma = \frac{\ker\, d_A : \Omega^1(\Sigma, \mathfrak{g}_P) \to \Omega^2(\Sigma, \mathfrak{g}_P)}{\ker\, d_A : \Omega^0(\Sigma, \mathfrak{g}_P) \to \Omega^1(\Sigma, \mathfrak{g}_P)}.$$

If we fix a metric on Σ then this quotient can be identified with the space

$$H^1_A := \ker\, d_A \cap \ker d_A{}^*$$

of harmonic 1-forms on Σ with values in $\mathfrak{g}_P$. With this description the Hodge $*$-operator $* : H^1_A \to H^1_A$ defines a complex structure on M_Σ which is compatible with the symplectic form. If $\mathrm{G} = \mathrm{SU}(2)$ then a theorem of Narasimhan-Seshadri asserts that the regular part of M_Σ can be identified, as a Kähler manifold, with the space of stable rank-2 bundles over Σ of degree zero (see [4]).

Remark 2.1. If $\mathrm{G} = \mathrm{U}(1)$ then the moduli space $M_\Sigma(\mathrm{U}(1))$ is the Jacobian torus. If G has a finite fundmental group then $M_\Sigma(\mathrm{G})$ is simply connected and the first Chern class of the tangent bundle $TM_\Sigma(\mathrm{G})$ is a positive integer multiple of the class of the symplectic form $\omega/2\pi$. The factor depends on the Lie group. In the case $\mathrm{G} = \mathrm{SU}(2)$ this factor is 8 and in the case $\mathrm{G} = \mathrm{SO}(3)$ with $w_2(P) \neq 0$ the factor is 4 (see [6]). In the case $\mathrm{G} = \mathrm{U}(2)$ with $c_1(P)$ odd, the moduli space $M_\Sigma(\mathrm{U}(2))$ fibers over the Jacobian with fibres $M_\Sigma(\mathrm{SO}(3))$.

2.2 The symplectic action functional

We examine the symplectic action on the moduli space of flat connections. Throughout we fix a base point $A_0 \in \mathcal{A}_\Sigma^{\mathrm{flat}}$. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ denote the closed unit disc in the complex plane and $D \to \mathcal{A}_\Sigma^{\mathrm{flat}} : x + iy \mapsto A(x, y)$ be any smooth function. Denote

$$A_1(t) = A(e^{2\pi i t}).$$

Then the integral of the symplectic form ω over the disc $A : D \to \mathcal{A}_\Sigma$ is given by

$$\int_D \int_\Sigma \left\langle \frac{\partial A}{\partial x} \wedge \frac{\partial A}{\partial y} \right\rangle dx \wedge dy = \frac{1}{2} \int_0^1 \int_\Sigma \left\langle (A_1(t) - A_0) \wedge \dot{A}_1(t) \right\rangle dt. \qquad (1)$$

Now suppose that $A(e^{2\pi i t}) = g(t)^* A$ for some smooth path of gauge transformations $g(t) = g(t+1) \in \mathcal{G}_\Sigma$. Then the integral (1) is independent of the choice of the connection A and depends only on the homotopy class of the loop $g : S^1 \to \mathcal{G}_\Sigma$. We claim that

$$\frac{1}{2}\int_0^1 \int_\Sigma \left\langle (g(t)^*A - A_0)\wedge\frac{\partial}{\partial t}g(t)^*A \right\rangle dt = 2\pi \deg(g : S^1 \to \mathcal{G}_\Sigma) \qquad (2)$$

for any two flat connections $A, A_0 \in \mathcal{A}_\Sigma^{\mathrm{flat}}$. Here the **degree of the loop** g is defined as follows. Think of g as a function $P \times S^1 \to \mathrm{G}$ and consider the pullback of the form $\tau \in \Omega^3(\mathrm{G})$ under this map. This form descends to $\Sigma \times S^1$ under the obvious projection

$$\pi : P \times S^1 \to \Sigma \times S^1$$

and the degree of g is defined as the integral of this form:

$$\deg(g : S^1 \to \mathcal{G}_\Sigma) = \int_{\Sigma\times S^1} \sigma, \qquad \pi^*\sigma = g^*\tau \in \Omega^3(P \times S^1).$$

By definition of τ, this number is always an integer.

Proof of (2). We first observe that the left hand side of (2) is independent of A_0 and hence we may assume $A = A_0$. Next we prove that it depends only on the homotopy class of the loop g. Hence choose a path of paths $s \mapsto g_s(t)$ with fixed endpoints and abbreviate $g = g_s(t)$, $\dot{g} = \partial_t g_s(t)$, and $\xi = g_s(t)^{-1}\partial_s g_s(t)$. Then $\partial_t(g^*A) = d_{g^*A}(g^{-1}\dot{g})$ and $\partial_s(g^*A) = d_{g^*A}\xi$. Hence

$$\begin{aligned}
&\frac{\partial}{\partial s}\frac{1}{2}\int_0^1 \int_\Sigma \left\langle (g^*A - A_0)\wedge\frac{\partial}{\partial t}g^*A \right\rangle \\
&= \frac{1}{2}\int_0^1 \int_\Sigma \left\langle d_{g^*A}\xi\wedge\frac{\partial}{\partial t}g^*A \right\rangle + \frac{1}{2}\int_0^1 \int_\Sigma \left\langle (g^*A - A_0)\wedge\frac{\partial}{\partial t}d_{g^*A}\xi \right\rangle \\
&= \int_0^1 \int_\Sigma \left\langle d_{g^*A}\xi\wedge\frac{\partial}{\partial t}g^*A \right\rangle \\
&= \int_0^1 \int_\Sigma \left\langle d_{g^*A}\xi\wedge d_{g^*A}(g^{-1}\dot{g}) \right\rangle \\
&= 0.
\end{aligned}$$

This proves that the left hand side of (2) depends only on the homotopy class of the path $t \mapsto g(t)$ with fixed endpoints.

Next we prove (2) in the case where $P = \Sigma \times \mathrm{G}$ is the product bundle and $A = A_0$ is the obvious product connection. Identify $\mathcal{A}_\Sigma$ with the space $\Omega^1(\Sigma, \mathfrak{g})$ of Lie algebra valued 1-forms so that $A_0 = 0$. Then

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \int_\Sigma \left\langle g^{-1}dg \wedge \frac{\partial}{\partial t} g^{-1}dg \right\rangle \, dt \\
&\quad = \frac{1}{2}\int_0^1 \int_\Sigma \left\langle g^{-1}dg \wedge \left(d(g^{-1}\dot{g}) + [g^{-1}dg, g^{-1}\dot{g}] \right) \right\rangle \, dt \\
&\quad = \frac{1}{4}\int_0^1 \int_\Sigma \left\langle [g^{-1}dg \wedge g^{-1}dg], g^{-1}\dot{g} \right\rangle \, dt \\
&\quad = 2\pi \deg(g).
\end{aligned}
$$

The last equality follows from the fact that the pullback of the integral differential form $\tau \in \Omega^3(G)$ under the map $g : \Sigma \times [0,1] \to \mathrm{G}$ is given by

$$
g^*\tau = \frac{1}{8\pi} \left\langle [g^{-1}dg \wedge g^{-1}dg], g^{-1}\dot{g} \right\rangle \wedge \, dt.
$$

By definition, the integral of this 3-form is the degree of g. This proves (2) in the case of the product bundle. In general, use the flat connection A_0 to trivialize the bundle P in a neighbourhood of a point. It follows from elementary arguments in homotopy theory that every loop $g : \mathbb{R}/\mathbb{Z} \to \mathcal{G}_\Sigma$ is homotopic to a loop which is equal to $\mathbb{1}$ outside our given neighbourhood in which the bundle has been trivialized. Hence (2) follows by combining the homotopy invariance of the left hand side in (2) with the result for the trivial bundle. □

2.3 A cocycle

The symplectic action functional gives rise to a cocycle $\theta : \mathcal{A}_\Sigma^{\mathrm{flat}} \times \mathcal{G}_\Sigma \to S^1$ given by

$$
\theta(A, g) = \exp\left(\frac{i}{2} \int_0^1 \int_\Sigma \left\langle (g(t)^*A - A_0) \wedge \frac{\partial}{\partial t} g(t)^*A \right\rangle \, dt \right), \tag{3}
$$

where $[0,1] \to \mathcal{G}_\Sigma : t \mapsto g(t)$ is a path of gauge transformations with $g(0) = \mathbb{1}$ and $g(1) = g$. The formula (2) shows that the right hand side of (3) depends only on the endpoint of the path $t \mapsto g(t)$. This is reminiscent of a construction by Ramdas, Singer, and Weitsman in [10]. The next lemma summarizes the basic properties of θ.

Lemma 2.2. **(i)** *The differential of θ is given by*

$$\frac{\delta\theta(A,g)}{\theta(A,g)} = \frac{i}{2}\int_\Sigma \langle \delta A\wedge(g_*A_0 - A_0)\rangle + \frac{i}{2}\int_\Sigma \left\langle (g^*A - A_0)\wedge d_{g^*A}(g^{-1}\delta g)\right\rangle.$$

(ii) *For $A, A_0 \in \mathcal{A}_\Sigma^{\mathrm{flat}}$ and $g \in \mathcal{G}_\Sigma$,*

$$\theta(A,g) = \theta(A_0,g)\exp\left(\frac{i}{2}\int_\Sigma \langle (A-A_0)\wedge(g_*A_0 - A_0)\rangle\right).$$

(ii) *For $A \in \mathcal{A}_\Sigma^{\mathrm{flat}}$ and $g, h \in \mathcal{G}_\Sigma$,*

$$\theta(A,gh) = \theta(A,g)\theta(g^*A,h).$$

Proof. The formula for the derivative of θ with respect to g is obvious from the definition. To differentiate θ with respect to A, we choose a smooth path $s \mapsto A_s$ of flat connections and abbreviate

$$\theta_s = \theta(A(s),g), \qquad \alpha = \delta A = \partial_s A_s.$$

Then

$$\begin{aligned}
\frac{\partial_s\theta_s}{\theta_s} &= \frac{i}{2}\frac{\partial}{\partial s}\int_0^1\int_\Sigma \left\langle (g(t)^*A - A_0)\wedge\frac{\partial}{\partial t}g(t)^*A\right\rangle dt \\
&= \frac{i}{2}\int_0^1\int_\Sigma \left\langle g(t)^{-1}\alpha g(t)\wedge\frac{\partial}{\partial t}g(t)^*A\right\rangle dt \\
&\quad + \frac{i}{2}\int_0^1\int_\Sigma \left\langle (g(t)^*A - A_0)\wedge\frac{\partial}{\partial t}g(t)^{-1}\alpha g(t)\right\rangle dt \\
&= i\int_0^1\int_\Sigma \left\langle g(t)^{-1}\alpha g(t)\wedge\frac{\partial}{\partial t}g(t)^*A\right\rangle dt \\
&\quad + \frac{i}{2}\int_\Sigma \langle (g^*A - A_0)\wedge g^{-1}\alpha g\rangle - \frac{i}{2}\int_\Sigma \langle (A - A_0)\wedge\alpha\rangle \\
&= i\int_0^1\int_\Sigma \langle \alpha\wedge d_A(\dot g(t)g(t)^{-1})\rangle\, dt + \frac{i}{2}\int_\Sigma \langle (A_0 - g_*A_0)\wedge\alpha\rangle \\
&= \frac{i}{2}\int_\Sigma \langle \alpha\wedge(g_*A_0 - A_0)\rangle.
\end{aligned}$$

The last but one identity uses the formula $g(g^*A - A_0)g^{-1} = A - g_*A_0$ and the last identity uses $d_A\alpha = 0$. Thus we have proved (i). Assertion (ii) follows by integrating (i). To prove (iii) use paths $g(t)$ and $h(t)$ such that $g(t) = g$ for $t \geq 1/2$ and $h(t) = \mathbb{1}$ for $t \leq 1/2$. Then the result follows easily. $\square$

The line bundle $\mathcal{L}_\Sigma \to M_\Sigma$ can be expressed as the quotient

$$\mathcal{L}_\Sigma = \frac{\mathcal{A}_\Sigma^{\mathrm{flat}} \times \mathbb{C}}{\mathcal{G}_\Sigma},$$

where the action of $\mathcal{G}_\Sigma$ on $\mathcal{A}_\Sigma^{\mathrm{flat}} \times \mathbb{C}$ is given by

$$g^*(A, z) = (g^*A, \theta(A, g)z).$$

The cocycle condition in Lemma 2.2 (iii) shows that this is a group action. In the next section we shall see that the bundle $\mathcal{L}_\Sigma$ carries a natural connection.

Remark 2.3. By Lemma 2.2 (i), the infinitesimal action of $\xi \in \mathrm{Lie}(\mathcal{G}_\Sigma) = \Omega^0(\Sigma, \mathfrak{g}_P)$ on the space $\mathcal{A}_\Sigma \times \mathbb{C}$ is given by the vector field

$$(A, z) \mapsto \left(d_A\xi, z\frac{i}{2}\int_\Sigma \langle (A - A_0)\wedge d_A\xi\rangle \right)$$

on $\mathcal{A}_\Sigma^{\mathrm{flat}} \times \mathbb{C}$.

Remark 2.4. Both the bundle $\mathcal{L}_\Sigma$ and the symplectic form on M_Σ depend on a choice of the inner product on $\mathfrak{g}$. If G is a simple Lie group then there is only one invariant inner product with the required properties, but in general there are many such inner products and this corresponds to the fact that $H^2(M_\Sigma; \mathbb{Z})$ is a $\mathbb{Z}$-module of rank bigger than 1. This will become more transparent in the next section where we discuss the first Chern class of the bundle $\mathcal{L}_\Sigma$.

