Long Paths and Cycles in Random Subgraphs of Graphs With Large Minimum Degree*

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ABSTRACT: For a given finite graph G of minimum degree at least k, let G_p be a random subgraph of G obtained by taking each edge independently with probability p. We prove that (i) if $p \geq \omega/k$ for a function $\omega = \omega(k)$ that tends to infinity as k does, then G_p asymptotically almost surely contains a cycle (and thus a path) of length at least $(1 - o(1))k$, and (ii) if $p \geq (1 + o(1)) \ln k/k$, then G_p asymptotically almost surely contains a path of length at least k. Our theorems extend classical results on paths and cycles in the binomial random graph, obtained by taking G to be the complete graph on $k + 1$ vertices. © 2013 Wiley Periodicals, Inc.   Random Struct. Alg., 46, 320–345, 2015

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1. INTRODUCTION

Paths and cycles are two of the most simple yet important structures in graph theory, and being such, the problem of finding conditions that imply the existence of paths and cycles of various lengths has attracted a lot of attention in the field for the past 60 years. For example,

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Dirac [8] proved that for $k \geq 2$, every graph of minimum degree at least k contains a path of length k and a cycle of length at least $k + 1$, and that every graph on n vertices of minimum degree at least $\frac{n}{2}$ is Hamiltonian, i.e., contains a cycle of length n. By considering a complete graph on $n = k + 1$ vertices, and two edge-disjoint complete graphs of the same size sharing a single vertex, one can see that these results are tight. One reason that finding such conditions is of considerable interest is because it is often the case that conditions implying the existence of paths and cycles can be generalized to other substructures such as trees, and general subgraphs. For example, Pósa [21] proved that expansion implies the existence of long paths, and later Friedman and Pippenger [10] generalized Pósa's approach to trees.

Given a graph G and a real $p \in [0, 1]$, let G_p be the probability space of subgraphs of G obtained by taking each edge of G independently with probability p. We sometimes use the notation $(G)_p$ to avoid ambiguity. For a given graph property $\mathcal{P}$, and sequences of graphs $\{G_i\}_{i=1}^{\infty}$ and of probabilities $\{p_i\}_{i=1}^{\infty}$, we say that $(G_i)_{p_i} \in \mathcal{P}$ *asymptotically almost surely*, or *a.a.s.* for brevity, if the probability that $(G_i)_{p_i} \in \mathcal{P}$ tends to 1 as i goes to infinity. In this paper, when G and p are parameterized by some parameter, we abuse notation and consider G and p as sequences obtained by taking the parameter to tend to infinity, and will say that G_p has $\mathcal{P}$ asymptotically almost surely if the sequence does.

The most studied case of the above model of random graphs is when G is a complete graph K_n on n vertices, where $(K_n)_p$ is known as the *binomial random graph* $G(n, p)$. This model has been first introduced in 1959 [12] and has been intensively studied thereafter. In a seminal paper, Ajtai et al. [1], confirming a conjecture of Erdős, proved in particular that for $p = \frac{c}{n}$ and $c > 1$, $G(n, p)$ a.a.s. contains a path of length at least $(1 - f(c))n$, where $f(c)$ is a function tending to zero as c goes to infinity; this was also proved independently by Fernandez de la Vega [9]. Analogous problem for cycles was studied by Bollobás et al. [5]. Frieze [11] later determined the asymptotics of the number of vertices not covered by a longest path and cycle in $G(n, p)$. Also, for Hamiltonian paths and cycles, i.e., paths and cycles which pass through every vertex of the graph, improving on results of Pósa [21] and Korshunov [14], Bollobás [3] and Komlós and Szemerédi [13] independently proved that for every fixed positive ε and $p \geq \frac{(1+\varepsilon)\log n}{n}$, the random graph $G(n, p)$ is a.a.s. Hamiltonian. See the book of Bollobás [4] for a comprehensive overview of results on paths and cycles in random graphs.

In this paper, we study generalizations of the above mentioned results. Our goal is to extend classical results on random graphs to a more general class of graphs. More precisely, we would like to replace the host graph, taken to be the complete graph in the classical setting, by a graph of large minimum degree, and to find a.a.s. long paths and cycles in random subgraphs of large minimum degree graphs. Throughout the paper, we will only consider finite graphs.

Our first two theorems study paths.

Theorem 1.1. *Let G be a finite graph with minimum degree at least k, and let $p = \frac{c}{k}$ for some positive c satisfying $c = o(k)$ (c is not necessarily fixed). Then a.a.s. G_p contains a path of length $(1 - 2c^{-1/2})k$.*

We can also a.a.s. find a path of length exactly k, given that p is sufficiently large.

