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The Linear–Quadratic Control Problem for Retarded Systems with Delays in Control and Observation

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1. Introduction

THE OBJECT of this paper is to solve the linear–quadratic control problem (LQCP) for retarded functional differential equations (RFDE) with delays in the input and output variables. This will be done within the general semigroup-theoretic framework which has been developed in [28].

For retarded systems with undelayed input and output variables the LQCP has been studied by various authors for about twenty years. We mention the work of Krasovskii [20], Kushner & Barnea [21], Alekal, Brunovskii, Chyung & Lee [1], Delfour & Mitter [14], Curtain [6], Manitius [25], Delfour, McCalla & Mitter [13], Delfour, Lee & Manitius [11], Delfour [8], Banks & Burns [2]. First results on systems with a single-point delay in the state and control variables can be found in Koivo & Lee [19] and Kwong [22]. Ichikawa [16] has developed a comprehensive evolution equation approach for the treatment of the LQCP for RFDEs with input delays. His idea was to include a past segment of the input function in the state of the system. A completely different approach to this problem has been developed by Vinter & Kwong [30] for RFDEs with distributed input delays. Their approach has been generalized to RFDEs with general delay, in the state and control variables by Delfour [9, 10] and to neutral systems by Karrakchu [18] in her recent thesis. The LQCP for neutral systems with output delays has been studied by Datko [7] and Ito & Tarn [17].

For RFDEs with general delays in state, control, and observation the LQCP has been an outstanding problem for many years. The only available papers on this subject seem to be those by Lee [24] and by Fernandez-Berdagler & Lee [15] which deal only with the finite time problem for a rather special class of systems, namely those with a single point delay. Furthermore, in [24] the optimal control is given in open loop form only and the proofs in [15] are rather complicated. In the present paper we fill this gap and present a general and—as we think—elegant solution of the LQCP for RFDEs with delays in state, control, and observation.

In Section 2 we develop a state-space approach for this class of system with a particular emphasis on the duality relationships (Section 2.2). These results are very much analogous to those in Delfour & Manitius [12] on retarded, and in Salamon [29] on neutral, systems. We also extend the concept of structural operators (Bernier & Manitius [3], Manitius [26], Delfour & Manitius [12], Vinter & Kwong [30], Delfour [10]) to RFDEs with delays in both input and output variables (Section 2.3). In Section 3 we then combine the results of Section 2 with those of [28] in order to solve the LQCP on the finite (Section 3.1) and on the infinite (Section 3.3) time interval. In Section 3.2 we collect some known and new results on stabilizability and detectability for retarded systems.

We begin with a brief resumé of the abstract results of [28]. The basic model is

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \tag{1.1}$$

$$y(t) = Cx(t) \tag{1.2}$$

for $t_0 \leqslant t \leqslant t_1$, where $u(\bullet) \in L^2[t_0, t_1; U]$, $y(\bullet) \in L^2[t_0, t_1; Y]$, and U and Y are Hilbert spaces. A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on a Hilbert space H and, in order to allow for unboundedness of the operators B and C, we assume that $B \in \mathfrak{L}(U, V)$ and $\mathfrak{L}(W, Y)$ where W and V are Hilbert spaces such that

$$W \subset H \subset V$$

with continuous dense injections. (1.1) is interpreted in the mild form

$$x(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-\sigma)Bu(\sigma)\,d\sigma \qquad (t_0 \leqslant t \leqslant t_1).$$

In order to make sure that the trajectories are well defined in all three spaces W, H, V we have to assume that $S(t)$ is a strongly continuous semigroup on W and V and the following hypotheses are satisfied.

(H1) There exists some constant $b > 0$ such that

$$\left\| \int_{t_0}^{t} S(t_1-\sigma)Bu(\sigma)\,d\sigma \right\|_W \leqslant b\, \|u(\bullet)\|_{L^2[t_0,t_1;U]} \quad \text{for all } u(\bullet) \in L^2[t_0, t_1; U].$$

(H2) There exists some constant $c > 0$ such that

$$\|CS(\bullet - t_0)x\|_{L^2[t_0,t_1,Y]} \leqslant c\, \|x\|_V \quad \text{for all } x \in W.$$

(H3) $Z = \mathfrak{D}_V(A) \subset W$ with continuous dense embedding where Z is endowed with the graph norm of A regarded as an unbounded closed operator on V.

Associated with the control system (1.1)–(1.2) is the performance index

$$J(u) = \langle x(t_1), Gx(t_1)\rangle_{V,V^*} + \int_{t_0}^{t_1} [\|Cx(t)\|_Y^2 + \langle u(t), Ru(t)\rangle_U]\,dt$$

where $G \in \mathfrak{L}(V, V^*)$ is nonnegative and $R \in \mathfrak{L}(U)$ satisfies $\langle u, Ru\rangle_U \geqslant \varepsilon\, \|u\|_U^2$ for some $\varepsilon > 0$ and every $u \in U$.

In [28] it is shown that the optimal control is given by

$$u(t) = -R^{-1}B^*P(t)x(t)$$

where $P(t) \in \mathfrak{L}(V, V^*)$ and for every $x \in Z$, the function $P(t)x$ is differentiable with values in Z^* and satisfies the differential Riccati equation

$$\frac{d}{dt} P(t)x + A^*P(t)x + P(t)Ax - P(t)BR^{-1}B^*P(t)x + C^*Cx = 0, \qquad P(t_1)x = Gx.$$

In this equation A is regarded as a bounded operator from Z into V.

We also considered the infinite-time problem where

$$J(u) = \int_0^\infty [\|y(t)\|_Y^2 + \langle u(t), Ru(t)\rangle_U] \, dt.$$

Under the additional assumption

(H4) for every $x_0 \in V$, there exists $u_{x_0}(\bullet) \in L^2[0, \infty; U]$ such that

$$J(u_{x_0}) < \infty$$

we showed that the optimal control was

$$u(t) = -R^{-1}B^*Px(t)$$

where $P \in \mathfrak{L}(V, V^*)$ satisfies the algebraic Riccati equation

$$A^*Px + PAx - PBR^{-1}B^*Px + C^*Cx = 0. \tag{1.3}$$

In the above $x \in Z$ and the equation holds in Z^*. With the following assumption:

(H5) if $x_0 \in V$ and $u(\bullet) \in L^2[0, \infty; U]$ are such that $J(u) < \infty$, then

$$x(\bullet) \in L^2[0, \infty; V],$$

the closed-loop semigroup generated by $A - BR^{-1}B^*P$ is exponentially stable where P is the (unique) nonnegative solution of (1.3).

2. State-space theory for retarded systems with delays in input and output

2.1 *Control Systems with Delays*

We consider the linear RFDE

$$\dot{x}(t) = Lx_t + Bu_t, \qquad y(t) = Cx_t, \tag{2.1a, b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and x_t and u_t are defined by $x_t(\tau) = x(t + \tau)$ and $u_t(\tau) = u(t + \tau)$ for $-h \leq \tau \leq 0$, with $0 < h < \infty$.

Correspondingly L, B, C are bounded linear functionals from $C(-h, 0; \mathbb{R}^n)$, $C(-h, 0; \mathbb{R}^m)$, $C(-h, 0; \mathbb{R}^n)$ into $\mathbb{R}^n$, $\mathbb{R}^n$, $\mathbb{R}^p$, respectively. These can be represented by matrix functions $\Lambda(\tau)$, $B(\tau)$, $\Gamma(\tau)$ in the following way

$$L\phi = \int_0^h [d\Lambda(\tau)\, \phi(-\tau)], \quad C\phi = \int_0^h [d\Gamma(\tau)\, \phi(-\tau)] \qquad (\phi \in C(-h, 0; \mathbb{R}^n)),$$

$$B\xi = \int_0^h [dB(\tau)\, \xi(-\tau)] \qquad (\xi \in C(-h, 0; \mathbb{R}^m)).$$

Without loss of generality we assume that the matrix functions A, B, and Γ are normalized, i.e. vanish for $\tau \leq 0$, are constant for $\tau \geq h$ and left continuous for $0<\tau<h$. A solution of (2.1a) is a function $x \in L^2_{loc}(-h, \infty; \mathbb{R}^n)$ which is absolutely continuous with L^2 derivative on every compact interval $[0, T]$ (with $T>0$) and satisfies (2.1a) for almost every $t \geq 0$. It is well known that (2.1a) admits a unique solution $x(t)=x(t; \phi, u)$ $(t \geq -h)$ for every input $u(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$ and every initial condition

$$x(0)=\phi^0, \qquad x(\tau)=\phi^1(\tau) \quad (-h \leq \tau < 0), \tag{2.2a}$$

$$u(\tau)=\phi^2(\tau) \qquad (-h \leq \tau < 0), \tag{2.2b}$$

where $\phi=(\phi^0, \phi^1, \phi^2) \in \mathfrak{X}=\mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n) \times L^2(-h, 0; \mathbb{R}^m)$. Moreover, $x(\bullet; \phi, u)$ depends continuously on ϕ and u on compact intervals, i.e. for any $T>0$ there exists a $K>0$ such that

$$\|x(\bullet; \phi, u)\|_{W^{1,2}(0,T;\mathbb{R}^n)} \leq K(\|\phi\| + \|u\|_{L^2(0,T;\mathbb{R}^m)})$$

where $\|\phi\|=(\|\phi^0\|^2+\|\phi^1\|^2_{L^2}+\|\phi^2\|^2_{L^2})^{\frac{1}{2}}$ for $\phi \in \mathfrak{X}$ (see e.g. Borisovic & Turbabin [4], Delfour & Manitius [12], Salamon [29]). The corresponding output $y(\bullet)=y(\bullet; \phi, u)$ is in $L^2_{loc}(0, \infty; \mathbb{R}^p)$ and depends—in this space—continuously on ϕ and u. The fundamental solution of (2.1a) will be denoted by $X(t)$ $(t \geq -h)$ and is the $n \times n$ matrix-valued solution of (2.1a) which corresponds to $u \equiv 0$ and satisfies $X(0)=I$ and $X(\tau)=0$ for $-h \leq \tau < 0$. Its Laplace transform is given by $\Delta^{-1}(\lambda)$, where

$$\Delta(\lambda)=\lambda I - L(e^{\lambda \cdot})=\lambda I - \int_0^h [dA(\tau)\, e^{-\lambda\tau}] \qquad (\lambda \in \mathbb{C}),$$

is the characteristic matrix of (2.1a). It is well known that the forced motions of (2.1a) can be written in the form

$$x(t; 0, u)=\int_0^t X(t-s)Bu_s\, ds \qquad (t \geq 0).$$

We also consider the transposed RFDE

$$\dot z(t)=L^T z_t + C^T v_t, \qquad w(t)=B^T z_t, \tag{2.3a, b}$$

with initial data

$$z(0)=\psi^0, \qquad z(\tau)=\psi^1(\tau) \quad (-h \leq \tau < 0) \tag{2.4a}$$

$$v(\tau)=\psi^2(\tau) \qquad (-h \leq \tau < 0), \tag{2.4b}$$

where $\psi=(\psi^0, \psi^1, \psi^2) \in \mathfrak{X}^T=\mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n) \times L^2(-h, 0; \mathbb{R}^p)$. The unique solution of (2.3a) and (2.4) will be denoted by $z(t)=z(t; \psi, v)$ $(t \geq -h)$ and the corresponding output by $w(t)=w(t; \psi, v)$ $(t \geq 0)$.

2.2 *State Concepts and Duality*

The 'classical' way of introducing the state of a delay system is to specify an initial function of suitable length which describes the past history of the solution.

This is due to the existence and uniqueness of the solution to the delay equation (in our case (2.1)) and its continuous dependence on the initial function (in our case (2.2)). Correspondingly, we may define the state of system (2.1) at time $t \geq 0$ to be the triple.

$$\hat{x}(t) = (x(t), x_t, u_t) \in \mathfrak{X}$$

and analogously, the state of the transposed system (2.3) at time $t \geq 0$ will be given by

$$\hat{z}(t) = (z(t), z_t, v_t) \in \mathfrak{X}^{\mathrm{T}}.$$

The idea of including the input segment in the state of the system was first suggested by Ichikawa [16].

In order to describe the duality relation between the systems (2.1) and (2.3), we need an alternative state concept. For this we replace the initial functions ϕ^1 and ϕ^2 of the state- and input-variables by additional forcing terms of suitable length on the right-hand side of both equations in (2.1). These terms completely determine the future behaviour of the solution and the output. More precisely, we rewrite system (2.1)–(2.2) as

$$\dot{x}(t) = \int_0^t [\mathrm{d}A(\tau)\, x(t-\tau)] + \int_0^t [\mathrm{d}B(\tau)\, u(t-\tau)] + f^1(t), \qquad x(0) = f^0, \tag{2.5a}$$

$$y(t) = \int_0^t [\mathrm{d}\Gamma(\tau)\, x(t-\tau)] + f^2(t), \tag{2.5b}$$

for $t \geq 0$, where the triple

$$f = (f^0, f^1, f^2) \in \mathfrak{X}^{\mathrm{T}*} = \mathbb{R}^n \times \mathrm{L}^2[0, h; \mathbb{R}^n] \times \mathrm{L}^2[0, h; \mathbb{R}^p]$$

is given by

$$f^0 = \phi^0, \tag{2.6a}$$

$$f^1(t) = \int_t^h [\mathrm{d}A(\tau)\, \phi^1(t-\tau)] + \int_t^h [\mathrm{d}B(\tau)\, \phi^2(t-\tau)] \qquad (0 \leq t \leq h), \tag{2.6b}$$

$$f^2(t) = \int_t^h [\mathrm{d}\Gamma(\tau)\, \phi^1(t-\tau)] \qquad (0 \leq t \leq h). \tag{2.6c}$$

Remarks 2.1

(i) The expressions on the right-hand sides of (2.6b) and (2.6c) are well defined as square-integrable functions on the interval $[0, h]$ (see e.g. Delfour & Manitius [12] or Salamon [29]). Each of them can be interpreted as the convolution of a Borel-measure on the interval $[0, h]$ with an L^2 function on the interval $[-h, 0]$.

