Systems & Control Letters 3 (1983) 91–99
North-Holland
July 1983

Observability of nontrivial small solutions for neutral systems

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Received 1 February 1983
Revised 17 April 1983

In this paper we introduce the concept of observability of nontrivial small solutions for neutral functional differential equations with a single point delay and prove a matrix type criterion.

Keywords: Neutral functional differential equations, Small solutions, Observability.

1. Introduction

We consider the neutral functional differential equation (NFDE)

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) - A_{-1}\dot{x}(t-h) \tag{1}$$

with output

$$y(t) = C_0 x(t) + C_1 x(t-h) + C_{-0}\dot{x}(t) + C_{-1}\dot{x}(t-h) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and $A_0, A_1, A_{-1}, C_0, C_1, C_{-0}, C_{-1}$ are real matrices of the appropriate size. It is easy to see that the NFDE (1) admits a unique solution $x \in W^{1,p}_{\mathrm{loc}}([-h, \infty); \mathbb{R}^n)$ for every initial condition of the form

$$x(\tau) = \varphi(\tau), \quad -h \leqslant \tau \leqslant 0,$$

where $\varphi \in W^{1,p} = W^{1,p}([-h, 0]; \mathbb{R}^n)$ and $1 < p < \infty$.

Small solutions

We say that $x(t)$, $t \geqslant -h$, is a *small solution* of (1) if

$$\lim_{t\to\infty} e^{\omega t} x(t) = 0$$

for every $\omega \geqslant 0$. This means that $x(t)$ tends to zero more rapidly than any exponential or equivalently its Laplace transform is an entire function. An important fact is that every small solution of (1) vanishes after a finite time $T \leqslant (n-1)h - \alpha$ where $\alpha \geqslant 0$ is the exponential type of the entire function $\det(sI - A_0 - e^{-sh}A_1 - s\,e^{-sh}A_{-1})$, $s \in \mathbb{C}$. This has first been shown by Henry [1] in the retarded case and later on by Kappel [2] for general neutral systems.

Observability of nontrivial small solutions

A small solution of (1) is said to be *trivial* if it vanishes for $t \geqslant 0$. We introduce the following important concept.

Definition (observability of nontrivial small solutions). The nontrivial small solutions of system (1), (2) are said to be *observable* if every nontrivial small solution has a nonzero output for some $t \geqslant 0$. This means that the solutions of (1), (2) have the following property for any $T \geqslant 0$:

$$(x(t)=0\ \forall t \geqslant T, y(t)=0\ \forall t \geqslant 0) \quad \Rightarrow \quad (x(t)=0\ \forall t \geqslant 0).$$

Such a concept has previously not been considered in the literature on control systems with delays except in Salamon [11]. However, it has turned out to provide a crucial tool for the study of controllability and observability properties of neutral systems as well as for the derivation of matrix type criteria for completeness and F-completeness of eigenfunctions (Salamon [11]). In particular, the above property is precisely the 'gap' between *final observability* (i.e. $y(t)=0$ implies $x(t)=0\ \forall t \geqslant T$) and the stronger concept of *observability* in the sense

$$(y(t)=0\ \forall t \geqslant 0) \quad \Rightarrow \quad (x(t)=0\ \forall t \geqslant 0).$$

The latter concept plays an important role in the control theory of functional differential systems (see e.g. Lee [4], Kwong [3], Lee–Olbrot [5], Olbrot [8,9], Manitius [7]). However, a satisfactory characterization of this property has only recently been given by Manitius [7] for retarded systems with undelayed output variables. Our result below allows a generalization of his criterion to neutral systems with output delays. This generalization will not be worked out here since it needs a lot of state space theory and would lead too far for this paper. The interested reader is refered to Salamon [11].

