Cauchy-Riemann operators, self-duality, and the spectral flow

Stamatis Dostoglou and Dietmar A. Salamon*
Mathematics Institute
University of Warwick
Coventry CV4 7AL
Great Britain

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1 Introduction

Atiyah, Patodi and Singer [3] observed that the Fredholm index of the operator

$$\mathcal{D}_A = \frac{d}{dt} + A(t)$$

with invertible limits $A^{\pm} = \lim_{t\to\pm\infty} A(t)$ is given by the spectral flow of the self-adjoint operator family $A(t)$ (the number of eigenvalues crossing 0 counted with signs). Such operators appear in infinite dimensional analogues of Morse theory as the linearisation of the gradient flow equation. The Fredholm index is the dimension of the space of gradient flow lines connecting two critical points. It can be thought of as the *relative Morse index* in cases where the *absolute Morse index* (the number of negative eigenvalues of the *Hessian*) is infinite.

The *Floer homology* groups of such a variational problem arise from a chain complex which is generated by the critical points and graded by the relative Morse index. The boundary operator is given by counting the number of connecting orbits (with appropriate signs) whenever the relative Morse index is 1. This approach to infinite dimensional Morse theory was discovered by Floer in his study of the gradient flow of the symplectic action [11]. In this theory the critical points are fixed points of a symplectomorphism and the connecting orbits are pseudoholomorphic curves in the sense of Gromov. Another version of Floer homology arises from Morse theory for the Chern-Simons functional and leads to invariants of 3-manifolds which play an important role in Donaldson's

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theory of 4-manifolds [10], [12], [7]. In this theory the critical points are flat SO(3)-connections over a 3-manifold M and the connecting orbits are self-dual instantons over the 4-manifold $M \times \mathbb{R}$ with finite Yang-Mills energy.

Both theories are related to Riemann surfaces as follows. On the one hand the moduli space of flat connections on a nontrivial SO(3)-bundle $P \to \Sigma$ is a symplectic manifold and the mapping class group of orientation preserving diffeomorphisms of Σ acts by symplectomorphisms $\phi_f : \mathcal{M}(P) \to \mathcal{M}(P)$. On the other hand an automorphism $f : P \to P$ determines a mapping cylinder P_f which is a principal SO(3)-bundle over a 3-manifold. Moreover, the flat connections on P_f correspond naturally to the fixed points of ϕ_f. Hence in this case there are two Floer homologies $HF_*^{\mathrm{inst}}(P_f)$ and $HF_*^{\mathrm{symp}}(\phi_f)$, both arising from the same chain complex, and it was conjectured by Atiyah [1] and Floer that there is a natural isomorphism

$$HF_*^{\mathrm{symp}}(\phi_f) \simeq HF_*^{\mathrm{inst}}(P_f).$$

We shall prove this conjecture in a forthcoming paper [9]. The main result of the present paper asserts that the relative Morse indices agree. If $a^{\pm}$ are nondegenerate flat connections on P_f or equivalently nondegenerate fixed points of ϕ_f then

$$\mu^{\mathrm{symp}}(a^-, a^+) = \mu^{\mathrm{inst}}(a^-, a^+).$$

This implies that the grading of the chain complex is the same in both theories. The proof requires a comparison of the spectral flow of the linearized Cauchy-Riemann equations in the symplectic theory with the spectral flow of the linearized self-duality equations in the Chern-Simons theory. In sections 2-4 we explain the necessary background about Floer homology and flat connections over Riemann surfaces. In section 5 we discuss the Atiyah-Floer conjecture. In sections 6 and 7 we state and prove our main theorem about the spectral flow.

2 Cauchy-Riemann operators

Let $(\mathcal{M}, \omega)$ be a $2n$-dimensional symplectic manifold and $\phi : \mathcal{M} \to \mathcal{M}$ be a symplectomorphism. This means that ω is a nondegenerate closed 2-form and $\phi^*\omega = \omega$. Let $\mathbb{R} \times \mathcal{M} \to \mathbb{R} : (s, p) \mapsto H_s(p)$ be a smooth time dependent Hamiltonian function such that $H_s = H_{s+1} \circ \phi$. Denote by $\psi_s : \mathcal{M} \to \mathcal{M}$ the corresponding Hamiltonian symplectomorphisms defined by

$$\frac{d}{ds}\psi_s = X_s \circ \psi_s, \qquad \psi_0 = \mathrm{id}, \qquad \iota(X_s)\omega = dH_s.$$

They satisfy

$$\psi_{s+1} \circ \phi_H = \phi \circ \psi_s$$

where $\phi_H := {\psi_1}^{-1} \circ \phi$. The fixed points of ϕ_H are in one-to-one correspondence with smooth curves $x : \mathbb{R} \to \mathcal{M}$ such that $x(s) = \psi_s(x_0)$ and $x(s+1) = \phi(x(s))$. For a generic perturbation H the fixed points of ϕ_H are all nondegenerate.

J-holomorphic curves

An almost complex structure $J : T\mathcal{M} \to T\mathcal{M}$ is called compatible with ω if $\langle v, w\rangle = \omega(v, Jw)$ is a Riemannian metric. Denote the space of such structures by $\mathcal{J}(\mathcal{M}, \omega)$. Let $J_s \in \mathcal{J}(\mathcal{M}, \omega)$ be a smooth 1-parameter family of almost complex structures such that $J_s = \phi^* J_{s+1}$. Consider the perturbed nonlinear Cauchy-Riemann equations

$$\frac{\partial u}{\partial t} + J_s(u)\left(\frac{\partial u}{\partial s} - X_s(u)\right) = 0 \tag{1}$$

with boundary condition

$$u(s + 1, t) = \phi(u(s, t)). \tag{2}$$

The solutions of (1) and (2) are the gradient flow lines of the symplectic action functional on the path space Ω_ϕ of all paths $\gamma : \mathbb{R} \to \mathcal{M}$ such that $\gamma(s + 1) = \phi(\gamma(s))$. In the case $X_s = 0$ these are Gromov's pseudoholomorphic curves [14]. If the fixed points of ϕ_H are all nondegenerate then any solution of (1) and (2) with finite energy has limits

$$\lim_{t\to\pm\infty} u(s, t) = \psi_s(x^\pm), \qquad x^\pm = \phi_H(x^\pm). \tag{3}$$

Moreover, $\partial_t u$ converges to zero and $\partial_s u$ converges to $X_s \circ \psi_s(x^\pm)$ as t tends to $\pm\infty$.

For any smooth function $u : \mathbb{R}^2 \to \mathcal{M}$ which satisfies (2) the Sobolev space

$$W^{k,2}_\phi(u^* T\mathcal{M})$$

is the completion of the space of smooth compactly supported vector fields $\xi(s, t) \in T_{u(s,t)}\mathcal{M}$ along u which satisfy $\xi(s + 1, t) = d\phi(u(s, t))\xi(s, t)$ with respect to the $W^{k,2}$-norm over $[0, 1] \times \mathbb{R}$. For $k = 0$ denote $L^2_\phi(u^* T\mathcal{M}) = W^{0,2}_\phi(u^* T\mathcal{M})$. Linearization of (1) gives rise to the operator

$$\mathcal{D}_u = W^{1,2}_\phi(u^* T\mathcal{M}) \to L^2_\phi(u^* T\mathcal{M})$$

defined by

$$\mathcal{D}_u\xi = \nabla_t\xi + J_s(u)(\nabla_s\xi - \nabla_\xi X_s(u)) + \nabla_\xi J_s(u)(\partial_s u - X_s(u))$$

where ∇ denotes the Levi-Civita connection associated to the metric $\langle v, w\rangle_s = \omega(v, J_s w)$. If the fixed points $x^\pm = \phi_H(x^\pm)$ are nondegenerate then $\mathcal{D}_u$ is a Fredholm operator and its index is given by the Maslov class of u.

The Maslov index for symplectic paths

Denote by $\mathrm{Sp}(2n)$ the group of symplectic matrices. These are matrices which preserve the standard symplectic structure $\omega_0 = \sum_j dx_j \wedge dy_j$ on $\mathbb{R}^{2n}$. In [5] Conley and Zehnder introduced a Maslov index for paths of symplectic matrices. Their index assigns an integer $\mu_{\mathrm{CZ}}(\Psi)$ to every path $\Psi : [0,1] \to \mathrm{Sp}(2n)$ such that $\Psi(0) = \mathbb{1}$ and $\det(\mathbb{1} - \Psi(1)) \neq 0$. Other expositions are given in [23] and [20]. Here we summarize the results which are needed in the sequel.

Denote by $\mathrm{Sp}^*(2n)$ the open and dense set of all symplectic matrices which do not have 1 as an eigenvalue. This set has two components distinguishes by the sign of $\det(\mathbb{1} - \Psi)$. Its complement is called the **Maslov cycle**. It is an algebraic variety of codimension 1 and admits a natural coorientation. The intersection number of a loop $\Phi : S^1 \to \mathrm{Sp}(2n)$ with the Maslov cycle is always even and the Maslov index $\mu(\Phi)$ is half this intersection number. Alternatively, the Maslov index can be defined as the degree

$$\mu(\Phi) = \deg(\rho \circ \Phi)$$

where $\rho : \mathrm{Sp}(2n) \to S^1$ is a continuous extension of the determinant map $\det : \mathrm{U}(n) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n) \to S^1$. The map ρ is not a homomorphism but can be chosen to be multiplicative with respect to direct sums, invariant under similarity, and taking the value ± 1 for symplectic matrices with no eigenvalues on the unit circle. These properties determine ρ uniquely [23].

Now denote by $\mathcal{SP}^*(n)$ the space of paths $\Psi : [0,1] \to \mathrm{Sp}(2n)$ with $\Psi(0) = \mathbb{1}$ and $\Psi(1) \in \mathrm{Sp}^*(2n)$. Any such path admits an extension $\Psi : [0,2] \to \mathrm{Sp}(2n)$, unique up to homotopy, such that $\Psi(s) \in \mathrm{Sp}^*(2n)$ for $s \geq 1$ and $\Psi(2)$ is one of the matrices $W^+ = -\mathbb{1}$ and $W^- = \mathrm{diag}(2, -1, \ldots, -1, 1/2, -1, \ldots, -1)$. Since $\rho(W^\pm) = \pm 1$ it follows that $\rho^2 \circ \Psi : [0,2] \to S^1$ is a loop and the **Conley-Zehnder index** of Ψ is defined as its degree

$$\mu_{\mathrm{CZ}}(\Psi) = \deg(\rho^2 \circ \Psi).$$

The Conley-Zehnder index has the following properties. It is uniquely determined by the homotopy, loop, and sigature properties [23].

(Naturality) For any path $\Phi : [0,1] \to \mathrm{Sp}(2n)$

$$\mu_{\mathrm{CZ}}(\Phi\Psi\Phi^{-1}) = \mu_{\mathrm{CZ}}(\Psi).$$

(Homotopy) The Conley-Zehnder index is constant on the components of $\mathcal{SP}^*(n)$

(Zero) If $\Psi(s)$ has no eigenvalue on the unit circle for $s > 0$ then $\mu_{\mathrm{CZ}}(\Psi) = 0$.