(ii) The product space $\mathfrak{X}^{\mathrm{T}*} = \mathbb{R}^n \times \mathrm{L}^2(0, h; \mathbb{R}^n) \times \mathrm{L}^2(0, h; \mathbb{R}^p)$ can be identified with the dual space of $\mathfrak{X}^{\mathrm{T}} = \mathbb{R}^n \times \mathrm{L}^2(-h, 0; \mathbb{R}^n) \times \mathrm{L}^2(-h, 0; \mathbb{R}^p)$ via the duality pairing

$$\langle \psi, f \rangle_{\mathfrak{X}^{\mathrm{T}}, \mathfrak{X}^{\mathrm{T}*}} = \psi^{0\mathrm{T}} f^0 + \int_0^h \psi^{1\mathrm{T}}(-s) f^1(s)\, \mathrm{d}s + \int_0^h \psi^{2\mathrm{T}}(-s) f^2(s)\, \mathrm{d}s$$

for $\psi \in \mathfrak{X}^{\mathrm{T}}$ and $f \in \mathfrak{X}^{\mathrm{T}*}$. In the same manner we can identify the product space $\mathfrak{X}^* = \mathbb{R}^n \times L^2(0, h; \mathbb{R}^n) \times L^2(0, h; \mathbb{R}^m)$ with the dual space of $\mathfrak{X} = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n) \times L^2(-h, 0; \mathbb{R}^m)$.

Now it is easy to see that the solution $x(t)$ and the output $y(t)$ of system (2.5) vanish for $t \geq 0$ if and only if $f = 0$. This fact motivates the definition of the initial state of system (2.5) to be the triple $f \in \mathfrak{X}^{\mathrm{T}*}$. Correspondingly the state of (2.5) at time $t \geq 0$ is the triple

$$\hat{x}(t) = (x(t), x^t, y^t) \in \mathfrak{X}^{\mathrm{T}*}$$

where the function components $x^t \in L^2(0, h; \mathbb{R}^n)$ and $y^t \in L^2(0, h; \mathbb{R}^p)$ are the forcing terms of systems (2.5) after a time shift. These are given by

$$x^t(s) = \int_s^{t+s} [\mathrm{d}A(\tau)\, x(t+s-\tau)] + \int_s^{t+s} [\mathrm{d}B(\tau)\, u(t+s-\tau)] + f^1(t+s), \tag{2.7a}$$

$$y^t(s) = \int_s^{t+s} [\mathrm{d}\Gamma(\tau)\, x(t+s-\tau)] + f^2(t+s), \tag{2.7b}$$

for $0 \leq s \leq h$, where $f^1(t)$ and $f^2(t)$ are defined to be zero if $t \notin [0, h]$.

The idea of defining the state of a delay equation through the forcing term rather than the solution segment was first suggested by Miller [27] for Volterra integrodifferential equations. The corresponding duality relation has been discovered by Burns & Herdman [5]. Further references in this direction can be found in Salamon [29].

The same ideas as above can be applied to the transposed equation (2.3). For this we rewrite system (2.3)–(2.4) in the following way.

$$\dot{z}(t) = \int_0^t [\mathrm{d}A^{\mathrm{T}}(\tau)\, z(t-\tau)] + \int_0^t [\mathrm{d}\Gamma^{\mathrm{T}}(\tau)\, v(t-\tau)] + g^1(t), \qquad z(0) = g^0, \tag{2.8a}$$

$$w(t) = \int_0^t [\mathrm{d}B^{\mathrm{T}}(\tau)\, z(t-\tau)] + g^2(t), \tag{2.8b}$$

for $t \geq 0$, where the triple

$$g = (g^0, g^1, g^2) \in \mathfrak{X}^* = \mathbb{R}^n \times L^2(0, h; \mathbb{R}^n) \times L^2(0, h; \mathbb{R}^m)$$

is given by

$$g^0 = \psi^0, \tag{2.9a}$$

$$g^1(t) = \int_t^h [\mathrm{d}A^{\mathrm{T}}(\tau)\, \psi^1(t-\tau)] + \int_t^h [\mathrm{d}\Gamma^{\mathrm{T}}(\tau)\, \psi^2(t-\tau)], \tag{2.9b}$$

$$g^2(t) = \int_t^h [\mathrm{d}B^{\mathrm{T}}(\tau)\, \psi^1(t-\tau)], \tag{2.9c}$$

for $0 \leq t \leq h$. The initial state of system (2.8) is the triple $g \in \mathfrak{X}^*$ and the state at time $t \geq 0$ is given by

$$\hat{z}(t) = (z(t), z^t, w^t) \in \mathfrak{X}^*$$

where the function components $z^t \in L^2(0, h; \mathbb{R}^n)$ and $w^t \in L^2(0, h; \mathbb{R}^m)$ are of the form

$$z^t(s) = \int_s^{t+s} [d\Lambda^T(\tau)\, z(t+s-\tau)] + \int_s^{t+s} [d\Gamma^T(\tau)\, v(t+s-\tau)] + g^1(t+s),$$

$$w^t(s) = \int_s^{t+s} [dB^T(\tau)\, z(t+s-\tau)] + g^2(t+s),$$

for $0 \leq s \leq h$. These expressions can be obtained from equation (2.8) through a time shift.

Summarizing our situation, we have introduced two different notions of the state both for the original RFDE (2.1) and for the transposed RFDE (2.3). A duality relation between these two equations involves both state concepts. The dual state concept (forcing terms) for the original system (2.1) is dual to the 'classical' state concept (solution segments) for the transposed system (2.3). More, precisely, we have the following result.

THEOREM 2.2 *Let* $u(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$ *and* $v(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^p)$ *be given.*

(i) *Let* $f \in \mathfrak{X}^{T*}$ *and* $\psi \in \mathfrak{X}^T$. *Moreover, suppose that* $\hat{x}(t) = (x(t), x^t, y^t) \in \mathfrak{X}^{T*}$ *is the corresponding state of* (2.5) *with output* $y(t)$ *and that* $\hat{z}(t) = (z(t), z_t, v_t) \in \mathfrak{X}^T$ *is the state of* (2.3)–(2.4) *at time* $t \geq 0$ *with output* $w(t)$. *Then*

$$\langle \psi, \hat{x}(t) \rangle_{\mathfrak{X}^T, \mathfrak{X}^{T*}} - \langle \hat{z}(t), f \rangle_{\mathfrak{X}^T, \mathfrak{X}^{T*}} = \int_0^t w^T(t-s) u(s)\, ds - \int_0^t v^T(t-s) y(s)\, ds$$

for $t \geq 0$.

(ii) *Let* $\phi \in \mathfrak{X}$ *and* $g \in \mathfrak{X}^*$. *Moreover, suppose that* $\hat{x}(t) = (x(t), x_t, u_t)$ *is the corresponding state of* (2.1)–(2.2) *with output* $y(t)$ *and that* $\hat{z}(t) = (z(t), z^t, w^t) \in \mathfrak{X}^*$ *is the state of* (2.8) *at time* $t \geq 0$ *with output* $w(t)$. *Then*

$$\langle g, \hat{x}(t) \rangle_{\mathfrak{X}^*, \mathfrak{X}} - \langle \hat{z}(t), \phi \rangle_{\mathfrak{X}^*, \mathfrak{X}} = \int_0^t w^T(t-s) u(s)\, ds - \int_0^t v^T(t-s) y(s)\, ds$$

for $t \geq 0$.

Proof. We will give a proof of statement (i) only. For this let us assume that $z(t)$ $(t \geq -h)$ is the unique solution of (2.3)–(2.4) with output $w(t)$ $(t \geq 0)$ and that $x(t)$ $(t \geq 0)$ is the unique solution of (2.5) with output $y(t)$ $(t \geq 0)$. Moreover, let $x^t \in L^2[0, h; \mathbb{R}^n]$ and $y^t \in L^2[0, h; \mathbb{R}^p]$ be given by (2.7) and define $x(t) = 0$ and $u(t) = 0$ for $t < 0$. Then it is easy to see that

$$\int_0^t [z^T(t-s) L x_s - (L^T z_{t-s})^T x(s)]\, ds = -\int_{\tau=0}^h \int_{s=0}^\tau \psi^{1T}(-s)[d\Lambda(\tau)\, x(t+s-\tau)]\, ds$$

and analogous equations hold for B and C. Moreover

$$\psi^{0\mathrm{T}}x(t) - z^{\mathrm{T}}(t)f^0 = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}[z^{\mathrm{T}}(t-s)x(s)]\,\mathrm{d}s$$
$$= \int_0^t z^{\mathrm{T}}(t-s)\dot{x}(s)\,\mathrm{d}s - \int_0^t \dot{z}^{\mathrm{T}}(t-s)x(s)\,\mathrm{d}s.$$

This implies

$$\langle \psi, \hat{x}(t)\rangle_{\mathfrak{X}^{\mathrm{T}},\mathfrak{X}^{\mathrm{T}*}} - \langle \hat{z}(t), f\rangle_{\mathfrak{X}^{\mathrm{T}},\mathfrak{X}^{\mathrm{T}*}}$$
$$= \psi^{0\mathrm{T}}x(t) - z^{\mathrm{T}}(t)f^0 + \int_0^h \psi^{1\mathrm{T}}(-s)x^t(s)\,\mathrm{d}s$$
$$+ \int_0^h \psi^{2\mathrm{T}}(-s)y^t(s)\,\mathrm{d}s - \int_0^h z^{\mathrm{T}}(t-s)f^1(s)\,\mathrm{d}s - \int_0^h v^{\mathrm{T}}(t-s)f^2(s)\,\mathrm{d}s$$
$$= \int_0^t z^{\mathrm{T}}(t-s)[Lx_s + Bu_s + f^1(s)]\,\mathrm{d}s - \int_0^t (L^{\mathrm{T}}z_{t-s} + C^{\mathrm{T}}v_{t-s})^{\mathrm{T}}x(s)\,\mathrm{d}s$$
$$+ \int_0^{h-t} \psi^{1\mathrm{T}}(-s)f^1(t+s)\,\mathrm{d}s + \int_{\tau=0}^h \int_{s=0}^{\tau} \psi^{1\mathrm{T}}(-s)[\mathrm{d}A(\tau)\,x(t+s-\tau)]\,\mathrm{d}s$$
$$+ \int_0^h \int_0^{\tau} \psi^{1\mathrm{T}}(-s)[\mathrm{d}B(\tau)\,u(t+s-\tau)]\,\mathrm{d}s$$
$$+ \int_0^h \int_0^{\tau} \psi^{2\mathrm{T}}(-s)[\mathrm{d}\Gamma(\tau)\,x(t+s-\tau)]\,\mathrm{d}s$$
$$+ \int_0^{h-t} \psi^{2\mathrm{T}}(-s)f^2(t+s)\,\mathrm{d}s - \int_0^t z^{\mathrm{T}}(t-s)f^1(s)\,\mathrm{d}s - \int_t^h \psi^{1\mathrm{T}}(t-s)f^1(s)\,\mathrm{d}s$$
$$- \int_0^t v^{\mathrm{T}}(t-s)f^2(s)\,\mathrm{d}s - \int_0^h \psi^{2\mathrm{T}}(t-s)f^2(s)\,\mathrm{d}s$$
$$= \int_0^t (B^{\mathrm{T}}z_{t-s})^{\mathrm{T}}u(s)\,\mathrm{d}s - \int_0^t v^{\mathrm{T}}(t-s)[Cx_s + f^2(s)]\,\mathrm{d}s$$
$$= \int_0^t w^{\mathrm{T}}(t-s)u(s)\,\mathrm{d}s - \int_0^t v^{\mathrm{T}}(t-s)y(s)\,\mathrm{d}s.$$

2.3 *Semigroups and Structural Operators*

Throughout this section we restrict our discussion to the homogeneous systems (2.1) and (2.5) (respectively (2.3) and (2.8)) which means that $u(t) = 0$ (respectively $v(t) = 0$) for $t \geq 0$.

