Symplectic Floer-Donaldson theory and quantum cohomology

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Abstract

The goal of this paper is to give in outline a new proof of the fact that the Floer cohomology groups of the loop space of a semi-positive symplectic manifold (M, ω) are naturally isomorphic to the ordinary cohomology of M. We shall then outline a proof that this isomorphism intertwines the quantum cup-product structure on the cohomology of M with the pair-of-pants product on Floer-homology. One of the key technical ingredients of the proof is a gluing theorem for J-holomorphic curves proved in [20]. In this paper we shall only sketch the proofs. Full details of the analysis will appear elsewhere.

1 Introduction

The Floer homology groups of a symplectic manifold (M, ω) can intuitively be described as the *middle dimensional homology groups* of the loop space. The boundary loops of J-holomorphic discs in the symplectic manifold with center in a given homology class $\alpha \in H_*(M)$ (integral homology modulo torsion) form a submanifold of the loop space of roughly half dimension and should therefore determine a Floer homology class. This determines a homomorphism $H_*(M) \to HF_*(M)$ which, in view of Poincaré duality, can also be described in terms of cohomology classes. The goal of this paper is to give a rigorous meaning to these heuristic ideas (Sections 2 and 3), prove that the resulting homomorphism is in fact an isomorphism (Section 4), and then show that it intertwines the respective product structures (Section 5).

The rigorous definition of Floer homology involves an infinite dimensional version of Morse theory for the perturbed symplectic action functional (on the space of contractible loops in M) which is due to Floer [11]. Here the perturbation is given by a Hamiltonian term and the critical points are the periodic solutions of the corresponding time dependent Hamiltonian system. The gradient flow of the action functional gives rise to a chain complex which is generated by these periodic solutions. The boundary operator is obtained by counting connecting orbits and these can be interpreted as perturbed J-holomorphic cylinders which connect two such periodic solutions. There is a relative Morse-type index for any pair of periodic solutions and this index determines the dimension of the space of J-holomorphic cylinders. If one considers the unperturbed symplectic action functional then the critical points are the constant loops and the connecting orbits are J-holomorphic spheres. Hence the unperturbed symplectic action is a (multi-valued) Morse-Bott function on the loop space of M whose critical manifolds (in the universal cover) are copies of M. This should imply that the Floer homology groups of M are isomorphic to the ordinary homology of M, tensored by a suitable Novikov ring associated to the group of covering transformations. In the previous literature (cf. [11], [16], [19], [21], [28]) this fact is proved by considering time independent Hamiltonian functions and then proving that in the case of index difference 1 all the connecting orbits are independent of the circle variable and hence correspond to ordinary gradient flow lines. It then follows from Morse theory that Floer homology is isomorphic to ordinary homology (cf. [26], [30]).

In this paper we give an alternative construction of this isomorphism which goes back to an idea of Graeme Segal. It is based on the study of J-holomorphic discs with infinite cylindrical ends which on these cylindrical ends satisfy the perturbed Cauchy-Riemann equations and have finite energy. We shall then outline a proof that this isomorphism intertwines the quantum cup-product in ordinary cohomology with the pair-of-pants product in Floer homology. This new approach also covers some cases (namely $N = n-1$ and $N = n-2$ in (c) in Section 2) for which the isomorphism had previously not been established.

Our approach is related to the ideas of a $(1+1)$-dimensional topological quantum field theory as follows. The general principle should be that invariants of a closed 2-dimensional manifold Σ should be obtained by integrating cohomology classes over certain moduli spaces $\mathcal{M}_\Sigma$ associated to the surface Σ. In our case this moduli space is the space of J-holomorphic maps $\Sigma \to M$ in a given homology class $A \in H_2(M)$. The relevant cohomology classes on $\mathcal{M}_\Sigma$ are pullbacks e^*a of cohomology classes $a \in H^*(M)$ under evaluation maps $e : \Sigma \to M$. Evaluating a top dimensional product of such classes on the fundamental class of $\mathcal{M}_\Sigma$ gives the **Gromov-Witten** invariants

$$\Phi_{\Sigma,A}(a_1, \ldots, a_p) = \langle {e_1}^* a_1 \wedge \cdots \wedge {e_p}^* a_p, [\mathcal{M}_\Sigma] \rangle.$$

This was made rigorous by Ruan–Tian in [24]. For the case $\Sigma = S^2$ another exposition can be found in McDuff–Salamon [20]. In particular, in the case $\Sigma = S^2$ and $p = 3$ these invariants can be interpreted as homomorphisms

$$H^*(M) \otimes H^*(M) \otimes H^*(M) \to \mathbb{C} : (a,b,c) \mapsto \sum_A \Phi_A(a,b,c) e^{-t\omega(A)}$$

and these determine Witten's deformed **quantum cup product** strucure on $H^*(M)$. The ordinary cup-product corresponds to the case $t = 0$.

A relative version of these invariants can intuitively be described along the following lines. Associated to a 1-dimensional compact manifold Γ is the space $\mathcal{N}_\Gamma$ of smooth maps $\Gamma \to M$. Now if Σ is a Riemann surface with boundary $\partial\Sigma = \Gamma$ then there is an obvious restriction map $\rho : \mathcal{M}_\Sigma \to \mathcal{N}_{\partial\Sigma}$ and hence there should be an induced map

$$\rho_* : H_*(\mathcal{M}_\Sigma) \to H_*(\mathcal{N}_{\partial\Sigma}).$$

The image of the fundamental class of $\mathcal{M}_\Sigma$ is just the set of maps $\partial\Sigma \to M$ which extend to J-holomorphic curves $\Sigma \to M$. This is a kind of nonlinear

Hardy space and is roughly a *middle dimensional* submanifold of $\mathcal{N}_{\partial\Sigma}$. The relevant homology theory for such objects is Floer homology. Thus if the boundary

$$\partial\Sigma = \Gamma_1 \cup \ldots \cup \Gamma_\ell$$

has ℓ components then we should obtain a homology class

$$\Psi_\Sigma = \rho_*([\mathcal{M}_\Sigma]) \in HF_*(\Gamma_1) \otimes \cdots \otimes HF_*(\Gamma_\ell)$$

The rigorous definition of these relative invariants will be given in Section 3. Intuitively, the Floer homology class Ψ_Σ should be obtained by *pushing down* the cycle $\rho(\mathcal{M}_\Sigma) \subset \mathcal{N}_{\partial\Sigma}$ using the gradient flow of the (unperturbed) action functional. But the gradient lines of the action functional (in each component of $\partial\Sigma$) are holomorphic cylinders and the critical points are constant loops. Hence the part of $\rho(\mathcal{M}_\Sigma)$ which under the gradient flow *gets stuck* on the critical manifolds should correspond exactly to those J-holomorphic maps $u : \Sigma \to M$ which extend over ℓ discs to give J-holomorphic maps $\tilde{u} : \widetilde{\Sigma} \to M$ on the corresponding closed Riemann surface. Thus the class Ψ_Σ should be determined by J-holomorphic maps on closed surfaces with ℓ marked points, which is precisely how the Gromov-Witten invariants are defined. In particular, when $\ell = p = 3$ and Σ has genus $g = 0$, then the homology class Ψ_Σ determines, via Poincaré duality, the *pair-of-pants* product on Floer cohomology $HF^*(M) = H^*(M)$. For the above reasons this product should agree with the quantum deformation of the cup-product defined in terms of the Gromov-Witten invariants. The deformation parameter t in Floer homology arises from the Novikov ring.

The difficulty with this approach is that the Floer homology groups have not been defined in a rigorous way with the unperturbed symplectic action functional but require a Hamiltonian perturbation. Thus there are two theories, one of which revolves around J-holomorphic curves, the Gromov-Witten invariants, and quantum cohomology, and can be interpreted as Morse thory for the unperturbed symplectic action, while the other revolves around Floer homology, Hamiltonian differential equations, and the pair-of-pants product, and can be interpreted as Morse thory for the perturbed symplectic action. The purpose of our paper is to show that both approaches are isomorphic and give the same invariants.

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2 Symplectic Floer homology

Throughout we shall assume that our symplectic manifold (M, ω) is compact and **semi-positive**. This means that it satisfies one of the following three conditions

(a) $\langle[\omega], A\rangle = \lambda\langle c_1, A\rangle$ for every $A \in \pi_2(M)$ where $\lambda \geq 0$ (M is **monotone**).

(b) $\langle c_1, A\rangle = 0$ for every $A \in \pi_2(M)$.

(c) The **minimal Chern number** $N \geq 0$ defined by $\langle c_1, \pi_2(M)\rangle = N\mathbb{Z}$ is greater than or equal to $n - 2$.

Here $c_1 = c_1(TM, J)$ is the first Chern class of the tangent bundle TM with an almost complex structure J which is compatible with ω in the sense that

$$g_J(v, w) = \omega(v, Jw)$$

defines a Riemannian metric on M. The space of such structures will be denoted by $\mathcal{J}(M, \omega)$. These assumptions guarantee that for a generic almost complex structure $J \in \mathcal{J}(M, \omega)$ there is no J-holomorphic curve with negative Chern number. At present this condition is required for the definition of both Floer homology and quantum cohomology.

We shall begin this section by discussing the relevant Novikov rings and then recalling the definition of Floer homology as given in [16] by the second author in collaboration with Hofer. In the Section 3 we shall discuss relative Donaldson-type invariants in symplectic Floer homology.