Note that Olbrot [9] has a different concept of final observability namely that $y(t)=0$ for $0 \leqslant t \leqslant T$ implies $x(T)=0$. However, in the retarded case it has been shown in [9] that these two definitions coincide if T is sufficiently large. For neutral systems, this equivalence is a consequence of Salamon [11, Lemma IV.1.10]. Again in the case that T is sufficiently large, it has been shown in Salamon [11, Theorem IV.1.11] that final observability is equivalent to spectral observability and hence can be characterized by the matrix type condition

$$\operatorname{rank}\begin{bmatrix} \lambda I - A_0 - e^{-\lambda h}A_1 - \lambda e^{-\lambda h}A_{-1} \\ C_0 + e^{-\lambda h}C_1 + \lambda C_{-0} + \lambda e^{-\lambda h}C_{-1} \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}. \tag{3}$$

2. The main result

Theorem. *The nontrivial small solutions of system* (1), (2) *are observable if and only if*

$$\max_{\lambda \in \mathbb{C}} \operatorname{rank}\begin{bmatrix} A_0 - \lambda I & A_1 + \lambda A_{-1} \\ A_1 + \lambda A_{-1} & 0 \\ C_0 + \lambda C_{-0} & C_1 + \lambda C_{-1} \\ C_1 + \lambda C_{-1} & 0 \end{bmatrix} = n + \max_{\lambda \in \mathbb{C}} \operatorname{rank}\begin{bmatrix} A_1 + \lambda A_{-1} \\ C_1 + \lambda C_{-1} \end{bmatrix}. \tag{4}$$

Proof. Let us introduce the matrices

$$A(\lambda) = \begin{bmatrix} A_0 - \lambda I & A_1 + \lambda A_{-1} \\ A_1 + \lambda A_{-1} & 0 \end{bmatrix}, \qquad C(\lambda) = \begin{bmatrix} C_0 + \lambda C_{-0} & C_1 + \lambda C_{-1} \\ C_1 + \lambda C_{-1} & 0 \end{bmatrix}$$

and define

$$K = \max_{\lambda \in \mathbb{C}} \operatorname{rank}\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}, \qquad k = \max_{\lambda \in \mathbb{C}} \operatorname{rank}\begin{bmatrix} A_1 + \lambda A_{-1} \\ C_1 + \lambda C_{-1} \end{bmatrix}.$$

Then K is always less than or equal to $n + k$.

Necessity

Suppose that $K < n + k$. Then we prove in three steps that there exists a nontrivial small solution of (1), (2) with zero output.

Step 1. The exist polynomials

$$p(\lambda) = \sum_{j=0}^{l} p_j \lambda^j, \qquad q(\lambda) = \sum_{j=0}^{l} q_j \lambda^j$$

in $\mathbb{R}^n[\lambda]$ such that $p(\lambda) \not\equiv 0$ and

$$A(\lambda)\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} = 0, \quad C(\lambda)\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix} = 0 \qquad \forall \lambda \in \mathbb{C}. \tag{5}$$

Proof. Let $M(\lambda)$ and $N(\lambda)$ be unimodular matrices of appropriate size such that

$$M(\lambda)\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} N(\lambda) = \begin{bmatrix} \alpha_1(\lambda) & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \alpha_K(\lambda) & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

is in Smith-form. Then the last $2n - K$ columns

$$\begin{pmatrix} p^j(\lambda) \\ q^j(\lambda) \end{pmatrix}, \quad j = K + 1, \dots, 2n,$$

of $N(\lambda)$ satisfy (5). Now suppose that the polynomials $p^j(\lambda)$ vanish identically. Then the $q^j(\lambda)$ are linearly independent (for every $\lambda \in \mathbb{C}$) and satisfy $[A_1 + \lambda A_{-1}]q^j(\lambda) \equiv 0$ as well as $[C_1 + \lambda C_{-1}]q^j(\lambda) \equiv 0$. This implies that

$$\max_{\lambda \in \mathbb{C}} \operatorname{rank}\begin{bmatrix} A_1 + \lambda A_{-1} \\ C_1 + \lambda C_{-1} \end{bmatrix} \leqslant n - (2n - K) = K - n < k$$

which is a contradiction. $\square$

Step 2. Let us define $p_j = q_j = 0$ for $j \in \mathbb{Z}$ and $j \notin \{0, \dots, l\}$. Then the following equations hold for all $j \in \mathbb{Z}$:

$$A_0 p_{j+1} - p_j + A_1 q_{j+1} + A_{-1} q_j = 0, \tag{6.1}$$

$$A_1 p_{j+1} - A_{-1} p_j = 0, \tag{6.2}$$

$$C_0 p_{j+1} + C_{-0} p_j + C_1 q_{j+1} + C_{-1} q_j = 0, \tag{6.3}$$

$$C_1 p_{j+1} + C_{-1} p_j = 0. \tag{6.4}$$

Proof. These equations follow from (5) by comparison of the coefficients. In particular, the following equation holds:

$$\begin{aligned} 0 &= [A_0 - \lambda I]p(\lambda) + [A_1 + \lambda A_{-1}]q(\lambda) \\ &= \sum_{j=0}^{l} (A_0 p_j + A_1 q_j)\lambda^j + \sum_{j=0}^{l} (A_{-1} q_j - p_j)\lambda^{j+1} \\ &= \sum_{j=0}^{l+1} (A_0 p_j - p_{j-1} + A_1 q_j + A_{-1} q_{j-1})\lambda^j. \end{aligned}$$

This proves (6.1). The equations (6.2–6.4) can be established analogously. □

Step 3. The function

$$x(t) = \begin{cases} \sum_{j=1}^{l+1} \left(q_{l+1-j}\frac{t^j}{j!} + p_{l+1-j}\frac{(t-h)^j}{j!} \right), & -h \leqslant t < 0, \\ \sum_{j=1}^{l+1} p_{l+1-j}\frac{(t-h)^j}{j!}, & 0 \leqslant t < h, \\ 0, & h \leqslant t < \infty, \end{cases}$$

defines a nontrivial small solution of (1), (2) with zero output.

Proof. First note that $x(t)$ does not vanish identically for $0 \leqslant t \leqslant h$ since $p(\lambda)$ is a nonzero polynomial. Secondly, it is easy to see that $x(t)$ is absolutely continuous for $t \geqslant -h$. Finally, it can be proved straightforwardly – by the use of (6) – that $x(t)$ satisfies the NFDE (1) for almost every $t \geqslant 0$ and that the output $y(t)$ – given by (2) – vanishes for $t \geqslant 0$. We will only show that (1) holds for $0 < t < h$.

$$\begin{aligned} \dot{x}(t) &= \sum_{j=0}^{l+1} p_{l-j}\frac{(t-h)^j}{j!} \\ &= \sum_{j=0}^{l+1} (A_0 p_{l+1-j} + A_1 q_{l+1-j} + A_{-1} q_{l-j})\frac{(t-h)^j}{j!} + \sum_{j=0}^{l+1} (A_1 p_{l+1-j} + A_{-1} p_{l-j})\frac{(t-2h)^j}{j!} \\ &= A_0 x(t) + A_1 x(t-h) + A_{-1}\dot{x}(t-h), \quad 0 < t < h. \end{aligned}$$

□

Sufficiency

Suppose that $K = n + k$ and let $x(t)$, $t \geqslant -h$, be a solution of (1), (2) such that $x(t) = 0$ for $t \geqslant h$ and $y(t) = 0$ for $t \geqslant 0$. Then we prove in four steps that $x(t) = 0$ for $t \geqslant 0$.

Making use of this fact one can easily show by induction that the nontrivial small solutions of system (1), (2) are observable.

Step 1. The complex functions

$$\hat{x}(\lambda) = \int_0^h e^{-\lambda t} x(t)\,dt, \quad \hat{\hat{x}}(\lambda) = \int_0^{2h} e^{-\lambda t} x(t-h)\,dt, \qquad \lambda \in \mathbb{C},$$

satisfy the equation

$$\begin{bmatrix} A_0 - \lambda I & A_1 + \lambda A_{-1} \\ A_1 + \lambda A_{-1} & 0 \\ C_0 + \lambda C_{-0} & C_1 + \lambda C_{-1} \\ C_1 + \lambda C_{-1} & 0 \end{bmatrix} \begin{pmatrix} \hat{x}(\lambda) \\ \hat{\hat{x}}(\lambda) \end{pmatrix} = \begin{pmatrix} A_{-1} x(-h) - x(0) \\ A_{-1} x(0) \\ C_{-0} x(0) + C_{-1} x(-h) \\ C_{-1} x(0) \end{pmatrix} =: x. \tag{7}$$