The evolution of the systems (2.1) and (2.3) in terms of the 'classical' state concept (solution segments) can be described by strongly continuous semigroups:

$$\mathcal{S}(t) : \mathfrak{X} \to \mathfrak{X}, \qquad \mathcal{S}^{\mathrm{T}}(t) : \mathfrak{X}^{\mathrm{T}} \to \mathfrak{X}^{\mathrm{T}}.$$

The semigroup $\mathcal{S}(t)$ on $\mathfrak{X}$ has first been introduced by Ichikawa [22]. It associated with every $\phi \in \mathfrak{X}$ the state

$$\mathcal{S}(t)\phi = \hat{x}(t) = (x(t), x_t, u_t) \in \mathfrak{X}$$

of (2.1)–(2.2) at time $t \geq 0$ which corresponds to the input $u(s) = 0$ $(s \geq 0)$. Its infinitesimal generator is given by

$$\mathfrak{D}(\mathcal{A}) = \{\phi \in \mathfrak{X} : \phi^1 \in W^{1,2}(-h, 0; \mathbb{R}^n), \quad \phi^2 \in W^{1,2}(-h, 0; \mathbb{R}^m),\ \phi^0 = \phi^1(0),\ \phi^2(0) = 0\},$$

$$\mathcal{A}\phi = (L\phi^1 + B\phi^2, \dot{\phi}^1, \dot{\phi}^2)$$

(Salamon [40, Theorem 1.2.6]). The semigroup $\mathcal{S}^{\mathrm{T}}(t)$ is defined analogously and generated by the operator

$$\mathfrak{D}(\mathcal{A}^{\mathrm{T}}) = \{\psi \in \mathfrak{X}^{\mathrm{T}} : \psi^1 \in W^{1,2}(-h, 0; \mathbb{R}^n), \quad \psi^2 \in W^{1,2}(-h, 0; \mathbb{R}^p);\ \psi^0 = \psi^1(0),\ \psi^2(0) = 0\},$$

$$\mathcal{A}^{\mathrm{T}}\psi = (L^{\mathrm{T}}\psi^1 + C^{\mathrm{T}}\psi^2, \dot{\psi}^1, \dot{\psi}^2).$$

An interpretation of the adjoint semigroups $\mathcal{S}^{\mathrm{T}*}(t) : \mathfrak{X}^{\mathrm{T}*} \to \mathfrak{X}^{\mathrm{T}*}$ and $\mathcal{S}^*(t) : \mathfrak{X}^* \to \mathfrak{X}^*$ can be given through the dual-state concept (forcing terms) for the systems (2.1) and (2.3). More precisely, we have the following result which is a direct consequence of Theorem 2.2.

COROLLARY 2.3

(i) *Let $f \in \mathfrak{X}^{\mathrm{T}*}$ be given and let $\hat{x}(t) = (x(t), x^t, y^t) \in \mathfrak{X}^{\mathrm{T}*}$ be the state of system (2.5) at time $t \geq 0$ corresponding to the input $u(\bullet) \equiv 0$. Then $\hat{x}(t) = \mathcal{S}^{\mathrm{T}*}(t)f$.*

(ii) *Let $g \in \mathfrak{X}^*$ be given and let $z(t) = (z(t), z^t, w^t) \in \mathfrak{X}^*$ be the state of system (2.8) at time $t \geq 0$ corresponding to the input $v(\bullet) \equiv 0$. Then $\hat{z}(t) = \mathcal{S}^*(t)g$.*

Our next result is an explicit characterization of the infinitesimal generators $\mathcal{A}^{\mathrm{T}*}$ and $\mathcal{A}^*$ of the semigroups $\mathcal{S}^{\mathrm{T}*}(t)$ and $\mathcal{S}^*(t)$.

PROPOSITION 2.4

(i) *Let $f, d \in \mathfrak{X}^{\mathrm{T}*}$ be given. Then $f \in \mathfrak{D}(\mathcal{A}^{\mathrm{T}*})$ and $\mathcal{A}^{\mathrm{T}*}f = d$ if and only if the following equations hold*

$$\Lambda(h)f^0 = d^0 + \int_0^h d^1(s)\,\mathrm{d}s, \tag{2.10a}$$

$$f^1(t) + [\Lambda(t) - \Lambda(h)]f^0 = -\int_t^h d^1(s)\,\mathrm{d}s \qquad (0 \leq t \leq h), \tag{2.10b}$$

$$f^2(t) + [\Gamma(t) - \Gamma(h)]f^0 = -\int_t^h d^2(s)\,\mathrm{d}s \qquad (0 \leq t \leq h). \tag{2.10c}$$

(ii) *Let $g, k \in \mathfrak{X}^*$ be given. Then $g \in \mathfrak{D}(\mathcal{A}^*)$ and $\mathcal{A}^*g = k$ if and only if*

$$\Lambda^{\mathrm{T}}(h)g^0 = k^0 + \int_0^h k^1(s)\,\mathrm{d}s \tag{2.11a}$$

$$g^1(t) + [\Lambda^{\mathrm{T}}(t) - \Lambda^{\mathrm{T}}(h)]g^0 = -\int_t^h k^1(s)\,\mathrm{d}s \qquad (0 \leq t \leq h), \tag{2.11b}$$

$$g^2(t) + [B^{\mathrm{T}}(t) - B^{\mathrm{T}}(h)]g^0 = -\int_t^h k^2(s)\,\mathrm{d}s \qquad (0 \leq t \leq h). \tag{2.11c}$$

Proof. Obviously it is enough to prove statement (i). First note that $f \in \mathfrak{D}(\mathscr{A}^{T*})$ and $\mathscr{A}^{T*}f = d$ if and only if $\langle \psi, d \rangle = \langle \mathscr{A}^T\psi, f \rangle$ for every $\psi \in \mathfrak{D}(\mathscr{A}^T)$. Hence statement (i) is a consequence of

$$\begin{aligned}\langle \psi, d \rangle &= \psi^{0T}d^0 + \int_0^h \psi^{1T}(-s)d^1(s)\,ds + \int_0^h \psi^{2T}(-s)d^2(s)\,ds \\ &= \psi^{1T}(0)\Big(d^0 + \int_0^h d^1(s)\,ds\Big) - \int_0^h \dot{\psi}^{1T}(-t)\int_t^h d^1(s)\,ds\,dt \\ &\quad - \int_0^h \dot{\psi}^{2T}(-t)\int_t^h d^2(s)\,ds\,dt\end{aligned}$$

and

$$\begin{aligned}\langle \mathscr{A}^T\psi, f \rangle &= \int_0^h \psi^{1T}(-\tau)\,dA(\tau)f^0 + \int_0^h \psi^{2T}(-\tau)\,d\Gamma(\tau)f^0 \\ &\quad + \int_0^h \dot{\psi}^{1T}(-s)f^1(s)\,ds + \int_0^h \dot{\psi}^{2T}(-s)f^2(s)\,ds \\ &= \psi^{1T}(-h)A(h)f^0 + \psi^{2T}(-h)\Gamma(h)f^0 + \int_0^h \dot{\psi}^{1T}(-s)A(s)f^0\,ds \\ &\quad + \int_0^h \dot{\psi}^{2T}(-s)\Gamma(s)f^0\,ds + \int_0^h \dot{\psi}^{1T}(-s)f^1(s)\,ds + \int_0^h \dot{\psi}^{2T}(-s)f^2(s)\,ds \\ &= \psi^{1T}(0)A(h)f^0 + \int_0^h \dot{\psi}^{1T}(-s)[f^1(s) + A(s)f^0 - A(h)f^0]\,ds \\ &\quad + \int_0^h \dot{\psi}^{2T}(-s)[f^2(s) + \Gamma(s)f^0 - \Gamma(h)f^0]\,ds. \quad \square\end{aligned}$$

The duality relation between the systems (2.1) and (2.3) can now be described through the following four semigroups:

$$\begin{aligned}&\mathscr{S}(t) : \mathfrak{X} \to \mathfrak{X}, \qquad &&\mathscr{S}^T(t) : \mathfrak{X}^T \to \mathfrak{X}^T, \\ &\mathscr{S}^{T*}(t) : \mathfrak{X}^{T*} \to \mathfrak{X}^{T*}, \qquad &&\mathscr{S}^*(t) : \mathfrak{X}^* \to \mathfrak{X}^*.\end{aligned}$$

The semigroups on the left-hand side correspond to the RFDE (2.1) and those on the right-hand side to the transposed RFDE (2.3). On each side the upper semigroup describes the respective equation within the 'classical' state concept (solution segments) and the semigroup below within the dual state concept (forcing terms). A diagonal relation is actually given by functional-analytic duality theory.

The relation between the two state concepts can be described by a so-called structural operator

$$\mathscr{F}\phi : \mathfrak{X} \to \mathfrak{X}^{T*}$$

which associates with every $\phi \in \mathfrak{X}$ the corresponding triple

$$\mathscr{F}\phi = f \in \mathfrak{X}^{T*} \quad (f \text{ given by (2.6)}).$$

It is easy to see that this operator maps every state $\hat{x}(t) \in \mathfrak{X}$ of system (2.1) into the corresponding state $\mathring{x}(t) \in \mathfrak{X}^{T*}$ of system (2.5) which is given by (2.7) and (2.6). This fact together with Corollary 2.3 shows that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathcal{S}(t)} & \mathfrak{X} \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathfrak{X}^{T*} & \xrightarrow{\mathcal{S}^{T*}(t)} & \mathfrak{X}^{T*}. \end{array}$$

Another important fact is that the adjoint operator $\mathcal{F}^* : \mathfrak{X}^T \rightarrow \mathfrak{X}^*$ plays the same role for the transposed RFDE (2.3) as the structural operator $\mathcal{F} : \mathfrak{X} \rightarrow \mathfrak{X}^{T*}$ does for the original RFDE (2.1). These properties are summarized in the theorem below.

THEOREM 2.5
(i) $\mathcal{F}\mathcal{S}(t) = \mathcal{S}^{T*}(t)\mathcal{F}$, *then* $\mathcal{F}^*\mathcal{S}^T(t) = \mathcal{S}^*(t)\mathcal{F}^*$.
(ii) *If* $\phi \in \mathfrak{D}(\mathcal{A})$, *then* $\mathcal{F}\phi \in \mathfrak{D}(\mathcal{A}^{T*})$ *and* $\mathcal{A}^{T*}\mathcal{F}\phi = \mathcal{F}\mathcal{A}\phi$.
(iii) *If* $\psi \in \mathfrak{D}(\mathcal{A}^T)$, *then* $\mathcal{F}^*\psi \in \mathfrak{D}(\mathcal{A}^*)$ *and* $\mathcal{A}^*\mathcal{F}^*\psi = \mathcal{F}^*\mathcal{A}^T\psi$.
(iv) *The adjoint operator* $\mathcal{F}^* : \mathfrak{X}^T \rightarrow \mathfrak{X}^*$ maps every $\psi \in \mathfrak{X}^T$ *into the triple* $\mathcal{F}^*\psi = g \in \mathfrak{X}^*$ *which is given by* (2.9).

Proof. Statement (i) follows from the above considerations, the statements (ii) and (iii) are immediate consequences of (i) and statement (iv) can be proved straightforwardly. □

A structural operator of the above type has first been introduced in Bernier & Manitius [3], Delfour & Manitus [12] for retarded systems with state delays only and later on by Vinter & Kwong [30], and Delfour [10] for RFDEs with delays in the state and control variables. An extension to neutral systems can be found in Salamon [29].

2.4 *Abstract Cauchy Problems*

In order to describe the action of the output operators for the RFDEs (2.1) and (2.3)—each within the two state concepts of Section 2.2—we introduce the following four subspaces

$$\mathfrak{W} = \{\phi \in \mathfrak{X} : \phi^1 \in W^{1,2}(-h, 0; \mathbb{R}^n),\ \phi^0 = \phi^1(0)\},$$
$$\mathfrak{W}^T = \{\psi \in \mathfrak{X}^T : \psi^1 \in W^{1,2}(-h, 0; \mathbb{R}^n),\ \psi^0 = \psi^1(0)\},$$
$$\mathfrak{V}^{T*} = \{f \in \mathfrak{X}^{T*} : \exists d^2 \in L^2(0, h; \mathbb{R}^p) \text{ s.t. (2.10c) holds}\},$$
$$\mathfrak{V}^* = \{g \in \mathfrak{X}^* : \exists k^2 \in L^2(0, h; \mathbb{R}^m) \text{ s.t. (2.11c) holds}\}.$$

These have the following properties.

Remarks 2.6. (i) The subspaces $\mathfrak{W}$, $\mathfrak{W}^T$, $\mathfrak{V}^{T*}$, and $\mathfrak{V}^*$ are dense in $\mathfrak{X}$, $\mathfrak{X}^T$, $\mathfrak{X}^{T*}$, and $\mathfrak{X}^*$, respectively. Moreover, $\mathfrak{W}$ and $\mathfrak{V}^{T*}$ become Hilbert spaces if they are endowed with the norms

$$\|\phi\|^2_{\mathfrak{W}} = \|\phi^0\|^2_{\mathbb{R}^n} + \int_{-h}^0 \|\dot\phi^1(\tau)\|^2_{\mathbb{R}^n}\,d\tau + \int_{-h}^0 \|\phi^2(\tau)\|^2_{\mathbb{R}^m}\,d\tau \qquad (\phi \in \mathfrak{W}),$$

$$\|f\|^2_{\mathfrak{V}^{T*}} = \|f^0\|^2_{\mathbb{R}^n} + \int_0^h \|f^1(s)\|^2_{\mathbb{R}^n}\,ds + \int_0^h \left\|\frac{d}{ds}[f^2(s) + \Gamma(s)f^0]\right\|^2_{\mathbb{R}^p} ds \qquad (f \in \mathfrak{V}^{T*})$$

Topologies on $\mathfrak{W}^T$ and $\mathfrak{V}^*$ can be defined analogously.

(ii) The dual spaces $\mathfrak{V}$, $\mathfrak{V}^T$, $\mathfrak{W}^{T*}$, and $\mathfrak{W}^*$ are extensions of $\mathfrak{X}$, $\mathfrak{X}^T$, $\mathfrak{X}^{T*}$, and $\mathfrak{X}^*$, respectively. Thus we obtain the inclusions

$$\mathfrak{W} \subset \mathfrak{X} \subset \mathfrak{V}, \qquad \mathfrak{W}^T \subset \mathfrak{X}^T \subset \mathfrak{V}^T, \qquad \mathfrak{V}^{T*} \subset \mathfrak{X}^{T*} \subset \mathfrak{W}^{T*}, \qquad \mathfrak{V}^* \subset \mathfrak{X}^* \subset \mathfrak{W}^*,$$

with continuous, dense embeddings.