Novikov rings

Throughout we shall identify $S^1 = \mathbb{R}/\mathbb{Z}$ and denote by $\mathcal{L}$ the space of contractible loops $x : S^1 \to M$. For every loop $x \in \mathcal{L}$ there exists a smooth map $v : B \to M$ defined on the unit disc $B = \{z \in \mathbb{C} \,|\, |z| \le 1\}$ which satisfies $v(e^{2\pi i t}) = x(t)$. Two such maps v_0 and v_1 are called **equivalent** if their sum $v_0 \# (-v_1)$ is a torsion class in $H_2(M, \mathbb{Z})$. We shall use the notation $[x, u_0] \equiv [x, u_1]$ for equivalent pairs and denote by $\widetilde{\mathcal{L}}$ the space of equivalence classes. The elements of $\widetilde{\mathcal{L}}$ will be denoted by $\tilde{x}$. The space $\widetilde{\mathcal{L}}$ is the unique covering space of $\mathcal{L}$ whose group of deck transformations is the image $\Gamma \subset H_2(M)$ of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$. Here $H_2(M)$ denotes integral homology modulo torsion. We denote by $\Gamma \times \widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}} : (A, \tilde{x}) \mapsto A \# \tilde{x}$ the obvious action of Γ on $\widetilde{\mathcal{L}}$.

Now consider the homomorphism $\omega : \Gamma \to \mathbb{R}$ defined by integrating the form ω over the class $A \in \Gamma$. Associated to this homomorphism is the Novikov ring $\Lambda = \Lambda_\omega$ whose elements are formal sums

$$\lambda = \sum_{A \in \Gamma} \lambda_A e^{2\pi i A}$$

with rational coefficients $\lambda_A \in \mathbb{Q}$ which satisfy the finiteness condition

$$\# \{A \in \Gamma \,|\, \lambda_A \ne 0,\, \omega(A) \le c\} < \infty$$

for every $c > 0$. The multiplication is given by

$$\lambda * \mu = \sum_{A,B} \lambda_A \mu_B e^{2\pi i (A+B)}.$$

This ring comes with a natural grading defined by

$$\deg(e^{2\pi i A}) = 2c_1(A)$$

and we shall denote by Λ_k the elements of degree k. Note in particular that Λ_0 is a subring and Λ_k will only be nonempty if k is an integer multiple of $2N$. Moreover, the multiplication maps $\Lambda_j \times \Lambda_k \to \Lambda_{j+k}$.

The **quantum cohomology groups** of a symplectic manifold are defined as the tensor product of ordinary cohomology with the Novikov ring. To be more precise we define

$$QH^k(M) = \bigoplus_j H^j(M) \otimes \Lambda_{k-j}$$

where $H^*(M)$ denotes the quotient of $H^*(M,\mathbb{Z})$ modulo torsion. Think of an element of $QH^k(M)$ as a formal sum of the form

$$a = \sum_{A\in\Gamma} a_A e^{2\pi i A}, \qquad a_A \in H^{k-2c_1(A)}(M,\mathbb{Q}),$$

with $\#\{A\in\Gamma\,|\,a_A\neq 0,\ \omega(A)\leq c\} < \infty$ for every c. The module structure over the Novikov ring Λ_ω is given by

$$\lambda * a = \sum_{A,B} \lambda_{A-B} a_B e^{2\pi i A}.$$

Similarly, consider the **quantum homology** $QH_*(M) = H_*(M)\otimes\Lambda$. It is convenient to write the elements of $QH_k(M)$ as a formal sums

$$\alpha = \sum_{A\in\Gamma} \alpha_A e^{2\pi i A}, \qquad \alpha_A \in H_{k+2c_1(A)}(M,\mathbb{Q}),$$

which satisfy $\#\{A\in\Gamma\,|\,\alpha_A\neq 0,\ \omega(A)\leq c\} < \infty$ for every c. The module structure over the Novikov ring is given by the same formula as above with a replaced by α. There is a natural pairing $QH_k(M)\times QH^k(M)\to\Lambda_0$ given by

$$\langle a,\alpha\rangle = \sum_{c_1(A)=0}\sum_B \langle a_{A-B},\alpha_B\rangle e^{2\pi i A}$$

for $a\in QH_k(M,\mathbb{Q})$ and $\alpha\in QH_k(M,\mathbb{Q})$. The Poincaré duality isomorphism $\mathrm{PD}: QH^k(M)\to QH_{2n-k}(M)$ is given by

$$\mathrm{PD}(a) = \sum_A \mathrm{PD}(a_A)e^{2\pi i A}.$$

Floer homology

Let $H_t = H_{t+1}: M\to\mathbb{R}$ be a smooth family of Hamiltonian functions and consider the time-dependent Hamiltonian differential equation

$$\dot{x}(t) = X_t(x(t)), \qquad \iota(X_t)\omega = dH_t. \tag{1}$$

Denote by $\mathcal{P}(H)$ the set of all contractible 1-periodic solutions $x(t)=x(t+1)$ of (1) and assume that these are all nondegenerate. Similarly, let $\widetilde{\mathcal{P}}(H)\subset\widetilde{\mathcal{L}}$ denote the set of those pairs $[x,u]\in\widetilde{\mathcal{L}}$ with $x\in\mathcal{P}(H)$. This can be interpreted as the set of critical points of the perturbed symplectic **action functional** $\mathcal{A}_H:\widetilde{\mathcal{L}}\to\mathbb{R}$ defined by

$$\mathcal{A}_H(\tilde{x}) = -\int_D u^*\omega - \int_0^1 H(t,x(t))\,dt$$

for $\tilde{x} = [x, u]$. Note that

$$\mathcal{A}_H(A\#\tilde{x}) = \mathcal{A}_H(\tilde{x}) - \omega(A)$$

for every $\tilde{x} \in \widetilde{\mathcal{L}}$ and every $A \in \Gamma$. The gradient flow lines of $\mathcal{A}_H$ with respect to the L^2-inner product induced by g_J are solutions $u : \mathbb{R}^2 \to M$ of the PDE

$$\partial_s u + J(u)\partial_t u - \nabla H_t(u) = 0 \tag{2}$$

which satisfy $u(s,t) = u(s,t+1)$ and the limit condition

$$\lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t) \tag{3}$$

with $x^{\pm} \in \mathcal{P}(H)$. Given $\tilde{x}^{\pm} \in \widetilde{\mathcal{P}}(H)$ we denote by $\mathcal{M}(\tilde{x}^-, \tilde{x}^+, H, J)$ the set of those solutions u of (2) and (3) for which $\tilde{x}^+\#u \equiv \tilde{x}^+$. The energy of such solutions is given by

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left(|\partial_s u|^2 + |\partial_t - X_t(u)|^2\right) \, dtds = \mathcal{A}_H(\tilde{x}^-) - \mathcal{A}_H(\tilde{x}^+).$$

The following result was proved in [28] based on a transversality theorem in [13].

Proposition 2.1 (Transversality) *For a generic pair $(H, J) \in \mathcal{HJ}_{\mathrm{reg}} = \mathcal{HJ}_{\mathrm{reg}}(M, \omega)$ the spaces $\mathcal{M}(\tilde{x}^-, \tilde{x}^+, H, J)$ are all finite dimensional manifolds of dimensions*

$$\dim \mathcal{M}(\tilde{x}^-, \tilde{x}^+, H, J) = \mu(\tilde{x}^-) - \mu(\tilde{x}^+)$$

Here the map $\mu : \widetilde{\mathcal{P}}(H) \to \mathbb{Z}$ is given by the Conley-Zehnder index. It satisfies

$$\mu(A\#\tilde{x}) = \mu(\tilde{x}) - 2c_1(A).$$

Moreover, $\mu([x, u])$ agrees with the Morse coindex (i.e. $2n$ minus the Morse index) whenever $H : M \to \mathbb{R}$ is a (time-independent) Morse function with sufficiently small second derivatives, x is a critical point of H, and $u(z) \equiv x$ is the constant disc.

Proposition 2.2 (Compactness) *For a generic pair $(H, J) \in \mathcal{HJ}_{\mathrm{reg}} = \mathcal{HJ}_{\mathrm{reg}}(M, \omega)$ we have*

$$\sum_{\substack{\omega(A)\leq c \\ c_1(A)=0}} \# \left\{\mathcal{M}(\tilde{x}, A\#\tilde{y})/\mathbb{R}\right\} < \infty$$

for all $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{P}}(H)$ with $\mu(\tilde{y}) - \mu(\tilde{x}) = 1$ and every constant c.

To prove this, one has to show that the relevant moduli spaces are compact. This will be the case if no bubbling occurs. The key observation is that, for a generic almost complex structure J, the points lying on a J-holomorphic spheres of Chern number 0 form a set in M of codimension 4 and so, for a generic H, no such sphere will intersect an isolated connecting orbit. Thus, it follows from Gromov's compactness theorem that they cannot bubble off. Moreover, J-holomorphic spheres of negative Chern number do not exist by weak monotonicity. J-holomorphic spheres of Chern number at least 1 connot bubble off because otherwise in the limit there would be a connecting orbit with negative index difference but such orbits do not exist generically. This is the essence of the proof of Proposition 2.2. Details are carried out in [16].

Whenever $\mu(\tilde{x}) - \mu(\tilde{y}) = 1$ we denote

$$n(\tilde{x}, \tilde{y}) = \# \{\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}\},$$

where the connecting orbits are to be counted with appropriate signs determined by a system of coherent orientations of the moduli spaces of connecting orbits as in [12]. These numbers determine a cochain complex as follows. Define $CF_k = CF_k(H)$ as the set of formal sums

$$\xi = \sum_{\substack{\tilde{x} \in \widetilde{\mathcal{P}}(H) \\ \mu(\tilde{x}) = k}} \xi_{\tilde{x}} \langle \tilde{x} \rangle$$

with rational coefficients $\xi_{\tilde{x}} \in \mathbb{Q}$ which satisfy the finiteness condition

$$\left\{ \tilde{x} \in \widetilde{\mathcal{P}}(H) \,|\, \xi_{\tilde{x}} \neq 0,\, \mathcal{A}_H(\tilde{x}) \geq c \right\} < \infty$$

for every c. This complex CF_* is a module over the Novikov ring $\Lambda = \Lambda_\omega$ via

$$\lambda * \xi = \sum_{\tilde{x}} \sum_{A} \lambda_A \xi_{(-A)\# \tilde{x}} \langle \tilde{x} \rangle.$$

Note that the dimension of CF_k over Λ_ω is precisely the number of 1-periodic solutions of (1) with Conley-Zehnder index $\mu(x) = k (\mathrm{mod}\ 2N)$.