(Product) If $n' + n'' = n$ identify $\mathrm{Sp}(2n') \oplus \mathrm{Sp}(2n'')$ with a subgroup of $\mathrm{Sp}(2n)$ in the obvious way. Then

$$\mu_{\mathrm{CZ}}(\Psi' \oplus \Psi'') = \mu_{\mathrm{CZ}}(\Psi') + \mu_{\mathrm{CZ}}(\Psi'').$$

(Loop) If $\Phi : [0, 1] \to \mathrm{Sp}(2n, \mathbb{R})$ is a loop with $\Phi(0) = \Phi(1) = \mathbb{1}$ then

$$\mu_{\mathrm{CZ}}(\Phi\Psi) = \mu_{\mathrm{CZ}}(\Psi) + 2\mu(\Phi).$$

(Signature) If $S = S^T \in \mathbb{R}^{2n\times 2n}$ is a symmetric matrix with $\|S\| < 2\pi$ and $\Psi(s) = \exp(J_0 S s)$ then

$$\mu(\Psi) = \tfrac{1}{2}\mathrm{sign}\, S.$$

(Determinant)

$$(-1)^{\mu_{\mathrm{CZ}}(\Psi)+n} = \mathrm{sign}\det(\mathbb{1} - \Psi(1)).$$

(Inverse)

$$\mu_{\mathrm{CZ}}(\Psi^{-1}) = \mu_{\mathrm{CZ}}(\Psi^T) = -\mu_{\mathrm{CZ}}(\Psi).$$

The Maslov class for J-holomorphic curves

Given two nondegenerate fixed points $x^\pm$ of ϕ_H denote by

$$\mathcal{P}(x^-, x^+) = \mathcal{P}(x^-, x^+, \phi, H)$$

the space of all smooth functions $u : \mathbb{R}^2 \to \mathcal{M}$ which satisfy (2) and (3). Denote by $\mathcal{P}(\phi)$ the space of pairs (u, H) such that $u \in \mathcal{P}(x^-, x^+, \phi, H)$ for two nondegenerate fixed points $x^\pm \in \mathrm{Fix}(\phi_H)$. The Conley-Zehnder index determines a map

$$\mu : \mathcal{P}(\phi) \to \mathbb{Z}$$

as follows. Given $u \in \mathcal{P}(x^-, x^+)$ choose a trivialization $\Phi(s, t) : \mathbb{R}^{2n} \to T_{u(s,t)}\mathcal{M}$ such that

$$\Phi(s, t)^*\omega = \omega_0, \qquad \Phi(s + 1, t) = d\phi(u(s, t))\Phi(s, t),$$

and $\Phi(s, t)$ converges to $\Phi^\pm(s) : \mathbb{R}^{2n} \to T_{\psi_s(x^\pm)}\mathcal{M}$ as t tends to $\pm\infty$. Consider the symplectic paths

$$\Psi^\pm(s) = \Phi^\pm(s)^{-1} d\psi_s(x^\pm)\Phi^\pm(0). \tag{4}$$

Since $x^\pm$ are nondegenerate these are in $\mathcal{SP}^*(n)$ and the **Maslov class** of (u, H) is defined by

$$\mu(u, H) = \mu_{\mathrm{CZ}}(\Psi^+) - \mu_{\mathrm{CZ}}(\Psi^-).$$

By the naturality property of the Conley-Zehnder index the Maslov class is independent of the choice of the trivialization. If $u_{01} \in \mathcal{P}(x_0, x_1)$ and $u_{12} \in \mathcal{P}(x_1, x_2)$ such that $u_{01}(s, t) = \psi_s(x_1)$ for $t \geq 0$ and $u_{12}(s, t) = \psi_s(x_1)$ for $t \leq 0$ define the **catenation** $u_{01}\#u_{12} \in \mathcal{P}(x_0, x_2)$ by

$$u_{01}\#u_{12}(s, t) = \begin{cases} u_{01}(s, t), & t \leq 0, \\ u_{12}(s, t), & t \geq 0. \end{cases}$$

Proposition 2.1 *The Maslov class has the following properties.*

(Homotopy) *The Maslov class is constant on the components of* $\mathcal{P}(\phi)$*.*

(Zero) *If* $x^- = x^+ = x$ *and* $u(s,t) = \psi_s(x)$ *then* $\mu(u,H) = 0$*.*

(Catenation)

$$\mu(u_{01}\#u_{12}, H) = \mu(u_{01}, H) + \mu(u_{12}, H).$$

(Chern class) *If* $v : S^2 \to \mathcal{M}$ *then*

$$\mu(u\#v, H) = \mu(u, H) - 2\langle c_1, v\rangle. \tag{5}$$

(Morse index) *Assume* $\phi = \mathrm{id}$ *and* $H_s = H : \mathcal{M} \to \mathbb{R}$ *is a Morse function with sufficiently small second derivatives. Then the fixed points of* ϕ_H *are the critical points of* H *and*

$$\mu(u, H) = \mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-) \tag{6}$$

for every $u \in \mathcal{P}(x^-, x^+, \phi, H)$ *with* $u(s,t) \equiv \gamma(t)$*.*

(Fixed point index) *For* $u \in \mathcal{P}(x^-, x^+, \phi, H)$

$$(-1)^{\mu(u,H)} = \mathrm{sign}\det(\mathbb{1} - d\phi_H(x^-))\det(\mathbb{1} - d\phi_H(x^+)) \tag{7}$$

Proof: The homotopy and catenation properties are obvious. The equations (6) and (7) follow from the signature and determinant properties of the Conley-Zehnder index. We prove (5). Let $u_1 = u_0\#v$ and assume without loss of generality that $u_0(s,t) = u_1(s,t) = \psi_s(x^\pm)$ for $\pm t \geq 1$ and $u_0(0,t) = u_1(0,t)$ for all t. Decompose $S^2 = D_0 \cup D_1$ with $\partial D_0 = \partial D_1 = S^1$ and choose homeomorphisms $h_j : D_j \to [0,1] \times [-1,1]$ such that h_1 is orientation preserving and h_0 is orientation reversing. Let $v : S^2 \to \mathcal{M}$ be given by

$$v|_{D_j} = u_j \circ h_j, \qquad j = 0, 1.$$

Choose trivializations $\Phi_j(s,t)$ of $u_j^*T\mathcal{M}$ as above such that $\Phi_0(s,t) = \Phi_1(s,t)$ for $t \leq -1$ and $\Phi_0(0,t) = \Phi_1(0,t)$ for all t. Then the loop

$$\Phi(s) = \Phi_0(s,1)^{-1}\Phi_1(s,1) \in \mathrm{Sp}(2n, \mathbb{R})$$

determines the Chern class of v via $\langle c_1, v\rangle = \mu(\Phi)$. Moreover $\Psi_0^+ = \Phi\Psi_1^+$ and $\Psi_0^- = \Psi_1^-$. Hence

$$\mu(u_1, H) - \mu(u_0, H) = \mu_{\mathrm{CZ}}(\Psi_1^+) - \mu_{\mathrm{CZ}}(\Psi_0^+) = -2\mu(\Phi) = -2\langle c_1, v\rangle.$$

The second equality follows from the loop property of the Conley-Zehnder index.
□

Denote by $\widetilde{\Omega}_\phi$ the universal cover of Ω_ϕ. The cover group is $\pi_2(\mathcal{M})$. Let $\widetilde{\mathrm{Fix}}(\phi_H) \subset \widetilde{\Omega}_\phi$ denote the elements which cover curves of the form $\gamma(s) = \psi_s(x)$ where $x \in \mathrm{Fix}(\phi_H)$. Then there is a fibration of discrete sets

$$\pi_2(\mathcal{M}) \hookrightarrow \widetilde{\mathrm{Fix}}(\phi_H) \to \mathrm{Fix}(\phi_H).$$

Every function $u \in \mathcal{P}(x^-, x^+, \phi, H)$ and every lift $\tilde{x}^- \in \widetilde{\mathrm{Fix}}(\phi_H)$ of x^- determines a unique lift $\tilde{x}^+ = \tilde{x}^- \# u$ of x^+. By the homotopy and catenation properties of the Maslov class there exists a unique map $\mu^{\mathrm{symp}} : \widetilde{\mathrm{Fix}}(\phi_H) \times \widetilde{\mathrm{Fix}}(\phi_H) \to \mathbb{Z}$ such that

$$\mu(u, H) = \mu^{\mathrm{symp}}(\tilde{x}^-, \tilde{x}^+)$$

whenever $\tilde{x}^+ = \tilde{x}^- \# u$. Equation (5) now reads

$$\mu^{\mathrm{symp}}(\tilde{x}^-, v \# \tilde{x}^+) = \mu^{\mathrm{symp}}(\tilde{x}^-, \tilde{x}^+) - 2\langle c_1, v\rangle \tag{8}$$

for $v : S^2 \to \mathcal{M}$ and the catenation property can be written in the form

$$\mu^{\mathrm{symp}}(\tilde{x}_0, \tilde{x}_1) + \mu^{\mathrm{symp}}(\tilde{x}_1, \tilde{x}_2) = \mu^{\mathrm{symp}}(\tilde{x}_0, \tilde{x}_2).$$

By (8) the map μ^{symp} descends to a map $\mathrm{Fix}(\phi_H) \times \mathrm{Fix}(\phi_H) \to \mathbb{Z}_{2N}$ (still denoted by μ^{symp}) such that

$$\mu(u, H) = \mu^{\mathrm{symp}}(x^-, x^+) (\mathrm{mod}\ 2N)$$

for every $u \in \mathcal{P}(x^-, x^+, H)$. Here the integer N is the **minimal Chern number** defined by $\langle c_1, \pi_2(\mathcal{M})\rangle = N\mathbb{Z}$.

The Fredholm index

Identify $S^1 = \mathbb{R}/\mathbb{Z}$ and consider the Cauchy-Riemann operator $\bar{\partial}_S : W^{1,2}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^2(S^1 \times \mathbb{R}, \mathbb{R}^{2n})$ defined by

$$\bar{\partial}_S \zeta = \frac{\partial \zeta}{\partial t} + J_0 \frac{\partial \zeta}{\partial s} - S\zeta.$$

Here

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

is the standard complex structure on $\mathbb{R}^{2n}$ and the matrix function $S(s,t) = S(s+1,t) \in \mathbb{R}^{2n\times 2n}$ is continuous with symmetric limits

$$S^\pm(s) = \lim_{t\to\pm\infty} S(s,t).$$

These determine symplectic paths $\Psi^\pm : [0,1] \to \mathrm{Sp}(2n)$ via $\dot{\Psi}^\pm = J_0 S^\pm \Psi^\pm$. If these are in $\mathcal{SP}^*(n)$ then $\bar{\partial}_S$ is a Fredholm operator and in [23] it is proved that

$$\mathrm{index}\,\bar{\partial}_S = \mu_{\mathrm{CZ}}(\Psi^+) - \mu_{\mathrm{CZ}}(\Psi^-). \tag{9}$$

(See also [21].)

Theorem 2.2 *Assume that $x^\pm$ are nondegenerate fixed points of ϕ_H. Then the operator $\mathcal{D}_u$ is Fredholm for every $u \in \mathcal{P}(x^-, x^+, \phi, H)$ and its index is given by the Maslov class*

$$\operatorname{index} \mathcal{D}_u = \mu(u).$$

Proof: Choose a trivialization Φ of $u^*T\mathcal{M}$ as above such that in addition $\Phi(s,t) * J_s = J_0$. Then

$$\mathcal{D}_u \circ \Phi = \Phi \circ \bar{\partial}_S$$

where $\Phi S = \nabla_t \Phi + J_s(\nabla_s \Phi - \nabla_\Phi X_s) + (\nabla_\Phi J_s)(\partial_s u - X_s)$ The limit functions $S^\pm(s) = \lim_{t\to\pm\infty} S(s,t)$ are symmetric and the paths $\Psi^\pm : [0,1] \to \mathrm{Sp}(2n)$ defined by (4) satsfy $\dot{\Psi}^\pm = J_0 S^\pm \Psi^\pm$. Hence it follows from (9) that

$$\operatorname{index} \mathcal{D}_u = \operatorname{index} \bar{\partial}_S = \mu_{\mathrm{CZ}}(\Psi^+) - \mu_{\mathrm{CZ}}(\Psi^+) = \mu(u). \qquad \square$$

Floer homology

Assume that the symplectic manifold $(\mathcal{M}, \omega)$ is simply connected and monotone. This means that $\langle [\omega], A\rangle = \langle c_1, A\rangle$ for every $A \in \pi_2(\mathcal{M})$ and some positive constant $\lambda > 0$. Denote by $\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+, \phi, H, J)$ the space of all solutions $u : \mathbb{R}^2 \to \mathcal{M}$ of (1), (2) and (3). Call the triple (ϕ, H, J) **regular** if the fixed points of ϕ_H are all nondegenerate and the operator $\mathcal{D}_u$ is onto for all $x^\pm \in \mathrm{Fix}(\phi_H)$ and all $u \in \mathcal{M}(x^-, x^+, \phi, H, J)$. For such triples the spaces $\mathcal{M}(x^-, x^+)$ are finite dimensional manifolds of local dimension

$$\dim_u \mathcal{M}(x^-, x^+) = \mu(u).$$

In [11] Floer proved that for a dense set of pairs (H, J) the triple (ϕ, H, J) is regular. The arguments in [11] and in [23] for a generic Hamiltonian are carried out for the case $\phi = \mathrm{id}$ but generalize easily to arbitrary ϕ. The real numbers act on the space $\mathcal{M}(x^-, x^+)$ and it follows that the components of $\mathcal{M}$ with $\mu \le 0$ must be empty unless $x^- = x^+$. Moreover, it follows from Gromov's compactness for J-holomorphic curves that the 1-dimensional component of the space $\mathcal{M}(x^-, x^+)$ consists of finitely many connecting orbits whenever $\mu^{\mathrm{symp}}(x^-, x^+) \equiv 1 (\mathrm{mod}\ 2N)$. These can be used to construct a chain complex as follows. Define

$$C_k = C_k(\phi, H) = \bigoplus_{\substack{x = \phi_H(x) \\ \mu^{\mathrm{symp}}(x_0, x) = k (\mathrm{mod}\ 2N)}} \mathbb{Z}\, x.$$