(iii) It is easy to see that $\mathcal{S}(h) \in \mathfrak{L}(\mathfrak{X}, \mathfrak{W})$, $\mathcal{S}^T(h) \in \mathfrak{L}(\mathfrak{X}^T, \mathfrak{W}^T)$, $\mathcal{S}^{T*}(h) \in \mathfrak{L}(\mathfrak{X}^{T*}, \mathfrak{V}^{T*})$, and $\mathcal{S}^*(h) \in \mathfrak{L}(\mathfrak{X}^*, \mathfrak{V}^*)$. By duality, we obtain $\mathcal{S}(h) \in \mathfrak{L}(\mathfrak{V}, \mathfrak{X})$, $\mathcal{S}^T(h) \in \mathfrak{L}(\mathfrak{V}^T, \mathfrak{X}^T)$, $\mathcal{S}^{T*}(h) \in \mathfrak{L}(\mathfrak{W}^{T*}, \mathfrak{X}^{T*})$, and $\mathcal{S}^*(h) \in \mathfrak{L}(\mathfrak{W}^*, \mathfrak{X}^*)$.

Before introducing the input—and output—operators, we prove that the spaces $\mathfrak{W}$, $\mathfrak{W}^T$, $\mathfrak{V}^{T*}$, and $\mathfrak{V}^*$ are invariant under the semigroups $\mathcal{S}(t)$, $\mathcal{S}^T(t)$, $\mathcal{S}^{T*}(t)$, and $\mathcal{S}^*(t)$, respectively. For this we need the following preliminary result.

LEMMA 2.7 *Let $f \in \mathfrak{V}^{T*}$ be given and let $d \in L^2(0, h; \mathbb{R}^p)$ satisfy*

$$f^2(s) + [\Gamma(s) - \Gamma(h)]f^0 = -\int_s^h d(\sigma)\,d\sigma \qquad (0 \leqslant s \leqslant h).$$

Moreover, let $x(\bullet) \in W^{1,2}(0, t; \mathbb{R}^n)$ be chosen such that $x(0) = f^0$ and let $y^t \in L^2(0, h; \mathbb{R}^p)$ be defined by (2.7b). *Then*

$$y^t(s) + [\Gamma(s) - \Gamma(h)]x(t) = -\int_s^h \left(\int_\sigma^{t+\sigma} [d\Gamma(\tau)\,\dot x(t+\sigma-\tau)] + d(t+\sigma)\right) d\sigma \qquad (0 \leqslant s \leqslant h).$$

Proof. Let us define $x(s) = f^0$ for $s \leqslant 0$ and $d(\sigma) = 0$ for $\sigma \notin [0, h]$. Then the equation

$$\int_{t+s}^h [d\Gamma(\tau)\,x(t+s-\tau)] = [\Gamma(h) - \Gamma(t+s)]f^0$$

holds for all $t, s \geqslant 0$. This implies

$$\begin{aligned}
&\int_s^h \left(\int_\sigma^{t+\sigma} [d\Gamma(\tau)\,\dot x(t+\sigma-\tau)] + d(t+\sigma)\right) d\sigma \\
&= \int_s^h \int_\sigma^h [d\Gamma(\tau)\,\dot x(t+\sigma-\tau)]\,d\sigma + \int_s^{h-t} d(t+\sigma)\,d\sigma \\
&= \int_s^h \left(d\Gamma(\tau)\int_s^\tau \dot x(t+\sigma-\tau)\,d\sigma\right) + \int_{t+s}^h d(\sigma)\,d\sigma \\
&= \int_s^h \{d\Gamma(\tau)\,[x(t) - x(t+s-\tau)]\} - f^2(t+s) - [\Gamma(t+s) - \Gamma(h)]f^0
\end{aligned}$$

$$= [\Gamma(h) - \Gamma(s)]x(t) - \int_s^{t+s} [\mathrm{d}\Gamma(\tau)\, x(t+s-\tau)] - f^2(t+s)$$

$$= -y^t(s) - [\Gamma(s) - \Gamma(h)]x(t). \quad \square$$

Now we are in the position to prove the desired invariance properties of the subspaces $\mathfrak{W}$, $\mathfrak{W}^{\mathrm{T}}$, $\mathfrak{V}^{\mathrm{T}*}$, and $\mathfrak{V}^*$.

PROPOSITION 2.8

(i) $\mathcal{S}(t)$ is a strongly continuous semigroup on $\mathfrak{W}$ and $\mathfrak{V}$.
(ii) $\mathcal{S}^{\mathrm{T}}(t)$ is a strongly continuous semigroup on $\mathfrak{W}^{\mathrm{T}}$ and $\mathfrak{V}^{\mathrm{T}}$.
(iii) $\mathcal{S}^{\mathrm{T}*}(t)$ is a strongly continuous semigroup on $\mathfrak{V}^{\mathrm{T}*}$ and $\mathfrak{W}^{\mathrm{T}*}$.
(iv) $\mathcal{S}^*(t)$ is a strongly continuous semigroup on $\mathfrak{V}^*$ and $\mathfrak{W}^*$.
(v) $\mathcal{F} \in \mathfrak{L}(\mathfrak{W}, \mathfrak{V}^{\mathrm{T}*})$ and $\mathcal{F} \in \mathfrak{L}(\mathfrak{V}, \mathfrak{W}^{\mathrm{T}*})$.
(vi) $\mathcal{F}^* \in \mathfrak{L}(\mathfrak{W}^{\mathrm{T}}, \mathfrak{V}^*)$ and $\mathcal{F}^* \in \mathfrak{L}(\mathfrak{V}^{\mathrm{T}}, \mathfrak{W}^*)$.

Proof. First note that every solution $x(t)$ of (2.5) is absolutely continuous for $t \geq 0$ and that its L^2 derivative depends continuously on $f \in \mathfrak{X}^{\mathrm{T}*}$ [29, Theorem 1.2.3 (i)]. This shows that $\mathcal{S}(t)$ is a strongly continuous semigroup on $\mathfrak{W}$.

Now let $f \in \mathfrak{V}^{\mathrm{T}*}$ be given and let $x(\bullet) \in \mathrm{W}^{1,2}_{\mathrm{loc}}(0, \infty; \mathbb{R}^n)$ be the corresponding solution of (2.5) with $u(t) \equiv 0$. Moreover let $y(t)$ $(t \geq 0)$ be the output of (2.5) and let x^t and y^t be given by (2.7). Then $\mathcal{S}^{\mathrm{T}*}(t)f = (x(t), x^t, y^t)$ (Corollary 2.3) and hence it follows from Lemma 2.7 that the function $t \mapsto \mathcal{S}^{\mathrm{T}*}(t)f$ is continuous with values in $\mathfrak{V}^{\mathrm{T}*}$ and depends in this space continuously on $f \in \mathfrak{V}^{\mathrm{T}*}$.

The same considerations—applied to the transposed system (2.3)—show that $\mathcal{S}^{\mathrm{T}}(t)$ is a semigroup on $\mathfrak{W}^{\mathrm{T}}$ and that $\mathcal{S}^*(t)$ is a semigroup on $\mathfrak{V}^*$. The remaining assertations in (i), (ii), (iii), and (iv) follow by duality.

In order to prove (v) and (vi), let $\phi \in \mathfrak{W}$ be given. Then Lemma 2.7 —applied to $f = 0$, $t = h$, and $x(s) = \phi^1(s-h)$ for $0 \leq s \leq h$ —shows that $\mathcal{F}\phi \in \mathfrak{X}^{\mathrm{T}*}$ satisfies the equation

$$(\mathcal{F}\phi)^2(s) + [\Gamma(s) - \Gamma(h)](\mathcal{F}\phi)^0 = -\int_s^h \int_\sigma^h [\mathrm{d}\Gamma(\tau)\, \dot{\phi}^1(\sigma - \tau)]\, \mathrm{d}\sigma \qquad (0 \leq s \leq h).$$

Hence $\mathcal{F}\phi$ is in $\mathfrak{V}^{\mathrm{T}*}$ and depends in this space continuously on $\phi \in \mathfrak{W}$. We conclude that $\mathcal{F} \in \mathfrak{L}(\mathfrak{W}, \mathfrak{V}^{\mathrm{T}*})$. The remaining assertions of (v) and (vi) follow from this fact by analogy and duality. $\square$

Now let us introduce the output operators

$$\mathscr{C} : \mathfrak{W} \to \mathbb{R}^p, \qquad \mathscr{B}^{\mathrm{T}} : \mathfrak{W}^{\mathrm{T}} \to \mathbb{R}^m, \qquad \mathscr{C}^{\mathrm{T}*} : \mathfrak{V}^{\mathrm{T}*} \to \mathbb{R}^p, \qquad \mathscr{B}^* : \mathfrak{V}^* \to \mathbb{R}^m,$$

by defining

$$\mathscr{C}\phi = \int_0^h [\mathrm{d}\Gamma(\tau)\, \phi^1(-\tau)] \quad (\phi \in \mathfrak{W}), \qquad \mathscr{B}^{\mathrm{T}}\psi = \int_0^h [\mathrm{d}B^{\mathrm{T}}(\tau)\, \psi^1(-\tau)] \quad (\psi \in \mathfrak{W}^{\mathrm{T}}),$$

$$\mathscr{C}^{\mathrm{T}*}f = f^2(0) \quad (f \in \mathfrak{V}^{\mathrm{T}*}), \qquad \mathscr{B}^*g = g^2(0) \quad (g \in \mathfrak{V}^*).$$

Then the adjoint operators

$$\mathscr{B} : \mathbb{R}^m \to \mathfrak{V}, \quad \mathscr{C}^{\mathrm{T}} : \mathbb{R}^p \to \mathfrak{V}^{\mathrm{T}}, \qquad \mathscr{B}^{\mathrm{T}*} : \mathbb{R}^m \to \mathfrak{W}^{\mathrm{T}*}, \quad \mathscr{C}^* : \mathbb{R}^p \to \mathfrak{W}^*$$

describe the input action for the systems (2.1) and (2.3). More precisely, we have the following result for the RFDE (2.1). The corresponding statements for the transposed RFDE (2.3) can be formulated analogously.

THEOREM 2.9 *Let $u(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$ be given.*

(i) *Let $\phi \in \mathfrak{W}$ and let $\hat{x}(t) \in \mathfrak{X}$ be the corresponding state of* (2.1)–(2.2) *at time $t \geq 0$. Then $\hat{x}(t)$ $(t \geq 0)$ is a continuous function with values in $\mathfrak{W}$ and depends in this space continuously on $\phi \in \mathfrak{W}$ and $u(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$. Moreover*

$$\hat{x}(t) = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s)\mathcal{B}u(s)\,ds \qquad (t \geq 0) \tag{2.12a}$$

where the integral is to be understood in the Hilbert space $\mathfrak{V}$. The output $y(t)$ of (2.1) *is given by*

$$y(t) = \mathcal{C}\hat{x}(t) \qquad (t \geq 0). \tag{2.12b}$$

(ii) *Let $f \in \mathfrak{V}^{T*}$ and let $\hat{x}(t) \in \mathfrak{X}^{T*}$ be the corresponding state of* (2.5) *at time $t \geq 0$. Then $t \mapsto \hat{x}(t)$ $(t \geq 0)$ is a continuous function with values in $\mathfrak{V}^{T*}$ and depends in this space continuously on $f \in \mathfrak{V}^{T*}$ and $u(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$. Moreover*

$$\hat{x}(t) = \mathcal{S}^{T*}(t)f + \int_0^t \mathcal{S}^{T*}(t-s)\mathcal{B}^{T*}u(s)\,ds \qquad (t \geq 0) \tag{2.13a}$$

where the integral is to be understood in the Hilbert space $\mathfrak{W}^{T*}$. The output $y(t)$ of (2.5) *is given by*

$$y(t) = \mathcal{C}^{T*}\hat{x}(t) \qquad (t \geq 0). \tag{2.13b}$$

Proof. If $u(\bullet) \equiv 0$, then the statements of the theorem follow immediately from Proposition 2.8(i), (iii) together with the definition of the operators $\mathcal{C}$ and $\mathcal{C}^{T*}$. So we can restrict ourselves to the case $\phi = 0$ and $f = 0$.

First of all, the same arguments as in the beginning of the proof of Proposition 2.8 show that $\hat{x}(t)$ is continuous with values in $\mathfrak{W}$ and depends in this space continuously on $u(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$. Secondly, we establish equation (2.12a). For this let $g \in \mathfrak{V}^*$ be given, let $\hat{z}(t) \in \mathfrak{X}^*$ be the corresponding state of (2.8) with $v(\bullet) \equiv 0$ and let $w(t)$ $(t \geq 0)$ be the output of (2.8). Then $\hat{z}(t) = \mathcal{S}^*(t)g \in \mathfrak{V}^*$ (Corollary 2.3 and Proposition 2.8(iv)) and $w(t) = \mathcal{B}^*\hat{z}(t)$, by definition of the operator $\mathcal{B}^*$. Hence it follows from Theorem 2.2(ii) that the following equation holds for every $t \geq 0$:

$$\begin{aligned}
\langle g, \hat{x}(t)\rangle_{\mathfrak{V}^*, \mathfrak{V}} &= \langle g, \hat{x}(t)\rangle_{\mathfrak{X}^*, \mathfrak{X}} \\
&= \int_0^t w^T(t-s)u(s)\,ds \\
&= \int_0^t \langle \mathcal{B}^*\mathcal{S}^*(t-s)g, u(s)\rangle_{\mathbb{R}^m}\,ds \\
&= \left\langle g, \int_0^t \mathcal{S}(t-s)\mathcal{B}u(s)\,ds \right\rangle_{\mathfrak{V}^*, \mathfrak{V}}.
\end{aligned}$$

This proves statement (i).