Remark 2.5. The bundle $\mathcal{L}_\Sigma \to M_\Sigma$ extends naturally to the infinite dimensional quotient $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$. Correspondingly there is an extended cocycle

$$\theta : \mathcal{A}_\Sigma \times \mathcal{G}_\Sigma \to S^1$$

given by

$$\theta(A, g) = \exp\left(\frac{i}{2}\int_0^1 \int_\Sigma \left(\left\langle (A(t) - A_0)\wedge \dot{A}(t) \right\rangle - 2\left\langle F_{A(t)} - F_{A_0}, \Phi(t) \right\rangle \right) dt \right),$$

where

$$A(t) := g(t)^*A \in \mathcal{A}_\Sigma, \qquad \Phi(t) := g(t)^{-1}\dot{g}(t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$$

for all t, and $[0,1] \to \mathcal{G}_\Sigma : t \mapsto g(t)$ is a path of gauge transformations such that

$$g(0) = 1\!\!1, \qquad g(1) = g.$$

The exponent in the formula for θ is the Chern-Simons functional of the connection $A(t) + \Phi(t)\,dt$ over $\Sigma \times [0,1]$. This functional agrees with the symplectic action when A and A_0 are flat. In the case of a trivial bundle $P = \Sigma \times \mathrm{G}$ an alternative definition of θ can be given in terms of the Chern-Simons functional on a 3-manifold Y with boundary $\partial Y = \Sigma$ (see Ramadas-Singer-Weitsman [10]). In this case the Chern-Simons functional over Y can be viewed as a section of the pullback of the line bundle $\mathcal{L}_\Sigma \to \mathcal{A}_\Sigma$ under the obvious restriction map $\mathcal{A}_Y \to \mathcal{A}_\Sigma$.

2.4 A connection

Sections of $\mathcal{L}_\Sigma$ can be represented as smooth functions $u : \mathcal{A}_\Sigma^{\mathrm{flat}} \to \mathbb{C}$ which satisfy the condition

$$u(g^*A) = \theta(A,g)u(A) \tag{4}$$

for $A \in \mathcal{A}_\Sigma^{\mathrm{flat}}$ and $g \in \mathcal{G}_\Sigma$. An explicit formula for a connection on $\mathcal{L}_\Sigma$ is

$$\nabla_\alpha u(A) = du(A)\alpha - u(A)\frac{i}{2}\int_\Sigma \langle (A - A_0)\wedge\alpha\rangle \tag{5}$$

for $A \in \mathcal{A}_\Sigma^{\mathrm{flat}}$, $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$, and a section $u : \mathcal{A}_\Sigma^{\mathrm{flat}} \to \mathbb{C}$.

Lemma 2.6. *(5) defines a connection on $\mathcal{L}_\Sigma$ with curvature form $-i\omega$.*

Proof. That the right hand side of (5) is a 1-form on M_Σ with values in $\mathcal{L}_\Sigma$ is equivalent to the identities

$$\nabla_{g^{-1}\alpha g} u(g^*A) = \theta(A,g)\nabla_\alpha u(A), \qquad \nabla_{d_A\xi} u(A) = 0, \tag{6}$$

for $A \in \mathcal{A}_\Sigma^{\mathrm{flat}}$, $g \in \mathrm{G}$, $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_P)$, and $\xi \in \Omega^0(\Sigma, \mathfrak{g}_P)$. To prove the first identity in (6) differentiate (4) with respect to A and use Lemma 2.2 (i) to obtain

$$\begin{aligned}
&du(g^*A)g^{-1}\alpha g - \theta(A,g)du(A)\alpha \\
&= \frac{i}{2}\theta(A,g)u(A)\int_\Sigma \langle (A_0 - g_*A_0)\wedge\alpha\rangle \\
&= \frac{i}{2}u(g^*A)\int_\Sigma \left\langle (g^*A - A_0)\wedge g^{-1}\alpha g\right\rangle - \frac{i}{2}\theta(A,g)u(A)\int_\Sigma \langle (A - A_0)\wedge\alpha\rangle\,.
\end{aligned}$$

The last equality uses the formula $A - g_*A_0 = g(g^*A - A_0)g^{-1}$. This proves the first equation in (6). The second equation follows by differentiating (4) with respect to g and using Lemma 2.2 (i). Thus we have proved that (5) is a connection on $\mathcal{L}_\Sigma$. To examine the curvature we choose a 2-parameter family $(s,t) \mapsto A(s,t)$ of flat connections on P and a function $(s,t) \mapsto z(s,t)$, thought of as a section of the bundle $\mathcal{L}_\Sigma$ along $[A(s,t)]$. Then

$$\nabla_s z = \partial_s z - \frac{i}{2} \int_\Sigma \langle (A - A_0) \wedge \partial_s A \rangle \, z.$$

One checks easily, by direct computation, that the curvature of (5) is given by

$$\nabla_s \nabla_t z - \nabla_t \nabla_s z = -i \int_\Sigma \langle \partial_s A \wedge \partial_t A \rangle \, z.$$

This proves the lemma. □

Corollary 2.7. *The first Chern class of the line bundle $\mathcal{L}_\Sigma \to M_\Sigma$ is (an integral lift of) the cohomology class of the form $\omega/2\pi$.*

Proof. The first Chern class of a complex line bundle is represented by $i/2\pi$ times the curvature form. Hence the result follows from Lemma 2.6. □

Remark 2.8. The horizontal lift of a point $z_0 \in \mathbb{C}$ along a path $t \mapsto A(t)$ with respect to the connection (5) is given by

$$z(t) = z_0 \exp\left(\frac{i}{2} \int_0^t \int_\Sigma \left\langle (A(s) - A_0) \wedge \dot{A}(s) \right\rangle \, ds \right). \tag{7}$$

It follows from Lemma 2.2 (i), by direct calculation, that, for every path of gauge transformation $g(t) \in \mathcal{G}_\Sigma$, the curve $[A(t), z(t)]$ is horizontal if and only if the curve $[g(t)^*A(t), \theta(A(t), g(t))z(t)]$ is horizontal.

3 The loop group

The discussion of the previous section generalizes easily to Riemann surfaces with boundary. In this section we shall mainly consider the case where the Riemann surface in question is the unit disc D. Section 3.1 reviews the Kähler structure on the group of contractible based loops in G. Section 3.2 discusses the identification of the based loop group ΩG with the moduli space M_D of framed flat G-connections on the disc. Section 3.3 describes the central extension $\widetilde{\Omega \mathrm{G}}$ of the loop group in terms of a suitable group structure on the pre-quantum line bundle $\mathcal{L}_D \to M_D$ over the moduli space of framed flat G-connections over the disc. Section 3.4 discusses the one-parameter subgroups in $\widetilde{\Omega \mathrm{G}}$. Section 3.5 explains how the constructions of this section fit into the framework of Pressley and Segal [9].

3.1 The Kähler structure on the loop group

Denote by $L\mathrm{G}$ the group of contractible free loops in G and by

$$\Omega \mathrm{G} = \left\{ \gamma : S^1 \to \mathrm{G} \,|\, \gamma(1) = \mathbb{1},\, \gamma \sim \mathbb{1} \right\} = \mathrm{G} \backslash L\mathrm{G}$$

the group of contractible based loops in G. The tangent space of $\Omega \mathrm{G}$ at a loop γ is the space of vector fields along γ (vanishing at 1) and can be identified, by right translation, with the loop algebra $\Omega \mathfrak{g}$ of smooth functions $\zeta : S^1 \to \mathfrak{g}$ which satisfy $\zeta(1) = 0$. Thus we write a tangent vector to $\Omega \mathrm{G}$ at γ in the form $\zeta\gamma$ where $\zeta \in \Omega\mathfrak{g}$. With this notation the symplectic form on $\Omega \mathrm{G}$ is given by

$$\omega_\gamma(\zeta\gamma, \eta\gamma) = \int_{S^1} \langle d\zeta, \eta \rangle$$

for $\zeta, \eta \in \Omega\mathfrak{g}$. The complex structure on $\Omega\mathfrak{g}$ can be expressed in terms of the complexified Lie algebra $\mathfrak{g}^c := \mathfrak{g} \oplus i\mathfrak{g}$ as follows. Write a loop $\zeta \in \Omega\mathfrak{g}$ as a Fourier series

$$\zeta(e^{2\pi i t}) = \sum_{n>0} \left(\zeta_n e^{2\pi i n t} + \bar{\zeta}_n e^{-2\pi i n t} \right)$$

with $\zeta_n = \xi_n + i\eta_n \in \mathfrak{g}^c$ and $\bar{\zeta}_n = \xi_n - i\eta_n$. In this notation the complex structure on $\Omega\mathfrak{g}$ is given by the formula

$$(I\zeta)(e^{2\pi i t}) = \sum_{n>0} \left(i\zeta_n e^{2\pi i n t} - i\bar{\zeta}_n e^{-2\pi i n t} \right) \tag{8}$$

for $\zeta \in \Omega\mathfrak{g}$. At a general loop $\gamma \in \Omega \mathrm{G}$ the complex structure is given by $\gamma\zeta \mapsto \gamma I \zeta$.

Remark 3.1. Let $\|\cdot\|$ denote the norm on the loop algebra, induced by the symplectic form ω and the complex structure I. This is the $H^{1/2}$-norm. In terms of the Fourier coefficients it is given by

$$\|\zeta\|^2 = \omega(\zeta, I\zeta) = 4\pi \sum_{n>0} n|\zeta_n|^2. \qquad \square$$

Remark 3.2. There is a natural right action of $L\mathrm{G}$ on $\Omega\mathrm{G}$ by

$$g^*\gamma(z) = g(1)^{-1}\gamma(z)g(z) \tag{9}$$

for $g \in L\mathrm{G}$ and $\gamma \in \Omega\mathrm{G}$. This action preserves the symplectic and complex structures on $\Omega\mathrm{G}$. Thus, for every contractible free loop $g \in L\mathrm{G}$, the map $\Omega\mathrm{G} \to \Omega\mathrm{G} : \gamma \mapsto g^*\gamma$ is a Kähler isomorphism.

3.2 Flat connections over the disc

Every principal G-bundle over D admits a trivialization and we shall therefore only consider the product bundle $P = D \times \mathrm{G}$. Thus a G-connection over D is simply a Lie-algebra valued 1-form $A \in \Omega^1(D, \mathfrak{g})$. Consider the space

$$\mathcal{A}_D^{\mathrm{flat}} := \left\{A \in \Omega^1(D, \mathfrak{g}) \,|\, F_A = 0\right\}$$

of flat G-connections on D under the action of the gauge group

$$\mathcal{G}_D := \{g \in C^\infty(D, \mathrm{G}) \,|\, g|_{\partial D} = 1\!\!1\}\,.$$

The quotient

$$M_D := \mathcal{A}_D^{\mathrm{flat}}/\mathcal{G}_D$$

can be identified with the loop group $\Omega\mathrm{G}$ as follows. Let

$$\mathrm{Map}_0(D, \mathrm{G}) := \{h \in C^\infty(D, \mathrm{G}) \,|\, h(1) = 1\!\!1\}$$

and suppose that the subgroup $\mathcal{G}_D$ acts on $\mathrm{Map}_0(D, \mathrm{G})$ on the left. Then there is a bijection

$$\mathrm{Map}_0(D, \mathrm{G}) \longrightarrow \mathcal{A}_D^{\mathrm{flat}} : h \mapsto A_h := h^{-1}dh.$$

The formula $A_{hg} = g^*A_h$ shows that this map is equivariant and hence induces a bijection of quotient spaces

$$\Omega\mathrm{G} \cong \mathcal{G}_D\backslash\mathrm{Map}_0(D, \mathrm{G}) \longrightarrow M_D.$$

In [3] Thomas Davies proved that the diffeomorphism $\Omega\mathrm{G} \to M_D$ identifies the natural Kähler structures up to a sign.