Here $x_0 \in \mathrm{Fix}(\phi_H)$ is reference point such that $\det(\mathbb{1} - d\phi_H(x_0)) > 0$. The boundary operator $\partial : C_{k+1} \to C_k$ is defined by taking the sum of the numbers $\nu(u)$ over all 1-dimensional components of $\mathcal{M}(x^-, x^+)$. These numbers are defined by comparing the flow orientation of u with the coherent orientation of $\mathcal{M}(x^-, x^+)$ as in [13]. In [11] Floer proved that $\partial^2 = 0$ (again for the case

$\phi = \mathrm{id}$.) The **Floer homology** groups are the homology groups of this chain complex

$$HF_*^{\mathrm{symp}}(\mathcal{M}, \phi, H, J) = \ker \partial / \mathrm{im}\, \partial.$$

Floer also proved in [11] that the Floer homology groups are independent of the almost complex structures J_s and the perurbation H used to define them. They depend on ϕ only up to Hamiltonian isotopy. (For different choices of ϕ, H, and J there is a natural isomorphism of Floer homologies.) By (5) the Floer homology groups are graded modulo $2N$ where N is the minimal Chern numbber of $\mathcal{M}$. By (7) the Euler characteristic is the Lefschetz number of ϕ:

$$\chi(HF_*^{\mathrm{symp}}(\mathcal{M}, \phi)) = \sum_{x=\phi_H(x)} \mathrm{sign}\ \det(\mathbb{1} - d\phi_H(x)) = L(\phi).$$

Remark 2.3 The proof that the Floer homology groups are independent of J and H is as in the case $\phi = \mathrm{id}$ and we refer for details to [11] and [23]. That they depend on ϕ only up to Hamiltonian isotopy follows from the fact that $HF_*^{\mathrm{symp}}(\mathcal{M}, \phi, H, J_s)$ is naturally isomorphic to $HF_*^{\mathrm{symp}}(\mathcal{M}, \phi_H, 0, \psi_s^* J_s)$. To see this consider the function $v(s,t) = \psi_s^{-1}(u(s,t))$ where $u(s,t)$ is a solution of (1) and (2).

Remark 2.4 In [11] Floer proved that if $\phi = \mathrm{id}$ then there is a natural isomorphism

$$HF_*^{\mathrm{symp}}(\mathcal{M}, \mathrm{id}) \simeq H_*(\mathcal{M}, \mathbb{Z}).$$

The proof involves homotoping the Hamiltonian to a time independent Morse function $H_s = H : \mathcal{M} \to \mathbb{R}$, proving that all the solutions of (1) are independent of s when H is sufficiently small and using the formula (6) which relates the Maslov index with the Morse index. It follows that the fixed points of ϕ_H are related to the $2N$-periodic Betti numbers of $\mathcal{M}$ by the Morse inequalities. In particular the number of fixed points can be estimated from below by the sum of the Betti numbers as was conjectured by Arnold.

Remark 2.5 Floer's original theory for the case $\phi = \mathrm{id}$ does not require the manifold $\mathcal{M}$ to be simply connected. In [15] his work has been extended to some classes of compact symplectic manifolds $\mathcal{M}$ which are not monotone. In this case the Floer homology groups are modules over a suitable Novikov ring. This can be extended further to the case where ϕ is not the identity. There are then Floer homology groups for every component of Ω_ϕ.

Remark 2.6 For every symplectomorphism ψ there is a natural isomorphism of Floer homologies

$$HF_*^{\mathrm{symp}}(\mathcal{M}, \phi) = HF_*^{\mathrm{symp}}(\mathcal{M}, \psi^{-1} \circ \phi \circ \psi).$$

To see this consider the function $v(s,t) = \psi^{-1}(u(s,t))$ where $u(s,t)$ is a solution of (1) and (2).

Remark 2.7 Donaldson has suggested the construction of a homomorphism

$$HF_*^{\mathrm{symp}}(\mathcal{M},\psi)\otimes HF_*^{\mathrm{symp}}(\mathcal{M},\phi)\to HF_*^{\mathrm{symp}}(\mathcal{M},\psi\circ\phi)$$

using Moduli spaces of J-holomorphic curves with three cylindrical ends (the *pair-of-pants construction*). If $\psi=\mathrm{id}$ then this determines an action of the homology of $\mathcal{M}$ on the Floer homology groups of ϕ. If $\phi=\psi=\mathrm{id}$ then this should agree with the deformed cup-product of Witten.

3 Floer homology for 3-manifolds

Let $Q\to M$ be a principal bundle over a compact oriented 3-manifold with structure group $G=\mathrm{SO}(3)$. Denote by $\mathcal{A}(Q)$ the space of connections and by $\mathcal{G}_0(Q)$ the identity component of the group of gauge transformations. A gauge transformation $g:Q\to G$ is called **even** if it lifts to a map $\tilde{g}:Q\to\mathrm{SU}(2)$. The subgroup of even gauge transformations is denoted by $\mathcal{G}^{\mathrm{ev}}(Q)$. The **degree** of a gauge transformation is the integer $\deg(g)\in\mathbb{Z}$ determined by the induced map on homology $H_3(M)=\mathbb{Z}\to H_3(G)=\mathbb{Z}$.

The Chern-Simons functional

The perturbed Chern-Simons functional $\mathcal{CS}_H:\mathcal{A}(Q)/\mathcal{G}_0(Q)\to\mathbb{R}$ is defined by

$$\mathcal{CS}_H(a_0+\alpha)=\frac{1}{2}\int_M\left(\langle d_{a_0}\alpha\wedge\alpha\rangle+\frac{1}{3}\langle[\alpha\wedge\alpha]\wedge\alpha\rangle\right)-H(a_0+\alpha)$$

for $\alpha\in\Omega^1(\mathfrak{g}_Q)$ and a fixed flat connection $a_0\in\mathcal{A}_{\mathrm{flat}}(Q)$. Here $\langle\,,\,\rangle$ denotes the invariant inner product on $\mathfrak{g}$ given by minus the Killing form (in the case $G=\mathrm{SO}(3)$ this is 4 times the trace). The perturbation $H:\mathcal{A}(Q)\to\mathbb{R}$ is a function of the SU(2)-valued holonomy of the connection along finitely many *thickened* loops in M. Thus H is invariant under the action of $\mathcal{G}^{\mathrm{ev}}(Q)$. The perturbed Chern-Simons functional satisfies the identity [3]

$$\mathcal{CS}_H(a)-\mathcal{CS}_H(g^*a)=8\pi^2\deg(g). \tag{10}$$

The gradient of $\mathcal{CS}_H$ is given by $\operatorname{grad}\mathcal{CS}_H(a)=*F_a-*Y(a)$ where F_a is the curvature and $Y:\mathcal{A}(Q)\to\Omega^2(\mathfrak{g}_Q)$ represents the differential of H. The critical points of $\mathcal{CS}_H$ are called H**-flat** connections and the space of such connections is denoted by $\mathcal{A}_{\mathrm{flat}}(Q,H)$. An H-flat connection $a\in\mathcal{A}_{\mathrm{flat}}(Q,H)$ is both a **regular** point for the action of $\mathcal{G}_0(Q)$ and a **nondegenerate** critical point of $\mathcal{CS}_H$ iff the **extended Hessian**

$$D_a=\begin{pmatrix} *d_a-*dY(a) & d_a\\ d_a^* & 0\end{pmatrix}$$

of $\mathcal{CS}_H$ is a nonsingular operator on $\Omega^1(\mathfrak{g}_Q)\oplus\Omega^0(\mathfrak{g}_Q)$.

Lemma 3.1 *If there exists an oriented embedded Riemann surface $\Sigma \subset M$ such that $w_2(Q|_\Sigma) \neq 0$ then every flat connection on Q is regular. Moreover, there exists a gauge transformation of degree* 1.

The second statement is proved in [12] and [8]. The first implies that the perturbation H can be chosen such that every H-flat connection is regular and nondegenerate. Throughout we shall assume that the bundle Q satisfies the requirements of this lemma. In particular, this excludes the case of homology-3-spheres which is treated in Floer's original work [10]. The present case is treated in [12].

Self-dual instantons

The gradient flow of $\mathcal{CS}_H$ takes the form

$$\dot{a} + *F_a - *Y(a) = 0. \tag{11}$$

The smooth path $a : \mathbb{R} \to \mathcal{A}(Q)$ can be regarded as a connection on the bundle $Q \times \mathbb{R}$ over the 4-manifold $M \times \mathbb{R}$ and in the case $Y = 0$ equation (11) states that this connection is self-dual. If a satisfies (11) and has finite Yang-Mills energy then $a(t)$ converges to H-flat connections on Q as t tends to $\pm\infty$ provided that the H-flat connections on Q are all nondegenerate [16]. Fix $a^\pm \in \mathcal{A}_{\text{flat}}(Q, H)$ and consider the solutions of (11) with limits

$$\lim_{t \to \pm\infty} a(t) = g_\pm^* a^\pm \in \mathcal{A}_{\text{flat}}(Q, H) \tag{12}$$

for some $g_\pm \in \mathcal{G}_0(Q)$. Denote by $\mathcal{A}_{\text{SD}}(a^-, a^+, H)$ the space of smooth solutions of (11) and (12). The group $\mathcal{G}_0(Q)$ acts on this space and the quotient is denoted by

$$\mathcal{M}(a^-, a^+, H) = \frac{\mathcal{A}_{\text{SD}}(a^-, a^+, H)}{\mathcal{G}_0(Q)}.$$

Linearization of the self-duality equation (11) gives rise to the operator

$$\mathcal{D}_a = \frac{\partial}{\partial t} + D_{a(t)}.$$

It was proved by Atiyah, Patodi, and Singer [3] that if $a^\pm \in \mathcal{A}_{\text{flat}}(Q, H)$ are regular and nondegenerate then $\mathcal{D}_a$ is a Fredholm operator and

$$\operatorname{index} \mathcal{D}_a = \mu^{\text{inst}}(a^-, a^+) = \mu(a^+, H) - \mu(a^-, H)$$

where $\mu(a_0, H) = \frac{1}{2}\eta(D_{a_0}) - \mathcal{CS}_H(a_0)/2\pi^2$ and

$$\mu^{\text{inst}}(a_0, g^* a_0) = 4 \deg(g) \tag{13}$$

for $a_0 \in \mathcal{A}_{\text{flat}}(Q, H)$ and $g \in \mathcal{G}(Q)$. For $a \in \mathcal{A}(Q)$ the η-invariant $\eta(D_a)$ is the real number $\eta(D_a) = \eta_0(D_a)$ where $\eta_s(D_a) = \sum_\lambda |\lambda|^{-s} \operatorname{sign} \lambda$. The sum runs over all nonzero eigenvalues $\lambda \in \sigma(D_a)$. The function $s \mapsto \eta_s(D_a)$ is meromorphic and 0 is not a pole. With this definition the number $\mu(a, H)$ is not necessarily an integer.