Now recall that $\hat{x}(t) = \mathcal{F}\hat{x}(t)$ as long as $f=0$ and $\phi = 0$. Hence it follows from (i) and Proposition 2.8(v) that $\hat{x}(t)$ $(t \geq 0)$ is continuous with values in $\mathfrak{W}^{T*}$ and depends in this space continuously on $u(\bullet) \in L^2_{loc}(0,\infty;\mathbb{R}^m)$. Finally, equation (2.13a) can be established in an analogous manner as (2.21a). □

The previous theorem shows that the evolution of the state $\hat{x}(t)$ of the RFDE (2.1) in terms of the 'classical' state concept can be formally described through the abstract Cauchy problem

$$\Sigma: \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = \mathcal{A}\hat{x}(t) + \mathcal{B}u(t), \quad \hat{x}(0) = \phi, \qquad y(t) = \mathcal{C}\hat{x}(t),$$

in the Hilbert space $\mathfrak{W}$ respectively $\mathfrak{V}$.

Analogously, the state $\hat{x}(t) \in \mathfrak{X}^{T*}$ of equation (2.5) in terms of the dual state concept defines a mild solution of the abstract Cauchy problem.

$$\Sigma^{T*}: \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = \mathcal{A}^{T*}\hat{x}(t) + \mathcal{B}^{T*}u(t), \quad \hat{x}(0) = f, \qquad y(t) = \mathcal{C}^{T*}\hat{x}(t),$$

in the Hilbert space $\mathfrak{V}^{T*}$ respectively $\mathfrak{W}^{T*}$.

If we consider the Cauchy problem Σ (respectively Σ^{T*}) in the smaller state space $\mathfrak{W}$ (respectively $\mathfrak{V}^{T*}$), then the output operator $\mathcal{C}$ (respectively $\mathcal{C}^{T*}$) will be bounded and the input operator $\mathcal{B}$ (respectively $\mathcal{B}^{T*}$) unbounded. Nevertheless, the solution of Σ (respectively Σ^{T*}) in the state space $\mathfrak{W}$ (respectively $\mathfrak{V}^{T*}$) is well defined, since the input operator satisfies the hypothesis (H1) of [28]. More precisely, the operator $\mathcal{B}$ (respectively $\mathcal{B}^{T*}$) has the following property which follows directly from Theorem 2.9.

Remark 2.10. For every $T>0$ there exists some constant $b_T > 0$ such that the inequalities

$$\left\| \int_0^T \mathcal{S}(T-s)\mathcal{B}u(s)\,\mathrm{d}s \right\|_{\mathfrak{W}} \leq b_T \, \|u(\bullet)\|_{L^2(0,T;\mathbb{R}^m)},$$

$$\left\| \int_0^T \mathcal{S}^{T*}(T-s)\mathcal{B}^{T*}u(s)\,\mathrm{d}s \right\|_{\mathfrak{V}^{T*}} \leq b_T \, \|u(\bullet)\|_{L^2(0,T;\mathbb{R}^m)}$$

hold for every $u(\bullet) \in L^2(0,T;\mathbb{R}^m)$.

If we consider the Cauchy problem Σ (respectively Σ^{T*}) in the larger space $\mathfrak{V}$ (respectively $\mathfrak{W}^{T*}$), then the input operator will be bounded and the output operator unbounded. Nevertheless, the output of the system is well defined as a locally square-integrable function since the output operator satisfies the hypothesis (H2) [28]. More precisely, the operator $\mathcal{C}$ (respectively $\mathcal{C}^{T*}$) has the following property.

Remark 2.11. For every $T>0$ there exists some constant $c_T > 0$ such that the inequalities

$$\|\mathcal{C}\mathcal{S}(\bullet)\phi\|_{L^2(0,T;\mathbb{R}^p)} \leq c_T \, \|\phi\|_{\mathfrak{W}}, \qquad \|\mathcal{C}^{T*}\mathcal{S}^{T*}(\bullet)f\|_{L^2(0,T;\mathbb{R}^p)} \leq c_T \, \|f\|_{\mathfrak{V}^{T*}}$$

hold for every $\phi \in \mathfrak{W}$ and every $f \in \mathfrak{V}^{T*}$.

This follows by duality from the fact that the adjoint operators $\mathcal{C}^T$ and $\mathcal{C}^*$ are the input operators of the transposed equation (2.3) and hence satisfy analogous inequalities as those in Remark 2.10.

Now let us apply Theorem 2.9 to the transposed RFDE (2.3). Then we obtain that the state $\hat{z}(t) \in \mathfrak{X}^{\mathrm{T}}$ of (2.3) in terms of the 'classical' state concept defines a mild solution of the Cauchy problem

$$\Sigma^{\mathrm{T}}: \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{z}(t) = \mathscr{A}^{\mathrm{T}}\hat{z}(t) + \mathscr{C}^{\mathrm{T}}v(t), \quad \hat{z}(0) = \psi, \qquad w(t) = \mathscr{B}^{\mathrm{T}}\hat{z}(t)$$

(to be considered in the Hilbert spaces $\mathfrak{W}^{\mathrm{T}}$ and $\mathfrak{V}^{\mathrm{T}}$) whereas as the state $\hat{\hat{z}}(t) \in \mathfrak{X}^*$ of (2.8) in terms of the dual state concept defines a mild solution of the Cauchy problem

$$\Sigma^*: \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{\hat{z}}(t) = \mathscr{A}^*\hat{\hat{z}}(t) + \mathscr{C}^*v(t), \quad \hat{\hat{z}}(0) = g, \qquad w(t) = \mathscr{B}^*\hat{\hat{z}}(t)$$

(to be considered in the Hilbert spaces $\mathfrak{V}^*$ and $\mathfrak{W}^*$).

Summarizing our situation we have to deal with the four Cauchy problems

$$\begin{matrix} \Sigma & \Sigma^{\mathrm{T}} \\ \Sigma^{\mathrm{T}*} & \Sigma^* \end{matrix}.$$

These are related in the same manner as the semigroups $\mathscr{S}(t)$, $\mathscr{S}^{\mathrm{T}}(t)$, $\mathscr{S}^{\mathrm{T}*}(t)$, and $\mathscr{S}^*(t)$. More precisely, the Cauchy problems on the left-hand side correspond to the RFDE (2.1) and those on the right-hand side to the transposed RFDE (2.3). On each side the upper Cauchy problem describes the respective equation with the 'classical' state concept (solution segments) and the Cauchy problem below within the dual state concept (forcing terms). A diagonal relation is actually given by functional-analytic duality theory.

The vertical relations between the four Cauchy problems above may also be described through the structural operators $\mathscr{F}$ and $\mathscr{F}^*$. In particular it follows from Theorem 2.9 that $\hat{x}(t) = \mathscr{F}\tilde{x}(t)$ $(t \geq 0)$ defines a mild solution of $\Sigma^{\mathrm{T}*}$ if $\tilde{x}(t)$ $(t \geq 0)$ is a mild solution of Σ. This fact is also a consequence of Theorem 2.5 together with the following relations between the various input/output operators by means of the structural operator $\mathscr{F}$.

PROPOSITION 2.12

$$\mathscr{B}^{\mathrm{T}*} = \mathscr{F}\mathscr{B}, \qquad \mathscr{C}^* = \mathscr{F}^*\mathscr{C}^{\mathrm{T}}, \qquad \mathscr{C} = \mathscr{C}^{\mathrm{T}*}\mathscr{F}, \qquad \mathscr{B}^{\mathrm{T}} = \mathscr{B}^*\mathscr{F}^*.$$

Proof. Let us first consider $\mathscr{F}$ as an operator from $\mathfrak{W}$ into $\mathfrak{V}^{\mathrm{T}*}$ (Proposition 2.8) and let $\phi \in \mathfrak{W}$. Then $\mathscr{F}\phi \in \mathfrak{V}^{\mathrm{T}*}$ and

$$\mathscr{C}^{\mathrm{T}*}\mathscr{F}\phi = (\mathscr{F}\phi)^2(0) = \int_0^h [\mathrm{d}\Gamma(\tau)\,\phi^1(-\tau)] = C\phi^1 = \mathscr{C}\phi.$$

The equation $\mathscr{B}^{\mathrm{T}} = \mathscr{B}^*\mathscr{F}^*$ can be established analogously by the use of Theorem 2.5(iv) and the remaining assertions of the proposition follow by duality. □

Finally, note that the Cauchy problems Σ, Σ^{T}, $\Sigma^{\mathrm{T}*}$, and Σ^* may also be understood in a strong sense. In particular, if $\phi \in \mathfrak{W}$ and $u(\cdot) \in \mathrm{L}^2_{\mathrm{loc}}[0, \infty; \mathbb{R}^m]$, then it can be shown that the corresponding mild solution $\hat{x}(t)$ of Σ is in fact a strong solution. This means that $\hat{x}(t)$ $(t \geq 0)$, is a continuous function with values in $\mathfrak{W}$, that its derivative exists as a locally square- (Bochner-) integrable function with values in the larger space $\mathfrak{V}$, and that the first equation in Σ is satisfied in the Hilbert space $\mathfrak{V}$ for almost every $t \geq 0$ [29, Theorem 1.3.4]. In order to make this rigorous, we need the fact that $\mathcal{A}$ can be interpreted as a bounded operator from $\mathfrak{W}$ to $\mathfrak{V}$. This means that $\mathfrak{W}$ is the domain of $\mathcal{A}$ when $\mathcal{A}$ is regarded as an unbounded, closed operator on $\mathfrak{V}$.

PROPOSITION 2.13

$$\mathfrak{W} = \mathfrak{D}_{\mathfrak{V}}(\mathcal{A}), \qquad \mathfrak{W}^{\mathrm{T}} = \mathfrak{D}_{\mathfrak{V}^{\mathrm{T}}}(\mathcal{A}^{\mathrm{T}}), \qquad \mathfrak{V}^{\mathrm{T}*} = \mathfrak{D}_{\mathfrak{W}^{\mathrm{T}*}}(\mathcal{A}^{\mathrm{T}*}), \qquad \mathfrak{V}^* = \mathfrak{D}_{\mathfrak{W}*}(\mathcal{A}^*).$$

Proof. First note that $\mathfrak{Z}^{\mathrm{T}} := \mathfrak{D}_{\mathfrak{X}^{\mathrm{T}}}(\mathcal{A}^{\mathrm{T}}) \subset \mathfrak{W}^{\mathrm{T}}$ and hence $\mathfrak{D}_{\mathfrak{W}^{\mathrm{T}*}}(\mathcal{A}^{\mathrm{T}*}) \subset \mathfrak{D}_{\mathfrak{Z}^{\mathrm{T}*}}(\mathcal{A}^{\mathrm{T}*}) = \mathfrak{X}^{\mathrm{T}*}$ [28, Remark 2.3]. Now let $f \in \mathfrak{X}^{\mathrm{T}*}$. Then $f \in \mathfrak{D}_{\mathfrak{W}^{\mathrm{T}*}}(\mathcal{A}^{\mathrm{T}*})$ if and only if the map

$$\psi \mapsto \langle \mathcal{A}^{\mathrm{T}}\psi, f\rangle_{\mathfrak{V}^{\mathrm{T}},\mathfrak{V}^{\mathrm{T}*}} \quad (\psi \in \mathfrak{D}_{\mathfrak{V}^{\mathrm{T}}}(\mathcal{A}^{\mathrm{T}}))$$

extends to a bounded linear functional on $\mathfrak{W}^{\mathrm{T}}$. But $\psi \in \mathfrak{D}_{\mathfrak{V}^{\mathrm{T}}}(\mathcal{A}^{\mathrm{T}})$ if and only if $\psi^1 \in \mathrm{W}^{2,2}(-h, 0; \mathbb{R}^n)$, $\psi^2 \in \mathrm{W}^{1,2}(-h, 0; \mathbb{R}^p)$, $\psi^0 = \psi^1(0)$, $\psi^2(0) = 0$, and $\dot{\psi}^1(0) = L^{\mathrm{T}}\psi^1 + C^{\mathrm{T}}\psi^2$; and the following equation holds for every $\psi \in \mathfrak{D}_{\mathfrak{V}^{\mathrm{T}}}(\mathcal{A}^{\mathrm{T}})$:

$$\begin{aligned}
\langle \mathcal{A}^{\mathrm{T}}\psi, f\rangle_{\mathfrak{V}^{\mathrm{T}},\mathfrak{V}^{\mathrm{T}*}} &= \langle \mathcal{A}^{\mathrm{T}}\psi, f\rangle_{\mathfrak{X}^{\mathrm{T}},\mathfrak{X}^{\mathrm{T}*}} \\
&= \int_0^h \psi^{1\mathrm{T}}(-\tau)\,\mathrm{d}\eta(\tau)f^0 + \int_0^h \psi^{2\mathrm{T}}(-\tau)\,\mathrm{d}\gamma(\tau)f^0 \\
&\quad + \int_0^h \dot{\psi}^{1\mathrm{T}}(-s)f^1(s)\,\mathrm{d}s + \int_0^h \dot{\psi}^{2\mathrm{T}}(-s)f^2(s)\,\mathrm{d}s \\
&= \int_0^h \psi^{1\mathrm{T}}(-\tau)\,\mathrm{d}\eta(\tau)f^0 + \int_0^h \dot{\psi}^{1\mathrm{T}}(-s)f^1(s)\,\mathrm{d}s \\
&\quad + \int_0^h \dot{\psi}^{2\mathrm{T}}(-s)[f^2(s) + \Gamma(s)f^0 - \Gamma(h)f^0]\,\mathrm{d}s
\end{aligned}$$