Proposition 3.3 (Davies). *The map $\Omega\mathrm{G} \to M_D : [h] \mapsto [A_h]$ is a Kähler anti-isomorphism.*

Proof. The tangent space $T_{[A]}M_D$ is the quotient

$$T_{[A]}M_D = \frac{\{d\xi \,|\, \xi : D \to \mathfrak{g},\, \xi(1) = 0\}}{\{d\xi \,|\, \xi : D \to \mathfrak{g},\, \xi|_{\partial D} = 0\}}.$$

This space can be identified with the space of harmonic functions $\xi : D \to \mathfrak{g}$ which vanish at 1, i.e.

$$T_{[A]}M_D \cong \{d\xi \,|\, \xi : D \to \mathfrak{g},\, d^*d\xi = 0,\, \xi(1) = 0\}.$$

With this identification the complex structure on $T_{[A]}M_D$ is given by the Hodge $*$-operator $d\xi \mapsto *d\xi$. Given a loop $\zeta \in \Omega\mathfrak{g}$ with Fourier coefficients $\zeta_n \in \mathfrak{g}^c$ denote by $\xi_\zeta : D \to \mathfrak{g}$ the unique harmonic extension, given by

$$\xi_\zeta(z) = \sum_{n>0} \left(\zeta_n z^n + \bar{\zeta}_n \bar{z}^n\right).$$

The function

$$\Omega\mathfrak{g} \to T_{[0]}M_D : \zeta \mapsto -d\xi_\zeta$$

is the differential of our diffeomorphism at $\mathbb{1}$. We must prove that it intertwines the two complex structures. Since $*d\xi = -d\xi \circ i$ and $\bar{\partial}\xi = \frac{1}{2}(d\xi + i d\xi \circ i)$ we obtain

$$*\bar{\partial}\xi = i\bar{\partial}\xi, \qquad *\partial\xi = -i\partial\xi.$$

This implies

$$*d\xi_\zeta = i\bar{\partial}\xi_\zeta - i\partial\xi_\zeta = -\bar{\partial}\xi_{I\zeta} - \partial\xi_{I\zeta} = -d\xi_{I\zeta}.$$

Here the second identity follows from (8). This proves that the diffeomorphism $[h] \mapsto [A_h]$ reverses the complex structures. To prove that it reverses the symplectic structures consider the differential of the diffeomorphism $h \mapsto A_h$ before taking the quotient. At a general point h this differential is given by

$$\xi h \mapsto d_{A_h}(h^{-1}\xi h) = h^{-1}(d\xi)h$$

for $h \in \mathrm{Map}_0(D, \mathrm{G})$ and $\xi \in \mathrm{Map}_0(D, \mathfrak{g})$. Hence the formula

$$\int_D \langle h^{-1}(d\xi)h \wedge h^{-1}(d\eta)h\rangle = \int_D \langle d\xi \wedge d\eta\rangle = \int_{\partial D} \langle \xi, d\eta\rangle = -\omega_h(\xi h, \eta h)$$

shows that $[h] \mapsto [A_h]$ is an anti-symplectomorphism. This proves the proposition. □

In [5] Donaldson proved that, for general Riemann surfaces Σ with boundary, the (infinite dimensional) moduli space M_Σ of framed flat G-connections over Σ can be naturally identified with the space of isomorphism classes of holomorphic G^c-bundles over Σ with trivialization over the boundary. This theorem will be discussed further in Section 4.1. In the case $\Sigma = D$ it reduces to the well known factorization theorem $\Omega\mathrm{G} = \mathrm{Hol}(D, \mathrm{G}^c)\backslash L\mathrm{G}^c$ (cf. Pressley-Segal [9]).

3.3 The central extension

In [9] Pressley and Segal describe a natural central extension $\widetilde{\Omega \mathrm{G}}$ of the loop group $\Omega\mathrm{G}$:

$$1 \to S^1 \hookrightarrow \widetilde{\Omega \mathrm{G}} \to \Omega\mathrm{G} \to 1.$$

The purpose of this section is to give an explicit presentation of the central extension as the circle bundle associated to the line bundle

$$\mathcal{L}_D = \mathcal{A}_D \times_{\mathcal{G}_D} \mathbb{C} \to M_D$$

over the moduli space M_D of framed flat G-connections over the disc. Recall that the action of $\mathcal{G}_D$ on the product $\mathcal{A}_D \times \mathbb{C}$ is determined by the cocycle

$$\theta : \mathrm{Map}_0(D, \mathrm{G}) \times \mathcal{G}_D \to S^1$$

given by

$$\theta(h, g) = \exp\left(\frac{i}{2}\int_0^1 \int_D \left\langle A_{hg(t)} \wedge \frac{\partial}{\partial t} A_{hg(t)} \right\rangle dt\right) \tag{10}$$

for $h \in \mathrm{Map}_0(D, \mathrm{G})$ and $g \in \mathcal{G}_D$. Here $[0,1] \to \mathcal{G}_D : t \mapsto g(t)$ is a smooth path of gauge transformations such that $g(0) = \mathbb{1}$ and $g(1) = g$. Note that this formula coincides with (3) and that we have chosen $A_0 = 0$. Note also that the path $t \mapsto g(t)$ is required to satisfy the boundary condition $g(t)|_{\partial D} = \mathbb{1}$ for all t. With this understood, all the assertions of Lemma 2.2 continue to hold for Riemann surfaces with boundary. In particular, θ satisfies the cocycle condition

$$\theta(h, g_1 g_2) = \theta(h, g_1)\theta(hg_1, g_2) \tag{11}$$

for $h \in \mathrm{Map}_0(D, \mathrm{G})$ and $g_1, g_2 \in \mathcal{G}_D$, and hence defines an action of $\mathcal{G}_D$ on $\mathrm{Map}_0(D, \mathrm{G}) \times S^1$ via

$$g^*(h, z) := (hg, \theta(h, g)z).$$

We denote by

$$\widetilde{\Omega \mathrm{G}} := \frac{\mathrm{Map}_0(D, \mathrm{G}) \times S^1}{\mathcal{G}_D}$$

the corresponding circle bundle over $\Omega\mathrm{G}$. By Corollary 2.7 and Proposition 3.3, its Euler class is the cohomology class of the symplectic form multiplied by $1/2\pi$.

The total space $\mathrm{Map}_0(D, \mathrm{G}) \times S^1$ carries a natural group structure which descends to $\widetilde{\Omega \mathrm{G}}$ and thus identifies this space with the central extension of $\Omega\mathrm{G}$. An explicit formula for the group operation is

$$(h_0, z_0) \cdot (h_1, z_1) = (h_0 h_1, z_0 z_1 \lambda(h_0, h_1)), \tag{12}$$

where the function $\lambda : \mathrm{Map}_0(D, \mathrm{G}) \times \mathrm{Map}_0(D, \mathrm{G}) \to S^1$ is given by

$$\lambda(h_0, h_1) = \exp\left(\frac{i}{2} \int_D \left\langle A_{h_0} \wedge A_{{h_1}^{-1}} \right\rangle\right). \tag{13}$$

The next proposition asserts that (12) is a group operation which descends to the quotient $\widetilde{\Omega \mathrm{G}}$.

Proposition 3.4. *(12) defines a group operation on* $\mathrm{Map}_0(D, \mathrm{G}) \times S^1$ *which commutes with the action of* $\mathcal{G}_D$.

Proof. Associativity of (12) is equivalent to the identity

$$\lambda(h_0, h_1)\lambda(h_0 h_1, h_2) = \lambda(h_0, h_1 h_2)\lambda(h_1, h_2) \tag{14}$$

for $h_0, h_1, h_2 \in \mathrm{Map}_0(D, \mathrm{G})$. To prove this we consider logarithms. Namely

$$\begin{aligned}
&\int_D \left\langle A_{h_0} \wedge A_{{h_1}^{-1}} \right\rangle + \int_D \left\langle A_{h_0 h_1} \wedge A_{{h_2}^{-1}} \right\rangle \\
&= -\int_D \left\langle {h_0}^{-1} dh_0 \wedge dh_1 \cdot {h_1}^{-1} \right\rangle \\
&\quad - \int_D \left\langle {h_1}^{-1} dh_1 + {h_1}^{-1}({h_0}^{-1} dh_0) h_1 \wedge dh_2 \cdot {h_2}^{-1} \right\rangle \\
&= -\int_D \left\langle {h_0}^{-1} dh_0 \wedge dh_1 \cdot {h_1}^{-1} \right\rangle - \int_\Sigma \left\langle {h_1}^{-1} dh_1 \wedge dh_2 \cdot {h_2}^{-1} \right\rangle \\
&\quad - \int_D \left\langle {h_0}^{-1} dh_0 \wedge h_1 (dh_2 \cdot {h_2}^{-1}) {h_1}^{-1} \right\rangle \\
&= \int_D \left\langle A_{h_1} \wedge A_{{h_2}^{-1}} \right\rangle + \int_D \left\langle A_{h_0} \wedge A_{(h_1 h_2)^{-1}} \right\rangle .
\end{aligned}$$

This proves (14) and hence associativity. Since

$$\lambda(h, \mathbb{1}) = \lambda(\mathbb{1}, h) = 1$$

it follows that the pair $(\mathbb{1}, 1) \in \mathrm{Map}_0(D, \mathrm{G}) \times S^1$ is the neutral element. Since

$$\lambda(h, h^{-1}) = 1$$

for all $h \in \mathrm{Map}_0(D, \mathrm{G})$, the inverse of (h, z) is given by (h^{-1}, z^{-1}). This proves the first assertion.

Next we prove that (12) commutes with the action of $\mathcal{G}_D$. One checks easily that this is equivalent to the identity

$$\theta(h_0 h_1, {h_1}^{-1} g_0 h_1 g_1)\lambda(h_0, h_1) = \theta(h_0, g_0)\theta(h_1, g_1)\lambda(h_0 g_0, h_1 g_1). \tag{15}$$

We shall first establish the formula

$$\theta(h_0 h, h^{-1} g_0 h) = \theta(h_0, g_0) \frac{\lambda(h_0 g_0, h)}{\lambda(h_0, h)} \tag{16}$$

for $h, h_0 \in \mathrm{Map}_0(D, \mathrm{G})$ and $g_0 \in \mathcal{G}_D$. To see this choose a path $[0,1] \to \mathcal{G}_D : t \mapsto g_0(t)$ such that $g_0(0) = \mathbb{1}$ and $g_0(1) = g_0$. Then

$$\begin{aligned}
&\int_0^1 \int_D \left\langle A_{h_0 g_0(t) h} \wedge \frac{\partial}{\partial t} A_{h_0 g_0(t) h} \right\rangle dt \\
&= \int_0^1 \int_D \left\langle h^{-1} dh + h^{-1} A_{h_0 g_0(t)} h \wedge h^{-1} \left(\frac{\partial}{\partial t} A_{h_0 g_0(t)} \right) h \right\rangle dt \\
&= \int_0^1 \int_D \left\langle A_{h_0 g_0(t)} \wedge \frac{\partial}{\partial t} A_{h_0 g_0(t)} \right\rangle dt + \int_0^1 \int_D \left\langle dh \cdot h^{-1} \wedge \frac{\partial}{\partial t} A_{h_0 g_0(t)} \right\rangle dt \\
&= \int_0^1 \int_D \left\langle A_{h_0 g_0(t)} \wedge \frac{\partial}{\partial t} A_{h_0 g_0(t)} \right\rangle dt + \int_0^1 \int_D \frac{\partial}{\partial t} \left\langle A_{h_0 g_0(t)} \wedge A_{h^{-1}} \right\rangle dt \\
&= \int_0^1 \int_D \left\langle A_{h_0 g_0(t)} \wedge \frac{\partial}{\partial t} A_{h_0 g_0(t)} \right\rangle dt + \int_D \Big(\left\langle A_{h_0 g_0} \wedge A_{h^{-1}} \right\rangle - \left\langle A_{h_0} \wedge A_{h^{-1}} \right\rangle \Big).
\end{aligned}$$

This proves (16). Moreover, by Lemma 2.2 (ii), $\theta(h, g) = \theta(\mathbb{1}, g)\lambda(h, g)$ and hence

$$\frac{\theta(h, g)}{\lambda(h, g)} = \frac{\theta(h', g)}{\lambda(h', g)} \tag{17}$$

for $h, h' \in \mathrm{Map}_0(D, \mathrm{G})$ and $g \in \mathcal{G}_D$.

Using (11), (16), (17), and (14), in that order, we find

$$\begin{aligned}
&\theta(h_0h_1, {h_1}^{-1}g_0h_1g_1)\lambda(h_0,h_1)\\
&\quad = \theta(h_0h_1, {h_1}^{-1}g_0h_1)\theta(h_0g_0h_1, g_1)\lambda(h_0,h_1)\\
&\quad = \theta(h_0,g_0)\lambda(h_0g_0,h_1)\theta(h_0g_0h_1,g_1)\\
&\quad = \theta(h_0,g_0)\theta(h_1,g_1)\frac{\lambda(h_0g_0,h_1)\lambda(h_0g_0h_1,g_1)}{\lambda(h_1,g_1)}\\
&\quad = \theta(h_0,g_0)\theta(h_1,g_1)\lambda(h_0g_0,h_1g_1).
\end{aligned}$$

This proves (15) and the proposition. $\square$

Proposition 3.4 gives rise to a group structure on the quotient

$$\widetilde{\Omega\mathrm{G}} = \frac{\mathrm{Map}_0(D,\mathrm{G})\times S^1}{\mathcal{G}_D}.$$

Moreover, there are obvious homomorphisms

$$S^1 \to \widetilde{\Omega\mathrm{G}} \to \Omega\mathrm{G}$$

given by $z \mapsto [\mathbb{1}, z]$ and $[h,z] \mapsto h|_{S^1}$. It is easy to see that this sequence is exact. In particular, the map $[h,z] \mapsto h|_{S^1}$ is surjective since, by assumption, $\Omega\mathrm{G}$ is the group of contractible based loops. That the kernel is the image of the homomorphism $S^1 \to \widetilde{\Omega\mathrm{G}}$ follows from the assertion $[h,z] \equiv [\mathbb{1}, \theta(h,h^{-1})z]$ whenever $h \in \mathcal{G}_D$. The next proposition characterizes the Lie algebra of $\widetilde{\Omega\mathrm{G}}$.

Proposition 3.5. *The Lie algebra* $\widetilde{\Omega\mathfrak{g}} = \mathrm{Lie}(\widetilde{\Omega\mathrm{G}})$ *is naturally isomorphic to* $\Omega\mathfrak{g}\times\mathbb{R}$ *with Lie bracket given by*

$$[(\xi,s),(\eta,t)] = ([\xi,\eta],\omega(\xi,\eta)) \tag{18}$$

for $\xi,\eta\in\Omega\mathfrak{g}$ *and* $s,t\in\mathbb{R}$.