Floer homology

Now for a generic perturbation H the operator $\mathcal{D}_a$ is onto for every solution $a \in \mathcal{A}_{\mathrm{SD}}(a^-, a^+, H)$ and all $a^\pm \in \mathcal{A}_{\mathrm{flat}}(Q, H)$. For such perturbations the moduli space $\mathcal{M}(a^-, a^+, H)$ is a finite dimensional manifold of dimension $\mu^{\mathrm{inst}}(a^-, a^+)$. As in section 2 the moduli spaces determine a boundary operator on the chain complex

$$C_k = C_k(Q, H) = \bigoplus_{\substack{[a] \in \mathcal{A}_{\mathrm{flat}}(Q)/\mathcal{G}_0(Q) \\ \mu^{\mathrm{inst}}(a_0, a) = k}} \mathbb{Z}[a].$$

Choose coherent orientations of the moduli spaces $\mathcal{M}(a^-, a^+, H)$ as in [10] and [13]. Whenever $a \in \mathcal{M}(a^-, a^+, H)$ with $\mu^{\mathrm{inst}}(a^-, a^+) = 1$ define $\nu(a) = \pm 1$ according to whether the natural flow orientation of $a(t)$ (given by time shift) agrees with this coherent orientation or not. The (a^+, a^-)-entry of the boundary operator

$$\partial : C_{k+1} \to C_k$$

is defined by taking the sum of the numbers $\nu(a)$ over all instantons $[a] \in \mathcal{M}(a^-, a^+, H)/\mathbb{R}$. In [10] Floer proved that $\partial^2 = 0$. The **Floer homology** groups of the pair (M, Q) are the homology groups of this chain complex

$$HF_*^{\mathrm{inst}}(M, Q) = \ker \partial / \operatorname{im} \partial$$

They are independent of the metric on M and the perurbation H used to define them [10], [12]. (Different choices of metric and perturbation give rise to natural isomorphisms.) Since there exists a gauge transformation of degree 1 it follows from (13) that the Floer homology groups are graded modulo 4. Casson's invariant of the pair (M, Q) appears as the Euler characteristic of Floer homology with respect to the mod 4 grading

$$\lambda(M, Q) = \chi(HF_*^{\mathrm{inst}}(M, Q)) = \tfrac{1}{2} \sum_{[a]} (-1)^{\mu^{\mathrm{inst}}(a_0, a)}.$$

The sum runs over all even gauge equivalence classes $[a] \in \mathcal{A}_{\mathrm{flat}}(Q, H)/\mathcal{G}^{\mathrm{ev}}(Q)$. By Lemma 3.1 there exists a gauge transformation of degree 1. Such a gauge transformation is not even and hence each flat connection appears twice as an even gauge equivalence class. This shows that the Casson invariant is an integer. It is independent of the choice of the perturbation.

Remark 3.2 If H can be chosen invariant under all gauge transformations (not just the even ones) then the group Γ of components of the space of degree-0 gauge transformations acts on $HF_k^{\mathrm{inst}}(M; Q)$ for every k. This requires an equivariant perturbation theory which takes account of the action of a finite group.

4 Flat connections over Riemann surfaces

Let $\pi : P \to \Sigma$ be a principal bundle over a compact oriented Riemann surface of genus $k \geq 2$ with structure group $G = \mathrm{SO}(3)$. A gauge transformation $g \in \mathcal{G}(P)$ is homotopic to $\mathbb{1}$ iff it is even. The moduli space

$$\mathcal{M}(P) = \mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}_0(P)$$

of flat connections modulo even gauge equivalence is a compact symplectic manifold of dimension $6k - 6$ with symplectic structure

$$\omega_A(a, b) = \int_\Sigma \langle a \wedge b \rangle$$

for $a, b \in H^1_A = T_{[A]}\mathcal{M}(P)$. Every conformal structure on the Riemann surface Σ determines a Kähler structure on $\mathcal{M}(P)$. The Hodge-$*$-operators on the spaces $H^1_A = \ker\, d_A \cap \ker\, d_A^*$ of harmonic forms determine an integrable complex structure on $\mathcal{M}(P)$ which is compatible with ω. (See [2].)

Via the holonomy the space $\mathcal{M}(P)$ can be identified with the odd representations of the fundamental group of P in $\mathrm{SU}(2)$

$$\mathcal{M}(P) = \frac{\mathrm{Hom}^{\mathrm{odd}}(\pi_1(P), \mathrm{SU}(2))}{\mathrm{SU}(2)}.$$

A homomorphism $\rho : \pi_1(P) \to \mathrm{SU}(2)$ is called **odd** if $\rho(\gamma) = -\mathbb{1}$ for a nontrivial loop in the fibre. Every odd homomorphism arises as the holonomy of a flat connection on P and two flat connections are gauge equivalent by a gauge transformation in $\mathcal{G}_0(P)$ iff their holonomy representations are conjugate.

Symplectomorphisms

Every orientation preserving diffeomorphism $h : \Sigma \to \Sigma$ lifts (in a nonunique way) to an automorphism $f : P \to P$. Every such automorphism $f : P \to P$ determines a symplectomorphism

$$\phi_f : \mathcal{M}(P) \to \mathcal{M}(P)$$

defined by $[A] \mapsto [f^*A]$. In terms of representations of the fundamental group this symplectomorphism is given by $\rho \mapsto \rho \circ f_*$. Hence the symplectomorphism ϕ_f depends only on the homotopy class of h. In other words the homomorphism $f \mapsto \phi_f$ determines an action of the extended mapping class group

$$\mathrm{Aut}^+(P)/\mathrm{Aut}_0(P) \to \mathrm{Symp}(\mathcal{M}(P)).$$

Here $\mathrm{Aut}^+(P)$ denotes the group of those automorphisms of P which cover orientation preserving diffeomorphisms of Σ and $\mathrm{Aut}_0(P)$ denotes the component of the identity. It is an open question whether the induced homomorphism $\pi_0(\mathrm{Aut}^+(P)) \to \pi_0(\mathrm{Symp}(\mathcal{M}(P)))$ is onto or injective. The fixed points of ϕ_f will in general not be all nondegenerate. The presence of degenerate fixed points requires a Hamiltonian perturbation.

Remark 4.1 Let $\mathcal{G}_f(P)$ denote the subgroup of gauge transformations $g \in \mathcal{G}(P)$ such that $g \circ f \sim g$. Then the finite group $\Gamma_f = \mathcal{G}_f(P)/\mathcal{G}_0(P)$ acts on $\mathcal{M}(P)$ by symplectomorphism $[A] \mapsto [g^*A]$ which commute with ϕ_f.

Mapping cylinders

An automorphism $f : P \to P$ also determines a principal bundle $P_f \to \Sigma_h$ where $h : \Sigma \to \Sigma$ is the diffeomorphism induced by f via $\pi \circ f = h \circ \pi$ and P_f and Σ_h denote the mapping cylinders. We assume throughout that h is orientation preserving. A connection $a \in \mathcal{A}(P_f)$ is a 1-form $a = A + \Phi\, ds$ where $A(s) \in \mathcal{A}(P)$, $\Phi(s) \in \Omega^0(\mathfrak{g}_P)$ and

$$A(s+1) = f^*A(s), \qquad \Phi(s+1) = \Phi(s) \circ f.$$

The group $\mathcal{G}(P_f)$ of gauge transformations of P_f consists of smooth maps $g : \mathbb{R} \to \mathcal{G}(P)$ such that $g(s+1) = g(s) \circ f$. It acts on $\mathcal{A}(P_f)$ by

$$g^*a = g^*A + \left(g^{-1}\dot{g} + g^{-1}\Phi g\right)\, ds.$$

Here the notation g^* is used ambiguously: g^*a denotes the action of $g \in \mathcal{G}(P_f)$ on $a \in \mathcal{A}(P_f)$ whereas g^*A denotes the pointwise action of $g(s) \in \mathcal{G}(P)$ on $A(s) \in \mathcal{A}(P)$.

The Chern-Simons functional on P_f takes the form

$$\mathcal{CS}(a) = \int_0^1 \int_\Sigma \left(\tfrac{1}{2}\langle \dot{A} \wedge (A - A_0)\rangle + \langle F_A \wedge \Phi\rangle \right) ds$$

for $a = A + \Phi\, ds$ where $A_0 = f^*A_0$ is a fixed flat connection on P. A connection $A + \Phi\, ds$ is a critical point of $\mathcal{CS}$, that is a flat connection on P_f iff

$$F_A = 0, \qquad \dot{A} - d_A\Phi = 0.$$

The flat connections on P_f correspond naturally to the fixed points of the symplectomorphism ϕ_f. Moreover, a flat connection $a = A + \Phi\, ds$ is nondegenerate as a critical point of the Chern-Simons functional if and only if $A(0)$ represents a nondegenerate fixed point of ϕ_f.

Remark 4.2 It follows from results in [2] and [6] that the space $\mathcal{A}_{\mathrm{flat}}(P)$ is simply connected and hence the space $\mathcal{A}_\Sigma(P_f)$ of all $A + \Phi\, ds \in \mathcal{A}(P_f)$ such that $F_A = 0$ and $d_A *_s (dA/ds - d_A\Phi) = 0$ is connected. Here $*_s$ denotes the Hodge-$*$-operator corresponding to a family of conformal structures on Σ such that $*_{s+1} \circ f^* = f^* \circ *_s$. The section $\Phi(s) \in \Omega^0(\mathfrak{g}_P)$ is uniquely determined by the condition $d_{A(s)} *_s d_{A(s)} \Phi(s) = d_{A(s)} *_s \dot{A}(s)$ and hence the elements of $\mathcal{A}_\Sigma(P_f)$ can be thought of as paths in $\mathcal{A}_{\mathrm{flat}}(P)$. Hence there is a natural bijection

$$\Omega_{\phi_f} \simeq \mathcal{A}_\Sigma(P_f)/\mathcal{G}_\Sigma(P_f)$$

where $\mathcal{G}_\Sigma(P_f)$ denotes the subgroup of all $g \in \mathcal{G}(P_f)$ such that $g(s) \in \mathcal{G}_0(P)$ for all s. The universal cover of Ω_{ϕ_f} is the quotient $\mathcal{A}_\Sigma(P_f)/\mathcal{G}_0(P_f)$. The restriction of the Chern-Simons functional to the space $\mathcal{A}_\Sigma(P_f)$ agrees with the symplectic action. It is invariant under the action of the finite group $\Gamma_f = \mathcal{G}_f(P)/\mathcal{G}_0(P) = \mathcal{G}(P_f)/\mathcal{G}_\Sigma(P_f)$ of components of degree-0 gauge transformations of P_f.

Perturbations

If there are degenerate flat connections on P_f respectively degenerate fixed points of the symplectomorphism ϕ_f then we must perturb the Chern-Simons functional to obtain nondegenerate critical points. As in [7] and [24] such perturbations arise from the holonomy and they can be interpreted in terms of Hamiltonian dynamics on the infinite dimensional symplectic manifold $\mathcal{A}(P)$.

Choose $2k$ embeddings $\gamma_j : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to P$ of the annulus such that the projections $\pi \circ \gamma_j$ are orientation preserving, generate the fundamental group of Σ, and satisfy $\gamma_j(0, \lambda) = p_\lambda$ for every j. Denote by $\rho_\lambda : \mathcal{A}(P) \to \mathrm{SU}(2)^{2k}$ the holonomy along the loops $\theta \mapsto \gamma_j(\theta, \lambda)$ for $\lambda \in \mathbb{R}$. Now choose a smooth family of functions $h_s : \mathrm{SU}(2)^{2k} \to \mathbb{R}$ which are invariant under conjugacy and vanishes for s near 0 and 1. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function supported in $[-1, 1]$ with mean value 1 and define $H_s : \mathcal{A}(P) \to \mathbb{R}$ by

$$H_s(A) = \int_{-1}^{1} \beta(\lambda) h_s(\rho_\lambda(A))\, d\lambda$$

for $A \in \mathcal{A}(P)$ and $0 \le s \le 1$. Any smooth Hamiltonian on the moduli space $\mathcal{M}(P)$ can be represented in this form

The functions are invariant under the action of $\mathcal{G}_0(P)$. They can be extended to $s \in \mathbb{R}$ such that

$$H_s(g^*A) = H_s(A) = H_{s+1}(f^*A) \tag{14}$$

for $A \in \mathcal{A}(P)$ and $g \in \mathcal{G}_0(P)$. The differential of H_s can be represented by a smooth map $X_s : \mathcal{A}(P) \to \Omega^1(\mathfrak{g}_P)$ such that

$$dH_s(A)\alpha = \int_\Sigma \langle X_s(A) \wedge \alpha \rangle.$$

In other words $X_s : \mathcal{A}(P) \to \Omega^1(\mathfrak{g}_P)$ is the Hamiltonian vector field on $\mathcal{A}(P)$ corresponding to the Hamiltonian function H_s. These vector fields satisfy $X_{s+1}(f^*A) = f^* X_s(A)$ and, since H_s is invariant under $\mathcal{G}_0(P)$,

$$X_s(g^*A) = g^{-1} X_s(A) g, \qquad d_A X_s(A) = 0 \tag{15}$$

for $g \in \mathcal{G}_0(P)$ and $A \in \mathcal{A}(P)$. The vector fields X_s that arise from the holonomy will be smooth with respect to the $W^{k,p}$-norm for all k and p and hence give rise to a Hamiltonian flow $\psi_s : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\frac{d}{ds}\psi_s = X_s \circ \psi_s, \qquad \psi_0 = \mathrm{id}.$$

The diffeomorphisms ψ_s preserve the symplectic structure and the curvature and are equivariant under the action of $\mathcal{G}_0(P)$. Moreover,

$$\psi_{s+1} \circ \phi_{f,H} = \phi_f \circ \psi_s, \qquad \phi_{f,H}(A) = {\psi_1}^{-1}(f^*A),$$

as in section 2. For a generic Hamiltonian H the fixed points of $\phi_{f,H}$ on the moduli space $\mathcal{M}(P)$ are nondegenerate.