The above numbers $n(\tilde{x}, \tilde{y})$ determine a boundary map $\partial_k : CF_k(H) \to CF_{k-1}(H)$ defined by

$$\partial \langle \tilde{x} \rangle = \sum_{\mu(\tilde{y}) = k-1} n(\tilde{x}, \tilde{y}) \langle \tilde{y} \rangle$$

for $\tilde{x} \in \widetilde{\mathcal{P}}(H)$ with $\mu(\tilde{x}) = k$. Proposition 2.2 guarantees the finiteness condition required for $\partial\langle \tilde{x} \rangle \in CF_{k-1}(H)$. Floer's proof that the square of this operator is zero carries over to the semi-positive case. Here the key observation is that 1-parameter families of connecting orbits with index difference 2 will still avoid the J-holomorphic spheres of Chern number 0 because they form a 3-dimensional set in M while these J-holomorphic spheres form a set of codimension 4. Similarly, holomorphic spheres of Chern number 1 can only bubble off if they intersect a periodic solution, and this does not happen for a generic H because the points on these spheres form a set in M of codimension 2 while the periodic orbits form 1 dimensional sets. For J-holomorphic spheres with Chern number at least 2 the same argument as above applies. Hence no bubbling occurs for connecting orbits with index difference 2 and hence such orbits can only degenerate by splitting into a pair of orbits each with index difference 1. As in the standard theory (cf. [11], [19], [28]) this shows that

$$\partial \circ \partial = 0.$$

Hence the solutions of (2) determine a chain complex (CF_*, ∂) and its homology groups

$$HF_*(H, J) = \frac{\ker \partial}{\operatorname{im} \partial}$$

are called the **Floer homology groups** of the pair (H, J). Because the coboundary map is linear over Λ_ω it follows that the Floer cohomology groups form a module over Λ_ω. In [16] it is proved that the Floer cohomology groups are independent of the almost complex structure J and the Hamiltonian H used to define them. This is stated more precisely in the next theorem.

Theorem 2.3 *Given regular pairs $(H^\alpha, J^\alpha), (H^\beta, J^\beta) \in \mathcal{HJ}_{\mathrm{reg}}$ there exists a natural isomorphism*

$$\Phi^{\beta\alpha} : HF_*(M, \omega, H^\alpha, J^\alpha) \to HF_*(M, \omega, H^\beta, J^\beta)$$

which preserves the grading by the Conley-Zehnder index. If $(H^\gamma, J^\gamma) \in \mathcal{HJ}_{\mathrm{reg}}$ is another such pair then

$$\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha} = \Phi^{\gamma\alpha}, \qquad \Phi^{\alpha\alpha} = \mathrm{id}.$$

These isomorphisms $\Phi^{\beta\alpha}$ are linear over Λ_ω.

Remark 2.4 (Poincaré duality) Consider the Floer cochain complex

$$CF^k(H) \simeq \mathrm{Hom}(CF_k(H), \Lambda_0) \simeq CF_{2n-k}(\bar{H})$$

with the the Hamiltonian $\bar{H}_t = -H_{-t}$. First define $CF^k(H)$ as the space of formal sums

$$\eta = \sum_{\mu(\tilde{x})=k} \eta_{\tilde{x}} \, \langle \tilde{x} \rangle$$

which satisfy the opposite finiteness condition

$$\left\{ \tilde{x} \in \widetilde{\mathcal{P}}(H) \,|\, \eta_{\tilde{x}} \neq 0, \, \mathcal{A}_H(\tilde{x}) \leq c \right\} < \infty$$

for every c. The action of the Novikov ring on this group is given by

$$\lambda * \eta = \sum_{\tilde{x}} \sum_A \lambda_A \eta_{A\#\tilde{x}} \, \langle \tilde{x} \rangle.$$

for $\lambda \in \Lambda_\omega$ and $\eta \in CF^k(H)$. There is a pairing $CF^k \times CF_k \to \Lambda_0$ defined by

$$\langle \eta, \xi \rangle = \sum_A \left(\sum_{\tilde{x}} \eta_{\tilde{x}} \xi_{A\#\tilde{x}} \right) e^{2\pi i A}$$

where the sum is over all $A \in \Gamma$ with $c_1(A) = 0$. This determines the isomorphism $CF^k(H) \cong \mathrm{Hom}(CF_k(H), \Lambda_0)$. Secondly, note that there is a one-to-one correspondence of periodic solutions $x \in \mathcal{P}(H)$ with $\bar{x} \in \mathcal{P}(\bar{H})$ via $\bar{x}(t) = x(-t)$. In the universal cover the element $[x, v] \in \widetilde{\mathcal{P}}(H)$ corresponds to $[\bar{x}, \bar{v}] \in \widetilde{\mathcal{P}}(\bar{H})$ where $\bar{v}(z) = v(\bar{z})$ and

$$\mathcal{A}_{\bar{H}}([\bar{x}, \bar{v}]) = -\mathcal{A}_H([x, v]), \qquad \mu([\bar{x}, \bar{v}]) = 2n - \mu([x, v])$$

(with the index conventions of Proposition 2.1). This shows that $CF^k(H) = CF_{2n-k}(\bar{H})$. Since the solutions of (2) for the two Hamiltonians are related by $\bar{u}(s, t) = u(-s, -t)$ it follows that the two boundary operators are also the same. Hence there is a natural isomorphism $HF^k(H, J) \cong HF_{2n-k}(\bar{H}, J)$.

Now use the isomorphism of Theorem 2.3 to obtain a **Poincaré duality isomorphism**

$$PD_F : HF^k(H,J) \to HF_{2n-k}(H,J).$$

Now there is an obvious pairing $HF_{2n-k}(H,J) \times HF^{2n-k}(H,J) \to \Lambda_0$ in view of the identification $CF^* = \mathrm{Hom}(CF_*, \Lambda_0)$. Combining this with the Poincaré duality isomorphism we obtain a **Poincaré duality pairing**

$$HF^k(H,J) \times HF^{2n-k}(H,J) \to \Lambda_0$$

and similarly for homology.

Remark 2.5 (Products) Even though the product

$$M = M_1 \times \cdots \times M_\ell$$

of semi-positive symplectic manifolds is not, in general, semi-positive its Floer homology groups are well-defined for every product almost complex structure

$$J = J_1 \times \cdots \times J_\ell$$

(compatible with the product symplectic form) and arbitrary Hamiltonian functions $H_t : M \to \mathbb{R}$. To see this one only needs to examine the proof of the compactness result Proposition 2.2 and observe that the J-holomorphic curves are all products of J_i-holomorphic curves in M_i and so cannot have negative Chern number for generic J_i's.

It is interesting to discuss the case where $M_1 = \cdots M_\ell = M$ and H is a sum of Hamiltonian functions $H_i = H_i(t,x)$ on M in more detail. The tensor product over the Novikov ring Λ_ω of the Floer chain complexes $CF_*(H_i)$ can be described as follows. Denote by $\widetilde{\mathcal{P}}(H_1, \ldots, H_\ell)$ the set of all equivalence classes

$$\tilde{x} = [\tilde{x}_1, \ldots, \tilde{x}_\ell]$$

where $\tilde{x}_i \in \widetilde{\mathcal{P}}(H_i)$ and the equivalence relation is given by

$$[\tilde{x}_1, \ldots, \tilde{x}_\ell] \sim [A_1\#\tilde{x}_1, \ldots, A_\ell\#\tilde{x}_\ell]$$

whenever $A_i \in \Gamma$ and $A_1 + \cdots + A_\ell$ is a torsion class. Then Γ acts on $\widetilde{\mathcal{P}}(H_1, \ldots, H_\ell)$ by

$$A\#[\tilde{x}_1, \ldots, \tilde{x}_\ell] = [A\#\tilde{x}_1, \ldots, \tilde{x}_\ell].$$