Proof. Differentiating the equation

$$(h,1)\cdot(\exp(t\eta),1)\cdot(h,1)^{-1} = (h\exp(t\eta)h^{-1}, \lambda(h,\exp(t\eta))\lambda(h\exp(t\eta),h^{-1}))$$

with respect to t we find that the adjoint action of $(h,1)\in\mathrm{Map}_0(D,\mathrm{G})\times S^1$ on the Lie algebra $\mathrm{Map}_0(D,\mathfrak{g})\times i\mathbb{R}$ is given by

$$(h,1)\cdot(\eta,0)\cdot(h,1)^{-1} = \left(h\eta h^{-1}, \frac{i}{2}\int_D \langle d_{A_h}\eta\wedge A_h\rangle - \frac{i}{2}\int_D\langle A_h\wedge d\eta\rangle\right). \tag{19}$$

Inserting $h(t) = \exp(t\xi)$ and differentiating with respect to t we find that the Lie bracket of two pairs $(\xi, ir), (\eta, is) \in \mathrm{Map}_0(D, \mathfrak{g}) \times i\mathbb{R}$ is given by

$$[(\xi, ir), (\eta, is)] = \left([\xi, \eta], -i \int_D \langle d\xi \wedge d\eta \rangle \right).$$

Hence the map $\mathrm{Map}_0(D, \mathfrak{g}) \times i\mathbb{R} \to \Omega\mathfrak{g} \times \mathbb{R} : (\xi, ir) \mapsto (\xi|_{\partial D}, r)$ is a Lie algebra homomorphism with respect to (18). Moreover, this homomorphism is surjective, and its kernel is the space of all functions $\xi : D \to \mathfrak{g}$ which vanish on the boundary, i.e. the tangent space of the orbit of $(\mathbb{1}, 1)$ under the action of $\mathcal{G}_D$. Hence the above map induces a Lie algebra isomorphism

$$\mathrm{Lie}(\widetilde{\Omega \mathrm{G}}) = \frac{\mathrm{Map}_0(D, \mathfrak{g}) \times i\mathbb{R}}{\{\xi : D \to \mathfrak{g} \,|\, \xi|_{\partial D} = 0\}} \to \Omega\mathfrak{g} \times \mathbb{R}.$$

This proves the proposition. □

3.4 The exponential map

The circle bundle $\widetilde{\Omega \mathrm{G}} \to \Omega \mathrm{G}$ carries a canonical connection. The horizontal lift of a point $[h_0, z_0]$ along a path $[h(t)] \in \mathrm{Map}_0(D, \mathrm{G})/\mathcal{G}_D$ is given by $[h(t), z(t)]$ where

$$z(t) = z_0 \exp \left(\frac{i}{2} \int_0^t \int_D \left\langle A_{h(s)} \wedge \frac{\partial}{\partial s} A_{h(s)} \right\rangle ds \right). \tag{20}$$

(See Section 2.4.) To compute the curvature of this connection it is useful to consider the associated line bundle $\mathcal{L}_D = \mathrm{Map}_0(D, \mathrm{G}) \times_{\mathcal{G}_D} \mathbb{C}$ and work with covariant derivatives. It then follows as in Lemma 2.6 that the curvature form of the canonical connection is equal to $-i\omega$. Hence the Euler class of the circle bundle $\widetilde{\Omega \mathrm{G}} \to \Omega \mathrm{G}$ is (an integral lift of) the cohomology class $\omega/2\pi$.

Remark 3.6. Let $\mathbb{R} \to \mathcal{G}_D : t \mapsto g(t)$ be a path of gauge transformations. Then a path $[h(t), z(t)] \in \mathrm{Map}_0(D, \mathrm{G}) \times S^1$ is horizontal for the canonical connection if and only if the path $[h(t)g(t), \theta(h(t), g(t))z(t)]$ is horzontal. This follows by direct calculation as in Remark 2.8. □

The next proposition characterizes the one-parameter subgroups in the infinite dimensional Lie group $\mathrm{Map}_0(D, \mathrm{G}) \times S^1$ as horizontal lifts of the one-parameter subgroups of $\mathrm{Map}_0(D, \mathrm{G})$.

Proposition 3.7. *The one-parameter subgroups in* $\mathrm{Map}_0(D, \mathrm{G}) \times S^1$ *are given by*

$$\exp(t(\eta, ir)) = (\exp(t\eta), e^{irt} z(t; \eta))$$

for $r, t \in \mathbb{R}$ *and* $\eta \in \mathrm{Map}_0(D, \mathfrak{g})$*, where*

$$z(t; \eta) = \exp\left(\frac{i}{2}\int_0^t \int_D \left\langle A_{\exp(s\eta)} \wedge \frac{\partial}{\partial s} A_{\exp(s\eta)}\right\rangle ds\right). \tag{21}$$

Proof. One checks easily by direct calculation that the left and right actions of $(\eta, 0) \in \mathrm{Map}_0(D, \mathfrak{g}) \times i\mathbb{R}$ on the group $\mathrm{Map}_0(D, \mathrm{G}) \times S^1$ are given by

$$(\eta, 0) \cdot (h, z) = \left(\xi h, z\frac{i}{2}\int_D \langle d\eta \wedge A_{h^{-1}}\rangle\right),$$

$$(h, z) \cdot (\eta, 0) = \left(h\eta, -z\frac{i}{2}\int_D \langle A_h \wedge d\eta\rangle\right).$$

Hence the curve $\exp(t(\eta, 0)) = (h(t), z(t)) \in \mathrm{Map}_0(D, \mathrm{G}) \times S^1$ is the unique solution of the ordinary differential equation

$$\dot{h}(t) = h(t)\eta = \eta h(t), \qquad \frac{\dot{z}(t)}{z(t)} = \frac{i}{2}\int_D \langle d\eta \wedge A_{h^{-1}}\rangle = -\frac{i}{2}\int_D \langle A_h \wedge d\eta\rangle. \tag{22}$$

In particular, $h(t) = \exp(\eta t)$. Since $\eta = h^{-1}\dot{h} = \dot{h}h^{-1}$ we have

$$h^{-1}(d\eta)h = \frac{\partial}{\partial t} h^{-1} dh, \qquad h(d\eta)h^{-1} = \frac{\partial}{\partial t} dh \cdot h^{-1}.$$

This implies

$$\int_D \left\langle dh \cdot h^{-1} + h^{-1} dh \wedge d\eta\right\rangle = 0$$

(differentiate with respect to t), and hence

$$-\int_D \langle A_h \wedge d\eta\rangle = -\int_D \langle A_{h^{-1}} \wedge d\eta\rangle = \int_D \left\langle A_h \wedge h^{-1}(d\eta)h\right\rangle = \int_D \left\langle A_h \wedge \frac{\partial}{\partial t} A_h\right\rangle.$$

Here we have used the formula $A_h = -h^{-1} A_{h^{-1}} h$. This shows that the solution of (22) is given by (21) as claimed. □

We would expect that the equivalence class $[(\exp(t\eta), z(t; \eta)]$ in the quotient $\widetilde{\Omega \mathrm{G}} = \mathrm{Map}_0(D, \mathrm{G}) \times_{\mathcal{G}_D} S^1$ depends only on the boundary values of η. Explicitly, this can be expressed in the formula

$$z(t; \eta + \xi) = \theta(\exp(t\eta), \exp(-t\eta)\exp(t(\eta + \xi)))z(t; \eta) \tag{23}$$

for $\xi, \eta : D \to \mathfrak{g}$ with $\eta(1) = 0$ and $\xi|_{\partial D} = 0$. This formula is a direct consequence of Lemma 3.8 below with $h(t) = \exp(t\eta)$ and $h(t)g(t) = \exp(t(\eta + \xi))$.

3.5 Relation with the approach of Pressley–Segal

In [9] Pressley and Segal give an explicit presentation of an extension $\widetilde{\Gamma}$ of a group Γ in terms of the action of Γ on a simply connected symplectic manifold (X, ω) with an integral symplectic form. Their construction is as follows. Fix a base point $x_0 \in X$ and denote by

$$\mathcal{P} \subset \Gamma \times \mathrm{Map}([0,1], X)$$

the set of all pairs (γ, p) where $p : [0,1] \to X$ is a path connecting $p(0) = x_0$ with $p(1) = \gamma x_0$. The symplectic action gives rise to an equivalence relation on $\mathcal{P} \times S^1$ given by

$$[\gamma, p, u] \equiv [\gamma', p', u'] \qquad \Longleftrightarrow \qquad \gamma = \gamma', \quad u = \exp\left(i \int \sigma^* \omega\right) u', \tag{24}$$

where $\sigma \subset X$ is a surface with boundary $p' \# p^{-1}$. The central extension of Γ is the group $\widetilde{\Gamma}$ of equivalence classes in $\mathcal{P} \times S^1$. The group operation is given by catenation of paths and multiplication in S^1. Pressley and Segal suggest as an example the action of $\Gamma = \Omega \mathrm{G}$ on itself. Our description of $\widetilde{\Omega \mathrm{G}}$ is simply an explicit formula for this construction in the case $X = M_D$.

Consider the space

$$\mathcal{H} \subset \mathrm{Map}([0,1] \times D, \mathrm{G})$$

of smooth maps $h : [0,1] \times D \to \mathrm{G}$ which satisfy $h(0,z) = h(t,1) = \mathbb{1}$ for $z \in D$ and $0 \leq t \leq 1$, and are locally independent of t near the two ends. This space carries a binary operation

$$h_0 \# h_1(t,z) = \begin{cases} h_1(2t, z), & 0 \leq t \leq 1/2, \\ h_0(2t-1, z) h_1(1, z), & 1/2 \leq t \leq 1. \end{cases}$$

This operation is associative on the quotient space $\mathcal{H}/\sim$, where the equivalence relation is given by reparametrization. Now think of h as a smooth function $[0,1] \to \mathrm{Map}_0(D, \mathrm{G})$ and consider the symplectic action of the corresponding path $[0,1] \to \mathcal{A}_D^{\mathrm{flat}} : t \mapsto A_{h(t)}$. This gives rise to a smooth map

$$\Theta : \mathcal{H} \to S^1$$

given by

$$\Theta(h) = \exp\left(\frac{i}{2} \int_0^1 \int_D \left\langle A_{h(t)} \wedge \partial_t A_{h(t)} \right\rangle dt\right).$$

This function descends to the quotient $\mathcal{H}/\sim$ but is not a semigroup homomorphism. There is a correction term in the product formula for catenations, which involves the function λ introduced in (13). This is in fact how I discovered the formula for the group operation (12).

Lemma 3.8. *If $h_0, h_1 \in \mathcal{H}$ then*

$$\Theta(h_0 \# h_1) = \Theta(h_0)\Theta(h_1)\lambda(h_0(1), h_1(1)). \tag{25}$$

If $g, h \in \mathcal{H}$ and $g|_{[0,1]\times\partial D} = \mathbb{1}$ then

$$\Theta(hg) = \Theta(h\#g) = \Theta(h)\theta(h(1), g(1)). \tag{26}$$

Proof. If h is independent of t then one shows as in the proof of (16) that

$$\int_0^1 \int_D \left\langle A_{h_0(t)h} \wedge \partial_t A_{h_0(t)h} \right\rangle = \int_0^1 \int_D \left\langle A_{h_0(t)} \wedge \partial_t A_{h_0(t)} \right\rangle + \int_D \left\langle A_{h_0(1)} \wedge A_{h^{-1}} \right\rangle.$$

With $h = h_1(1)$ this implies (25). The second equality in (26) follows from (25) and the fact that

$$\Theta(g)\lambda(h(1), g(1)) = \theta(\mathbb{1}, g(1))\lambda(h(1), g(1)) = \theta(h(1), g(1)).$$

Next we prove that, if $\mathbb{R} \to \mathcal{H} : s \mapsto h_s$ is a smooth path such that $\partial_s h = 0$ on $\partial([0,1] \times D)$, then $\Theta(h_s)$ is independent of s. To see this note first that, if $\partial_s h = 0$ on $[0,1] \times \partial D$, then

$$\begin{aligned}
\int_0^1 \int_D \langle \partial_s A_h \wedge \partial_t A_h \rangle \, dt &= \int_0^1 \int_D \left\langle d_{A_h}(h^{-1}\partial_s h) \wedge \partial_t A_h \right\rangle \, dt \\
&= \int_0^1 \int_{\partial D} \left\langle h^{-1}\partial_s h \wedge \partial_t A_h \right\rangle \, dt \\
&= 0.
\end{aligned}$$

Here we have used the fact that $d_{A_h}\partial_t A_h = \partial_t F_{A_h} = 0$. If $\partial_s h = 0$ on $\{0,1\} \times D$ then the previous equation implies

$$\frac{\partial}{\partial s} \int_0^1 \int_D \langle A_h \wedge \partial_t A_h \rangle \, dt = \int_0^1 \frac{\partial}{\partial t} \int_D \langle A_h \wedge \partial_s A_h \rangle \, dt = 0.$$

Hence $\Theta(h_s)$ is independent of s as claimed. Now the path $h\#g$ is homotopic, by means of a homotopy which satisfies $\partial_s h = 0$ on $\partial([0,1] \times D)$, to the path which is equal to $\mathbb{1}$ for $0 \le t \le 1/2$ and equal to $h(2t-1)g(2t-1)$ for $1/2 \le t \le 1$. The value of Θ on this path is obviously equal to $\Theta(hg)$. This proves the first equality in (26). $\square$

Consider the space $\mathcal{H} \times S^1$ with the equivalence relation

$$(h, u) \equiv (h', u') \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \pi(h) = \pi(h'), \\ \Theta(h')u' = \theta(h(1), h(1)^{-1}h'(1))\Theta(h)u, \end{array} \right. \tag{27}$$

where $\pi(h) \in \Omega\mathrm{G}$ denotes the loop $t \mapsto h(1, e^{2\pi i t})$. This equivalence relation is motivated by (24). Geometrically, the term $\theta(h(1), h(1)^{-1}h'(1))$ is the symplectic action of the path running from $h(1)$ to $h'(1)$ with time independent boundary condition. Now one can define the central extension of $\Omega\mathrm{G}$ as the set of equivalence classes $(h, u) \in \mathcal{H} \times S^1$ under the above equivalence relation. The group operation on this quotient space is given by

$$[h_0, u_0] \cdot [h_1, u_1] = [h_0 \# h_1, u_0 u_1].$$

There is a natural homomorphism from $\mathcal{H} \times S^1$ to $\mathrm{Map}_0(D, \mathrm{G}) \times S^1$ given by

$$(h, u) \mapsto (h(1), \Theta(h)u).$$

This map identifies the two quotient spaces.

