Heat kernels on metric measure spaces

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The classical heat kernel is the fundamental solution of the heat equation in $\mathbb{R}^n$ given by the Gauss-Weierstrass formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Many needs of analysis, geometry and stochastic analysis require the study of the heat equation on Riemannian manifolds. For certain classes of manifolds, the behavior of the heat kernel is well-studied and is described by the Gaussian estimates:

$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \exp\left(-c\frac{d^2(x,y)}{t}\right),$$

where $d(x,y)$ is the geodesic distance and C, c are positive constants. On the other hand, the recent development of analysis on fractal spaces by M.Barlow, R.Bass, J.Kigami et al. has lead to construction of diffusion processes on such spaces, whose heat kernels satisfy sub-Gaussian estimates:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right)$$

with some parameters $\alpha > 0$ and $\beta > 2$. These striking results have motivated the study of abstract heat kernels on metric measure spaces. Let (M, d, μ) be a metric measure space. An abstract heat kernel is a function $p_t(x,y)$ where $x, y \in M$ and $t > 0$, that satisfies certain standard properties of the classical heat kernel, such as positivity, symmetry, the semi-group property, approximation of identity, and stochastic completeness. Such a heat kernel is associated with a Markov process on the space in question and is, in fact, its transition density.

Another example of a heat kernel is the transition density of the β-stable process in $\mathbb{R}^n$, which satisfies the estimate

$$p_t(x,y) \asymp \frac{C}{t^{n/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(n+\beta)}.$$

One of the main directions of the course is the study of heat kernels that satisfy the estimates of self-similar type:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right),$$

where α, β are positive constants and Φ is a decreasing function. The questions to be answered are as follows: what are possible values of the parameters α, β and what functions Φ can occur in such estimates.

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