Remark 4.3 We do not assume here that H_s is invariant under the action of Γ_f. This would require a perturbation theorem which takes account of the action of a finite group.

The perturbed Chern-Simons functional $\mathcal{CS}_H : \mathcal{A}(P_f) \to \mathbb{R}$ is defined by

$$\mathcal{CS}_H(a) = \mathcal{CS}(a) - \int_0^1 H_s(A(s))\, ds.$$

for $a = A + \Phi\, ds \in \mathcal{A}(P_f)$. Its critical point are the solutions of

$$F_A = 0, \qquad \dot{A} - d_A\Phi - X_s(A) = 0.$$

A connection $A + \Phi\, ds \in \mathcal{A}(P_f)$ which satisfies these equations is called ***H*-flat**. The space of H-flat connections is denoted by $\mathcal{A}_{\mathrm{flat}}(P_f, H)$.

Proposition 4.4 *There is a natural bijection*

$$\mathcal{A}_{\mathrm{flat}}(P_f, H)/\mathcal{G}_\Sigma(P_f) \simeq \mathrm{Fix}(\phi_{f,H})$$

induced by $A + \Phi\, ds \mapsto A(0)$. A connection $A + \Phi\, ds \in \mathcal{A}_{\mathrm{flat}}(P_f, H)$ is nondegenerate as a critical point of the perturbed Chern-Simons functional $\mathcal{CS}_H$ iff $A(0)$ represents a nondegenerate fixed point of $\phi_{f,H}$.

Proof: Assume $A + \Phi ds \in \mathcal{A}_{\mathrm{flat}}(P_f, H)$ and denote $A_0 = A(0)$. Then $A(s) \in \mathcal{A}_{\mathrm{flat}}(P)$ for every s and $\dot{A} = d_A\Phi + X_s(A)$. Let $g(s) \in \mathcal{G}(P)$ be the unique solution of the ordinary differential equation $\dot{g} + \Phi g = 0$ with $g(0) = \mathbb{1}$. Then $\frac{d}{ds}g^*A = X_s(g^*A)$ and hence $\psi_1(A_0) = g(1)^*A(1) = g(1)^*f^*A_0$. Hence A_0 represents a fixed point of $\phi_{f,H}$. This shows that there is a map $\mathcal{A}_{\mathrm{flat}}(P_f, H)/\mathcal{G}_\Sigma(P_f) \to \mathrm{Fix}(\phi_{f,H})$ induced by $A + \Phi\, ds \mapsto A(0)$. We must prove that it is bijective.

To prove surjectivity, suppose that $A_0 \in \mathcal{A}_{\mathrm{flat}}(P)$ represents a fixed point of $\phi_{f,H}$ and let $g_1 \in \mathcal{G}_0(P)$ such that $g_1^* f^* A_0 = \psi_1(A_0)$. Choose a smooth map $g : \mathbb{R} \to \mathcal{G}_0(P)$ such that $g(0) = 1$ and $g(s+1) = (g(s) \circ f) g_1$. Let $A(s) \in \mathcal{A}_{\mathrm{flat}}(P)$ and $\Phi(s) \in \Omega^0(\mathfrak{g}_P)$ be defined by $g(s)^* A(s) = A_0$ and $\Phi(s) = -\dot{g}(s) g(s)^{-1}$. Then $A + \Phi\, ds \in \mathcal{A}_{\mathrm{flat}}(P_f, H)$.

To prove injectivity, let $a = A + \Phi\, ds$ and $a' = A' + \Phi'\, ds$ be H-flat connections and suppose that $A'(0) = g_0^* A(0)$ for some $g_0 \in \mathcal{G}_0(P)$. Define $g(s) \in \mathcal{G}_0(P)$ to be the unique solution of the ordinary differential equation $\dot{g} = g\Phi' - \Phi g$ with $g(0) = g_0$. Then $\frac{d}{ds} g^* A = d_{g^*A} \Phi' + X_s(g^*A)$, and $g(0)^* A(0) = A'(0)$. This implies $g(s)^* A(s) = A'(s)$ for all s. Since $A(s+1) = f^* A(s)$ and $A'(s+1) = f^* A'(s)$ it follows that $g(s+1)^* A(s+1) = (g(s) \circ f)^* A(s+1)$ and hence $g(s+1) = g(s) \circ f$. This shows that g is a gauge transformation of P_f and $a' = g^* a$.

Now let $a = A + \Phi\, ds \in \mathcal{A}_{\mathrm{flat}}(P_f, H)$ and define $A_0 = A(0) \in \mathcal{A}_{\mathrm{flat}}(P)$. Assume without loss of generality that $\Phi \equiv 0$ so that $A(s) = \psi_s(A_0)$ and $f^* A_0 = \psi_1(A_0)$. We prove that the linear map

$$d\psi_1(A_0) - f^* : H^1_{A_0}(\Sigma) \to H^1_{f^*A_0}(\Sigma)$$

is injective if and only if every infinitesimal connection $\alpha + \phi\, ds \in \Omega^1(\mathfrak{g}_{P_f})$ which satisfies

$$d_A \alpha = 0, \qquad \dot{\alpha} - d_A \phi - dX_s(A)\alpha = 0 \tag{16}$$

is of the form

$$\alpha = d_A \xi, \qquad \phi = \dot{\xi} \tag{17}$$

for some $\xi \in \Omega^0(\mathfrak{g}_{P_f})$. The former means that A_0 represents a non-degenerate fixed point of $\phi_{f,H}$ and the latter means that a is a nondegenerate critical point of $\mathcal{CS}_H$.

Assume first that $d\psi_1(A_0) - f^*$ is injective and $\alpha + \phi\, ds$ satisfies (16). Denote $\alpha_0 = \alpha(0)$. Then

$$\alpha(s) = d\psi_s(A_0)\alpha_0 + d_{A(s)} \int_0^s \phi(\theta)\, d\theta.$$

(Differentiate and use (15).) Hence $f^*\alpha_0 - d\psi_1(A_0)\alpha_0 \in \operatorname{im} d_{f^*A_0}$ and since A_0 is a nondegenerate fixed point of $\phi_{f,H}$ there exists a $\xi_0 \in \Omega^0(\mathfrak{g}_P)$ such that $\alpha_0 = d_{A_0}\xi_0$. Define $\xi(s) \in \Omega^0(\mathfrak{g}_P)$ by $\xi(0) = \xi_0$ and $\dot{\xi}(s) = \phi(s)$. Then

$$\frac{d}{ds}(\alpha - d_A\xi) = \dot{\alpha} - d_A\dot{\xi} - [\dot{A} \wedge \xi] = dX_s(A)(\alpha - d_A\xi)$$

and hence $\alpha = d_A\xi$.

Now suppose that $f^* - d\psi_1(A_0) : H^1_{A_0} \to H^1_{f^*A_0}$ is not injective. Then there exist $\alpha_0 \in \Omega^1(\mathfrak{g}_P)$ and $\xi_0 \in \Omega^0(\mathfrak{g}_P)$ such that

$$f^*\alpha_0 - d\psi_1(A_0)\alpha_0 = d_{f^*A_0}\xi_0, \qquad d_{A_0}\alpha_0 = 0, \qquad \alpha_0 \notin \mathrm{im} d_{A_0}.$$

Choose any function $\phi(s) \in \Omega^0(\mathfrak{g}_P)$ satisfying

$$\xi_0 = \int_0^1 \phi(s)\, ds, \qquad \phi(s+1) = \phi(s) \circ f,$$

and let $\alpha(s) \in \Omega^1(\mathfrak{g}_P)$ be the unique solution of $\dot{\alpha} = d_A\phi + dX_s(A)\alpha$ with $\alpha(0) = \alpha_0$. Then α and ϕ satisfy (16) but are not of the form (17) since $\alpha_0 \notin \mathrm{im} d_{A_0}$. □

5 The Atiyah-Floer conjecture

For every orientation preserving automorphism of the bundle $P \to \Sigma$ there are two Floer homology groups, one for the symplectomorphism ϕ_f as in section 2 and one for the mapping cylinder P_f as in section 3. In a forthcomin paper [9] we prove that there is a natural isomorphism

$$HF_*^{\mathrm{symp}}(\mathcal{M}(P), \phi_f) \simeq HF_*^{\mathrm{inst}}(\Sigma_h, P_f).$$

A similar statement in the context of Heegard splittings and Lagrangian intersections was conjectured by Atiyah [1] and Floer. The present version was suggested to us by Floer.

The proof relies on a comparison of self-dual instantons on the 4-manifold $\Sigma_h \times \mathbb{R}$ with J-holomorphic curves in the moduli space $\mathcal{M}(P)$. Fix two nondegenerate H-flat connections $a^\pm = A^\pm + \Phi^\pm\, ds \in \mathcal{A}_{\mathrm{flat}}(P_f, H)$ and choose smooth functions $A : \mathbb{R}^2 \to \mathcal{A}_{\mathrm{flat}}(P)$ and $\Phi, \Psi : \mathbb{R}^2 \to \Omega^0(\mathfrak{g}_P)$ which satisfy

$$A(s+1,t) = f^*A(s,t), \quad \Phi(s+1,t) = \Phi(s,t)\circ f, \quad \Psi(s+1,t) = \Phi(s,t)\circ f. \tag{18}$$

and

$$\lim_{t\to\pm\infty} A(s,t) = A^\pm(s), \quad \lim_{t\to\pm\infty} \Phi(s,t) = \Phi^\pm(s), \quad \lim_{t\to\pm\infty} \Psi(s,t) = 0 \tag{19}$$

Now choose a smooth family of conformal structures on Σ depending on a real parameter s such that $*_{s+1} \circ f^* = f^* \circ *_s$. This corresponds to the condition $J_s = \phi^* J_{s+1}$ of section 2. The perturbed Cauchy-Riemann equations take the form

$$\frac{\partial A}{\partial t} - d_A\Psi + *_s\left(\frac{\partial A}{\partial s} - X_s(A) - d_A\Phi\right) = 0. \tag{20}$$

For solutions of (20) the functions Φ and Ψ are uniquely determined by A. Think of $\Xi = A + \Phi\, ds + \Psi\, dt$ as a connection on $P_f \times \mathbb{R}$. Then the perturbed self-duality equations on $\mathbb{R} \times P_f$ take the form

$$\begin{aligned} \frac{\partial A}{\partial t} - d_A\Psi + *_s\left(\frac{\partial A}{\partial s} - X_s(A) - d_A\Phi\right) &= 0, \\ \frac{\partial \Phi}{\partial t} - \frac{\partial \Psi}{\partial s} - [\Phi, \Psi] + \frac{1}{\varepsilon^2} *_s F_A &= 0. \end{aligned} \tag{21}$$

The factor $1/\varepsilon^2$ arises from conformally rescaling the metric on Σ by the factor ε^2. Solutions of (21) with $0 \le s \le 1$ are equivalent to solutions with $\varepsilon = 1$ on the long cylinder $0 \le s \le 1/\varepsilon$. In the context of Heegard splittings this corresponds to Atiyah's suggestion to *stretch the neck*.

Note that the first equation in (21) agrees with (20) while the second equation replaces the condition on $A(s,t)$ to be flat. Thus equation (20) can be viewed as a singular limit of (21) analogous to singular perturbations of fast-slow systems in ordinary differential equations. In [9] we shall prove that the solutions of (21) shall in fact converge to those of (20) as ε tends to zero and thus for ε sufficiently small there is a one-to-one correspondence of self-dual instantons with J-holomorphic curves. The main result of the present paper asserts that the two Fredholm operators arising from linearizing (20) and (21) have the same index. This implies that the chain complexes of the two Floer homology groups have the same grading.