Remark 3.9. It follows from Lemma 3.8 and the formulae in the proof of Proposition 3.4 that the operation

$$(h_0, u_0) \cdot (h_1, u_1) = (h_0 \# h_1, u_0 u_1)$$

on $\mathcal{H} \times S^1$ preserves the equivalence relation (27).

Remark 3.10. The correction term in (25) is reminiscent of similar terms which appear in catenation formulas for the Maslov index and for generating functions in symplectic geometry. (See for example [11, 12].) This is not surprising. In all three contexts these terms arise from the symplectic action functional.

4 Holomorphic curves and instantons

The correspondence between instantons over the 4-sphere and holomorphic maps into the loop group was first observed by Atiyah [1]. In [8] Jarvis and Norbury found a new approach to this correspondence. They use the fact that the complement $S^4 - S^1$ is conformally diffeomorphic to the product $S^2 \times D$ of the 2-sphere and the 2-disc with the round and hyperbolic metrics, respectively. Below we review their result in the light of the Kähler anti-isomorphism between the loop group and the moduli space of framed flat connections on the disc. From this point of view the Jarvis-Norbury theorem appears as a counterpart to a construction in [7] which relates anti-self-dual instantons on $S^2 \times \Sigma$ with holomorphic curves in M_Σ, where Σ is a closed Riemann surface. In the present context the metric on the 2-disc is multiplied by a small constant ε. In the $\varepsilon \to 0$ limit the anti-self-dual instantons over the 4-sphere degenerate to anti-holomorphic maps from the 2-sphere into the loop group. The study of such an adiabatic limit was suggested by Donaldson in [5].

4.1 Framed holomorphic bundles

The correspondence between flat connections and the loop group was considered by Donaldson in [5] and he observed the analogy between the Narasimhan-Seshadri theorem for closed Riemann surfaces and factorization theorems for loop groups (cf. Pressley–Segal [9]). Before explaining this analogy we discuss a more general theorem in [5] about the relation between Hermitian Yang-Mills G-connections over Kähler manifolds with boundary and holomorphic G^c-bundles.

Let Z be a connected Kähler manifold with nonempty boundary. We assume that the Lie group G is embedded in the unitary group $\mathrm{U}(n)$ so that the complexified group G^c is embedded in $\mathrm{GL}(n, \mathbb{C})$ and its Lie algebra $\mathfrak{g}^c = \mathfrak{g} + i\mathfrak{g}$ is embedded in $\mathbb{C}^{n \times n}$. Let us assume for simplicity that our G^c-bundle over Z is topologically trivial. For example, this is true whenever Z is a Riemann surface, or whenever Z has real dimension 4 and G is simply connected. Under this assumption the set of G-connections over Z can be identified with the space $\mathcal{A}_Z = \Omega^1(Z, \mathfrak{g})$ of Lie algebra valued 1-forms. Denote by

$$\mathcal{A}_Z^{0,2} = \left\{ A \in \Omega^1(Z, \mathfrak{g}) \,|\, F_A^{0,2} = 0 \right\}$$

the subspace of those connections whose curvature has vanishing $(0,2)$-part. This space can be naturally identified with the space of holomorphic G^c-bundles over Z via the Cauchy Riemann operator $\bar{\partial}_A = \bar{\partial} + A^{0,1}$. The condition $F_A^{0,2} = 0$ is equivalent to $\bar{\partial}_A \circ \bar{\partial}_A = 0$. Consider the gauge group

$$\mathcal{G}_Z = \{ g : Z \to \mathrm{G} \,|\, g|_{\partial Z} = \mathbb{1} \}.$$

This group acts freely on $\mathcal{A}_Z^{0,2}$ and the action preserves the symplectic and complex structures. As in the case of manifolds without boundary (cf.

Atiyah-Bott [2] for Riemann surfaces), the moment map of this action is the part of the curvature parallel to the Kähler form ω:

$$\mathcal{A}_Z^{0,2} \to \Omega^0(Z, \mathfrak{g}) : A \mapsto \langle \omega, F_A \rangle.$$

Thus the zero set of the moment map is the set of connections $A \in \Omega^1(Z, \mathfrak{g})$ which satisfy

$$F_A^{0,2} = 0 \qquad \langle \omega, F_A \rangle = 0. \tag{28}$$

These are the Hermitian Yang-Mills G-connections over Z. In the case of Riemann surfaces the solutions of (28) are the flat G-connections and in the case of 4-manifolds they are the anti-self-dual Yang-Mills G-connections. In general, we denote the symplectic quotient by

$$\mathcal{M}^{\mathrm{HYM}}(Z, \mathrm{G}) = \mathcal{A}_Z^{\mathrm{HYM}} / \mathcal{G}_Z = \mathcal{A}_Z^{0,2} /\!\!/ \mathcal{G}_Z.$$

Note that this quotient is infinite dimensional whenever the boundary of Z is nonempty. If $Z = X$ is a 4-manifold we write $\mathcal{M}^{\mathrm{asd}}(X, \mathrm{G}) = \mathcal{M}^{\mathrm{HYM}}(X, \mathrm{G})$ and if $Z = \Sigma$ is a Riemann surface we write $M_\Sigma = \mathcal{M}^{\mathrm{HYM}}(\Sigma, \mathrm{G})$.

Exercise 4.1. Prove that every 2-form $\tau \in \Omega^{1,1}(Z, \mathfrak{g})$ with $d_A \tau = 0$ satisfies ${d_A}^* \tau = (d_A \langle \omega, \tau \rangle) \circ J$. In particular, for every $A \in \mathcal{A}_Z^{0,2}$,

$${d_A}^* F_A = (d_A \langle \omega, F_A \rangle) \circ J$$

and hence the solutions of (28) satisfy the Yang-Mills equation ${d_A}^* F_A = 0$.

The complexified gauge group $\mathcal{G}_Z^c = \{ g : Z \to \mathrm{G}^c \,|\, g|_{\partial Z} = \mathbb{1} \}$ acts on $\mathcal{A}_Z^{0,2}$ by

$$(g^* A)^{0,1} = g^{-1} \bar{\partial} g + g^{-1} A^{0,1} g. \tag{29}$$

The quotient $\mathcal{A}_Z^{0,2} / \mathcal{G}_Z^c$ is the space of holomorphic G^c-bundles over Z, up to isomorphisms which are the identity over the boundary. In [5] Donaldson proved the following result, which can be viewed as the analogue, for manifolds with boundary, of the Narasimhan-Seshadri theorem which, for closed Riemann surfaces, relates *irreducible* flat connections to *stable* bundles.

Theorem 4.2 (Donaldson). *Let Z be a ccompact connected Kähler manifold with nonempty boundary and $\mathrm{G} \subset \mathrm{U}(n)$ be a compact connected Lie group. Then the inclusion $\mathcal{A}_Z^{\mathrm{HYM}} \to \mathcal{A}_Z^{0,2}$ induces a bijection $\mathcal{A}_Z^{\mathrm{HYM}} / \mathcal{G}_Z \cong \mathcal{A}_Z^{0,2} / \mathcal{G}_Z^c$.*

In [5] Donaldson phrased the result in a slightly different form. He proved that, if $E \to Z$ is a holomorphic vector bundle with structure group G^c, then every Hermitian G^c-structure over the boundary extends uniquely to a Hermitian G^c-structure H over Z in such a way that the resulting Hermitian G-connection A_H on E (determined by the Hermitian form and the holomorphic structure) is a Hermitian Yang-Mills connection. We now explain the relation between these two formulations.

A holomorphic G^c-structure on $E = Z \times \mathbb{C}^n$ is a Cauchy-Riemann operator of the form $\bar\partial + \alpha$ where $\alpha \in \Omega^{0,1}(\Sigma, \mathfrak{g}^c)$ satisfies

$$\bar\partial\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \tag{30}$$

A Hermitian G^c-structure is a function $H : Z \to \mathrm{G}^c$ such that $H(z) = H(z)^*$ is a Hermitian matrix for every $z \in Z$. Any such pair α, H determines a unique connection 1-form $A = A_{H,\alpha} \in \Omega^1(Z, \mathfrak{g}^c)$ such that

$$F_A^{0,2} = 0, \qquad A^{0,1} = \alpha, \qquad dH = A^*H + HA. \tag{31}$$

To see this consider the action of a complex gauge transformation $\gamma : Z \to \mathrm{G}^c$ on triples (A, α, H) via

$$A \mapsto \gamma^{-1}d\gamma + \gamma^{-1}A\gamma, \qquad \alpha \mapsto \gamma^{-1}\bar\partial\gamma + \gamma^{-1}\alpha\gamma, \qquad H \mapsto \gamma^*H\gamma.$$

This action preserves (31) and the group acts transitively on the set of Hermitian structures. For $H = \mathbb{1}$ the unique solution of (31) is the G-connection $A_{\mathbb{1},\alpha} = \alpha - \alpha^* \in \mathcal{A}_Z^{0,2}$. Hence A is uniquely determined by α for any H.

Donaldson's theorem now asserts that, for every $\alpha \in \Omega^{0,1}(Z, \mathfrak{g}^c)$ which satisfies (30), there exists a unique Hermitian G^c-structure H such that

$$H|_{\partial Z} = \mathbb{1}, \qquad \langle \omega, F_{A_{H,\alpha}} \rangle = 0. \tag{32}$$

This can be rephrased in terms of the action of $\mathcal{G}_Z^c$ on $\mathcal{A}_Z^{0,2}$ as follows. Instead of fixing the Cauchy-Riemann operator $\bar\partial + \alpha$ and varying the Hermitian structure H we fix $H = \mathbb{1}$ and vary α. But $\Omega^{0,1}(Z, \mathfrak{g}^c)$ can be identified with $\mathcal{A}_Z^{0,2}$ via

$$\Omega^{0,1}(Z, \mathfrak{g}^c) \to \mathcal{A}_Z^{0,2} : \alpha \mapsto A_{\mathbb{1},\alpha} = \alpha - \alpha^*.$$

With this identification the natural conjugation action of $\mathcal{G}_Z^c$ on Cauchy-Riemann operators corresponds to the action on $\mathcal{A}_Z^{0,2}$ via (29). Given α, let H

be the Hermitian G^c-structure which satisfies (32). Choose $g \in \mathcal{G}^c$ such that $g^*Hg = \mathbb{1}$. Then

$$g^* A_{\mathbb{1},\alpha} = A_{\mathbb{1},g^{-1}\bar{\partial}g + g^{-1}\alpha g} = A_{g^*Hg,g^{-1}\bar{\partial}g+g^{-1}\alpha g} = g^{-1}dg + g^{-1}A_{H,\alpha}g$$

is a Hermitian Yang-Mills G-connection on Z. Thus, for every G-connection $A = A_{\mathbb{1},\alpha} \in \mathcal{A}_Z^{0,2}$ there exists a $g \in \mathcal{G}_Z^c$ such that g^*A is a Hermitian Yang-Mills connection. On the other hand, if both $A = A_{\mathbb{1},\alpha}$ and g^*A are HYM connections, then the previous equation shows that $A_{H,\alpha}$ is HYM for $H = (gg^*)^{-1}$. Hence the uniqueness part of Donaldson's theorem asserts that $H = \mathbb{1}$ and hence $g \in \mathcal{G}_Z$. This shows that Theorem 4.2 is equivalent to Theorem 1 in [5].

Remark 4.3. The idea of the proof of Theorem 4.2 is to use the gradient flow of the Yang-Mills functional

$$\mathcal{YM}(A) = \frac{1}{2}\int_Z |F_A|^2 \frac{\omega^n}{n!}$$

on $\mathcal{A}_Z^{0,2}$. A gradient flow line with fixed boundary values is a path of connections

$$[0,\infty) \to \mathcal{A}_Z^{0,2} : t \mapsto B(t)$$

which satisfies the PDE

$$\dot{B} + d_B{}^* F_B = 0, \qquad \dot{B}|_{\partial Z} = 0.$$

Using Exercise 4.1, one checks easily that the solutions of this equation have the form $B(t) = g(t)^*A$ where the path $g(t) \in \mathcal{G}_Z^c$ of complex gauge transformations satisfies

$$g^{-1}\dot{g} = i\langle \omega, F_{g^*A}\rangle.$$

In terms of the Hermitian structure $H = (gg^*)^{-1}$ this differential equation can be expressed in the form

$$\dot{H}H^{-1} = 2i\left\langle \omega, \partial_A(\bar{\partial}H \cdot H^{-1} - HA^{0,1}H^{-1}) - \bar{\partial}A^{1,0}\right\rangle. \tag{33}$$

In [5] Donaldson proved that this equation has a unique solution on the time interval $[0,\infty)$ with initial condition $H(0) = \mathbb{1}$ and boundary condition $H(t)|_{\partial Z} = \mathbb{1}$ for all t. He also proved that the limit $H_\infty = \lim_{t\to\infty} H(t)$ exists for every $A \in \mathcal{A}_Z^{0,2}$ and this limit is the required Hermitian metric and the unique stationary solution of (33).