Proof. For every $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
[A_1 + \lambda A_{-1}]\hat{x}(\lambda) &= \int_0^h e^{-\lambda t} A_1 x(t)\, dt + \int_0^h \lambda e^{-\lambda t} A_{-1} x(t)\, dt \\
&= \int_0^h e^{-\lambda t}(A_1 x(t) + A_{-1}\dot{x}(t))\, dt + A_{-1}x(0) \\
&= \int_0^h e^{-\lambda t}(\dot{x}(t+h) - A_0 x(t+h))\, dt + A_{-1}x(0) \\
&= A_{-1}x(0),
\end{aligned}$$

$$\begin{aligned}
&[A_0 - \lambda I]\hat{x}(\lambda) + [A_1 + \lambda A_{-1}]\hat{\hat{x}}(\lambda) \\
&\quad = \int_0^h e^{-\lambda t}(A_0 x(t) + A_1 x(t-h))\, dt + \int_0^h \lambda e^{-\lambda t}(A_{-1}x(t-h) - x(t))\, dt \\
&\qquad + [A_1 + \lambda A_{-1}] \int_h^{2h} e^{-\lambda t} x(t-h)\, dt \\
&\quad = \int_0^h e^{-\lambda t}(A_0 x(t) + A_1 x(t-h) + A_{-1}\dot{x}(t-h) - \dot{x}(t))\, dt \\
&\qquad - e^{-\lambda h}A_{-1}x(0) + A_{-1}x(-h) - x(0) + [A_1 + \lambda A_{-1}]e^{-\lambda h}\hat{x}(\lambda) \\
&\quad = A_{-1}x(-h) - x(0).
\end{aligned}$$

The remaining equations in (7) can be proved analogously. □

Step 2. There exist matrices $A_1(\lambda) \in \mathbb{R}^{n \times k}[\lambda]$ and $C_1(\lambda) \in \mathbb{R}^{m \times k}[\lambda]$ such that

$$\max_{\lambda \in \mathbb{C}} \operatorname{rank} \begin{bmatrix} A_0 - \lambda I & A_1(\lambda) \\ A_1 + \lambda A_{-1} & 0 \\ C_0 + \lambda C_{-0} & C_1(\lambda) \\ C_1 + \lambda C_{-1} & 0 \end{bmatrix} = n + k, \tag{8}$$

$$\max_{\lambda \in \mathbb{C}} \operatorname{rank} \begin{bmatrix} A_1(\lambda) \\ C_1(\lambda) \end{bmatrix} = k \tag{9}$$

and for almost every $\lambda \in \mathbb{C}$

$$\operatorname{range} \begin{bmatrix} A_1 + \lambda A_{-1} \\ C_1 + \lambda C_{-1} \end{bmatrix} = \operatorname{range} \begin{bmatrix} A_1(\lambda) \\ C_1(\lambda) \end{bmatrix}. \tag{10}$$

Proof. By assumption the rank of the matrix $[\begin{smallmatrix} A(\lambda_0) \\ C(\lambda_0) \end{smallmatrix}]$ is equal to $n + k$ for some $\lambda_0 \in \mathbb{C}$. Hence this matrix has $n + k$ linearly independent columns. Precisely k of these are contained in the right $(2n + 2m) \times n$-block of this matrix which is given by

$$\begin{bmatrix} A_1 + \lambda_0 A_{-1} \\ 0 \\ C_1 + \lambda_0 C_{-1} \\ 0 \end{bmatrix}.$$

Now let the matrices $A_1(\lambda)$, $C_1(\lambda)$ consist of the corresponding columns of $A_1 + \lambda A_{-1}$, $C_1 + \lambda C_{-1}$. Then $A_1(\lambda)$ and $C_1(\lambda)$ have the desired properties. □

Step 3. There exists a rational matrix $T(\lambda) \in \mathbb{R}^{k \times n}(\lambda)$ such that

$$A_1 + \lambda A_{-1} = A_1(\lambda) T(\lambda), \qquad C_1 + \lambda C_{-1} = C_1(\lambda) T(\lambda) \tag{11}$$

for almost every $\lambda \in \mathbb{C}$.