(compare the proof of Proposition 2.4). The latter expression defines a bounded linear functional on $\mathfrak{W}^{\mathrm{T}} = \{\psi \in \mathfrak{X}^{\mathrm{T}} : \psi^1 \in \mathrm{W}^{1,2}(-h, 0; \mathbb{R}^n),\ \psi^0 = \psi^1(0)\}$ if and only if there exists a $d \in \mathrm{L}^2(0, h; \mathbb{R}^p)$ such that the following equation holds for every $\psi^2 \in \mathrm{W}^{1,2}(-h, 0; \mathbb{R})$ satisfying $\psi^2(0) = 0$

$$\begin{aligned}
&\int_0^h \dot{\psi}^{2\mathrm{T}}(-s)[f^2(s) + \Gamma(s)f^0 - \Gamma(h)f^0]\,\mathrm{d}s \\
&\qquad = \int_0^h \psi^2(-s)d(s)\,\mathrm{d}s = -\int_0^h \dot{\psi}^{2\mathrm{T}}(-s)\int_s^h d(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}$$

This is equivalent to $f \in \mathfrak{V}^{\mathrm{T}*}$. We conclude that $\mathfrak{V}^{\mathrm{T}*} = \mathfrak{D}_{\mathfrak{W}^{\mathrm{T}*}}(\mathcal{A}^{\mathrm{T}*})$.

Analogous arguments show that $\mathfrak{V}^* = \mathfrak{D}_{\mathfrak{W}*}(\mathcal{A}^*)$. The remaining assertion of Proposition 2.13 follows by duality. □

3. The linear–quadratic optimal control problem

3.1 *The Finite-time Case*

In the previous section we have developed two state-space descriptions for the RFDE (2.1). Moreover, we have shown that the corresponding Cauchy problems Σ and Σ^{T*} both satisfy the hypotheses (H1), (H2), and (H3) of [28] in suitably chosen Hilbert spaces (Remarks 2.10 and 2.11). This allows us to apply the results of [28] to the RFDE (2.1) within the state concepts of Sub-section 2.2. For this sake we consider

$$J_T(u) = \int_0^T [\|y(t)\|^2_{\mathbb{R}^p} + \langle u(t), Ru(t)\rangle]\,dt \tag{3.1}$$

associated with the systems (2.1) and (2.5) where $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix.

Remarks 3.1. For simplicity, we assume there is no weight on the final state $\hat{x}(T)$, respectively $\hat{x}(T)$, in the cost functional $J_T(u)$. Such a weight could be introduced by means of a non-negative operator $G: \mathfrak{W}^{T*} \to \mathfrak{W}^T$ leading to the additional term $\langle \hat{x}(t), G\hat{x}(t)\rangle_{\mathfrak{W}^{T*},\mathfrak{W}}$ in the performance index $J_T(u)$. However, such a term could never be of the form $x^T(T)G_0x(t)$ for some nonnegative definite matrix $G_0 \in \mathbb{R}^{n\times n}$ since the map $f \to f^0$ from $\mathfrak{X}^{T*}$ into $\mathbb{R}^n$ cannot be extended to a bounded linear functional on $\mathfrak{W}^{T*}$.

The following result is now a direct consequence of [28, Theorem 2.7 and Proposition 2.8].

THEOREM 3.2

(i) *There exists a unique, strongly continuous operator family* $\pi(t) \in \mathfrak{L}(\mathfrak{V}, \mathfrak{V}^*)$ $(0 \le t \le T)$ *such that the function* $\pi(t)\phi$ is continuously differentiable in $\mathfrak{W}^*$ for every $\phi \in \mathfrak{W}$ and satisfies the equation

$$\frac{d}{dt}\pi(t)\phi + \mathscr{A}^*\pi(t)\phi + \pi(t)\mathscr{A}\phi - \pi(t)\mathscr{B}R^{-1}\mathscr{B}^*\pi(t)\phi + \mathscr{C}^*\mathscr{C}\Phi = 0,$$
$$\pi(T)\phi = 0. \tag{3.2}$$

(ii) *There exists a unique strongly continuous operator family* $\mathscr{P}(t) \in \mathfrak{L}(\mathfrak{W}^{T*}, \mathfrak{W}^T)$ $(0 \le t \le T)$ *such that the function* $\mathscr{P}(t)f$ *is continuously differentiable in* $\mathfrak{V}^T$ *for every* $f \in \mathfrak{V}^{T*}$ *and satisfies the equation*

$$\frac{d}{dt}\mathscr{P}(t)f + \mathscr{A}^T\mathscr{P}(t)f + \mathscr{P}(t)\mathscr{A}^{T*}f - \mathscr{P}(t)\mathscr{B}^{T*}R^{-1}\mathscr{B}^T\mathscr{P}(t)f + \mathscr{C}^T\mathscr{C}^{T*}f = 0,$$
$$\mathscr{P}(T)f = 0. \tag{3.3}$$

(iii) *If* $\pi(t) \in \mathfrak{L}(\mathfrak{V}, \mathfrak{V}^*)$ *and* $\mathscr{P}(t) \in \mathfrak{L}(\mathfrak{W}^{T*}, \mathfrak{W}^T)$ $(0 \le t \le T)$ *are the solution operators of* (3.2) *and* (3.3), *then*

$$\pi(t) = \mathscr{F}^*\mathscr{P}(t)\mathscr{F} \qquad (0 \le t \le T). \tag{3.4}$$

(iv) *There exists a unique optimal control which minimizes the performance index*

(3.1) *subject to* (2.1) *and* (2.2). *This optimal control is given by the feedback control law*

$$\begin{aligned} u(t) &= -R^{-1}\mathscr{B}^*\pi(t)\hat{x}(t) \\ &= -R^{-1}\mathscr{B}^{\mathrm{T}}\mathscr{P}(t)\hat{x}(t) \\ &= -R^{-1}\mathscr{B}^{\mathrm{T}}\mathscr{P}(t)\mathscr{F}\hat{x}(t), \end{aligned} \tag{3.5}$$

where $\pi(t) \in \mathfrak{L}(\mathfrak{V}, \mathfrak{V}^*)$ *and* $\mathscr{P}(t) \in \mathfrak{L}(\mathfrak{W}^{\mathrm{T}*}, \mathfrak{W}^{\mathrm{T}})$ *are given by* (3.2) *and* (3.3). *The optimal cost corresponding to the initial state* $\phi \in \mathfrak{X}$ *is*

$$\begin{aligned} J_T(u) &= \langle \phi, \pi(0)\phi \rangle_{\mathfrak{X},\mathfrak{X}^*} \\ &= \langle f, \mathscr{P}(0)f \rangle_{\mathfrak{X}^{\mathrm{T}*},\mathfrak{X}^{\mathrm{T}}} \end{aligned}$$

where $f = \mathscr{F}\phi \in \mathfrak{X}^{\mathrm{T}*}$ *is the initial state of* (3.5).

Proof. The statements (i), (ii), and (iv) follow immediately from [28, Theorem 2.7 and Proposition 2.8]. In order to prove (iii), let $\mathscr{P}(t) \in \mathfrak{L}(\mathfrak{W}^{\mathrm{T}*}, \mathfrak{W}^{\mathrm{T}})$ be the unique solution of (3.3) and let $\pi(t) \in \mathfrak{L}(\mathfrak{V}, \mathfrak{V}^*)$ be defined by (3.4). Moreover, let $\phi \in \mathfrak{W}$ and $f := \mathscr{F}\phi \in \mathfrak{V}^{\mathrm{T}*}$. Then, since $\mathscr{F}^* \in \mathfrak{L}(\mathfrak{V}^{\mathrm{T}}, \mathfrak{W}^*)$, the function $\pi(t)\phi = \mathscr{F}^*\mathscr{P}(t)f$ $(0 \leq t \leq T)$ is continuously differentiable with values in $\mathfrak{W}^*$ and satisfies the following equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\pi(t)\phi + \mathscr{A}^*\pi(t)\phi + \pi(t)\mathscr{A}\phi - \pi(t)\mathscr{B}R^{-1}\mathscr{B}^*\pi(t)\phi + \mathscr{C}^*\mathscr{C}\phi \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}^*\mathscr{P}(t)f + \mathscr{A}^*\mathscr{F}^*\mathscr{P}(t)f + \mathscr{F}^*\mathscr{P}(t)\mathscr{F}\mathscr{A}\phi \\ &\quad - \mathscr{F}^*\mathscr{P}(t)\mathscr{F}\mathscr{B}R^{-1}\mathscr{B}^*\mathscr{F}^*\mathscr{P}(t)\mathscr{F}\phi + \mathscr{F}^*\mathscr{C}^{\mathrm{T}}\mathscr{C}^{\mathrm{T}*}\mathscr{F}\phi \\ &= \mathscr{F}^*\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{P}(t)f + \mathscr{A}^{\mathrm{T}}\mathscr{P}(t)f + \mathscr{P}(t)\mathscr{A}^{\mathrm{T}*}f \\ &\quad - \mathscr{P}(t)\mathscr{B}^{\mathrm{T}*}R^{-1}\mathscr{B}^{\mathrm{T}}\mathscr{P}(t)f + \mathscr{C}^{\mathrm{T}}\mathscr{C}^{\mathrm{T}*}f\Big) \\ &= 0. \end{aligned}$$

(See Theorem 2.5 and Proposition 2.12.) Now statement (iii) follows from the uniqueness of the solution of (3.2). □

Note that an analogous relation as (3.4) has been shown in Delfour, Lee & Manitius [11] and Vinter & Kwong [30] for RFDEs with undelayed input/output variables.

3.2 *Stabilizability and Detectability*

In this section we investigate the sufficient conditions (H4) and (H5) of [28] for the unique solvability of the *algebraic Riccati equation* in the case of the systems Σ and $\Sigma^{\mathrm{T}*}$. We will not consider these hypotheses in their weakest form but have a look at the slightly stronger properties of stabilizability and detectability.

DEFINITION 3.3

(i) System Σ is said to be *stabilizable* if there exists a feedback operator $\mathcal{K} \in \mathfrak{L}(\mathfrak{V}, \mathbb{R}^m)$ such that the closed-loop semigroup $\mathcal{S}_{\mathcal{K}}(t) \in \mathfrak{L}(\mathfrak{V})$ defined by

$$\mathcal{S}_{\mathcal{K}}(t)\phi = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s)\mathcal{B}\mathcal{S}_{\mathcal{K}}(s)\phi \, ds \tag{3.6}$$

for $t \geq 0$ and $\phi \in \mathfrak{V}$ is exponentially stable.

(ii) System Σ^{T*} is said to be *stabilizable* if there exists a feedback operator $\mathcal{K}^{T*} \in \mathfrak{L}(\mathfrak{W}^{T*}; \mathbb{R}^m)$ such that the closed-loop semigroup $\mathcal{S}_{\mathcal{K}}^{T*}(t) \in \mathfrak{L}(\mathfrak{W}^{T*})$ defined by

$$\mathcal{S}_{\mathcal{K}}^{T*}(t)f = \mathcal{S}^{T*}(t)f + \int_0^t \mathcal{S}^{T*}(t-s)\mathcal{B}^{T*}\mathcal{K}^{T*}\mathcal{S}_{\mathcal{K}}^{T*}(s)f \, ds, \tag{3.7}$$

for $t \geq 0$ and $f \in \mathfrak{W}^{T*}$, is exponentially stable.

Remarks 3.4. (i) Note that the integral term in (3.6) is a bounded linear operator from $\mathfrak{V}$ to $\mathfrak{W}$ (Remark 2.10) and hence $\mathcal{S}_{\mathcal{K}}(t)$ is also a strongly continuous semigroup on $\mathfrak{X}$ and $\mathfrak{W}$.

(ii) It follows from Remark (2.6)(iii) that for every $t \geq h$,

$$\mathcal{S}_{\mathcal{K}}(t) \in \mathfrak{L}(\mathfrak{V}, \mathfrak{X}) \cap \mathfrak{L}(\mathfrak{X}, \mathfrak{W}).$$

(iii) The stability of the semigroup $\mathcal{S}_{\mathcal{K}}(t)$ is independent of the choice of the state space $\mathfrak{V}$, $\mathfrak{X}$, or $\mathfrak{W}$. In order to see this, note that the operator $\mu I - \mathcal{A} - \mathcal{B}\mathcal{K} : \mathfrak{W} \to \mathfrak{V}$ provides a similarity action between $\mathcal{S}_{\mathcal{K}}(t) \in \mathfrak{L}(\mathfrak{W})$ and $\mathcal{S}_{\mathcal{K}}(t) \in \mathfrak{L}(\mathfrak{V})$ if $\mu > 0$ is sufficiently large. Moreover, it follows from (ii) that the stability of $\mathcal{S}_{\mathcal{K}}(t)$ on the Hilbert space $\mathfrak{W}$ implies the stability on $\mathfrak{X}$ and the stability on $\mathfrak{X}$ implies the stability on $\mathfrak{V}$.