Exercise 4.4. Let $A \in \mathcal{A}_Z^{0,2}$ and $g \in \mathcal{G}_Z^c$ be given. Prove that

$$(g^*)^{-1} F_{g^*A} g^* = \partial_A(\bar\partial H \cdot H^{-1} - HA^{0,1}H^{-1}) - \bar\partial A^{1,0},$$

where $H = (gg^*)^{-1}$.

Remark 4.5. It was observed by Jarvis and Norbury [8] that Theorem 4.2 extends to the case $Z = S^2 \times D$ equipped with the product of the round metric on the 2-sphere and the hyperbolic metric on the 2-disc. To see this one can choose a sequence of Kähler forms ω_ε converging to the given singular Kähler form and examine the corresponding sequence of Hermitian forms H_ε which satisfy

$$\left\langle \omega_\varepsilon, \partial_A(\bar\partial H_\varepsilon \cdot H_\varepsilon^{-1} - H_\varepsilon A^{0,1} H_\varepsilon^{-1}) - \bar\partial A^{1,0} \right\rangle = 0, \qquad H_\varepsilon|_{\partial Z} = \mathbb{1},$$

for some fixed connection $A \in \mathcal{A}_Z^{0,2}$. It is fairly easy to establish estimates which guarantee the existence of the limit $H_0 = \lim_{\varepsilon \to 0} H_\varepsilon$. This limit is the required solution of (33). Uniqueness follows from the maximum principle. The details are in [8] and are similar to the work of Donaldson in [5], Section 2.3.

In the case of the 2-disc Theorem 4.2 reduces to a well known factorization theorem for loop groups (see Pressley–Segal [9]) which we explain next. Every holomorphic G^c-bundle $E \to D$ can be holomorphically trivialized. This means that for every $\alpha \in \Omega^{0,1}(D, \mathfrak{g}^c)$ there exists a function $\gamma : D \to \mathrm{G}^c$ such that $\alpha = \gamma^{-1}\bar\partial\gamma$. But this function γ is not unique. If $f : D \to \mathrm{G}^c$ is any holomorphic function then $f\gamma$ and γ determine the same $(0,1)$-form. Let $A_\gamma \in \Omega^1(D, \mathfrak{g})$ denote the unique G-connection on D with $(0,1)$-part $\gamma^{-1}\bar\partial\gamma$:

$$A_\gamma^{0,1} = \gamma^{-1}\bar\partial\gamma.$$

Then the correspondence $\gamma \mapsto A_\gamma$ induces a bijection

$$\mathrm{Hol}(D, \mathrm{G}^c) \backslash \mathrm{Map}(D, \mathrm{G}^c) \to \mathcal{A}_D.$$

The formula $g^* A_\gamma = A_{\gamma g}$ for $g \in \mathcal{G}_D^c$ shows that this bijection is equivariant under the right action of $\mathcal{G}_D^c$ and hence induces a bijection of quotient spaces

$$\mathrm{Hol}(D, \mathrm{G}^c) \backslash \mathrm{Map}(D, \mathrm{G}^c) / \mathcal{G}_D^c \cong \mathcal{A}_D / \mathcal{G}_D^c \cong \mathcal{A}_D^{\mathrm{flat}} / \mathcal{G}_D.$$

The last isomorphism is from Theorem 4.2. In explicit terms, for every function $\gamma : D \to \mathrm{G}^c$, there exists a holomorphic function $f : D \to \mathrm{G}^c$, a complex gauge transformation $g \in \mathcal{G}_D^c$ (with $g|_{\partial D} = \mathbb{1}$), and a function $h \in \mathrm{Map}_0(D, \mathrm{G})$ such that

$$\gamma = fhg.$$

Moreover, in any two such factorizations $\gamma_0 = f_0 h_0 g_0$ and $\gamma_1 = f_1 h_1 g_1$ with $\gamma_0|_{\partial D} = \gamma_1|_{\partial D}$, the function ${f_0}^{-1} f_1 : D \to \mathrm{G}^c$ is holomorphic with boundary values in G. Hence ${f_0}^{-1} f_1$ is constant and hence $h_0 {h_1}^{-1}$ is constant on the boundary. But since $h_0(1) = h_1(1) = \mathbb{1}$, this implies $h_0|_{\partial D} = h_1|_{\partial D}$ and $f_0 = f_1$. In other words, every loop $\gamma : S^1 \to \mathrm{G}^c$ decomposes uniquely as a product $\gamma = fh$ where $h : S^1 \to \mathrm{G}$ is a based loop with $h(1) = \mathbb{1}$ and $f : S^1 \to \mathrm{G}^c$ extends to a holomorphic function $D \to \mathrm{G}^c$. Thus Theorem 4.2 implies the following well-known factorization result for loop groups (see Pressley-Segal [9]).

Theorem 4.6. *Let* G *be a compact Lie group. Then the inclusion* $\Omega\mathrm{G} \hookrightarrow L\mathrm{G}^c$ *of the based loop group into the complexified free loop group induces a bijection*

$$\Omega\mathrm{G} \cong \mathrm{Hol}(D, \mathrm{G}^c) \backslash L\mathrm{G}^c,$$

where $\mathrm{Hol}(D, \mathrm{G}^c)$ *denotes the group of holomorphic maps* $g : D \to \mathrm{G}^c$.

As pointed out by Donaldson [5], this factorization theorem can be proved easier directly, without relying on Theorem 4.2. On the other hand it is not clear if these direct arguments generalize to arbitrary Riemann surfaces.

4.2 Holomorphic curves in the loop group

An anti-holomorphic function $u : S^2 = \mathbb{C} \cup \{\infty\} \to \Omega\mathrm{G}$ is a solution of the partial differential equation

$$\partial_s u - I \partial_t u = 0 \tag{34}$$

which has finite energy

$$E(u) = \frac{1}{2} \int_{\mathbb{C}} \left(\|\partial_s u\|^2 + \|\partial_t u\|^2 \right) dsdt < \infty. \tag{35}$$

Here $s + it$ denotes the complex coordinate, I denotes the complex structure on $\Omega\mathrm{G}$ discussed in Section 3.1, and

$$\|\xi\| = \omega(\xi, I\xi)$$

is the $H^{1/2}$-norm on the loop algebra, induced by the symplectic and complex structures. The finite energy condition guarantees that the function

$$\mathbb{C} \setminus \{0\} \to \Omega \mathrm{G} : z \mapsto u(1/z)$$

extends to an anti-holomorphic curve $\mathbb{C} \to \Omega\mathrm{G}$ and hence u is an anti-holomorphic sphere.

Lemma 4.7. *Let* $u : S^2 \to \Omega\mathrm{G}$ *be an anti-holomorphic curve. Then*

$$\deg(u) = -\frac{1}{2\pi} E(u).$$

Proof. Let $s + it$ denote the coordinate on $\mathbb{C} \subset S^2$ and θ the coordinate on $S^1 \cong \mathbb{R}/\mathbb{Z}$. Then the energy of u is given by

$$\begin{aligned}
E(u) &= -\int_{\mathbb{C}} \omega_u(\partial_s u, \partial_t u)\, ds dt \\
&= \frac{1}{2}\int_{\mathbb{C}}\int_0^1 \Big(-\langle \partial_\theta(\partial_s u \cdot u^{-1}), \partial_t u \cdot u^{-1}\rangle + \langle \partial_\theta(\partial_t u \cdot u^{-1}), \partial_s u \cdot u^{-1}\rangle \Big) \\
&= \frac{1}{2}\int_{\mathbb{C}}\int_0^1 \Big(-\langle \partial_s(\partial_\theta u \cdot u^{-1}), \partial_t u \cdot u^{-1}\rangle + \langle \partial_t(\partial_\theta u \cdot u^{-1}), \partial_s u \cdot u^{-1}\rangle \Big) \\
&\quad - \int_{\mathbb{C}}\int_0^1 \langle \partial_\theta u \cdot u^{-1}, [\partial_s u \cdot u^{-1}, \partial_t u \cdot u^{-1}]\rangle \\
&= \frac{1}{2}\int_{\mathbb{C}}\int_0^1 \langle \partial_\theta u \cdot u^{-1}, \partial_s(\partial_t u \cdot u^{-1}) - \partial_t(\partial_s u \cdot u^{-1})\rangle \\
&\quad - \int_{\mathbb{C}}\int_0^1 \langle \partial_\theta u \cdot u^{-1}, [\partial_s u \cdot u^{-1}, \partial_t u \cdot u^{-1}]\rangle \\
&= -\frac{1}{2}\int_{\mathbb{C}}\int_0^1 \langle \partial_\theta u \cdot u^{-1}, [\partial_s u \cdot u^{-1}, \partial_t u \cdot u^{-1}]\rangle \\
&= -2\pi \deg(u).
\end{aligned}$$

The last equation uses the formula

$$u^*\tau = (4\pi)^{-1}\langle [u^{-1}\partial_s u, u^{-1}\partial_t u], u^{-1}\partial_\theta u\rangle ds \wedge dt \wedge d\theta.$$

This proves the lemma. □

Let us denote by

$$\widetilde{\mathrm{Hol}}_k(S^2, \Omega\mathrm{G}) = \{u : \mathbb{C} \to \Omega\mathrm{G} \,|\, u \text{ satisfies (34) and } \deg(u) = -k\}$$

the space of anti-holomorphic functions $u : S^2 \to \Omega\mathrm{G}$ of degree $-k$. Note that the latter condition is equivalent to $E(u) = 2\pi k$. It is important to note that the group $L\mathrm{G}$ acts on the space $\widetilde{\mathrm{Hol}}_k(S^2, \Omega\mathrm{G})$ by (9). Namely, if $u : S^2 \to \Omega\mathrm{G}$ is an anti-holomorphic curve then so is g^*u for any contractible free loop $g \in L\mathrm{G}$ and both curves have obviously the same degree. We denote the quotient by

$$\mathrm{Hol}_k(S^2, \Omega\mathrm{G}) = \widetilde{\mathrm{Hol}}_k(S^2, \Omega\mathrm{G})/L\mathrm{G}.$$

The goal of this and the following two sections is to identify this quotient with the space of charge-k instantons over the 4-sphere.

It is interesting to rewrite (34) in a more explicit form. Recall from the proof of Proposition 3.3 that $*d\xi_\zeta = \xi_{I\zeta}$ for every $\zeta \in \Omega\mathfrak{g}$ where $\xi_\zeta : D \to \mathfrak{g}$ denotes the harmonic extension of ζ. Hence (34) is equivalent to

$$d\phi + *d\psi = 0, \qquad \phi|_{\partial D} = -\partial_s u \cdot u^{-1}, \qquad \psi|_{\partial D} = -\partial_t u \cdot u^{-1}. \tag{36}$$

Here $*$ denotes the Hodge $*$-operator on D with respect to the standard metric, $\phi(s,t) : D \to \mathfrak{g}$ is the harmonic extension of $-\partial_s u \cdot u^{-1}$ for all s and t, and similarly for ψ. Here the minus sign has been introduced for notational convenience. Note that ϕ and ψ extend to smooth maps $S^2 \times D \to \mathfrak{g}$ which are harmonic over $\{z\} \times D$ for every $z \in S^2$.

Proposition 3.3 suggests an alternative way of rewriting (34), namely as a holomorphic curve $f : S^2 \to M_D$ into the moduli space of framed flat connections over the 2-disc. Any such holomorphic curve f can be represented by a smooth function $\mathbb{C} \to \mathcal{A}_D^{\mathrm{flat}} : s+it \mapsto A(s,t)$, and two Lie algebra valued functions $\Phi, \Psi : \mathbb{C} \times D \to \mathfrak{g}$ which satisfies the partial differential equation

$$\partial_s A - d_A\Phi + *(\partial_t A - d_A\Psi) = 0, \tag{37}$$

and the boundary condition

$$\Phi(s,t)|_{\partial D} = \Psi(s,t)|_{\partial D} = 0, \tag{38}$$

and have finite energy

$$E(A, \Phi, \Psi) = \frac{1}{2}\int_{\mathbb{C}}\int_D \Big(|\partial_s A - d_A\Phi|^2 + |\partial_t A - d_A\Psi|^2\Big) < \infty. \tag{39}$$

For each $s+it \in \mathbb{C}$ we think of $\Phi(s,t)$ and $\Psi(s,t)$ as Lie algebra valued functions on the 2-disc D which vanish on the boundary. In other words, Φ and Ψ are functions from $\mathbb{C} \to \mathrm{Lie}(\mathcal{G}_D)$. They satisfy the condition

$$d_A{}^* d_A \Phi = d_A{}^* \partial_s A, \qquad d_A{}^* d_A \Psi = d_A{}^* \partial_t A. \tag{40}$$

The next remark shows that Φ and Ψ are uniquely determined by A and condition (40).

Remark 4.8. Given a smooth function $\mathbb{C} \to \mathrm{Map}_0(D, \mathrm{G}) : s+it \mapsto h(s,t)$ denote $A := A_h := h^{-1}dh$ and, for all s and t, let $\phi(s,t)$ be the unique harmonic extension of $-\partial_s h \cdot h^{-1}|_{\partial D}$. Then

$$\Phi_h := h^{-1}\partial_s h + h^{-1}\phi h$$

is the unique solution of (40) with $\Phi|_{\partial D} = 0$.