6 An index theorem

Fix a pair of H-flat connections $a^\pm = A^\pm(s) + \Phi^\pm(s)\,ds \in \mathcal{A}_{\rm flat}(P_f, H)$. For every such pair there are two integers. The Maslov index $\mu^{\rm symp}(a^-, a^+)$ of section 2 and the index $\mu^{\rm inst}(a^-, a^+)$ of section 3. Our main theorem asserts that both indices agree

Theorem 6.1 *For every pair $a^\pm \in \mathcal{A}_{\rm flat}(P_f, H)$ of nondegenerate H-flat connections $\mu^{\rm symp}(a^-, a^+) = \mu^{\rm inst}(a^-, a^+)$.*

Choose $\Xi = A + \Phi\,ds + \Psi\,dt$ such that $A(s,t)$ is flat for all s and t and (18) and (19) are satisfied. Such a connection always exists since the space $\mathcal{A}_{\rm flat}(P)$ is simply connected [2], [6]. Linearizing (20) gives rise to the perturbed Cauchy-Riemann operator $\mathcal{D}_0$ as in section 2. The bundle u^*TM of section 2 corresponds here to the bundle $H_A \to S^1 \times \mathbb{R}$ whose fiber over (s,t) is the space $H^1_{A(s,t)}$ of harmonic 1-forms on $\mathfrak{g}_P$ with respect to the connection $A(s,t)$ and the s-metric on Σ. The operator $\mathcal{D}_0 : W^{1,2}_f(H_A) \to L^2_f(H_A)$ is given by

$$\mathcal{D}_0(\alpha_0) = \pi_A\left(\nabla_t \alpha_0 + *_s \nabla_s \alpha_0 - *_s dX_s(A)\alpha_0\right)$$

where $\nabla_s = \partial_s + \Phi$, $\nabla_t = \partial_t + \Psi$ and $\pi_{A(s,t)}(\alpha)$ denotes the harmonic part of α. The subscript f refers to the periodic boundary condition $\alpha(s+1,t) = f^*\alpha(s,t)$. The Fredholm index of $\mathcal{D}_0$ is

$$\mu^{\rm symp}(a^-, a^+) = \operatorname{index} \mathcal{D}_0.$$

Linearizing (21) gives rise to the Fredholm operator $\mathcal{D}_\varepsilon : W^{1,2}_f \to L^2_f$ defined by

$$\mathcal{D}_\varepsilon = \nabla_t + \begin{pmatrix} *_s\nabla_s & 0 & 0 \\ 0 & 0 & -\nabla_s \\ 0 & *_s\nabla_s *_s & 0 \end{pmatrix} - \begin{pmatrix} *_s dX_s(A) & *_s d_A & d_A \\ -\varepsilon^{-2} *_s d_A & 0 & 0 \\ -\varepsilon^{-2} *_s d_A *_s & 0 & 0 \end{pmatrix}$$

as in section 3. Here $W^{k,2}_f = W^{k,2}_f(\mathbb{R}^2 \times T^*\Sigma \otimes \mathfrak{g}_P \oplus \mathfrak{g}_P \oplus \mathfrak{g}_P)$. This is a kind of infinite dimensional analogue of the Cauchy-Riemann operator. The Fredholm index of $\mathcal{D}_\varepsilon$ is independent of $\varepsilon > 0$. It is the relative Morse index of instanton homology

$$\mu^{\text{inst}}(a^-, a^+) = \operatorname{index} \mathcal{D}_\varepsilon.$$

The main result of this paper asserts that the two Fredholm indices agree. Here are two corollaries concerning Casson's invariant of the manifold Σ_h and the first Chern class of $\mathcal{M}(P)$. In the context of Heegard splittings and Lagrangian intersections the analogue of the first was proved by Taubes [24].

Corollary 6.2 *Casson's invariant of (Σ_h, P_f) is the Lefschetz number of ϕ_f*

$$\lambda(\Sigma_h, P_f) = L(\phi_f)$$

In particular, when $f = \mathrm{id}$,

$$\lambda(\Sigma \times S^1, P \times S^1) = \chi(\mathcal{M}(P)).$$

Proof: The Euler charcteristic of Floer homology agrees with the Euler charcteristic of the chain complex. By Theorem 6.1 both chain complexes are isomorphic. Hence they have the same Euler characteristic. $\square$

Atiyah and Bott [2] observed that the first Chern class of the tangent bundle of $\mathcal{M}(P)$ determines an isomorphism of $\pi_2(\mathcal{M}(P))$ with the even integers. We give an alternative proof of this result as a corollary of Theorem 6.1. Since the space of flat connections $\mathcal{A}_{\text{flat}}(P)$ (**not** modulo gauge equivalence) is simply connected and $\pi_2(\mathcal{A}_{\text{flat}}(P)) = \{0\}$ (see [2] and [6]) the homotopy exact sequence of the fibration $\mathcal{G}_0 \hookrightarrow \mathcal{A}_{\text{flat}}(P) \to \mathcal{M}(P)$ shows that

$$\pi_2(\mathcal{M}(P)) = \pi_1(\mathcal{G}_0(P)/\{-1\!\!1, +1\!\!1\}) = \mathbb{Z}.$$

The last isomorphism is given by the degree for gauge transformations of the bundle $Q = P \times S^1$ over the 3-manifold $M = \Sigma \times S^1$. (See Lemma 3.1.) Here is a more explicit description of the first isomorphism. Let $A_0 \in \mathcal{A}_{\text{flat}}(P)$ be a flat connection and let $g(s) \in \mathcal{G}_0(P)$ be a loop of gauge transformations such that $g(0) = 1\!\!1 = \pm g(1)$. Since $\mathcal{A}_{\text{flat}}(P)$ is simply connected there exists a map $A : D \to \mathcal{A}_{\text{flat}}(P)$ on the unit disc such that

$$A(e^{2\pi i s}) = g(s)^* A_0.$$

This map represents a sphere in the moduli space $\mathcal{M}(P)$ since the boundary of D is mapped to a point.

Corollary 6.3

$$\langle c_1, A \rangle = 2 \deg(g) = \frac{1}{4\pi^2} \langle [\omega], A \rangle.$$

Proof: Choose polar co-ordinates $\theta : [0,1]^2 \to D$ by $\theta(s,t) = te^{2\pi is}$ and denote $B = A \circ \theta$. Since θ is orientation reversing

$$\begin{aligned}\int_D \int_\Sigma \langle \partial_x A \wedge \partial_y A \rangle \, dxdy &= -\int_0^1 \int_0^1 \int_\Sigma \langle \partial_s B \wedge \partial_t B \rangle \, dsdt \\ &= -\frac{1}{2} \int_0^1 \int_\Sigma \left\langle \frac{d}{ds} g(s)^* A_0 \wedge g(s)^* A_0 - A_0 \right\rangle ds \\ &= 8\pi^2 \deg(g).\end{aligned}$$

The last identity follows from equation (10) with $Q = P \times S^1$.

Now let $f = \mathrm{id}$ and choose a Hamiltonian perturbation H such that the induced symplectomorphism ψ_1 has a nondegenerate fixed point $A_0 \in \mathcal{A}_{\mathrm{flat}}(P)$ such that $\psi_s(A_0) = A_0$ for all $s \in \mathbb{R}$. Assume that $A : D \to \mathcal{A}_{\mathrm{flat}}(P)$ satisfies $A(z) = A_0$ for $|z| < \varepsilon$ and $A(te^{2\pi is}) = g(s)^* A_0$ for $t \geq 1 - \varepsilon$. Extend $B = A \circ \theta$ to a smooth map $B : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathcal{A}_{\mathrm{flat}}(P)$ such that $B(s,t) = A_0$ for $t \leq 0$ and $B(s,t) = g(s)^* A_0$ for $t \geq 1$. Now choose $\Phi : \mathbb{R}^2 \to \Omega^0(\mathfrak{g}_P)$ such that $d_B *_s (\partial_s B - d_B \Phi) = 0$. Then the connection $B + \Phi \, ds$ satisfies (18) and (19) with $a^- = a_0$ and $a^+ = g^* a_0$. Hence

$$2\langle c_1, A \rangle = -2\langle c_1, B \rangle = \mu^{\mathrm{symp}}(a_0, g^* a_0) = \mu^{\mathrm{inst}}(a_0, g^* a_0) = 4 \deg(g).$$

The first identity follows from the fact that $\theta : [0,1]^2 \to D$ is orientation reversing, the second from (8), the third from Theorem 6.1 and the last from (13). $\square$

7 Proof of the main theorem

Spectral flow

Let H be a separable Hilbert space and $A(t)$ be a family of unbounded self adjoint operators on H with a dense domain $\mathrm{dom}\, A(t) = W$ which is independent of t. Then W is a Hilbert space in its own right. Assume throughout that the inclusion $W \subset H$ is a compact operator and that the resolvent set of $A(t)$ is nonempty for every t. Then $A(t)$ has a compact resolvent operator and hence its spectrum is discrete and consists of real eigenvalues of finite multiplicity. The operators $A(t)$ can also be considered as a bounded operator from W to H. Assume that the map $t \mapsto A(t) \in \mathcal{L}(W, H)$ is continuously differentiable with respect to the weak operator topology with invertible limits

$$A^{\pm} = \lim_{t \to \pm\infty} A(t).$$

Denote $\mathcal{W} = L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H)$ and $\mathcal{H} = L^2(\mathbb{R}, H)$ and consider the operator $\mathcal{D}_A : \mathcal{W} \to \mathcal{H}$ defined by

$$\mathcal{D}_A \xi = \frac{d\xi}{dt} + A\xi.$$

This is a Fredholm operator and its index is given by the spectral flow $\mu(A)$ of the operator family $A(t)$ [3]. To make this precise recall the following theorem of Kato. (For a proof see Theorem II.5.4 and Theorem II.6.8 in [17].)

Theorem 7.1 (Kato selection theorem) *Let $t_0 \in \mathbb{R}$ and $c_0 > 0$ such that $\pm c_0 \notin \sigma(A(t_0))$. Then there exists a constant $\varepsilon > 0$ and continuously differentiable functions $\lambda_j : (t_0 - \varepsilon, t_0 + \varepsilon) \to (-c_0, c_0)$, $j = 1, \ldots, N$ such that $\lambda_j(t) \in \sigma(A(t))$ and*

$$\dot{\lambda}_j(t) \in \sigma(P_j(t)\dot{A}(t)P_j(t))$$

where $P_j(t) : H \to H$ denotes the orthogonal projection onto $\ker(\lambda_j(t)\mathbb{1} - A(t))$. Moreover, if $\lambda \in \sigma(A(t)) \cap (-c_0, c_0)$ with corresponding spectral projection $P : H \to \ker(\lambda\mathbb{1} - A(t))$ and $\theta \in \sigma(P\dot{A}(t)P)$ is an eigenvalue of multiplicity m then there are precisely m indices $j_1, \ldots, j_m$ such that $\lambda_{j_\nu}(t) = \lambda$ and $\dot{\lambda}_{j_\nu}(t) = \theta$ for $\nu = 1, \ldots, m$.

By Theorem 7.1 cover the set $\{(t, \lambda) \mid \lambda \in \sigma(A(t))\}$ by countably many graphs of continuously differentiable curves $t \mapsto \lambda_j(t)$. By Sard's theorem the complement of the set of common regular values has measure zero. A number $\delta \in \mathbb{R}$ is such a common regular value if and only if there exist only finitely many times $t \in \mathbb{R}$ with $\delta \in \sigma(A(t))$ and for each such t the operator $P_\delta(t)\dot{A}(t)P_\delta(t)$ is nonsingular on the kernel of $\delta\mathbb{1} - A(t)$. Here $P_\delta(t)$ denotes the orthogonal projection of H onto the kernel of $\delta\mathbb{1} - A(t)$. For any sufficiently small regular value δ define

$$\mu(A) = \sum_t \operatorname{sign}(P_\delta(t)\dot{A}(t)P_\delta(t)) \tag{22}$$

where the summation runs over all $t \in \mathbb{R}$ with $\delta \in \sigma(A(t))$ and sign denotes the signature (the number of positive minus the number of negative eigenvalues). This number is obviously finite and by Kato's selection it is independent of δ for δ sufficiently small. It is called the **spectral flow** of the operator family $A(t)$. If $A(t)$ has only simple eigenvalues for every t then the spectral flow is given by

$$\mu(A) = \#\{j \mid \lambda_j(-T) < 0 < \lambda_j(T)\} - \#\{j \mid \lambda_j(-T) > 0 > \lambda_j(T)\}$$

for T sufficiently large. Intuitively speaking $\mu(A)$ counts the number of times an eigenvalue of $A(t)$ crosses 0. The following theorem was first stated by Atiyah, Patodi and Singer [3]. In the present form it is proved in [21].