(iv) The same arguments as above show that the closed-loop semigroup $\mathcal{S}_{\mathcal{K}}^{T*}(t) \in \mathfrak{L}(\mathfrak{W}^{T*})$ can be restricted to a semigroup on $\mathfrak{X}^{T*}$ or $\mathfrak{V}^{T*}$ and that its stability is independent of the choice of the state space $\mathfrak{W}^{T*}$, $\mathfrak{X}^{T*}$, or $\mathfrak{V}^{T*}$.

(v) Let $\mathcal{K}^{T*} \in \mathfrak{L}(\mathfrak{W}^{T*}, \mathbb{R}^m)$ be given and define

$$\mathcal{K} = \mathcal{K}^{T*}\mathcal{F} \in \mathfrak{L}(\mathfrak{V}, \mathbb{R}^m). \tag{3.8}$$

Then the following equation holds for every $t \geq 0$:

$$\mathcal{F}\mathcal{S}_{\mathcal{K}}(t) = \mathcal{S}_{\mathcal{K}}^{T*}(t)\mathcal{F}. \tag{3.9}$$

In fact, it follows from Theorem 2.5 and Proposition 2.12 that for every $\phi \in \mathfrak{V}$ the function $\hat{x}(t) = \mathcal{F}\mathcal{S}_{\mathcal{K}}(t)\phi \in \mathfrak{W}^{T*}$ $(t \geq 0)$ defines a solution of (3.7) with $f = \mathcal{F}\phi$.

(vi) Every $\mathcal{K}^{T*} \in \mathfrak{L}(\mathfrak{W}^{T*}, \mathbb{R}^m)$ can be represented as

$$\mathcal{K}^{T*}f = K_0 f^0 + \int_0^h K_1(-s)f(s)\, ds + \int_0^h K_2(-s)f^2(s)\, ds$$

where $K_2(\bullet) \in L^2(-h, 0; \mathbb{R}^{m\times p})$, $K_1(\bullet) \in W^{1,2}(-h, 0; \mathbb{R}^{m\times n})$, and $K_0 = K_1(0)$.

Moreover, let us again suppose that $\mathcal{K} = \mathcal{K}^{T*}\mathcal{F}$ and consider the control law

$$\begin{aligned} u(t) &= \mathcal{K}^{T*}\mathcal{F}\hat{x}(t) \\ &= K_0x(t) + \int_{\tau=0}^{h}\int_{s=0}^{\tau} K_1(s-\tau)[\mathrm{d}A(\tau)\,x(t-s)]\,\mathrm{d}s \\ &\quad + \int_0^h\int_0^{\tau} K_2(s-\tau)[\mathrm{d}\Gamma(\tau)\,x(t-s)]\,\mathrm{d}s \\ &\quad + \int_0^h\int_0^{\tau} K_1(s-\tau)[\mathrm{d}B(\tau)\,u(t-s)]\,\mathrm{d}s \end{aligned} \tag{3.10}$$

for system (2.1). Then it follows from equation (3.6) and Theorem 2.9 that for every solution pair, $x(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^n) \cap W^{1,2}_{loc}(0, \infty; \mathbb{R}^n)$ with $u(\bullet) \in L^2_{loc}(0, \infty; \mathbb{R}^m)$, of (2.1), (2.2), (3.10), the corresponding state $\hat{x}(t) = (x(t), x_t, u_t) \in \mathfrak{X}$ at time $t \geq 0$ is given by

$$\hat{x}(t) = \mathcal{S}_{\mathcal{K}}(t)\phi.$$

By (3.9), this implies that $\hat{f}(t) = \mathcal{F}\hat{x}(t) \in \mathfrak{X}^{T*}$ is given by $\hat{f}(t) = \mathcal{S}^{T*}_{\mathcal{K}}(t)\mathcal{F}\phi$.

(vii) If $\mathcal{K} \in \mathfrak{L}(\mathfrak{B}, \mathbb{R}^m)$ is given by (3.8) then the exponential stability of $\mathcal{S}_{\mathcal{K}}(t)$ on $\mathfrak{X}$ is equivalent to that of $\mathcal{S}^{T*}_{\mathcal{K}}(t)$ on $\mathfrak{X}^{T*}$. In fact, it follows from equation (3.6) and Theorem 2.9 that

$$\operatorname{range} \mathcal{S}^{T*}_{\mathcal{K}}(h) \subset \operatorname{range} \mathcal{F}$$

and hence equation (3.9) shows that the stability of $\mathcal{S}_{\mathcal{K}}(t)$ implies that of $\mathcal{S}^{T*}_{\mathcal{K}}(t)$. The converse implication is a consequence of the fact that $\mathcal{S}_{\mathcal{K}}(t)\phi = (x(t), x_t, u_t)$ and

$$x(t) = [\mathcal{S}^{T*}_{\mathcal{K}}(t)\mathcal{F}\phi]^0, \qquad u(t) = \mathcal{K}^{T*}\mathcal{S}^{T*}_{\mathcal{K}}(t)\phi,$$

($t \geq 0$) for every solution pair $x(t)$, $u(t)$ ($t \geq -h$) of the closed-loop system (2.1), (2.2), (3.10) with $\phi \in \mathfrak{X}$.

Having collected the basic properties of the feedback semigroups $\mathcal{S}_{\mathcal{K}}(t)$ and $\mathcal{S}^{T*}_{\mathcal{K}}(t)$, we are now in the position to prove the following stabilizability criterion.

THEOREM 3.5

The following statements are equivalent.

(i) *System Σ is stabilizable.*

(ii) *There exists a feedback operator $\mathcal{K} \in \mathfrak{L}(\mathfrak{W}, \mathbb{R}^m)$ such that the closed-loop semigroup $\mathcal{S}_{\mathcal{K}}(t) \in \mathfrak{L}(\mathfrak{W})$ defined by (3.6) for $t \geq 0$ and $\phi \in \mathfrak{W}$ is exponentially stable.*

(iii) *System Σ^{T*} is stabilizable.*

(iv) *There exists a feedback operator $\mathcal{K}^{T*} \in \mathfrak{L}(\mathfrak{B}^{T*}, \mathbb{R}^m)$ such that the closed-loop semigroup $\mathcal{S}^{T*}_{\mathcal{K}}(t) \in \mathfrak{L}(\mathfrak{B}^{T*})$, defined by (3.7) for $t \geq 0$ and $f \in \mathfrak{B}^{T*}$ is exponentially stable.*

(v) *For every $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda \geq 0 \operatorname{rank}[\Delta(\lambda), B(e^{\lambda\cdot})] = n$.

Proof. The implications "(iii)⇒(i)⇒(ii)" and "(iii)⇒(iv)" follow from Remark 3.4.

Now we will prove that (v) implies (iii). Note that it has been shown [29, Theorem 5.2.11 and Corollary 5.3.3] that (v) implies the existence of a stabilizing control law of the form (3.10) for the system (2.1) where $K_1(\bullet) \in W^{1,2}[-h, 0; \mathbb{R}^{m\times n}]$, $K_0 = K_1(0)$, and $K_2(\tau) \equiv 0$. This means that every solution pair $x(t), u(t)$ $(t \geq -h)$ of (2.1), (2.2), (3.10) with $\phi \in \mathfrak{X}$ tends to zero with an exponential decay rate which is independent of ϕ. This shows that the semigroup $\mathcal{S}_{\mathcal{K}}^{T*}(t)$ is stable on $\mathfrak{X}^{T*}$ (Remark 3.4(vi)) and hence on $\mathfrak{W}^{T*}$ (Remark 3.4(iv)).

It remains to show that (ii) and (iv) imply (v). For this sake assume that there exists a $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$, and a nonzero vector $x_0 \in \mathbb{C}^n$, such that $x_0^T\Delta(\lambda) = 0$ and $x_0^T B(e^{\lambda\cdot}) = 0$ and define $\psi := (x_0, e^{\lambda\cdot}x_0, 0) \in \mathfrak{W}^T$. Then it is easy to see that $\mathcal{A}^T\psi = \lambda\psi$ and $\mathcal{B}^T\psi = 0$ and hence $\mathcal{A}^*\mathcal{F}^*\psi = \lambda\mathcal{F}^*\psi$ and $\mathcal{B}^*\mathcal{F}^*\psi = 0$ (Theorem 2.5 and Proposition 3.12). Now equations (3.7) and (3.6) show that $\mathcal{S}_{\mathcal{K}}^T(t)\psi = \mathcal{S}^T(t)\psi = e^{\lambda t}\psi$ and $\mathcal{S}_{\mathcal{K}}^*(t)\mathcal{F}^*\psi = \mathcal{S}^*(t)^*\psi = e^{\lambda t}\mathcal{F}^*\psi$ for every $\mathcal{K}^{T*} \in \mathfrak{L}(\mathfrak{V}^{T*}, \mathbb{R}^m)$ and every $\mathcal{K} \in \mathfrak{L}(\mathfrak{W}, \mathbb{R}^m)$. Since $\psi \neq 0$ and $\mathcal{F}^*\psi \neq 0$, this shows that (ii) and (iv) are not satisfied. □

The next result is obtained by dualizing Theorem 3.5.

COROLLARY 3.6 *The following statements are equivalent.*

(i) *System* Σ *is* detectable *in the sense that there exists an output injection operator* $\mathcal{H} \in \mathfrak{L}(\mathbb{R}^p, \mathfrak{V})$ *such that the closed-loop semigroup* $\mathcal{S}_{\mathcal{H}}(t) \in \mathfrak{L}(\mathfrak{V})$ *generated by* $\mathcal{A} + \mathcal{H}\mathcal{C} : \mathfrak{W} \to \mathfrak{V}$ *is exponentially stable.*

(ii) *System* Σ^{T*} *is* detectable *in the sense that there exists an output injection operator* $\mathcal{H}^{T*} \in \mathfrak{L}(\mathbb{R}^p, \mathfrak{W}^{T*})$ *such that the closed-loop semi-group* $\mathcal{S}_{\mathcal{H}}^{T*}(t) \in \mathfrak{L}(\mathfrak{W}^{T*})$ *generated by* $\mathcal{A}^{T*} + \mathcal{H}^{T*}\mathcal{C}^{T*} : \mathfrak{V}^{T*} \to \mathfrak{W}^{T*}$ *is exponentially stable.*

(iii) *For every* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\,\lambda \geq 0$ *we have* $\mathrm{rank}\begin{bmatrix}\Delta(\lambda)\\ C(e^{\lambda\cdot})\end{bmatrix} = n$. *Note that* $\mathcal{S}_{\mathcal{H}}(t) \in \mathfrak{L}(\mathfrak{V})$ *satisfies the integral equation*

$$\mathcal{S}_{\mathcal{H}}(t)\phi = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}_{\mathcal{H}}(t-s)\mathcal{H}\mathcal{C}\mathcal{S}(s)\phi\, ds$$

for every $t \geq 0$ *and every* $\phi \in \mathfrak{W}$ [29, *Theorem* 1.3.9] *and hence can be restricted to a semigroup on* $\mathfrak{X}$ *if* $\mathcal{H} \in \mathfrak{L}(\mathbb{R}^p, \mathfrak{X})$. *At the end of this section we give a concrete representation of the output injection semigroup* $\mathcal{S}_{\mathcal{H}}(t) \in \mathfrak{L}(\mathfrak{X})$ *by means of a closed-loop functional differential equation. For this purpose, note that every* $\mathcal{H} \in \mathfrak{L}(\mathbb{R}^p, \mathfrak{X})$ *can be represented as*

$$\mathcal{H}y = (H_0 y, H_1(\bullet)y, H_2(\bullet)y) \in \mathfrak{X} \qquad (y \in \mathbb{R}^p)$$

where $H_0 \in \mathbb{R}^{n\times p}$, $H_1(\bullet) \in L^2(-h, 0; \mathbb{R}^{n\times p})$, $H_2(\bullet) \in L^2(-h, 0; \mathbb{R}^{m\times p})$. *Moreover, we introduce the abbreviating notation*

$$H_i \star \phi^2(\tau) = \int_\tau^0 H_i(\tau - \sigma)\phi^2(\sigma)\, ds \qquad (-h \leq \tau \leq 0)$$

for $i = 1, 2$ *and* $\phi^2 \in L^2(-h, 0; \mathbb{R}^p)$.