We abbreviate $H = (H_1, \ldots, H_\ell)$ and

$$\mu(\tilde{x}) = \sum_{i=1}^{\ell} \mu(\tilde{x}_i), \qquad \mathcal{A}_H(\tilde{x}) = \sum_{i=1}^{\ell} \mathcal{A}_{H_i}(\tilde{x}_i)$$

for $\tilde{x} = [\tilde{x}_1, \ldots \tilde{x}_\ell] \in \widetilde{\mathcal{P}}(H_1, \ldots, H_\ell)$. With this notation the tensor product $CF_*(H_1, \ldots, H_\ell) = CF_*(H_1) \otimes \cdots \otimes CF_*(H_\ell)$ can be defined exactly as in the case $\ell = 1$. Namely, $CF_k(H_1, \ldots, H_\ell)$ is the set of formal sums

$$\xi = \sum_{\substack{\tilde{x} \in \widetilde{\mathcal{P}}(H_1, \ldots, H_\ell) \\ \mu(\tilde{x}) = k}} \xi_{\tilde{x}} \langle \tilde{x} \rangle$$

with rational coefficients $\xi_{\tilde{x}} \in \mathbb{Q}$ which satisfy

$$\left\{\tilde{x} \in \widetilde{P}(H_1, \ldots, H_\ell) \,|\, \xi_{\tilde{x}} \neq 0, \, \mathcal{A}_H(\tilde{x}) \geq c\right\} < \infty$$

for all c. Also the boundary operator is given by the same formula. More precisely, let $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{P}}(H_1, \ldots, H_\ell)$ with $\mu(\tilde{x}) - \mu(\tilde{y}) = 1$. Then the corresponding entry of the boundary operator can only be nonzero if there exist representatives $\tilde{x} = [\tilde{x}_1, \ldots, \tilde{x}_\ell]$ and $\tilde{y} = [\tilde{y}_1, \ldots, \tilde{y}_\ell]$ of these equivalence classes such that $\mu(\tilde{x}_j) - \mu(\tilde{y}_j) = 1$ for some j and $\tilde{x}_i = \tilde{y}_i$ for all $i \neq j$. In this case the entry of the boundary map is given by

$$n(\tilde{x}, \tilde{y}) = n(\tilde{x}_j, \tilde{y}_j).$$

Note that the right hand side is independent of the choice of the representatives of $\tilde{x}$ and $\tilde{y}$ with the above properties because $n(A\#\tilde{x}_j, A\#\tilde{y}_j) = n(\tilde{x}_j, \tilde{y}_j)$ for all $A \in \Gamma$. The resulting Floer homology groups are

$$HF_*(H_1, \ldots, H_\ell, J) = HF_*(H_1, J) \otimes \cdots \otimes HF^*(H_\ell, J)$$

where the right hand side is the graded tensor product over the Novikov ring Λ_0. At this place the choice of rational coefficients is essential. In the case of integer coefficients the Floer homology of the product is given by the Künneth formula. The corresponding Floer cohomology groups are

$$HF_*(H_1, \ldots, H_\ell, J) = HF_*(H_1, J) \otimes \cdots \otimes HF^*(H_\ell, J).$$

As before they are generated by the cochain complex $CF^k(H_1, \ldots, H_\ell) = \mathrm{Hom}(CF_k(H_1, \ldots, H_\ell), \Lambda_0)$.

3 Relative Donaldson type invariants

We shall now consider J-holomorphic curves $u : \Sigma \to M$ defined on a Riemann surface Σ of genus g with ℓ cylindrical ends $Z_i = \phi_i((0, \infty) \times S^1) \subset \Sigma$. We shall fix an almost complex structure j on Σ such that $\phi_i^* j$ agrees with the standard structure on the cylinders. We shall also fix ℓ time dependent Hamiltonian functions $H_i = H_i(s, t, x) = H_i(s, t+1, x)$ which vanish near $s = 0$ and are independent of the s-variable for $s \geq 1$. Assume that the periodic solutions of the Hamiltonian differential equation $\dot{x} = X_i(1, t, x)$ with $\iota(X_i)\omega = dH_i$ are all nondegenerate and denote by $\widetilde{\mathcal{P}}(H_i) \subset \widetilde{\mathcal{L}}$ the lift of the set of such periodic solutions. Given $\tilde{x}_i = [x_i, v_i] \in \widetilde{\mathcal{P}}(H_i)$ we shall consider the space

$$\mathcal{M}_\Sigma(\tilde{x}_1, \ldots, \tilde{x}_\ell) = \mathcal{M}_\Sigma(\tilde{x}_1, \ldots, \tilde{x}_\ell, H_1, \ldots, H_\ell, J)$$

of all smooth maps $u : \Sigma \to M$ which satisfy the following conditions.

(a) u is J-holomorphic on the complement

$$\Sigma_0 = \Sigma - \bigcup_i Z_i.$$

(b) The maps $u_i = u \circ \phi_i$ satisfy

$$\partial_s u_i + J(u)\partial_t u_i - \nabla H_i(s,t,u_i) = 0,$$

$$x_i(t) = \lim_{s\to\infty} u_i(s,t).$$

(c) The map u capped off by the discs v_i (with opposite orientations) represents a torsion homology class in $H_2(M,\mathbb{Z})$.

Note that the condition (c) and hence the space $\mathcal{M}_\Sigma(\tilde{x}_1,\ldots,\tilde{x}_\ell)$ depends only on the equivalence class of the ℓ-tuple $[\tilde{x}_1,\ldots,\tilde{x}_\ell]$ as defined in Remark 2.5. The space $\mathcal{M}_\Sigma(\tilde{x}_1,\ldots,\tilde{x}_\ell)$ is a finite dimensional manifold for a generic choice of Hamiltonian functions H_i. Under the index conventions of Proposition 2.1 its dimension is given by

$$\dim \mathcal{M}_\Sigma(\tilde{x}_1,\ldots,\tilde{x}_\ell) = 2n(1-g) - \sum_{i=1}^{\ell} \mu(\tilde{x}_i).$$

The details are carried out in [31].

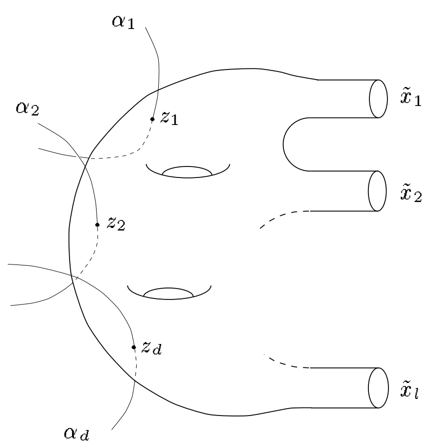

Figure 1: J-holomorphic curves with marked points

Now fix d distinct points $z_1,\ldots,z_d \in \Sigma$ and homology classes $\alpha_1,\ldots,\alpha_d \in H_*(M)$ such that

$$\sum_{i=1}^{\ell} \mu(\tilde{x}_i) = 2n(1-g) - \sum_{\nu=1}^{d} (2n - \deg(\alpha_\nu)). \tag{4}$$

Represent these classes by generic cycles (still denoted by α_ν) and define

$$\mathcal{M}_\Sigma(\alpha_1,\ldots,\alpha_d,\tilde{x}_1,\ldots,\tilde{x}_\ell)$$

to be the set of all curves $u \in \mathcal{M}_\Sigma(\tilde{x}_1, \dots, \tilde{x}_\ell)$ with $u(z_\nu) \in \alpha_\nu$ (see Figure 1). This is a finite set (for generic choices) and we denote

$$n_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_\ell) = \#\mathcal{M}_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_\ell)$$

where the points are counted with appropriate signs. Here we suppress the dependence on J and H_i in the notation. Now the numbers will in general depend on the choices. However they define a Floer homology class which is independent of the choices. From this point of view Floer homology can in fact be interpreted as a framework to extract the invariant information from these moduli spaces. More precisely we define the cycle

$$\psi_\Sigma(\alpha_1, \dots, \alpha_d) = \sum_{\tilde{x}_i} n_\Sigma(\alpha_1, \dots, \alpha_d, \tilde{x}_1, \dots, \tilde{x}_\ell)\langle \tilde{x}_1, \dots, \tilde{x}_\ell \rangle$$

in $CF_*(H_1, \dots, H_\ell) = CF_*(H_1) \otimes \cdots \otimes CF_*(H_\ell)$. Here the sum runs over all equivalence classes of ℓ-tuples $[\tilde{x}_1, \dots, \tilde{x}_\ell]$ (as in Remark 2.5 above) which satisfy the dimension condition (4). The following theorem shows that these cycles determine a Floer homology class which is independent of the choices. In summary, the marked surface Σ determines a multi-linear map

$$\Psi_\Sigma : H_*(M) \otimes \cdots \otimes H_*(M) \to HF_*(H_1) \otimes \cdots \otimes HF_*(H_\ell).$$

This can be interpreted as a symplectic version of a **relative Donaldson invariants** which in their original context are defined for 4-manifolds X with boundary and take values in the Floer homology groups of ∂X. In the symplectic case the relative Donaldson invariants of Σ take values in the Floer homology of $\partial\Sigma$.

Theorem 3.1 **(i)** *The above chain $\psi_\Sigma(\alpha_1, \dots, \alpha_d)$ is a Floer homology cycle. The corresponding Floer homology class is denoted by*

$$\Psi_\Sigma(\alpha_1, \dots, \alpha_d) \in HF_*(H_1) \otimes \cdots \otimes HF_*(H_\ell).$$

It has degree $2n(1-g) - \sum_{\nu=1}^d (2n - \deg(\alpha_\nu))$.

(ii) *If one of the cycles α_ν is a boundary then $\psi_\Sigma(\alpha_1, \dots, \alpha_d)$ is a boundary in the Floer chain complex. Moreover, the Floer homology class $\Psi_\Sigma(\alpha_1, \dots, \alpha_d)$ is independent of the choice of the marked points z_ν used to define it.*

(iii) *The Floer homology class $\Psi_\Sigma(\alpha_1, \dots, \alpha_d)$ is natural under variation of the Hamiltonian functions and the almost complex structure. This means that for two choices $(J^\beta, H_1^\beta, \dots, H_\ell^\beta)$ and $(J^\gamma, H_1^\gamma, \dots, H_\ell^\gamma)$ the corresponding classes are related by*

$$\Psi_\Sigma^\gamma(\alpha_1, \dots, \alpha_d) = \Phi^{\gamma\beta}\Psi_\Sigma^\beta(\alpha_1, \dots, \alpha_d)$$

where $\Phi^{\gamma\beta} : HF_*(H^\beta, J^\beta) \to HF_*(H^\gamma, J^\gamma)$ is the isomorphism of Theorem 2.3.