Theorem 7.2 *The operator $\mathcal{D}_A : \mathcal{W} \to \mathcal{H}$ is Fredholm and* $\operatorname{index} \mathcal{D}_A = \mu(A)$.

Selfadjoint operators

The Sobolev space $W^{k,2}_f(\mathbb{R} \times T^*\Sigma \otimes \mathfrak{g}_P)$ is the completion of the space of smooth maps $\alpha : \mathbb{R} \to \Omega^1(\mathfrak{g}_P)$ which satisfy $\alpha(s+1) = f^*\alpha(s)$ with respect to the $W^{k,2}$-norm over Σ_h. The space $W^{k,2}_f(\mathbb{R} \times \mathfrak{g}_P)$ is defined similarly. For any smooth map

$A:\mathbb{R}^2\to\mathcal{A}_{\mathrm{flat}}(P)$ which satisfies (18) the closed linear subspace $W^{k,2}_f(H_A(t))\subset W^{1,2}_f(\mathbb{R}^2\times T^*\Sigma\otimes\mathfrak{g}_P)$ consists of those α_0 such that $\alpha_0(s)\in H^1_{A(s,t)}$ for every s.

Assume that $A:\mathbb{R}^2\to\mathcal{A}_{\mathrm{flat}}(P)$ and $\Phi:\mathbb{R}^2\to\Omega^0(\mathfrak{g}_P)$ satisfy (18) and (19). Then both operators $\mathcal{D}_0$ and $\mathcal{D}_\varepsilon$ are well defined. Their Fredholm indices are given by the spectral flow of suitable families of self adjoint operators. These are obtained by removing the derivatives with respect to t from the operators $\mathcal{D}_0$ and $\mathcal{D}_\varepsilon$. Hence consider the operator $D_0(t):W^{1,2}_f(H_A(t))\to L^2_f(H_A(t))$ defined by

$$D_0(t)\alpha_0=*_s\pi_A\left(\nabla_s\alpha_0-dX_s(A)\alpha_0\right)$$

where $\pi_{A(s,t)}(\alpha)$ denotes the harmonic part of the 1-form α with respect to the connection $A(s,t)$ and the s-metric on Σ. Abbreviate

$$\xi=(\alpha,\phi,\psi)\in W^{k,2}_f=W^{k,2}_f(\mathbb{R}\times T^*\Sigma\otimes\mathfrak{g}_P\oplus\mathfrak{g}_P\oplus\mathfrak{g}_P).$$

Consider the operator $D_\varepsilon(t):W^{1,2}_f\to L^2_f$ defined by

$$D_\varepsilon(t)=\left(\begin{array}{ccc} *_s\nabla_s & 0 & 0\\ 0 & 0 & -\nabla_s\\ 0 & *_s\nabla_s*_s & 0\end{array}\right)-\left(\begin{array}{ccc} *_s dX_s(A) & *_s d_A & d_A\\ -\varepsilon^{-2}*_s d_A & 0 & 0\\ -\varepsilon^{-2}*_s d_A*_s & 0 & 0\end{array}\right)$$

This operator is self-adjoint with respect to the ε-inner product on L^2_f. We must prove that the spectral flows of the operator families $D_0(t)$ and $D_\varepsilon(t)$ agree. The key point is that the first term in $D_\varepsilon(t)$ is the dominating one whenever $\varepsilon>0$ is sufficiently small.

Elliptic estimates

It is convenient to use the ε-dependent Hilbert space norms

$$\|(\alpha,\phi,\psi)\|^2_{0,\varepsilon}=\|\alpha\|^2+\varepsilon^2\|\phi\|^2+\varepsilon^2\|\psi\|^2.$$

on L^2_f and

$$\begin{aligned}\|(\alpha,\phi,\psi)\|^2_{1,\varepsilon} &= \|\alpha\|^2+\|d_A\alpha\|^2+\|d_A*_s\alpha\|^2+\varepsilon^2\|\nabla_s\alpha\|^2\\ &\quad+\varepsilon^2\|d_A\phi\|^2+\varepsilon^4\|\nabla_s\phi\|^2+\varepsilon^2\|d_A\psi\|^2+\varepsilon^4\|\nabla_s\psi\|^2.\end{aligned}$$

on $W^{1,2}_f$. The norms on the right are L^2-norms on Σ_h. The next lemma is a refinement of the elliptic estimate for the operator $D_\varepsilon(t)$.

Lemma 7.3 *For every $c_0>0$ there exist constants $\varepsilon_0>0$ and $c>0$ such that*

$$\|\xi-\pi_A(\xi)\|_{1,\varepsilon}\le c\varepsilon\left(\|D_\varepsilon(t)\xi\|_{0,\varepsilon}+\|\pi_A(\xi)\|_{L^2}\right)$$

for $\xi=(\alpha,\phi,\psi)\in W^{1,2}_f$ where $\pi_A(\xi):=(\pi_A(\alpha),0,0)$.

Proof: Throughout we shall repeatedly use the formula

$$\nabla_s d_A \eta - d_A \nabla_s \eta = [B \wedge \eta], \qquad B = \partial_s A - d_A \Phi.$$

Denote $\tilde{\xi} = (\tilde{\alpha}, \tilde{\phi}, \tilde{\psi}) = D_\varepsilon(t)\xi$ so that

$$\tilde{\alpha} = *_s \nabla_s \alpha - *_s dX_s(A)\alpha - *_s d_A \phi - d_A \psi.$$

Apply d_A to $\tilde{\alpha}$ to obtain

$$\begin{aligned} d_A *_s d_A \phi &= d_A *_s (\nabla_s \alpha - dX_s(A)\alpha) - d_A \tilde{\alpha} \\ &= \nabla_s d_A *_s \alpha - d_A \tilde{\alpha} - [B \wedge *_s \alpha] - d_A(\dot{*}_s \alpha + *_s dX_s(A)\alpha) \\ &= \varepsilon^2 \nabla_s *_s (\tilde{\psi} - *_s \nabla_s *_s \phi) - d_A \tilde{\alpha} \\ &\quad -[B \wedge *_s \alpha] - d_A(\dot{*}_s \alpha + *_s dX_s(A)\alpha). \end{aligned}$$

Here we have used the identity

$$\tilde{\psi} = \varepsilon^{-2} *_s d_A *_s \alpha + *_s \nabla_s *_s \phi.$$

Now take the exterior product with ϕ and integrate over Σ_h to obtain

$$\begin{aligned} &\|d_A \phi\|^2 + \varepsilon^2 \|\nabla_s \phi\|^2 \\ &\quad = \langle d_A \phi, *_s \tilde{\alpha} \rangle + \langle d_A \phi, *_s \dot{*}_s \alpha - dX_s(A)\alpha \rangle + \langle \phi, *_s [B \wedge *_s \alpha] \rangle \\ &\quad +\varepsilon^2 \langle \nabla_s \phi, \tilde{\psi} \rangle - \varepsilon^2 \langle \nabla_s \phi, *_s \dot{*}_s \phi \rangle. \end{aligned} \tag{23}$$

Similarly, apply $d_A *_s$ to $\tilde{\alpha}$ to obtain

$$\begin{aligned} d_A *_s d_A \psi &= -d_A \nabla_s \alpha + d_A dX_s(A)\alpha - d_A *_s \tilde{\alpha} \\ &= -\nabla_s d_A \alpha - d_A *_s \tilde{\alpha} + [B - X_s(A) \wedge \alpha] \\ &= -\varepsilon^2 \nabla_s *_s (\tilde{\phi} + \nabla_s \psi) - d_A *_s \tilde{\alpha} + [B \wedge \alpha]. \end{aligned}$$

Here we have used (15) and the identity

$$\tilde{\phi} = \varepsilon^{-2} *_s d_A \alpha - \nabla_s \psi.$$

Now take the exterior product with ψ and integrate over Σ_h to obtain

$$\|d_A \psi\|^2 + \varepsilon^2 \|\nabla_s \psi\|^2 = -\langle d_A \psi, \tilde{\alpha} \rangle - \langle \psi, *_s [B - X_s(A) \wedge \alpha] \rangle - \varepsilon^2 \langle \nabla_s \psi, \tilde{\phi} \rangle. \tag{24}$$

Since every flat connection on the bundle P is regular there is an estimate

$$\|\eta\|_{L^2(\Sigma)} \le c_1 \|d_A \eta\|_{L^2(\Sigma)}$$

for $\eta \in \Omega^0(\mathfrak{g}_P)$. Hence it follows from (23) and (24) with a repeated use of the inequality $xy \le \delta x^2/2 + y^2/2\delta$ that

$$\varepsilon^2 \|d_A \phi\|^2 + \varepsilon^4 \|\nabla_s \phi\|^2 + \varepsilon^2 \|d_A \psi\|^2 + \varepsilon^4 \|\nabla_s \psi\|^2 \le c_2 \varepsilon^2 \left(\left\| \tilde{\xi} \right\|_{0,\varepsilon}^2 + \|\alpha\|^2 \right).$$

Here all norms are L^2-norms on Σ_h. By definition of $\tilde{\alpha}$, $\tilde{\phi}$, and $\tilde{\psi}$

$$\|d_A\alpha\|^2 + \|d_A *_s \alpha\|^2 + \varepsilon^2\|\nabla_s\alpha\|^2 \le c_3\varepsilon^2 \left(\left\|\tilde{\xi}\right\|_{0,\varepsilon}^2 + \|\alpha\|^2 \right).$$

This implies the required estimate. □

Lemma 7.4 *For every $c_0 > 0$ there exist constants $\varepsilon_0 > 0$ and $c > 0$ such that the following holds for every $t \in \mathbb{R}$ and every $|\lambda| \le c_0$. If*

$$\|\alpha_0\|_{L^2} \le c_0 \|D_0(t)\alpha_0 - \lambda\alpha_0\|_{L^2}$$

for every $\alpha_0 \in W_f^{1,2}(H_A(t))$ then

$$\|\xi\|_\varepsilon \le c\,\|D_\varepsilon(t)\xi - \lambda\xi\|_\varepsilon$$

for $0 < \varepsilon < \varepsilon_0$ and $\xi = (\alpha, \phi, \psi) \in W_f^{1,2}$.

Proof: Let $c_1, c_2, \ldots$ denote positive constants which depend only on A and Φ but not on ε. Using the Hodge decomposition on the surface Σ write

$$\alpha = \pi_A(\alpha) + d_A\zeta + *_s d_A\eta$$

with $\zeta, \eta \in \Omega^0(\mathfrak{g}_P)$. A simple calculation shows that

$$\pi_A(D_\varepsilon(t)\xi) = D_0(t)\pi_A(\xi) + \pi_A(\theta) \tag{25}$$

where

$$\theta = *_s[B - X_s(A) \wedge \zeta] - [B \wedge \eta] + *_s\dot{*}_s d_A\eta - *_s dX_s(A) *_s d_A\eta$$

and $B = \partial_s A - d_A\Phi$. Since $d_A *_s d_A\zeta = d_A *_s \alpha$ and $d_A *_s d_A\eta = d_A\alpha$ there is an estimate

$$\|\theta\|_{L^2} \le c_1 \left(\|d_A\alpha\|_{L^2} + \|d_A *_s \alpha\|_{L^2}\right) \le c_2 \|\xi - \pi_A(\xi)\|_{0,\varepsilon}.$$

Now it follows from (25) that

$$\begin{aligned}
\|\pi_A(\xi)\|_{L^2} &\le c_0 \|(\lambda \mathbb{1} - D_0(t))\pi_A(\xi)\|_{L^2} \\
&= c_0 \|\pi_A(\lambda\xi - D_\varepsilon(t)\xi) + \pi_A(\theta)\|_{L^2} \\
&\le c_0 \|D_\varepsilon(t)\xi - \lambda\xi\|_{0,\varepsilon} + c_0c_2 \|\xi - \pi_A(\xi)\|_{0,\varepsilon}.
\end{aligned}$$

Hence the statement follows from Lemma 7.3. □

Denote by

$$R_0 = \{(t, \lambda) \in \mathbb{R} \times \mathbb{C} : \lambda \notin \sigma(D_0(t))\}$$

the resolvent set of the operator family $D_0(t)$. Similarly, let R_ε denote the resolvent set of the operator family $D_\varepsilon(t)$. Lemma 7.4 states that for every compact subset $K \subset R_0$ there exists a constant $\varepsilon_0 > 0$ such that $K \subset R_\varepsilon$ for $0 < \varepsilon < \varepsilon_0$. In fact we shall prove that the harmonic part of the operator $(\lambda\mathbb{1} - D_\varepsilon(t))^{-1}$ converges to $(\lambda\mathbb{1} - D_0(t))^{-1}$ uniformly on every compact subset of R_0.