Remark 4.9. The gauge group

$$\mathcal{G} = \{g : \mathbb{C} \times D \to \mathrm{G} \,|\, \partial_s g|_{\mathbb{C}\times\partial D} = \partial_t g|_{\mathbb{C}\times\partial D} = 0\}. \tag{41}$$

acts on the space of solutions of (37) and (38) via

$$\tilde{A} = g^* A, \qquad \tilde{\Phi} = g^{-1}\partial_s g + g^{-1}\Phi g, \qquad \tilde{\Psi} = g^{-1}\partial_t g + g^{-1}\Psi g. \tag{42}$$

These equations are to be understood pointwise for all s and t. Note that the transformed solution has the same energy as the original one. If we think of $\Xi = A + \Phi\, ds + \Psi\, dt$ as a connection over $\mathbb{C} \times D$, then this corresponds to the action of the gauge group. Note also that $g(s,t)$ need not be in $\mathcal{G}_D$ and hence $g(s,t)^* A(s,t)$ will not, in general, represent the same point in $\Omega\mathrm{G}$. However, for any two maps $g, h : D \to \mathrm{G}$ we have

$$g^* A_h = A_{g^* h}, \qquad g^* h(z) := g(1)^{-1} h(z) g(z).$$

Hence the action of an element $g \in \mathcal{G}$ which is not equal to 1 on $\mathbb{C} \times \partial D$ corresponds to the action of $L\mathrm{G}$ on $\widetilde{\mathrm{Hol}}_k(S^2, \Omega\mathrm{G})$.

Remark 4.10. Every finite energy solution $\Xi = A + \Phi\, ds + \Psi\, dt$ of (37) and (38) extends to S^2, modulo gauge equivalence. More explicitly, this means that there exists a gauge transformation $g : \mathbb{C} \setminus \{0\} \to \mathcal{G}_D$ such that the transformed solution (42) extends smoothly to $S^2 \setminus \{0\}$. Equivalently, the function $\mathbb{C} \setminus \{0\} \to \mathcal{A}_D^{\mathrm{flat}} : z \mapsto \tilde{A}(1/z)$ extends smoothly to $\mathbb{C}$.

Remark 4.11. Note that the solutions of (36) extends smoothly to $S^2 \times D$ while those of (37) and (38) only do so after applying a prior gauge transformation as in Remark 4.10. The reason lies in the boundary condition. By assumption our anti-holomorphic curve $u : S^2 \times S^1 \to \mathrm{G}$ has nonzero degree $-k$ and hence does not extend smoothly to $S^2 \times D$. Hence our function $h : \mathbb{C} \times D \to \mathrm{G}$ does not extend smoothly to $S^2 \times D$. To understand this phenomenon more precisely we examine the correspondence between the solutions of (36) and (37). Given a holomorphic sphere $u : S^2 \to \Omega\mathrm{G}$, every extension $h : \mathbb{C} \times D \to \mathrm{G}$ of u gives rise to a solution $\Xi_h = A_h + \Phi_h\, ds + \Psi_h\, dt$ of (37) and (38), given by

$$A_h = h^{-1}dh, \qquad \Phi_h = h^{-1}\partial_s h + h^{-1}\phi h, \qquad \Psi_h = h^{-1}\partial_t h + h^{-1}\psi h, \quad (43)$$

where (ϕ, ψ) denotes the unique solution of (36) (see Remark 4.8). Moreover, any two extensions h and $h' = hg$ correspond to gauge equivalent solutions Ξ_h and $\Xi_{hg} = g^*\Xi_h$. Note that one can think of $h : \mathbb{C} \times D \to \mathrm{G}$ as a gauge transformation (which is not equal to the identity on the boundary) and of $\Xi_h := h^*(\phi\, ds + \psi\, dt)$ as the transformed solution. In other words, $\phi\, ds + \psi\, dt$ is a special solution of (37) with $A = 0$, but which does not, of course, satisfy the boundary condition (38). While the solution $\phi\, ds + \psi\, dt$ extends smoothly to $S^2 \times D$, neither h nor the transformed solution Ξ_h extend smoothly to $S^2 \times D$.

Remark 4.12. Denote by $\mathcal{A}_0(\mathrm{G}, k)$ the space of solutions of (37) and (38) with $F_{A(s,t)} = 0$ for all s and t and $E(A, \Phi, \Psi) = 2\pi k$. The gauge group (41) acts on this space and we denote the quotient by $\mathcal{M}_0(\mathrm{G}, k) = \mathcal{A}_0(\mathrm{G}, k)/\mathcal{G}$. By Remarks 4.9 and 4.10 there is a natural bijection

$$\mathrm{Hol}_k(S^2, \Omega\mathrm{G}) \cong \mathcal{A}_0(\mathrm{G}, k)/\mathcal{G} = \mathcal{M}_0(\mathrm{G}, k).$$

Explicitly, this bijection can be described as follows. Given an anti-holomorphic map $u : S^2 \to \Omega\mathrm{G}$, lift it over $\mathbb{C} \subset S^2$ to a map $\mathbb{C} \to \mathrm{Map}_0(D, \mathrm{G}) : s + it \mapsto h(s,t)$ and define the image of u to be the gauge equivalence class of the connection $\Xi_h = A_h + \Phi_h\, ds + \Psi_h\, dt$, defined by (43). It follows from the above discussion that the map $u \mapsto [\Xi_h]$ is a bijection of the respective quotient spaces.

Exercise 4.13. Let $\mathbb{C} \to \mathrm{Map}_0(D, \mathrm{G}) : s + it \mapsto h(s,t)$ represent an anti-holomorphic sphere in the loop group and denote $\Xi = \Xi_h \in \mathcal{A}_0(\mathrm{G}, k)$. Prove that the energy of this sphere is given by

$$E(\Xi_h) = \int_{\mathbb{C}} \int_D \langle \partial_s(h^{-1}dh) \wedge \partial_t(h^{-1}dh) \rangle = -2\pi \deg(h : S^2 \times \partial D \to \mathrm{G}).$$

Note here that the restriction $h|_{\mathbb{C}\times\partial D}$ extends smoothly to $S^2 \times \partial D$. Compare this with Lemma 4.7.

4.3 Instantons on the four-sphere

Let B be an anti-self-dual G-connection on the 4-sphere. Explicitly, one can think of B as a Lie algebra valued 1-form

$$B = \sum_{i=0}^{3} B_i \, dx_i \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$$

on $\mathbb{R}^4$ whose curvature form

$$F_B = \sum_{i<j} F_{ij} \, dx_i \wedge dx_j, \qquad F_{ij} = \partial_i B_j - \partial_j B_i + [B_i, B_j]$$

satisfies the anti-self-duality equation

$$F_{01} + F_{23} = F_{02} + F_{31} = F_{03} + F_{12} = 0,$$

and has finite Yang-Mills action

$$\mathcal{YM}(B) = \frac{1}{2} \int_{\mathbb{R}^4} \sum_{i<j} |F_{ij}|^2 < \infty.$$

Now let us denote by $\mathbb{H} \subset \mathbb{C}$ the closed upper half plane and by $\mathrm{int}(\mathbb{H})$ the open upper half plane. Consider the map $S^2 \times \mathbb{H} \to \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3 : (y, u + iv) \mapsto x = (u, vy)$, i.e.

$$x_0 = u, \qquad x_1 = vy_1, \qquad x_2 = vy_2, \qquad x_3 = vy_3.$$

This map is an orientation preserving diffeomorphism from $S^2 \times \mathrm{int}(\mathbb{H})$ onto the complement of the x_0-axis. It extends smoothly to a surjection $S^2 \times \mathbb{H} \to \mathbb{R}^4$ which maps the boundary $S^2 \times \partial\mathbb{H}$ onto the x_0-axis. The pullback of the standard metric on $\mathbb{R}^4$, multiplied by the function $({x_1}^2 + {x_2}^2 + {x_3}^2)^{-1}$, agrees with the product of the standard metric on S^2 and the hyperbolic metric on $\mathrm{int}(\mathbb{H})$.

Remark 4.14. Denote by

$$D = \left\{ z = x + iy \in \mathbb{C} \,|\, x^2 + y^2 \leq 1 \right\}, \qquad \mathbb{H} = \{ w = u + iv \in \mathbb{C} \,|\, v \geq 0 \}$$

the closed unit disc and the closed upper half space, respectively. Consider the standard hyperbolic metric on $\mathrm{int}(\mathbb{H})$ and the standard hyperbolic metric on $\mathrm{int}(D)$ multiplied by a factor ε^2. These metrics are given by

$$g_D = \frac{4\varepsilon^2(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}, \qquad g_{\mathbb{H}} = \frac{du^2 + dv^2}{v^2}.$$

The diffeomorphism $\mathrm{int}(D) \to \mathrm{int}(\mathbb{H}) : z \mapsto w = i\varepsilon^{-1}(1+z)(1-z)^{-1}$ identifies these two metrics. Explicitly, the image point $w = u + iv \in \mathbb{H}$ is given by

$$u = \frac{-2y}{\varepsilon((x-1)^2 + y^2)}, \qquad v = \frac{1 - x^2 - y^2}{\varepsilon((x-1)^2 + y^2)}. \qquad \square$$

Remark 4.15. The 2-sphere

$$S^2 = \{ y \in \mathbb{R}^3 \,|\, |y| = 1 \}$$

with its standard metric is isometric to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with the metric

$$g_{\mathbb{C}} = \frac{4(ds^2 + dt^2)}{(1 + s^2 + t^2)^2}.$$

An explicit formula for the isometry $\mathbb{C} \to S^2 : s + it \mapsto (y_1, y_2, y_3)$ is

$$y_1 = \frac{2s}{1 + s^2 + t^2}, \qquad y_2 = \frac{2t}{1 + s^2 + t^2}, \qquad y_3 = \frac{1 - s^2 - t^2}{1 + s^2 + t^2}.$$

This corresponds to stereographic projection from the south pole.

Let g_ε denote the product metric, on $S^2 \times \mathrm{int}(D)$, of the round metric on S^2 and the hyperbolic metric on $\mathrm{int}(D)$, rescaled by the factor ε^2. On $\mathbb{C} \times \mathrm{int}(D)$ with coordinates $s + it \in \mathbb{C}$ and $x + iy \in \mathrm{int}(D)$ this metric is given by

$$g_\varepsilon = \frac{4(ds^2 + dt^2)}{(1 + s^2 + t^2)^2} + \frac{4\varepsilon^2(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}. \tag{44}$$

Denote by $f_\varepsilon : \mathbb{C} \times \mathrm{int}(D) \to \mathbb{R}^4$ the composition of the above map $S^2 \times \mathrm{int}(\mathbb{H}) \to \mathbb{R}^4$ with the isometries of Remarks 4.14 and 4.15. Then the image point

$$(x_0, x_1, x_2, x_3) = f_\varepsilon(s + it, x + iy)$$

is given by

$$
\begin{aligned}
x_0 &= \frac{-2y}{\varepsilon((x-1)^2+y^2)}, \\
x_1 &= \frac{2s(1-x^2-y^2)}{\varepsilon(1+s^2+t^2)((x-1)^2+y^2)}, \\
x_2 &= \frac{2t(1-x^2-y^2)}{\varepsilon(1+s^2+t^2)((x-1)^2+y^2)} \\
x_3 &= \frac{(1-s^2-t^2)(1-x^2-y^2)}{\varepsilon(1+s^2+t^2)((x-1)^2+y^2)}.
\end{aligned}
\tag{45}
$$

This map is an orientation preserving conformal diffeomorphism from $S^2 \times \mathrm{int}(D)$ with the metric g_ε onto $\mathbb{R}^4 \setminus \mathbb{R}e_0$ with its standard metric. It extends to a smooth surjection $S^2 \times (D \setminus \{1\}) \to \mathbb{R}^4$ which maps $S^2 \times (S^1 \setminus \{1\})$ onto the x_0-axis. It is useful to introduce another conformal transformation $\phi : S^4 \to S^4$ which maps $(0,0,0,-1)$ to ∞. Then the composition $\phi \circ f_\varepsilon$ maps $\mathbb{C} \times D$ to $\mathbb{R}^4 = S^4 \setminus \{\infty\}$ and hence the pullback of any anti-self-dual connections 1-form over $\mathbb{R}^4$ under $\phi \circ f_\varepsilon$ is smooth over $\mathbb{C} \times D$. Note, however, that it will not in general extend to a smooth connection 1-form over $S^2 \times D$ unless we perform a prior gauge transformation as in Remark 4.10. Now our pullback connection ${f_\varepsilon}^* \phi^* B$ is anti-self-dual over $\mathbb{C} \times \mathrm{int}(D)$ with respect to g_ε. Any such connection can be written in the form

$$\Xi = A + \Phi\, ds + \Psi\, dt,$$

where $A = A(s,t) \in \Omega^1(D, \mathfrak{g})$ and $\Phi = \Phi(s,t)$ and $\Psi = \Psi(s,t)$ are Lie algebra valued functions on D which satisfy the boundary condition (38). The curvature of this connection is given by

$$
\begin{aligned}
F_\Xi &= F_A + (d_A\Phi - \partial_s A) \wedge ds + (d_A\Psi - \partial_t A) \wedge dt \\
&\quad + (\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi])\, ds \wedge dt.
\end{aligned}
$$

Let us denote by $*_\varepsilon$ the Hodge $*$-operator on $\mathbb{C} \times \mathrm{int}(D)$ with respect to the metric (44). On 2-forms this operator is given by

$$*_\varepsilon ds \wedge dt = \frac{\varepsilon^2(1+s^2+t^2)^2}{(1-x^2-y^2)^2} dx \wedge dy,$$

$$*_\varepsilon ds \wedge dx = -dt \wedge dy, \qquad *_\varepsilon ds \wedge dy = dt \wedge dx.$$