THEOREM 3.7

(i) *Let* $x(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^n) \cap W^{1,2}_{loc}(0, \infty; \mathbb{R}^n)$ *satisfy the* RFDE

$$\dot{x}(t) = L(x_t + H_1 \star y_t) + B(u_t + H_2 \star y_t) + H_0 y(t)$$

where $u(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^m)$ *and* $y(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^p)$. *Then*

$$\hat{x}(t) = (x(t), x_t + H_1 \star y_t, u_t + H_2 \star y_t) \in \mathfrak{X} \qquad (t \geq 0) \tag{3.11}$$

is given by the variation-of-constants formula

$$\hat{x}(t) = \mathcal{S}(t)\hat{x}(0) + \int_0^t \mathcal{S}(t-s)\mathcal{B}u(s)\,ds + \int_0^t \mathcal{S}(t-s)\mathcal{H}y(s)\,ds.$$

(ii) *Let* $x(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^n) \cap W^{1,2}_{loc}(0, \infty; \mathbb{R}^n)$ *and* $y(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^p)$ *satisfy the equations*

$$\dot{x}(t) = L(x_t + H_1 \star y_t) + B(H_2 \star y_t) + H_0 y(t), \qquad y(t) = C(x_t + H_1 \star y_t), \tag{3.12a, b}$$

for $t \geq 0$, *and let* $\hat{x}(t) \in \mathfrak{X}$ $(t \geq 0)$ *be defined by* (3.11) *with* $u(t) \equiv 0$. *Then*

$$\hat{x}(t) = \mathcal{S}_{\mathcal{K}}(t)\hat{x}(0). \tag{3.13}$$

Proof. In order to prove statement (i), let us first assume that $y(t) = 0$ for $t \geq 0$ and define $z(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^m$ for $t \geq -h$ by $z(t) = x(t)$ and $v(t) = u(t)$, for $t \geq 0$, and

$$z(\tau) = x(\tau) + \int_\tau^0 H_1(\tau - \sigma)y(\sigma)\,d\sigma, \qquad v(\tau) = u(\tau) + \int_\tau^0 H_2(\tau - \sigma)y(\sigma)\,d\sigma,$$

for $-h \leq \tau \leq 0$. Then it is easy to see that $\hat{x}(t) = (z(t), z_t, v_t)$ for all $t \geq 0$ and hence the following equation holds:

$$\begin{aligned}\dot{z}(t) &= \dot{x}(t) \\ &= L(x_t + H_1 \star y_t) + B(u_t + H_2 \star y_t) + H_0 y(t) \\ &= Lz_t + Bv_t \qquad (t \geq 0).\end{aligned}$$

This implies $\hat{x}(t) = (z(t), z_t, v_t) = \mathcal{S}(t)\hat{x}(0) + \int_0^t \mathcal{S}(t-s)\mathcal{B}u(s)\,ds$.

Secondly, let $u(t) \equiv 0$ and let $x(\tau) = 0$ and $y(\tau) = 0$ for $-h \leq \tau \leq 0$. Moreover, let $Z(t) \in \mathbb{R}^{n \times p}$ with $V(t) \in \mathbb{R}^{m \times p}$ $(t \leq -h)$ be the unique solution of $\dot{Z}(t) = LZ_t + BV_t$ corresponding to the input $V(t) = 0$ $(t \geq 0)$ and the initial conditions $Z(0) = H_0$, $Z(\tau) = H_1(\tau)$, and $V(\tau) = H_2(\tau)$ $(-h \leq \tau < 0)$. Then

$$(Z(t), Z_t, V_t) = \mathcal{S}(t)\mathcal{H} \in \mathfrak{L}(\mathbb{R}^p, \mathfrak{X}) \qquad (t \geq 0). \tag{3.14}$$

Now let us define

$$z(t) = \int_0^t Z(t-s)y(s)\,ds, \qquad z(\tau) = 0,$$

$$z(t, \tau) = \int_0^t Z(t - s + \tau)y(s)\,ds, \qquad v(t, \tau) = \int_0^t V(t - s + \tau)y(s)\,ds,$$

for $t \geq 0$ and $-h \leq \tau \leq 0$. Then we obtain

$$z(t, \bullet) = z_t + H_1 \star y_t \in C(-h, 0; \mathbb{R}^n), \qquad v(t, \bullet) = H_2 \star y_t \in L^2(-h, 0; \mathbb{R}^m), \tag{3.15}$$

and hence

$$\begin{aligned} \dot{z}(t) &= \int_0^t \dot{Z}(t-s)y(s)\,ds + Z(0)y(t) \\ &= \int_0^h \left(dA(\tau) \int_0^t Z(t-s-\tau)y(s)\,ds \right) \\ &\quad + \int_0^h \left(dB(\tau) \int_0^t V(t-s-\tau)y(s)\,ds \right) + Z(0)y(t) \\ &= L(z_t + H_1 \star y_t) + B(H_2 \star y_t) + H_0 y(t) \end{aligned}$$

for $t \geq 0$. This implies that $x(t) = z(t)$ for $t \geq -h$. Thus it follows from (3.14) and (3.15) that

$$\begin{aligned} \hat{x}(t) &= (x(t), x_t + H_1 \star y_t, H_2 \star y_t) \\ &= (z(t), z(t, \bullet), v(t, \bullet)) \\ &= \int_0^t (Z(t-s), Z_{t-s}, V_{t-s}) y(s)\,ds \\ &= \int_0^t \mathcal{S}(t-s)\mathcal{H}y(s)\,ds. \end{aligned}$$

This proves statement (i).

In order to prove statement (ii), let us assume that $x(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^n) \cap W^{1,2}_{loc}(0, \infty; \mathbb{R}^n)$ and $y(\bullet) \in L^2_{loc}(-h, \infty; \mathbb{R}^p)$ satisfy (3.12) and that $\hat{x}(t) \in \mathfrak{X}$ is defined by (3.11) with $u(\bullet) \equiv 0$. Moreover suppose that $\phi = \hat{x}(0) \in \mathfrak{W}$. Then $y(t)$ $(t \geq 0)$ satisfies the Volterra integral equation

$$\begin{aligned} y(t) &= C(x_t + H_1 \star y_t) \\ &= \int_t^h [d\Gamma(\tau)\, \phi^1(t-\tau)] \int_0^t [d\Gamma(\tau)\, x(t-\tau)] \\ &\quad + \int_0^t \int_s^t [d\Gamma(\tau)\, H_1(s-\tau)] y(t-s)\,ds \end{aligned}$$

$(t \geq 0)$ with forcing term in $W^{1,2}_{loc}(0, \infty; \mathbb{R}^p)$. This implies that $y(\bullet) \in W^{1,2}_{loc}(0, \infty; \mathbb{R}^p)$ and hence, by (i)

$$\hat{x}(t) = \mathcal{S}(t)\hat{x}(0) + \int_0^t \mathcal{S}(t-s)\mathcal{H}y(s)\,ds \in \mathfrak{W} = \mathfrak{D}_{\mathfrak{V}}(\mathcal{A}),$$

for every $t \geq 0$. Moreover it follows from a general semigroup-theoretic result that $\hat{x}(t)$ is continuously differentiable in $\mathfrak{V}$ and satisfies

$$\frac{d}{dt}\hat{x}(t) = \mathcal{A}\hat{x}(t) + \mathcal{H}y(t) = (\mathcal{A} + \mathcal{H}\mathcal{C})\hat{x}(t) \qquad (t \geq 0).$$

This proves (3.13) for the case $\hat{x}(0) \in \mathfrak{W}$. In general (3.13) follows from the fact that both sides of this equation depend continuously on the initial functions $(x(0), x_0, y_0) \in \mathfrak{X}^{\mathrm{T}}$ of (3.12). (For existence and uniqueness results for this type of equation see Salamon [29, Section 1.2].)

Finally, note that the transposed equation of (3.12) takes the form

$$\dot{z}(t) = L^{\mathrm{T}} z_t + C^{\mathrm{T}} v_t,$$

$$\begin{aligned} v(t) = H_0^{\mathrm{T}} z(t) &+ \int_{\tau=0}^{h} \int_{s=0}^{\tau} H_1^{\mathrm{T}}(-s)[\mathrm{d}A^{\mathrm{T}}(\tau)\, z(t+s-\tau)]\, \mathrm{d}s \\ &+ \int_0^h \int_0^{\tau} H_2^{\mathrm{T}}(-s)[\mathrm{d}B^{\mathrm{T}}(\tau)\, z(t+s-\tau)]\, \mathrm{d}s \\ &+ \int_0^h \int_0^{\tau} H_1^{\mathrm{T}}(-s)[\mathrm{d}\Gamma^{\mathrm{T}}(\tau) v(t+s-\tau)]\, \mathrm{d}s. \end{aligned}$$

This is nothing more than the transposed RFDE (2.3) with a control law which is analogous to (3.10).

3.3 *The Infinite-time Case*

In this subsection we consider the performance index

$$J(u) = \int_0^{\infty} [\|y(t)\|^2 + u^{\mathrm{T}}(t) R u(t)]\, \mathrm{d}t \tag{3.16}$$

associated with the Cauchy problems Σ and $\Sigma^{\mathrm{T}*}$ where $R \in \mathbb{R}^{m \times m}$ is a positive definite matrix.

Combining the results of [28, Theorem 3.3 and Theorem 3.4] and of the previous subsection (Theorem 3.5 and Corollary 3.6) we obtain the facts which are summarized in the theorem below.

THEOREM 3.8

(i) *If*

$$\operatorname{rank}[\Delta(\lambda), B(\mathrm{e}^{\lambda \cdot})] = n \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re} \lambda \geq 0, \tag{3.17}$$

then there exist positive semidefinite operators $\pi \in \mathfrak{L}(\mathfrak{V}, \mathfrak{V}^*)$ *and* $\mathcal{P} \in \mathfrak{L}(\mathfrak{W}^{\mathrm{T}*}, \mathfrak{W}^{\mathrm{T}})$ *satisfying the algebraic Riccati equations*

$$\mathcal{A}^* \pi \phi + \pi \mathcal{A} \phi - \pi \mathcal{B} R^{-1} \mathcal{B}^* \pi \phi + \mathcal{C}^* \mathcal{C} \phi = 0 \qquad (\phi \in \mathfrak{W}) \tag{3.18}$$

and

$$\mathcal{A}^{\mathrm{T}} \mathcal{P} f + \mathcal{P} \mathcal{A}^{\mathrm{T}*} f - \mathcal{P} \mathcal{B}^{\mathrm{T}*} R^{-1} \mathcal{B}^{\mathrm{T}} \mathcal{P} f + \mathcal{C}^{\mathrm{T}} \mathcal{C}^{\mathrm{T}*} f = 0 \qquad (f \in \mathfrak{V}^{\mathrm{T}*}), \tag{3.19}$$

respectively. The minimal solutions π of (3.18) *and* $\mathcal{P}$ *of* (3.19) *satisfy the relation*

$$\pi = \mathcal{F}^* \mathcal{P} \mathcal{F}.$$

(ii) *If* (3.17) *is satisfied, then there exists a unique optimal control* $u(\cdot) \in \mathrm{L}^2_{\mathrm{loc}}(0, \infty; \mathbb{R}^m)$ *which minimizes the performance index* (3.16) *subject to* (2.1)–

(2.2). *This optimal control is given by the feedback law*

$$u(t) = -R^{-1}\mathcal{B}^*\pi\hat{x}(t) = -R^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{F}\hat{x}(t) = -R^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P}\hat{\mathfrak{f}}(t)$$

where π (*respectively* $\mathcal{P}$) *is the minimal solution of* (3.18) (*respectively* (3.19)). *The optimal cost corresponding to the initial state* $\phi \in \mathfrak{X}$ *is given by*

$$J(u) = \langle \phi, \pi\phi \rangle = \langle f, \mathcal{P}f \rangle$$

where $f = \mathcal{F}\phi \in \mathfrak{X}^{\mathrm{T}*}$ *is the initial state of* (2.5).

(iii) *If*

$$\operatorname{rank}\begin{bmatrix} \Delta(\lambda) \\ C(e^{\lambda\cdot}) \end{bmatrix} = n \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geq 0,$$

then the algebraic Riccati equation (3.18) (*respectively* (3.19)) *has at most one self-adjoint, nonnegative solution* $\pi \in \mathfrak{L}(\mathfrak{V}, \mathfrak{V}^*)$ (*respectively* $\mathcal{P} \in \mathfrak{L}(\mathfrak{W}^{\mathrm{T}*}, \mathfrak{W}^{\mathrm{T}})$). *Moreover, if* π (*respectively* $\mathcal{P}$) *is such a solution, then the closed-loop semigroup* $\mathcal{S}_\pi(t) \in \mathfrak{L}(\mathfrak{V})$, *generated by* $\mathcal{A} - \mathcal{B}R^{-1}\mathcal{B}^*\pi$ (*respectively* $\mathcal{S}_{\mathcal{P}}^{\mathrm{T}*}(t) \in \mathfrak{L}(\mathfrak{W}^{\mathrm{T}*})$ *generated by* $\mathcal{A}^{\mathrm{T}*} - \mathcal{B}^{\mathrm{T}*}R^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P}$) *is exponentially stable.*

We have derived the solution to our infinite-time optimal-control problem via the positive semidefinite solution $\mathcal{P} \in \mathfrak{L}(\mathfrak{W}^{\mathrm{T}*}, \mathfrak{W}^{\mathrm{T}})$ (respectively $\pi \in \mathfrak{L}(\mathfrak{V}, \mathfrak{V}^*)$) of the algebraic Riccati equation (3.19) (respectively (3.18)). Therefore it would be extremely interesting to have a detailed characterization of the structure of the operators $\mathcal{P}$ and π, which arises from the product-space structure of the state space. In the case of RFDEs with state delays only, such a characterization has been given in Kwong [23] and Vinter & Kwong [30] for the operator $\mathcal{P}$, and in Delfour, McCalla & Mitter [13] for the operator π (note that in this special case the operators $\mathcal{P}$ and π may be defined on the state space $\mathfrak{M}^2 = \mathbb{R}^n \times L^2[-h, 0; \mathbb{R}^n]$). An analogous result for general systems of the type (2.1) seems to be unknown since the Riccati equations (3.18) and (3.19) are apparently new. The difference between (3.18) and Ichikawa's result [16] is that (3.18) allows for output delays, which leads to an unbounded output operator $\mathcal{C}$.

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