Proof: The theorem follows from the standard gluing and compactness arguments in Floer homology. The first statement is proved by considering the ends of the one-dimensional moduli spaces where at one of the ends $\tilde{x}_i \in \widetilde{\mathcal{P}}(H_i)$ is replaced by $\tilde{y}_i \in \widetilde{\mathcal{P}}(H_i)$ with $\mu(\tilde{y}_i) = \mu(\tilde{x}_i) - 1$. The second statement follows also by considering one-dimensional moduli spaces, but in this case the dimension is increased because the cycle $\alpha = \alpha_1 \times \cdots \times \alpha_d$ is replaced by a chain β in M^d with

$$\partial\beta = \alpha.$$

More precisely, choose an equivalence class $\tilde{x} = [\tilde{x}_1, \dots, \tilde{x}_n]$ as in Remark 2.5 with $\tilde{x}_i \in \widetilde{\mathcal{P}}(H_i)$ and

$$\mu(\tilde{x}) = 2n(1-g) - 2nd - \deg(\alpha).$$

Consider the 1-dimensional moduli space $\mathcal{M}_\Sigma(\beta, \tilde{x})$ with boundary

$$\partial\mathcal{M}_\Sigma(\beta, \tilde{x}) = \mathcal{M}_\Sigma(\partial\beta, \tilde{x}).$$

This moduli space will not be compact, in general, but solutions could *break up* into pairs $u \in \mathcal{M}_\Sigma(\beta, \tilde{y})$, $v \in \mathcal{M}_\Sigma(\tilde{y}, \tilde{x})$ with $\mu(\tilde{y}) = \mu(\tilde{x}) + 1$. The dimension formula shows that dim $\mathcal{M}_\Sigma(\beta, \tilde{y}) = 0$ and hence there are finitely many such pairs. The total number of boundary points and such ends is even, or zero when counted with appropriate signs. Hence

$$n_\Sigma(\alpha, \tilde{x}) = \sum_{\tilde{y}} n_\Sigma(\beta, \tilde{y}) n(\tilde{y}, \tilde{x})$$

where the sum runs over all $\tilde{y}$ with $\mu(\tilde{y}) = 2n(1-g) - 2nd - \deg(\alpha) + 1$. This can be abbreviated in the form

$$\psi_\Sigma(\partial\beta) = \partial\psi_\Sigma(\beta)$$

where the boundary operator an the right is the one in the Floer chain complex. This shows that the Floer homology class $\Psi_\Sigma(\alpha)$ depends only on the homology class of α and not on the cycle by which it is represented. This proves (ii). Statement (iii) is proved by an obvious gluing argument. The details will be carried out elsewhere. □

Remark 3.2 This homomorphism extends naturally, as a multi-linear map over the Novikov ring, to the quantum homology of M. This can be explicitly expressed as follows. The degree-k-part of the tensor product $QH_*(M)^{\otimes d}$ over Λ can be written as the space of formal sums

$$\alpha = \sum_A \alpha_A e^{2\pi i A}, \qquad \alpha_A \in H_*(M, \mathbb{Q})^{\otimes d}, \qquad \deg(\alpha_A) = k + 2c_1(A)$$

with $\#\{A \in \Gamma \,|\, \alpha_A \neq 0,\, \omega(A) \leq c\} < \infty$ for all c. Now if we abbreviate $\tilde{x} = [\tilde{x}_1, \dots, \tilde{x}_\ell]$ as in Remark 2.5 then we can write

$$\psi(\alpha) = \sum_{A,\tilde{x}} n_\Sigma(\alpha_A, (-A)\#\tilde{x})\langle\tilde{x}\rangle$$

where the sum runs over all $A \in \Gamma$ and $\tilde{x} \in \widetilde{\mathcal{P}}(H_1, \dots, H_\ell)$ with $\mu(\tilde{x}) = k = \deg(\alpha)$. Moreover, the numbers $n_\Sigma(\alpha_A, (-A)\#\tilde{x})$ are now rational and are obtained from the integer versions in the obvious way (some integer multiple of α_A lifts to an integral class). It is easy to see that this map ψ is linear over the Novikov ring and extends the map of Theorem 3.1. This gives rise to relative Donaldson type invariants of the form

$$\Psi_\Sigma : QH_*(M)^{\otimes d} \to HF_*(H_1) \otimes \cdots \otimes HF_*(H_\ell).$$

Example 3.3 The first case is where $\Sigma = \mathbb{C}$ is the complex plane or, in other words, a disc with a cylindrical end. In this case the space

$$\mathcal{M}(\tilde{x}) = \mathcal{M}(\tilde{x}, H, J)$$

consists of all perturbed J-holomorphic curves $u : \mathbb{C} \to M$ such that the map $(s,t) \mapsto u(e^{2\pi(s+it)})$ satisfies the perturbed equation for $s \geq 0$ with limit $\tilde{x} \in \widetilde{\mathcal{P}}(H)$. This space is a manifold of dimension $2n - \mu(\tilde{x})$. For a generic cycle α representing a homology class of degree $\deg(\alpha) = \mu(\tilde{x})$ we denote by $n(\alpha, \tilde{x})$ the number of curves $u \in \mathcal{M}(\tilde{x})$ with $u(0) \in \alpha$, counted with appropriate signs (see Figure 2). These numbers determine a homomorphism $\phi : QH_k(M) \to CF_k(H)$ defined by

$$\phi(\alpha) = \sum_{A,\tilde{x}} n(\alpha_A, (-A)\#\tilde{x})\langle\tilde{x}\rangle$$

for $\alpha = \sum_A \alpha_A e^{2\pi i A} \in QH_k(M)$ where the sum runs over all $A \in \Gamma$ and all $\tilde{x} \in \widetilde{\mathcal{P}}(H)$ with $\mu(\tilde{x}) = k$. This chain $\phi(\alpha) \in CF_k(H)$ is a Floer homology cycle and, while the chain $\phi(\alpha)$ itself will depend on the cycles representing the classes α_A, the resulting Floer homology class $\Phi(\alpha) \in HF_*(M)$ is independent of these choices and so depends only on the homology class of α.

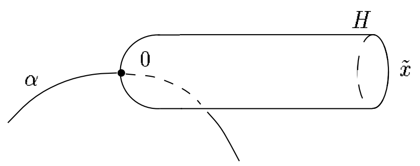

Figure 2: The homomorphism $QH_*(M) \to HF_*(M)$

This homomorphism $\Phi : QH_k(M) \to HF_k(H,J)$ is of course a special case of the map Ψ_Σ constructed in Theorem 3.1 and Remark 3.2. In particular, if (H^β, J^β) and (H^γ, J^γ) are two regular pairs and

$$\Phi^\beta : QH_*(M) \to HF_*(H^\beta, J^\beta), \qquad \Phi^\gamma : QH_*(M) \to HF_*(H^\gamma, J^\gamma)$$

denote the above homomorphism then, by Theorem 3.1, we have

$$\Phi^\gamma = \Phi^{\gamma\beta} \circ \Phi^\beta$$

where $\Phi^{\beta\alpha}$ is the isomorphism of Theorem 2.3. The dual homomorphism

$$\mathrm{PD}_F \circ \Phi \circ \mathrm{PD} : QH^*(M) \to HF^*(H,J)$$

of cohomologies will also be denoted by Φ. We shall prove in Section 4 that Φ is indeed an isomorphism

Example 3.4 (Cap product) Consider now the case where $\Sigma = Z = \mathbb{R} \times S^1$ is a cylinder and $d = 1$. Fix a homology class $\alpha \in H_*(M)$ of fixed degree. Then

$$\Psi_Z(\alpha) \in HF_*(H_1) \otimes HF_*(H_2)$$

is a class of degree $\deg(\alpha)$. Via Poincaré duality this map determines a homomorphism

$$HF_k(\bar{H}_1) = HF^{2n-k}(H_1) \to HF_{k-\operatorname{codim}\alpha}(H_2)$$

which is given by contracting a cohomology class in $HF^*(H_1)$ with the first factor of $\Psi_Z(\alpha) \in HF_*(H_1) \otimes HF_*(H_2)$. With $\bar{H}_1 = H_2 = H$ these maps can be interpreted as a kind of cap product with the cohomology class $a = \mathrm{PD}(\alpha)$ with Floer homology:

$$H^j(M) \otimes HF_k(H, J) \to HF_{k-j}(H, J) : (a, \xi) \mapsto a \cap_F \xi.$$

Geometrically, this product can be interpreted in terms of counting the connecting orbits which pass through the Poincaré dual $\alpha = \mathrm{PD}(a)$ (see Figure 3). If there are no J-holomorphic spheres, then one can prove that this pairing corresponds to the ordinary cap-product under the isomorphism of Example 3.3. This was exploited by Floer, and more recently by Ono and LeHong, to prove cup-length estimates for periodic solutions of Hamiltonian systems.

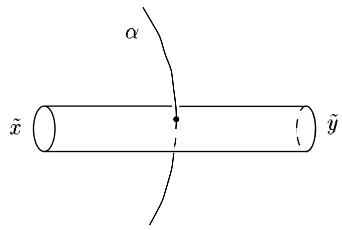

Figure 3: Cap-product on Floer homology

Example 3.5 (Natural isomorphisms) In the case where $\alpha = [M]$ is the fundamental class we obtain homomorphisms $HF_k(\bar{H}_1) \to HF_k(H_2)$ in Example 3.4. These are precisely the isomorphisms of Theorem 2.3.