Proof. By (9), there exist matrices $A_2 \in \mathbb{R}^{n\times(n+m-k)}$ and $C_2 \in \mathbb{R}^{m\times(n+m-k)}$ such that

$$\det\begin{bmatrix} A_1(\lambda) & A_2 \\ C_1(\lambda) & C_2 \end{bmatrix} \not\equiv 0. \tag{12}$$

Now let $T(\lambda) \in \mathbb{R}^{k\times n}(\lambda)$, $R(\lambda) \in \mathbb{R}^{(n+m-k)\times n}(\lambda)$ be defined by

$$\begin{bmatrix} T(\lambda) \\ R(\lambda) \end{bmatrix} = \begin{bmatrix} A_1(\lambda) & A_2 \\ C_1(\lambda) & C_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 + \lambda A_{-1} \\ C_1 + \lambda C_{-1} \end{bmatrix}.$$

Then

$$\begin{bmatrix} A_1(\lambda) \\ C_1(\lambda) \end{bmatrix} T(\lambda) + \begin{bmatrix} A_2 \\ C_2 \end{bmatrix} R(\lambda) = \begin{bmatrix} A_1 + \lambda A_{-1} \\ C_1 + \lambda C_{-1} \end{bmatrix}.$$

By (10), this implies

$$\operatorname{range}\begin{bmatrix} A_2 \\ C_2 \end{bmatrix} R(\lambda) \subset \operatorname{range}\begin{bmatrix} A_1(\lambda) \\ C_1(\lambda) \end{bmatrix} \cap \operatorname{range}\begin{bmatrix} A_2 \\ C_2 \end{bmatrix}$$

for almost every $\lambda \in \mathbb{C}$. Hence it follows from (12) that $A_2 R(\lambda) \equiv 0$ and $C_2 R(\lambda) \equiv 0$. □

Step 4. $x(t) = 0$ for $t \geqslant 0$.

Proof. By (8), there exist unimodular matrices $M(\lambda)$, $N(\lambda)$ of appropriate size such that

$$M(\lambda)\begin{bmatrix} A_0 - \lambda I & A_1(\lambda) \\ A_1 + \lambda A_{-1} & 0 \\ C_0 + \lambda C_{-0} & C_1(\lambda) \\ C_1 + \lambda C_{-1} & 0 \end{bmatrix} N(\lambda) = \begin{bmatrix} \alpha_1(\lambda) & & \\ & \ddots & \\ & & \alpha_{n+k}(\lambda) \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

is in Smith-form where all the $\alpha_j(\lambda)$ are nonzero polynomials. Now let $\tilde{M}(\lambda)$ consist of the upper $n+k$ rows of $M(\lambda)$. Then we have

$$N(\lambda)\begin{bmatrix} \alpha_1(\lambda)^{-1} & & \\ & \ddots & \\ & & \alpha_{n+k}(\lambda)^{-1} \end{bmatrix} \tilde{M}(\lambda) \begin{bmatrix} A_0 - \lambda I & A_1(\lambda) \\ A_1 + \lambda A_{-1} & 0 \\ C_0 + \lambda C_{-0} & C_1(\lambda) \\ C_1 + \lambda C_{-1} & 0 \end{bmatrix} = I_{n+k}.$$

By (7) and (11), this implies

$$N(\lambda)\begin{bmatrix} \alpha_1(\lambda)^{-1} & & \\ & \ddots & \\ & & \alpha_{n+k}(\lambda)^{-1} \end{bmatrix} \tilde{M}(\lambda) x = \begin{pmatrix} \hat{x}(\lambda) \\ T(\lambda)\hat{x}(\lambda) \end{pmatrix}.$$

Now recall that by definition $\hat{x}(\lambda)$ is an entire function which is square integrable on the imaginary axis.