Hence the anti-self-duality equations on $\mathbb{C} \times D$ take the form

$$\partial_s A - d_A \Phi + *(\partial_t A - d_A \Psi) = 0, \tag{46}$$

and

$$\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \frac{(1-x^2-y^2)^2}{\varepsilon^2(1+s^2+t^2)^2} * F_A = 0. \tag{47}$$

Here $*$ denotes the Hodge $*$-operator on D with respect to the flat metric The functions A, Φ, Ψ are understood to be smooth over $\mathbb{C} \times D$, where $D \subset \mathbb{C}$ denotes the closed unit disc, and Φ and Ψ satisfy the boundary condition (38). The Yang-Mills action of Ξ with respect to the metric g_ε is given by

$$\begin{aligned} \mathcal{YM}_\varepsilon(\Xi) \quad = \quad & \frac{1}{2} \int_{\mathbb{C}} \int_D \Big(|\partial_s A - d_A \Phi|^2 + |\partial_t A - d_A \Psi|^2 \\ & + \frac{\varepsilon^2(1+s^2+t^2)^2}{(1-x^2-y^2)^2} \left|\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi]\right|^2 \\ & + \frac{(1-x^2-y^2)^2}{\varepsilon^2(1+s^2+t^2)^2} |F_A|^2 \Big) \, dxdydsdt. \end{aligned} \tag{48}$$

Let us denote by $\mathcal{A}_\varepsilon(\mathrm{G}, k)$ the space of solutions of the anti-self-dual Yang-Mills equations (46), (47) on $S^2 \times D$ with boundary condition (38), which have Yang-Mills energy $2\pi k$. The gauge group $\mathcal{G}$, given by (41), acts on this space as in Remark 4.9, and we denote the quotient by

$$\mathcal{M}_\varepsilon(\mathrm{G}, k) = \mathcal{A}_\varepsilon(\mathrm{G}, k)/\mathcal{G}.$$

We also denote by $\mathcal{M}_{\mathrm{asd}}(S^4; \mathrm{G}, k)$ the moduli space of gauge equivalence classes of anti-self-dual charge-k instantons on S^4. Note that this space can be identified with the quotient $\mathcal{A}_{\mathrm{asd}}(\mathbb{R}^4; \mathrm{G}, k)/\mathrm{Map}(\mathbb{R}^4, \mathrm{G})$.

Proposition 4.16. *Let $f_\varepsilon : \mathbb{C} \times \mathrm{int}(D) \to \mathbb{R}^4$ be the function given by (45) and $\phi : S^4 \to S^4$ be a conformal diffeomorphism such that $\phi(0,0,0,-1) = \infty$. Then, for every $\varepsilon > 0$, the map*

$$\mathcal{A}_{\mathrm{asd}}(\mathbb{R}^4; \mathrm{G}, k) \longrightarrow \mathcal{A}_\varepsilon(\mathrm{G}, k) : B \mapsto {f_\varepsilon}^* \phi^* B$$

induces a bijection from the moduli space $\mathcal{M}_{\mathrm{asd}}(S^4; \mathrm{G}, k)$ of anti-self-dual instantons over the 4-sphere to the moduli space $\mathcal{M}_\varepsilon(\mathrm{G}, k)$ of solutions of (46), (47), and (38).

Proof. Note first that

$$f_\varepsilon(\mathbb{C} \times D) = S^4 \setminus \{x \in \mathbb{R}^4 \,|\, x_1 = x_2 = 0,\, x_3 < 0\}.$$

Hence $\phi \circ f_\varepsilon(\mathbb{C} \times D) \subset S^4 - \{\infty\}$. This implies that the pullback connection ${f_\varepsilon}^* \phi^* B$ is smooth over $\mathbb{C}\times D$. The diffeomorphism (45) satisfies $\partial_s f_\varepsilon = \partial_t f_\varepsilon = 0$ on $\mathbb{C}\times\partial D$. Hence the pullback connection $A + \Phi\, ds + \Psi\, dt = {f_\varepsilon}^* \phi^* B$ of any connection 1-form $B \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ satisfies the boundary condition (38). That ${f_\varepsilon}^* \phi^* B$ satisfies (46) and (47) whenever B is anti-self-dual (with respect to the standard metric) follows from the conformal invariance of the anti-self-dual Yang-Mills equations. Since the pullback $g\circ\phi\circ f_\varepsilon$ of any gauge transformation $g : \mathbb{R}^4 \to \mathrm{G}$ lies in the gauge group $\mathcal{G}$, defined by (41), it follows that the map $B \mapsto {f_\varepsilon}^* \phi^* B$ descends to the quotient spaces. We prove that the induced map is injective. Hence suppose that B and B' are anti-self-dual charge-k instantons on $\mathbb{R}^4$ such that

$${f_\varepsilon}^* \phi^* B' = g^* {f_\varepsilon}^* \phi^* B$$

for some gauge transformation $g \in \mathcal{G}$. The condition $\partial_s g = \partial_t g = 0$ on $\mathbb{C} \times \partial D$ guarantees that there exists a gauge transformation $u : \mathbb{R}^4 \to \mathrm{G}$, smooth over $\mathbb{R}^4 - \mathbb{R}$ and continuous along $\mathbb{R}$, such that

$$g = u \circ \phi \circ f_\varepsilon.$$

This implies that $B' = u^* B$ over $\mathbb{R}^4 \setminus \mathbb{R}$. Since B and B' are smooth it follows that u is smooth over all of $\mathbb{R}^4$, and hence B and B' are gauge equivalent. This proves that our map is injective.

We prove that the map is onto. Let $\Xi = A + \Phi\, ds + \Psi\, dt \in \mathcal{A}_\varepsilon(\mathrm{G}, k)$ be given and define $B \in \Omega^1(\mathbb{R}^4 \setminus \mathbb{R}, \mathfrak{g})$ by $\Xi =: {f_\varepsilon}^* \phi^* B$. The boundary condition (38) guarantees that B extends to a continuous 1-form on all of $\mathbb{R}^4$. Moreover, B is an anti-self-dual charge-k instanton over $\mathbb{R}^4 \setminus \mathbb{R}$. Hence the removable singularity theorem for anti-self-dual instantons asserts that there exists a continuous gauge tranformation $u : \mathbb{R}^4 \to \mathrm{G}$, which is smooth over $\mathbb{R}^4 \setminus \mathbb{R}$, such that $u^* B$ is smooth over $\mathbb{R}^4$. Hence

$$f_\varepsilon^* \phi^* u^* B = \Xi' = A' + \Phi'\, ds + \Psi'\, dt$$

lies in the image of our map $\mathcal{A}_{\mathrm{asd}}(S^4; \mathrm{G}, k) \longrightarrow \mathcal{A}_\varepsilon(\mathrm{G}, k)$. Moreover,

$$\Xi' = g^* \Xi,$$

where

$$g = u \circ \phi \circ f_\varepsilon : \mathbb{C} \times D \to \mathrm{G}.$$

This gauge transformation is smooth over $\mathbb{C} \times \mathrm{int}(D)$, continuous over $\mathbb{C} \times D$, and independent of s and t over the boundary. However, the equation $\Xi' = g^*\Xi$, where Ξ and Ξ' are both smooth up to the boundary, implies that u is also smooth up to the boundary and hence Ξ is gauge equivalent to a connection Ξ' in the image of our map. Hence the induced map on quotient spaces is onto. This proves the proposition. $\square$

4.4 An adiabatic limit

The following theorem relates the moduli spaces $\mathcal{M}_\varepsilon(\mathrm{G}, k)$ and $\mathcal{M}_0(\mathrm{G}, k)$, and hence describes a correspondence between anti-self-dual instantons over the 4-sphere and anti-holomorphic spheres in the loop group. We use the notation

$$\mathcal{G}^c = \{g : \mathbb{C} \times D \to \mathrm{G}^c \,|\, \partial_s g|_{\mathbb{C}\times\partial D} = \partial_t g|_{\mathbb{C}\times\partial D} = 0\}.$$

Theorem 4.17 (Jarvis-Norbury). *For every $\Xi_\varepsilon \in \mathcal{A}_\varepsilon(\mathrm{G}, k)$ there exists a unique $\Xi_0 \in \mathcal{A}_0(\mathrm{G}, k)$, which is complex gauge equivalent to Ξ_ε via a Hermitian gauge tranformation $g \in \mathcal{G}^c$. This defines a bijection $\mathcal{T}_\varepsilon : \mathcal{M}_\varepsilon(\mathrm{G}, k) \to \mathcal{M}_0(\mathrm{G}, k)$ for every $\varepsilon > 0$.*

The Definition of $\mathcal{T}_\varepsilon$ is obvious from Theorem 4.2. Namely, given a solution $\Xi = A + \Phi\, ds + \Psi\, dt$ of (46), (47) and (38), choose a smooth family of complex gauge transformations $\mathbb{C} \to \mathcal{G}^c_D : s + it \mapsto g(s,t)$ such that $g(s,t)^*A(s,t)$ is flat for all s and t. Then define $\widetilde{A}(s,t) \in \mathcal{A}_D$, and $\widetilde{\Phi}(s,t), \widetilde{\Psi}(s,t) \in \mathrm{Lie}(\mathcal{G}_D)$ by

$$\widetilde{A} = g^*A, \qquad \widetilde{\Phi} + i\widetilde{\Psi} = g^{-1}(\partial_s g + i\partial_t g) + g^{-1}(\Phi + i\Psi)g. \tag{49}$$

It is easy to see that (46) is preserved under this transformation, and hence $\widetilde{\Xi} = \widetilde{A} + \widetilde{\Phi}\, ds + \widetilde{\Psi}\, dt$ is a solution of (37). Secondly, it follows from Theorem 4.2 that different choices of complex gauge transformations $g(s,t)$ give rise to (real) gauge equivalent solutions $\widetilde{\Xi}$. Hence the map $\mathcal{T}_\varepsilon$ is well defined.

Lemma 4.18. *Let $\Xi = A + \Phi\, ds + \Psi\, dt$ be a solution of (37) and (38) and $g = g(s,t) \in \mathcal{G}^c_D$ be a smooth family of complex gauge transformations. Let $\widetilde{\Xi} = \widetilde{A} + \widetilde{\Phi}\, ds + \widetilde{\Psi}\, dt$ be given by (49). Then $\widetilde{\Xi}$ is also a solution of (37).*

Proof. Equation (37) can be written in the form

$$\partial_s A^{0,1} + i\partial_t A^{0,1} = \bar{\partial}_A(\Phi + i\Psi).$$

That this equation is invariant under complex gauge transformations as in (49), follows directly from the identities

$$\partial_s (g^* A)^{0,1} = \bar{\partial}_{g^*A}(g^{-1}\partial_s g) + g^{-1}(\partial_s A^{0,1})g, \qquad \bar{\partial}_{g^*A}(g^{-1}\Phi g) = g^{-1}(\bar{\partial}_A \Phi)g$$

This proves the lemma. □

Proof of Theorem 4.17. Note that the (46) is equivalent to $F_\Xi^{0,2} = 0$ and (47) is equivalent to $\langle \omega_\varepsilon, F_\Xi \rangle = 0$ where ω_ε is the symplectic form on $S^2 \times D$ determined by g_ε and the standard complex structure. Hence Theorem 4.17 is a special case of Theorem 4.2 where the Kähler manifold Z is the product $S^2 \times D$. But the metric g_ε is singular near the boundary and so Theorem 4.17 follows from Remark 4.5. □

We point out here that the inverse map $\mathcal{T}_\varepsilon^{-1} : \mathcal{M}_0(\mathrm{G}, k) \longrightarrow \mathcal{M}_\varepsilon(\mathrm{G}, k)$ converges to the identity as $\varepsilon \to 0$ in the following sense. For every $\Xi_0 \in \mathcal{A}_0(\mathrm{G}, k)$ and every $p > 2$ there exists an $\varepsilon_0 > 0$ and a collection of g_ε-anti-self-dual connections $\Xi_\varepsilon \in \mathcal{A}_\varepsilon(\mathrm{G}, k)$ for $0 < \varepsilon < \varepsilon_0$ such that

$$\mathcal{T}_\varepsilon([\Xi_\varepsilon]) = [\Xi_0]$$

and Ξ_ε converges to Ξ_0 in the $W^{1,p}$-norm (with respect to the product of the round metric on S^2 and the hyperbolic metric on D). To see this note that the Yang-Mills energy of a connection $\Xi = A + \Phi\, ds + \Psi\, dt \in \mathcal{A}_0(G, k)$ with respect to the metric g_ε is given by

$$\begin{aligned} \mathcal{YM}_\varepsilon(\Xi) &= \frac{1}{2}\int_{\mathbb{C}}\int_D \biggl(|\partial_s A - d_A\Phi|^2 + |\partial_t A - d_A \Psi|^2 \\ &\quad + \frac{\varepsilon^2(1+s^2+t^2)^2}{(1-x^2-y^2)^2}\,|\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]|^2 \biggr)\, dxdydsdt. \end{aligned}$$

This converges to the minimum $E(\Xi) = 2\pi k$ of the Yang-Mills functional as $\varepsilon \to 0$. One can use this to prove that the complex gauge equivalent solution $\Xi_\varepsilon = g_\varepsilon^* \Xi$ of (46) and (47) is almost flat over each slice $\{s + it\} \times D$. This gives rise to an estimate for the difference $\Xi_\varepsilon - \Xi$. Alternatively, one can use the techniques in [7].

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