Lemma 7.5 *For every compact subset $K \subset R_0$ there exist constants $\varepsilon_0 > 0$ such that $K \subset R_\varepsilon$ for $0 < \varepsilon < \varepsilon_0$ and*

$$\left\|\pi_A((\lambda 1\!\!1 - D_\varepsilon(t))^{-1}\eta) - (\lambda 1\!\!1 - D_0(t))^{-1}\pi_A(\eta)\right\|_{L^2} \le c\varepsilon \left\|\eta\right\|_{0,\varepsilon}$$

for $(t,\lambda) \in K$ and $\eta \in L^2_f$.

Proof: The statement of the lemma is equivalent to the estimate

$$\|\pi_A(\xi) - \alpha_0\|_{L^2} \le \delta\|\lambda\xi - D_\varepsilon(t)\xi\|_{0,\varepsilon}$$

for all $\xi \in W^{1,2}_f$ where $\alpha_0 = (\lambda 1\!\!1 - D_0(t))^{-1}\pi_A(\lambda\xi - D_\varepsilon(t)\xi) \in W^{1,2}_f(H_A(t))$. By (25)

$$(\lambda 1\!\!1 - D_0(t))(\pi_A(\xi) - \alpha_0) = \pi_A(\theta)$$

where $\theta \in W^{1,2}_f(\mathbb{R} \times T^*\Sigma \otimes \mathfrak{g}_P)$ and

$$\left\|\theta\right\|_{L^2} \le c_1 \left\|\xi - \pi_A(\xi)\right\|_{0,\varepsilon}.$$

Choose a constant $c_0 > 0$ such that the assumptions of Lemma 7.4 are satisfied for every $(t,\lambda) \in K$. Then

$$\begin{aligned}
\left\|\pi_A(\xi) - \alpha_0\right\|_{L^2} &\le c_0 \left\|\pi_A(\theta)\right\|_{L^2} \\
&\le c_0 c_1 \left\|\xi - \pi_A(\xi)\right\|_{0,\varepsilon} \\
&\le c_2\varepsilon \left(\left\|\lambda\xi - D_\varepsilon(t)\xi\right\|_{0,\varepsilon} + \left\|\pi_A(\xi) - \alpha_0\right\|_{L^2} + \left\|\alpha_0\right\|_{L^2}\right) \\
&\le c_2\varepsilon \left\|\pi_A(\xi) - \alpha_0\right\|_{L^2} + (1 + c_0)c_2\varepsilon \left\|\lambda\xi - D_\varepsilon(t)\xi\right\|_{0,\varepsilon}.
\end{aligned}$$

The third inequality follows from Lemma 7.3. This proves the lemma. □

Lemma 7.6 *For every $c_0 > 0$ there exists a constant $\kappa_0 > 0$ such that the following holds for $t \in \mathbb{R}$ and $0 < \varepsilon^2 + \delta^2 < \kappa_0$. If $\lambda_0 \in \sigma(D_0(t))$ is an eigenvalue of multiplicity m_0 and*

$$\alpha_0 \perp \ker(\lambda_0 1\!\!1 - D_0(t)) \qquad \Longrightarrow \qquad \|\alpha_0\|_{L^2} \le c_0\|\lambda_0\alpha_0 - D_0(t)\alpha_0\|_{L^2}$$

for every $\alpha_0 \in W^{1,2}_f(H_A(t))$ then the total multiplicity of all eigenvalues $\lambda \in \sigma(D_\varepsilon(t))$ with $|\lambda - \lambda_0| \le \delta$ does not exceed m_0.

Proof: Suppose by contradiction that the statement were false. Then there would exist, for $\varepsilon > 0$ and $\delta > 0$ arbitrarily small, finitely many (distinct) eigenvalues $\lambda_j \in \sigma(D_\varepsilon(t))$ with eigenvectors $\xi_j \in W^{1,2}_f$ such that

$$\pi_A(\xi) \perp \ker(\lambda_0 - D_0(t)), \qquad \xi = \sum_{j=1}^N \xi_j \ne 0, \qquad |\lambda_j - \lambda_0| \le \delta.$$

Assume without loss of generality that $N \leq m_0 + 1 \leq 6k - 5$. (Each eigenvalue of $D_0(t)$ has multiplicity at most $6k - 6$ where k is the genus of Σ.) Recall from (25) that

$$\lambda_j \pi_A(\xi_j) = \pi_A(D_\varepsilon(t)\xi_j) = D_0(t)\pi_A(\xi_j) + \pi_A(\theta_j)$$

where $\theta_j \in W^{1,2}_f(\mathbb{R} \times T^*\Sigma \otimes \mathfrak{g}_P)$ and

$$\|\theta_j\|_{L^2} \leq c_1 \|\xi_j - \pi_A(\xi_j)\|_{0,\varepsilon}.$$

Hence

$$(\lambda_0 - D_0(t))\, \pi_A(\xi) = \sum_{j=1}^N (\lambda_0 - \lambda_j)\pi_A(\xi_j) + \sum_{j=1}^N \pi_A(\theta_j).$$

By assumption this implies

$$\begin{aligned}
\|\pi_A(\xi)\|_{L^2} &\leq c_0 \sum_{j=1}^N |\lambda_j - \lambda_0| \, \|\xi_j\|_{0,\varepsilon} + c_0 \sum_{j=1}^N \|\theta_j\|_{L^2} \\
&\leq c_0\delta \sum_{j=1}^N \|\xi_j\|_{0,\varepsilon} + c_0 c_1 \sum_{j=1}^N \|\xi_j - \pi_A(\xi_j)\|_{0,\varepsilon}.
\end{aligned}$$

Since $D_\varepsilon(t)\xi_j = \lambda_j \xi_j$ it follows from Lemma 7.3 that

$$\|\xi_j - \pi_A(\xi_j)\|_{0,\varepsilon} \leq c_2 \varepsilon \|\pi_A(\xi_j)\|_{L_2}$$

for every j and hence

$$\|\xi\|^2_{0,\varepsilon} \leq c_3 \left(\delta^2 + \varepsilon^2\right) \sum_{j=1}^N \|\xi_j\|^2_{0,\varepsilon} = c_3 \left(\delta^2 + \varepsilon^2\right) \|\xi\|^2_{0,\varepsilon}.$$

We have used the fact that $D_\varepsilon(t)$ is self-adjoint. With $c_3 \left(\delta^2 + \varepsilon^2\right) < 1$ it follows that $\xi = 0$, a contradiction. This proves the lemma. □

Proof of Theorem 6.1: In a suitable trivialization of the unitary vector bundle H_A over $S^1 \times \mathbb{R}$ the operator $D_0(t)$ takes the form

$$(D_0(t)\zeta)(s) = J_0\dot{\zeta}(s) - S(s,t)\zeta(s).$$

Here J_0 denotes the standard symplectic matrix given by $J_0(x, y) = (-y, x)$ and $S(s,t) = S(s+1,t)$ is a symmetric matrix valued function on $S^1 \times \mathbb{R}$. In particular, $D_0(t)$ is a smooth family of self adjoint operators on $H = L^2(S^1, \mathbb{R}^{2n})$ and satisfies all the requirements of Theorem 7.2 with $W = W^{1,2}(S^1, \mathbb{R}^{2n})$. Choose $\delta > 0$ as in (22), let $t_1, \ldots, t_\ell$ be the finitely many real numbers with $\delta \in \sigma(D_0(t_\nu))$, and let

$$P_\nu : L^2(S^1, \mathbb{R}^{2n}) \to \ker(\delta\mathbb{1} - D_0(t_\nu))$$

denote the corresponding spectral projections. Then the Fredholm index of the operator $\mathcal{D}_0$ is given by the spectral flow of the operator family $D_0(t)$, i.e.

$$\operatorname{index} \mathcal{D}_0 = \sum_\nu m_\nu$$

where

$$m_\nu = \operatorname{sign}(P_\nu \dot{D}_0(t_\nu) P_\nu).$$

The integer m_ν is the local spectral flow of the operator family $D_0(t)$ in the interval $(t_\nu - \tau, t_\nu + \tau)$ across the line $\lambda = \delta$. More precisely, choose $\kappa > 0$ such that $\lambda = \delta$ is the only eigenvalue of $D_0(t_\nu)$ in the interval $[\delta - \kappa, \delta + \kappa]$. Now choose $\tau > 0$ such that $\delta \pm \kappa \notin \sigma(D_0(t))$ for $t_\nu - \tau \leq t \leq t_\nu + \tau$. Then

$$\begin{aligned} m_\nu \quad = \quad & \#\left\{\lambda \in \sigma(D_0(t_\nu - \tau)) \,|\, \delta - \kappa < \lambda < \delta\right\} \\ & -\#\left\{\lambda \in \sigma(D_0(t_\nu + \tau)) \,|\, \delta - \kappa < \lambda < \delta\right\}. \end{aligned}$$

Here the λ_j are the eigenvalue families of Theorem 7.1.

By Lemma 7.4 there exists a constant $\varepsilon_0 > 0$ such that $\delta \notin \sigma(D_\varepsilon(t))$ for $t \notin (t_\nu - \tau, t_\nu + \tau)$ and $\delta \pm \kappa \notin \sigma(D_\varepsilon(t))$ for $t \in (t_\nu - \tau, t_\nu + \tau)$ whenever $0 < \varepsilon < \varepsilon_0$. We prove that

$$\begin{aligned} m_\nu \quad = \quad & \#\left\{\lambda \in \sigma(D_\varepsilon(t_\nu - \tau)) : \delta - \kappa < \lambda < \delta\right\} \\ & -\#\left\{\lambda \in \sigma(D_\varepsilon(t_\nu + \tau)) : \delta - \kappa < \lambda < \delta\right\}. \end{aligned} \tag{26}$$

To see this choose a curve Γ encircling the eigenvalues of $D_0(t_\nu - \tau)$ in the interval $(\delta - \kappa, \delta)$ and note that the corrsponding spectral projection on $L^2_f(H_A(t_\nu - \tau))$ is given by

$$P_0 = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - D_0(t_\nu - \tau))^{-1} dz.$$

Similarly, the operator

$$P_\varepsilon = \frac{1}{2\pi i} \int_\Gamma (z\mathbb{1} - D_\varepsilon(t_\nu - \tau))^{-1} dz.$$

on $L^2_f(\mathbb{R} \times T^*\Sigma \otimes \mathfrak{g}_P \oplus \mathfrak{g}_P \oplus \mathfrak{g}_P)$ is the spectral projection of $D_\varepsilon(t_\nu - \tau)$ corresponding to the eigenvalues in the interval $(\delta - \kappa, \delta)$. By Lemma 7.5

$$P_0 \pi_A = \lim_{\varepsilon \to 0} \pi_A P_\varepsilon$$

where the limit is to be understood in the uniform operator topology. Hence

$$\operatorname{rank} P_\varepsilon \geq \operatorname{rank} P_0$$

for ε sufficiently small. To see this choose an orthonormal basis $\alpha_1, \ldots, \alpha_m$ of $\operatorname{range} P_0$ and let $\xi_j \in L^2_f(\mathbb{R} \times T^*\Sigma \otimes \mathfrak{g}_P \oplus \mathfrak{g}_P \oplus \mathfrak{g}_P)$ such that $\pi_A(\xi_j) = \alpha_j$.

Then $\pi_A(P_\varepsilon \xi_j)$ converges to α_j and hence $\mathrm{rank} P_\varepsilon \geq m$. Now it follows from Lemma 7.6 that

$$\mathrm{rank} P_\varepsilon = \mathrm{rank} P_0$$

for $\varepsilon > 0$ sufficiently small. The same argument applies at $t = t_\nu + \tau$ and this proves (26). Hence

$$\mathrm{index}\, \mathcal{D}_\varepsilon = \sum_\nu m_\nu = \mathrm{index}\, \mathcal{D}_0.$$

This proves Theorem 6.1. □

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