Moreover, the left-hand side of the above equation shows that $\hat{x}(\lambda)$ is a rational function and thus of exponential growth zero. This means that for any $\varepsilon > 0$ there exists an $M > 0$ such that $|\hat{x}(\lambda)| \leqslant Me^{\varepsilon|\lambda|}$ for every $\lambda \in \mathbb{C}$. Hence it follows from a theorem of Paley and Wiener (see e.g. Rudin [10, Theorem 19.3]) that $x(t)$ vanishes for $t \geqslant 0$. □

Remarks. (i) The criterion of the previous theorem can be generalized to systems with commensurable delays, but we will not do this here. In a more general situation the derivation of an analogous result seems to be a hard problem.

(ii) For retarded systems with undelayed output variables (i.e. $A_{-1} = 0$ and $C_{-0} = C_1 = C_{-1}) = 0$) the criterion of the theorem above reduces to

$$\operatorname{rank}\begin{bmatrix} A_0 - \lambda I & A_1 \\ A_1 & 0 \\ C_0 & 0 \end{bmatrix} = n + \operatorname{rank} A_1 \tag{13}$$

for some $\lambda \in \mathbb{C}$. This is precisely the transposed version of a necessary condition for F-controllability which has been derived by Manitius [7].

(iii) System (1) has only trivial small solutions iff

$$\max_{\lambda \in \mathbb{C}} \operatorname{rank}\begin{bmatrix} A_0 - \lambda I & A_1 + \lambda A_{-1} \\ A_1 + \lambda A_{-1} & 0 \end{bmatrix} = n + \max_{\lambda \in \mathbb{C}} \operatorname{rank}[A_1 + \lambda A_{-1}]. \tag{14}$$

This follows from the theorem above in the case $C_0 = C_1 = C_{-0} = C_{-1} = 0$.

(iv) Note that (14) is a generalization of the necessary and sufficient condition for F-completeness which has been derived by Manitius [6] in the retarded case ($A_{-1} = 0$).

Examples. (i) The scalar n-th order differential-difference equation

$$z^{(n)}(t) = \sum_{j=0}^{n-1} \alpha_j z^{(j)}(t) + \sum_{j=0}^{n} \beta_j z^{(j)}(t-h) \tag{15}$$

can be rewritten as an n-dimensional system of the form (1). It is easy to see that the corresponding matrices A_0, A_1, A_{-1} satisfy condition (14). Hence the solutions of (15) have the property

$$(z(t) = 0 \;\; \forall t \geqslant T) \quad \Rightarrow \quad (z(t) = 0 \;\; \forall t \geqslant 0).$$

(ii) The two-dimensional system

$$\dot{x}(t) = x_1(t-h) - \dot{x}_2(t-h), \qquad \dot{x}_2(t) = x_1(t), \tag{16}$$

is described by the matrices

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

These matrices do not satisfy (14) since

$$\operatorname{rank}\begin{bmatrix} \lambda & 0 & 1 & -\lambda \\ -1 & \lambda & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2 \quad \forall \lambda \in \mathbb{C}.$$

Hence system (16) has nontrivial small solutions. These are not observable through the output