Example 3.6 (Pair-of-pants product) Consider the case where Σ is a **pair of pants**, i.e. a surface of genus zero with three cylindrical ends where $H_1 = H_2 = \bar{H}_3 = H$ (see Figure 4). Then

$$\Psi_\Sigma([M]) \in HF_*(H) \otimes HF_*(H) \otimes HF_*(\bar{H})$$

is a class of degree $2n$. In view of Poincaré duality this class can be interpreted as a map

$$HF^j(H) \otimes HF^k(H) \to HF^{j+k}(H) : (\eta, \zeta) \mapsto \eta \cup_F \zeta$$

obtained by contracting the cohomology classes $\eta, \zeta \in HF^*(H)$ with the first two factors in $\Psi_\Sigma([M])$. This is the pair-of-pants product on Floer homology.

Now all other relative Donaldson invariants can be computed from these examples by the gluing formula below. At this stage it is perhaps more enlightning to consider cylindrical ends with two possible orientations (as a

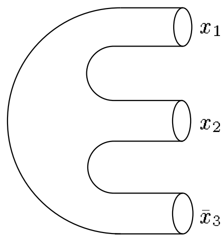

Figure 4: Cup-product on Floer homology

right or a left end) and write the relative Donldson invariants as an induced homomorphisms Ψ_Σ from the Floer homology of the left end to the Floer homology of the right end. The connected sum of two surfaces $\Sigma'' = \Sigma\#\Sigma'$ obtained from gluing the right ends of Σ to the left ends of Σ' then corresponds to the obvious composition (see Figure 5)

$$\Psi_{\Sigma\#\Sigma'} = \Psi_{\Sigma'} \circ \Psi_\Sigma.$$

Instead we shall phrase the result in terms of Poincaré duality. The proof will be carried out elsewhere.

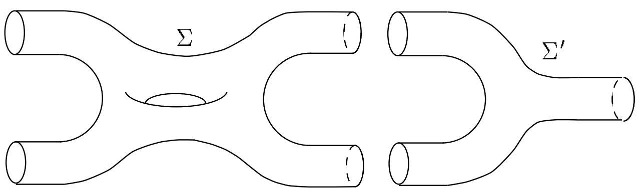

Figure 5: Composition

Theorem 3.7 *Let $\Sigma'' = \Sigma\#_p\Sigma'$ be the oriented connected sum of Σ_0 and Σ_1 over p of the boundary components. Then*

$$\Psi_{\Sigma''}(\alpha_1, \ldots, \alpha_{d+d'}) = \langle\Psi_\Sigma(\alpha_1, \ldots, \alpha_d), \Psi_{\Sigma'}(\alpha_{d+1}, \ldots, \alpha_{d+d'})\rangle_p$$

where the right hand side denotes the Poincaré duality pairing on $2p$ factors in Floer homology.

An interesting special case is where Σ'' is a closed surface. In this case the polynomials $\Psi_{\Sigma''}$ take values in $\mathbb{Z}$ and these are the Gromov-Witten invariants as defined by Ruan in [23]. Other references are [20] and [24].

4 Morse theory

Our goal in this section is to prove the following theorem. This result extends easily to integer coefficients.

Theorem 4.1 *The homomorphism $\Phi : QH_*(M) \to HF_*(H,J)$ defined in Example 3.3 is bijective.*

Proof: The key idea of the proof is to express the map Φ in an alternative way via Morse theory. Choose a generic Morse function $H_0 : M \to \mathbb{R}$ and consider the Morse complex $CM_*(H_0, \omega)$ defined as a module over the Novikov ring $\Lambda = \Lambda_\omega$ which is generated by the critical points of H_0. More explicitly, we think of the elements of $CM_k(H_0,\omega)$ as formal sums of the form

$$\xi = \sum_{x_0,A} \xi_{x_0,A}\langle x_0, A\rangle$$

with rational coefficients $\xi_{x_0,A}$. Here the sum is over all pairs $\langle x_0, A\rangle$ consisting of a critical point x_0 of H_0 and a homology classes $A \in \Gamma$ with

$$\mu(x_0, A) = \mathrm{ind}_{H_0}(x_0) - 2c_1(A) = k.$$

We impose the finiteness condition

$$\{\langle x_0, A\rangle \,|\, \xi_{x_0,A} \neq 0,\, \omega(A) \leq c\} < \infty.$$

The boundary operator on $CM_*(H_0)$ is defined by the gradient flow lines $\gamma : \mathbb{R} \to M$ of the gradient flow of H_0

$$\dot{\gamma} = -\nabla H_0(\gamma).$$

Denote by $n_0(x_0, y_0)$ the number of connecting orbits from x_0 to y_0 whenever $\mathrm{ind}_{H_0}(x_0) - \mathrm{ind}_{H_0}(y_0) = 1$ and define the boundary operator $\partial_M : CM_*(H_0) \to CM_{*-1}(H_0)$ by

$$\partial_M\langle x_0, A\rangle = \sum_{\mathrm{ind}_{H_0}(y_0)=k-1} n_0(x_0,y_0)\langle y_0, A\rangle$$

whenever $\mathrm{ind}_{H_0}(x_0) = k$. As an additive group the homology of this complex is naturally isomorphic to the quantum homology of M

$$HM_*(M,\omega) = QH_*(M) = H_*(M) \otimes \Lambda_\omega.$$

(See for example [26], [30], [34].)

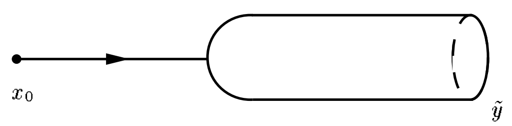

Figure 6: Unstable manifold and J-holomorphic disc

Now define a map $\phi : CM_*(H_0,\omega) \to CF_*(H)$ as follows. Given a critical point $x_0 \in \mathrm{Crit}(H_0)$ and a periodic orbit $\tilde{y} \in \widetilde{\mathcal{P}}(H)$ consider the space

$$\mathcal{M}(x_0, \tilde{y}) = \{u \in \mathcal{M}(\tilde{y}) \,|\, u(0) \in W^u(x_0)\}$$

where $\mathcal{M}(\tilde{y}) = \mathcal{M}(\tilde{y}, H, J)$ is defined as in Example 3.3. This space is a manifold of dimension

$$\dim \mathcal{M}(x_0, \tilde{y}) = \mathrm{ind}_{H_0}(x_0) - \mu(\tilde{y}).$$

In the case of index difference zero we obtain a finite set and denote the oriented number of elements by $n(x_0, \tilde{y}) = \#\mathcal{M}(x_0, \tilde{y})$ (see Figure 6). Now define

$$\phi \langle x_0, A \rangle = \sum_{\tilde{y}} n(x, (-A)\#\tilde{y})\langle \tilde{y} \rangle$$

where the sum is over all $\tilde{y} \in \widetilde{\mathcal{P}}(H)$ which satisfy $\mu(\tilde{y}) = \mathrm{ind}_{H_0}(x) - 2c_1(A)$. This map is linear over the Novikov ring. Moreover, it follows from the usual argument in Floer homology that ϕ intertwines the the boundary operators and so descends to a homomorphism of the homology groups

$$\Phi : HM_*(H_0, \omega) \to HF_*(H, J).$$

We first claim that this homomorphism agrees with the one of Example 3.3 under the above identification of $HM_*(H_0, \omega)$ with $QH_*(M)$. To see this just note that a homology class $\alpha \in H_*(M)$ can be represented by a chain $\xi = \sum_{x_0} \xi(\alpha, x_0)x_0$ in the Morse complex. Geometrically, this means that α can be represented by a cycle which is arbitarily close to the corresponding sum $W_\alpha = \sum_{x_0} \xi(\alpha, x_0)W^u(x_0)$ of unstable manifolds. For this cycle the Floer homology class $\phi(\xi)$ is precisely given by the intersection numbers of $\mathcal{M}(\tilde{y})$ with W_α. But this is the definition of the homomorphism in Example 3.3.

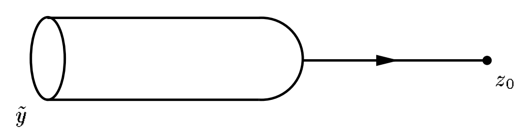

Figure 7: J-holomorphic disc and stable manifold

Now consider the inverse homomorphism

$$\Psi : HF_*(H, J) \to HM_*(H_0, \omega)$$

which is defined as follows. Denote

$$\mathcal{M}^-(\tilde{y}) = \mathcal{M}^-(\tilde{y}, H, J) = \mathcal{M}([\bar{y}, \bar{v}], \bar{H}, J)$$

where $\tilde{y} = [y, v]$, $\bar{y}(t) = y(-t)$ and $\bar{v}(z) = v(\bar{z})$ are defined as in Remark 2.4, and $\mathcal{M}(\tilde{y})$ is defined as in Example 3.3. Think of $\mathcal{M}^-(\tilde{y})$ as the space of perturbed J-holomorphic curves $u : \mathbb{C} \to M$ which have a cylindrical end to the left converging to $\tilde{y}$. Explicitly, the cylindrical end is given by the map $(s, t) \mapsto u(e^{-2\pi i(s+it)})$ for $s \le 0$. Note that

$$\dim \mathcal{M}^-(\tilde{y}) = \mu(\tilde{y}).$$

Now the space

$$\mathcal{M}^-(\tilde{y}, z_0) = \left\{ u \in \mathcal{M}^-(\tilde{y}) \,|\, u(0) \in W^s(z_0) \right\}$$

is a manifold of dimension

$$\dim \mathcal{M}^-(\tilde{y}, z_0) = \mu(\tilde{y}) - \mathrm{ind}_{H_0}(z_0).$$

Counting the number of elements in the case of index difference zero gives rise to integers $n(\tilde{y}, z_0) = \#\mathcal{M}^-(\tilde{y}, z_0)$ whenever $\mu(\tilde{y}) = \mathrm{ind}_{H_0}(z_0)$ (see Figure 7). These numbers determine a chain map $\psi : CF_*(H, J) \to CM_*(H_0, \omega)$ defined by

$$\psi(\tilde{y}) = \sum_{z_0, A} n((-A)\#\tilde{y}, z_0)\langle z_0, A\rangle.$$

Here the sum runs over all $(z_0, A) \in \mathrm{Crit}(H_0) \times \Gamma$ with $\mu(\tilde{y}) = \mathrm{ind}_{H_0}(z) - 2c_1(A)$. The induced map on homology is the above map Ψ.

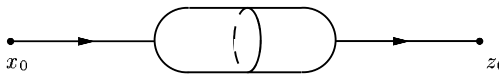

Figure 8: $\Psi \circ \Phi = \mathrm{id}$

We now prove that $\Psi \circ \Phi = \mathrm{id}$. On the chain level this composition is given by the numbers

$$n(x_0, z_0, A; H, J) = \sum_{\tilde{y}} n(x_0, \tilde{y})n((-A)\#\tilde{y}, z_0) \tag{5}$$

for triples (x_0, z_0, A) where x_0, z_0 are critical points of H_0 and $A \in \Gamma$ with

$$\mathrm{ind}_{H_0}(x_0) - \mathrm{ind}_{H_0}(z_0) + 2c_1(A) = 0.$$

In (5) the sum runs over all $\tilde{y} \in \widetilde{\mathcal{P}}(H)$ with $\mu(\tilde{y}) = \mathrm{ind}_{H_0}(x_0)$. The usual gluing and compactness arguments in Floer homology show that the integer $n(x_0, z_0, A; H, J)$ can be interpreted as the number of perturbed J-holomorphic spheres $u : S^2 \to M$ such that

$$u(0) \in W^u(x_0), \qquad u(\infty) \in W^s(z_0)$$

(see Figure 8). Here the perturbation is a Hamiltonian one with a very long neck and two J-holomorphic caps at the ends. Now choose a homotopy of perturbations from the given one to zero. For the zero perturbation there cannot be any solutions for dimensional reasons unless $A = 0$ and $x_0 = z_0$. Now the induced map on Floer homology is independent of the choice of the perturbation and for the zero perturbation we obtain

$$n(x_0, z_0, A; 0, J) = \begin{cases} 1 & \text{if } x_0 = z_0,\ A = 0, \\ 0 & \text{otherwise} \end{cases}$$

These numbers determine the identity homomorphism on $CM_*(H_0, \omega)$. Thus we have proved that on the chain level the map $\psi \circ \phi$ is chain homotopy equivalent to the identity and hence $\Psi \circ \Phi = \mathrm{id}$.

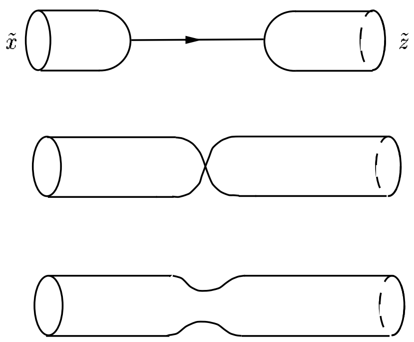

Figure 9: $\Phi \circ \Psi = \mathrm{id}$

The converse homomorphism $\Phi \circ \Psi$ is on the chain level given by the numbers

$$n(\tilde{x}, \tilde{z}) = \sum_{y_0, A} n((-A)\#\tilde{x}, y_0) n(y_0, (-A)\#\tilde{z}) \tag{6}$$

for $\tilde{x}, \tilde{z} \in \widetilde{\mathcal{P}}(H)$ with $\mu(\tilde{x}) = \mu(\tilde{z})$ where the sum runs over all all pairs $(y_0, A) \in \mathrm{Crit}(H_0) \times \Gamma$ with $\mathrm{ind}_{H_0}(y_0) = \mu(\tilde{x}) + 2c_1(A)$. A gluing argument in ordinary Morse theory now shows that the numbers $n(\tilde{x}, \tilde{z})$ can be interpreted geometrically as the number of triples (u^-, γ, u^+) where $u^- \in \mathcal{M}^-(\tilde{x})$, $u^+ \in \mathcal{M}(\tilde{z})$, and $\dot{\gamma} = -\nabla H_0(\gamma)$ with $\gamma(\pm T) = u^{\pm}(0)$. Now varying the parameter T will not change the induced map on Floer homology (see Figure 9). Hence we may take $T = 0$ and consider the map given by the numbers

$$n_0(\tilde{x}, \tilde{z}) = \#\{(u^-, u^+) \,|\, u^- \in \mathcal{M}^-(\tilde{x}), u^+ \in \mathcal{M}(\tilde{z}), u^-(0) = u^+(0)\}.$$

Now a gluing argument for J-holomorphic curves, as in the appendix of [20], shows that $n_0(\tilde{x}, \tilde{z})$ agrees with the number of perturbed J-holomorphic cylinders running from $\tilde{x}$ to $\tilde{z}$. A further homotopy argument, which is also used in the proof of Theorem 2.3, shows that the numbers $n_0(\tilde{x}, \tilde{y})$ induce the identity map on Floer homology. Thus we have proved that on the chain level the map $\phi \circ \psi$ is chain homotopy equivalent to the identity and hence $\Phi \circ \Psi = \mathrm{id}$. □

5 Quantum cohomology

The quantum cohomology ring of a semi-positive symplectic manifold is a deformation of the cup-product structure on ordinary cohomology, tensored by the Novikov ring, which in Section 2 was denoted by $QH^*(M) = H^*(M) \otimes \Lambda$. We have seen in Example 3.3 and Theorem 4.1 that there is a natural isomorphism $\Phi : QH^*(M) \to HF^*(H, J)$ and our goal in this section is to prove that this isomorphism intertwines the deformed cup-product on $QH^*(M)$ with the pair-of-pants product on $HF^*(H, J)$ defined in Example 3.6.

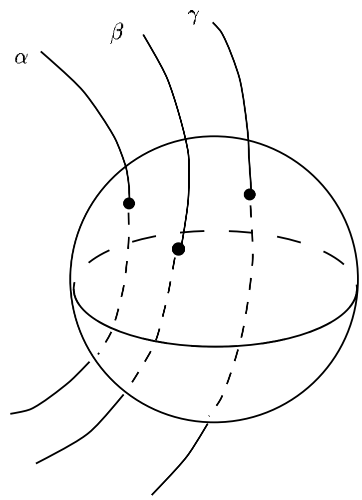

Figure 10: Witten's deformed cup-product

Let us first recall the definition of the deformed cup-product of Witten (cf. [20], [23], [35]). Given a spherical homology class $A \in H_2(M) = H_2(M, \mathbb{Z})/\text{torsion}$ consider the space $\mathcal{M}(A, J)$ of all simple J-holomorphic spheres $u : S^2 \to M$ representing the class A. For a generic almost complex structure $J \in \mathcal{J}(M, \omega)$ this is a manifold of dimension $2n + 2c_1(A)$. For three generic cycles α, β, γ in $H_*(M)$ with

$$\deg(\alpha) + \deg(\beta) + \deg(\gamma) = 4n - c_1(A)$$

define

$$\Phi_A(\alpha, \beta, \gamma) = \#\{u \in \mathcal{M}(A, J) \,|\, u(0) \in \alpha,\ u(1) \in \beta,\ u(\infty) \in \gamma\}$$

(see Figure 10). Here the number of points has to be counted with appropriate signs and the resulting integer is independent of the choice of the signs. This definition works unless A is a multiple class with Chern number $c_1(A) = 0$. In this case the compactness argument fails because of the possible presence of multiply covered curves with Chern number zero. This difficulty can be overcome by either considering J-holomorphic curves in $S^2 \times M$ as in [20] or by perturbing the nonlinear Cauchy-Riemann equations with a zero order term as in [23], [24]. Both modifications give rise the same invariant $\Phi_A(\alpha, \beta, \gamma)$. Moreover, if $A = 0$ is the class of constant curves, we define

$$\Phi_0(\alpha, \beta, \gamma) = \alpha \cdot \beta \cdot \gamma$$

to be the ordinary triple intersection number. Now the deformed cup product of two classe $a = \mathrm{PD}(\alpha) \in H^k(M)$ and $b = \mathrm{PD}(\beta) \in H^\ell(M)$ is defined by

$$a * b = \sum_A (a * b)_A e^{2\pi i A}$$

where $(a * b)_A \in H^{k+\ell-2c_1(A)}(M)$ is given by

$$\langle (a * b)_A, \gamma \rangle = \Phi_A(\alpha, \beta, \gamma).$$

for $\gamma \in H_{k+\ell-2c_1(A)}(M)$. Note that α, β, γ satisfy the dimension condition required for the definition of $\Phi_A(\alpha, \beta, \gamma)$. Note also that the 0-component of $a * b$ is the ordinary cup-product $(a * b)_0 = a \cup b$. It was proved by Ruan-Tian [24], Liu [18], and McDuff-Salamon [20] that the quantum cup-product is associative.

Theorem 5.1 *The isomorphism $\Phi : QH^*(M) \to HF^*(H, J)$ defined in Example 3.3 intertwines the quantum cup-product on $QH^*(M)$ with the pair-of-pants product of Example 3.6, i.e.*

$$\Phi(a * b) = \Phi(a) \cup_F \Phi(b)$$

for $a, b \in QH^*(M)$. It also intertwines the cap-product of Example 3.4 with the pair-of-pants product via

$$\mathrm{PD}_F(a \cap_F \mathrm{PD}_F(\Phi(b))) = \Phi(a * b)$$

for $a, b \in QH^*(M)$. Here $\mathrm{PD}_F : HF_*(H, J) \to HF^{2n-*}(H, J)$ denotes the Poincaré duality isomorphism of Floer homology (and its inverse) as defined in Remark 2.4.