$$y(t) = x_1(t) - \dot{x}_2(t). \tag{17}$$

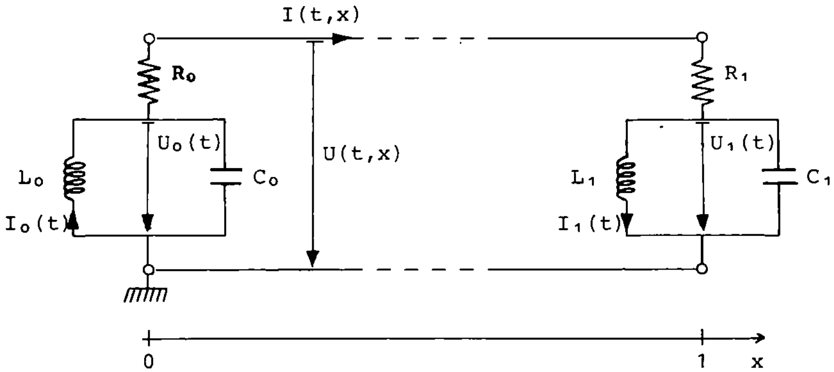

Fig. 1

However, they are observable if the output is given by

$$y(t)=x_1(t-h)-\dot{x}_2(t-h). \tag{18}$$

(iii) For the transposed system

$$\dot{x}_1(t)=x_1(t-h)+x_2(t), \qquad \dot{x}_2(t)=-\dot{x}_1(t-h), \tag{19}$$

condition (14) fails too. The nontrivial small solutions of (19) are observable through the output

$$y(t)=c_1x_1(t)+c_2x_2(t) \tag{20}$$

whenever $c_2 \neq 0$. In fact, in this case we have

$$\operatorname{rank}\begin{bmatrix} \lambda & -1 & 1 & 0 \\ 0 & \lambda & -\lambda & 0 \\ 1 & 0 & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \end{bmatrix} = 3 \quad \forall \lambda \in \mathbb{C}.$$

(iv) The lossless transmission line shown in Figure 1 can be described by the hyperbolic PDE

$$\frac{\partial U}{\partial x}=-L\frac{\partial L}{\partial t}, \qquad \frac{\partial I}{\partial x}=-C\frac{\partial U}{\partial t}, \tag{21}$$

with boundary conditions

$$U(t,0)=U_0(t)-R_0I(t,0), \qquad U(t,1)=U_1(t)+R_1I(t,1), \tag{22.1}$$

$$U_0(t)=-L_0\dot{I}_0(t), \qquad U_1(t)=L_1\dot{I}_1(t), \tag{22.2}$$

$$I(t,0)-I_0(t)=-C_0\dot{U}_0(t), \qquad I(t,1)-I_1(t)=C_1\dot{U}_1(t). \tag{22.3}$$

Integrating the PDE (21) we obtain

$$x_1(t)=\sqrt{C}\,U(t,0)+\sqrt{L}\,I(t,0)=\sqrt{C}\,U(t+h,1)+\sqrt{L}\,I(t+h,1),$$

$$x_2(t)=\sqrt{C}\,U(t,1)-\sqrt{L}\,I(t,1)=\sqrt{C}\,U(t+h,0)-\sqrt{L}\,I(t+h,0),$$

where $h=\sqrt{CL}$. Now let us introduce the variables $x_3(t)=2\sqrt{L}\,I_0(t)$, $x_4(t)=2\sqrt{L}\,I_1(t)$. Then the boundary conditions (22) lead to an NFDE of the form (1). The corresponding matrices are given by

$$A_0 = \begin{bmatrix} -\alpha_0 & 0 & \alpha_0 & 0 \\ 0 & -\alpha_1 & 0 & -\alpha_1 \\ -\alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \alpha_0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2\alpha_4 & 0 & 0 \\ -\alpha_3\alpha_5 & 0 & 0 & 0 \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} 0 & \alpha_4 & 0 & 0 \\ \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\alpha_2 = \frac{\sqrt{c}}{C_0}\frac{1}{R_0\sqrt{C} + \sqrt{L}}, \quad \alpha_2 = \frac{R_0\sqrt{C} + \sqrt{L}}{L_0\sqrt{C}}, \quad \alpha_4 = \frac{R_0\sqrt{C} - \sqrt{L}}{R_0\sqrt{C} + \sqrt{L}},$$

$$\alpha_1 = \frac{\sqrt{C}}{C_1}\frac{1}{R_1\sqrt{C} + \sqrt{L}}, \quad \alpha_3 = \frac{R_1\sqrt{C} + \sqrt{L}}{L_1\sqrt{C}}, \quad \alpha_5 = \frac{R_1\sqrt{C} - \sqrt{L}}{R_1\sqrt{C} + \sqrt{L}}.$$

In general, these matrices satisfy condition (14) and hence the corresponding neutral system has only trivial small solutions.

Acknowledgement

This work has been supported by the Forschungsschwerpunkt Dynamische Systeme.

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