Remark 5.2 Since Φ is an isomorphism (Theorem 4.1) it follows that the cap-product and the cup-product in Floer homology are related by

$$\mathrm{PD}_F(a \cap_F \xi) = \Phi(a) \cup_F \mathrm{PD}_F(\xi)$$

for $a \in QH^*(M)$ and $\xi \in HF_*(H, J)$.

Proof of Theorem 5.1: Let $a, b \in H^*(M)$ and denote $\alpha = \mathrm{PD}(A)$, $\beta = \mathrm{PD}(b)$. For $\tilde{x} \in \widetilde{\mathcal{P}}(H)$ with $\mu(\tilde{x}) = \deg(\alpha)$ denote

$$n(\alpha, \tilde{x}) = \#\mathcal{M}_\Sigma(\alpha, \tilde{x})$$

where $\Sigma = \mathbb{C}$ as in Section 3 and Example 3.3. Then the Floer homology class $\mathrm{PD}_F(\Phi(a) \cup_F \Phi(b))$ is given by

$$\mathrm{PD}_F(\Phi(a) \cup_F \Phi(b)) = \sum_{\tilde{x},\tilde{y},\tilde{z}} n(\alpha, \tilde{x}) n(\beta, \tilde{y}) n(\tilde{x}, \tilde{y}; \tilde{z}) \langle \tilde{z} \rangle.$$

Here $n(\tilde{x}, \tilde{y}; \tilde{z})$ denotes the oriented number of J-holomorphic *pants* with two limits $\tilde{x}, \tilde{y}$ on the left and one limit $\tilde{z}$ on the right. The sum is over all triples $\tilde{x}, \tilde{y}, \tilde{z}$ with $\mu(\tilde{z}) = \mu(\tilde{x}) + \mu(\tilde{y})$ and $\mu(\tilde{x}) = \deg(\alpha)$, $\mu(\tilde{y}) = \deg(\beta)$.

Now the usual gluing theorem in Floer homology shows that

$$\mathrm{PD}_F(\Phi(a) \cup_F \Phi(b)) = \sum_{\tilde{z}} n(\alpha, \beta, \tilde{z}) \langle \tilde{z} \rangle$$

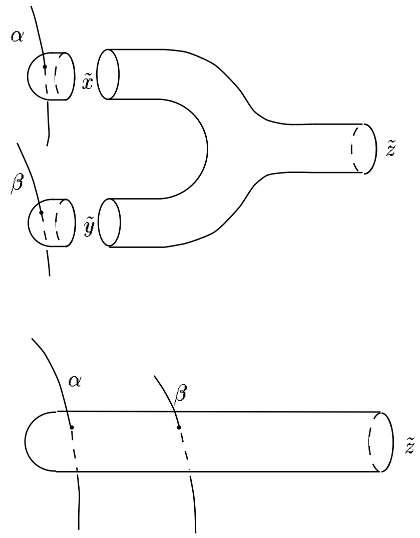

Figure 11: $\Phi(a) \cup_F \Phi(b)$

where $n(\alpha, \beta, \tilde{z}) = \#\mathcal{M}(\alpha, \beta, \tilde{z})$ is the number of J-holomorphic discs with one cylindrical end to the right converging to $\tilde{z}$ which intersect both α and β at two prescribed points $z_1, z_2 \in \mathbb{C}$ (see Figure 11). This is in fact a special case of Theorem 3.7 and can be rephrased as the formula

$$\mathrm{PD}_F(\Phi(a) \cup_F \Phi(b)) = \Psi_\Sigma(\alpha, \beta) \in HF_*(H, J)$$

where $\Sigma = \mathbb{C}$. Hence we must prove that

$$\Psi_\Sigma(\alpha, \beta) = \Phi(a * b). \tag{7}$$

To see this we first observe, as in [20], that the Poincaré dual

$$\xi_A = \mathrm{PD}((a * b)_A)$$

can be represented by the pseudo-cycle of all points which lie on J-holomorphic A-curves which intersect both $\alpha = \mathrm{PD}(a)$ and $\beta = \mathrm{PD}(b)$. Hence the right hand side of (7) is given by

$$\Phi(a * b) = \sum_{\tilde{x}} \sum_A n(\xi_A, (-A)\#\tilde{x})\langle \tilde{x} \rangle.$$

Thus we must prove that

$$n(\alpha, \beta, \tilde{x}) = \sum_A n(\xi_A, (-A)\#\tilde{x}). \tag{8}$$

For each class $A \in \Gamma$ the integer $n(\xi_A, (-A)\#\tilde{x})$ counts the number of J-holomorphic discs $u \in \mathcal{M}((-A)\#\tilde{x})$ (with cylindrical end to the right) such

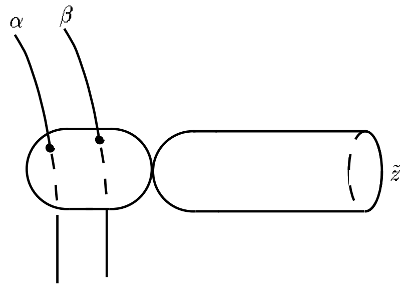

Figure 12: $\Phi(a * b)$

that $u(0)$ lies on a J-holomorphic A-curve which intersects both α and β (see Figure 12).

To prove (8) just consider the (perturbed) J-holomorphic planes $u_\varepsilon : \mathbb{C} \to M$ with cylindrical end $\tilde{x}$ and $u_\varepsilon(0) \in \alpha$, $u_\varepsilon(\varepsilon) \in \beta$. Now consider the limit $\varepsilon \to 0$. In the limit one obtains either a J-holomorphic plane with $u(0) \in \alpha \cap \beta$, and this contributes to the term $n(\xi_0, \tilde{x})$, or else a J-holomorphic A-sphere $v : S^2 \to M$ bubbles off at $z = 0$. On the complement of 0 the curve $u_\varepsilon(z)$ will then converge to a perturbed J-holomorphic plane $u : \mathbb{C} \to M$ with limit $(-A)\#\tilde{x}$ and $u(0) = v(\infty)$. Moreover, the bubble $v(S^2)$ must intersect both α and β. For a generic J these intersections must occur at distinct points. Thus the limit curves (u, v) contribute to the term $n(\xi_A, (-A)\#\tilde{x})$ on the right hand side of (8). This argument works only if there are no J-holomorphic curves of Chern number zero. If such spheres exist then the invariants $\Phi(\alpha, \beta, \gamma)$ and hence the class $\xi_A = \mathrm{PD}((a * b)_A)$ must in fact be defined in terms of perturbed J-holomorphic curves. In this case the proof can be modified by choosing an additional Hamiltonian perturbation for the function $\bar{u}(s,t) = u(e^{2\pi(s+it)})$ in the region $1/2 \leq s \leq 3/4$ and then stretching the neck $3/4 \leq s \leq 1$ on which we have the unperturbed Cauchy-Riemann equations. Details of this argument will be carried out elsewhere.

This proves the first statement in Theorem 5.1. The second statement follows by considering moduli spaces of the form $\mathcal{M}(\alpha, \tilde{x})$, $\mathcal{M}(\tilde{x}, \beta, \tilde{y})$, $\mathcal{M}(\tilde{y}, \gamma)$, and using the usual gluing techniques. □

Theorem 5.1 has important consequences because, in general, the pair-of-pants product on Floer cohomology is difficult to compute directly while recently the deformed cup-product on quantum cohomology has been computed in many cases. For complex Grassmanians see Vafa [33], Witten [36], Piunikhin [25], Siebert–Tian [32], for flag varieties see Givental–Kim [14], for generalized flag varieties see Astashkevich-Sadov [2] and for some other examples see Kontsevich–Manin [17]. In the next example we recall a recent computation by Donaldson [6].

Example 5.3 (Donaldson) Let Σ be a compact oriented Riemann surface of genus 2 and $P \to \Sigma$ be an $\mathrm{SO}(3)$-bundle with $w_2(P) \neq 0$. Denote by M_Σ the moduli space of flat connections on P. This space has real dimension 6 and, as a complex manifold, can be identified with the intersection of two quadrics in $\mathbb{C}P^5$. Its Betti-numbers are

$$b_0 = b_2 = b_4 = b_6 = 1, \quad b_1 = b_5 = 0, \quad b_3 = 4.$$

In particular, there is a natural isomorphism $\mu : H_1(\Sigma) \to H^3(M_\Sigma)$. Let $h_{2j} \in H^{2j}(M_\Sigma)$ denote the natural generators of the even dimensional cohomology for $j = 0, 1, 2, 3$ with $h_0 = \mathbb{1}$. Then the ordinary cohomology ring structure is given by

$$h_2 \cup h_2 = 4h_4, \qquad h_2 \cup h_4 = h_6, \qquad \mu(\gamma_1) \cup \mu(\gamma_2) = (\gamma_1 \cdot \gamma_2) h_6.$$

The minimal Chern number of M_Σ is 4 and so the quantum cohomology groups are graded modulo 4 with $QH^0 = H^0 \oplus H^4$, $QH^1 = H^1 \oplus H^5 = \{0\}$, $QH^2 = H^2 \oplus H^6$, $QH^3 = H^3$. According to Donaldson [6] the deformed cup-product is given by

$$\begin{aligned} h_2 * h_2 &= 4(h_4 + q), \\ h_2 * h_4 &= h_6 + 2h_2 q, \\ h_2 * h_6 &= 4(h_4 q + q^2), \\ \mu(\gamma_1) * \mu(\gamma_2) &= (\gamma_1 \cdot \gamma_2)(h_6 - h_2 q) \end{aligned}$$

where q denotes an auxiliary variable of degree 4.

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