Theorem 1.2. *Let ε be a fixed positive real. For a finite graph G of minimum degree at least k and a real $p \geq \frac{(1+\varepsilon)\log k}{k}$, G_p a.a.s. contains a path of length k.*

Note that we parameterize our graph in terms of its minimum degree. Hence it should be understood that there are underlying sequences of graphs $\{G_i\}_{i=1}^{\infty}$ and probabilities $\{p_i\}_{i=1}^{\infty}$ where the minimum degrees of graphs tend to infinity, and the statements above hold with probability tending to 1 as i tends to infinity.

Since we can take G to be the complete graph K_{k+1} on $k+1$ vertices, our theorems can be viewed as generalizations of classical results on the existence of long paths in $G(n, p)$. In particular, Theorem 1.1 generalizes the result of Ajtai, Komlós, and Szemerédi and of Fernandez de la Vega, and Theorem 1.2 generalizes the result of Bollobás, and of Komlós and Szemerédi.

Our theorem can also be placed in a slightly different context. Recently there has been a number of papers revisiting classical extremal graph theoretical results of the type "if a graph G satisfies certain condition, then it has some property $\mathcal{P}$," by asking the following question: "How strongly does G possess $\mathcal{P}$?" In other words, one attempts to measure the robustness of G with respect to the property $\mathcal{P}$. For example, call a graph on n vertices a *Dirac graph*, if it has minimum degree at least $\frac{n}{2}$. Consider the above mentioned theorem which asserts that all Dirac graphs are Hamiltonian. There are several possible ways one can measure the robustness of this theorem. Cuckler and Kahn [6], confirming a conjecture of Sárközy et al. [22], measured the robustness by counting the minimum number of Hamilton cycles in Dirac graphs and proved that all Dirac graphs contain at least $\frac{n!}{(2+o(1))^n}$ Hamilton cycles. In a recent paper [15], we measured the robustness by taking random subgraphs of Dirac graphs and proved that for every Dirac graph G on n vertices and $p \gg \frac{\log n}{n}$, a random subgraph G_p is a.a.s. Hamiltonian. In the same paper, we also discussed an alternative measure of robustness where one analyzes the biased Maker-Breaker Hamiltonicity game on the Dirac graph. The concept of *resilience* of graphs is another framework which allows one to measure robustness of graphs. See, e.g., the paper of Sudakov and Vu [23] for more details. Note that Theorem 1.2 in fact measures the robustness of graphs of minimum degree at least k with respect to containing paths of length k, by taking random subgraphs.

We can also a.a.s. find long cycles in random subgraphs of graphs with large minimum degree.

Theorem 1.3.   *Let ω be a function tending to infinity with k and let ε be a fixed positive real. For a finite graph G of minimum degree at least k and $p \geq \frac{\omega}{k}$, G_p a.a.s. contains a cycle of length at least $(1 - \varepsilon)k$.*

Similarly to above, Theorem 1.3 can be considered as a generalization of Bollobás, Fenner, and Frieze's result. Also note that this theorem implies a weak form of Theorem 1.1. The proof of this theorem is much more involved compared to the two previous theorems.

The main technique we use in proving our theorems is a technique recently developed in Refs. 2 and 16, based on the depth first search algorithm. In Section 2, we discuss this technique in detail and also provide some probabilistic tools that we will need later. Using these tools, in Section 3 we prove Theorems 1.1 and 1.2. Then in Section 4 we prove Theorem 1.3.

Notation. A graph $G = (V, E)$ is given by a pair of its vertex set $V = V(G)$ and edge set $E = E(G)$. We use $|G|$ or $|V|$ to denote the order of the graph. For a subset X of vertices, we use $e(X)$ to denote the number of edges spanned by X, and for two sets X, Y, we use $e(X, Y)$ to denote the number of pairs (x, y) such that $x \in X$, $y \in Y$, for which $\{x, y\}$ is an edge (note that $e(X, X) = 2e(X)$). $G[X]$ denotes the subgraph of G induced by a subset of vertices X. We use $N(X)$ to denote the collection of vertices which are adjacent to some vertex of X

(we do not require X and $N(X)$ to be disjoint). For two graphs G_1 and G_2 over the same vertex set V, we define their intersection as $G_1 \cap G_2 = (V, E(G_1) \cap E(G_2))$, their union as $G_1 \cup G_2 = (V, E(G_1) \cup E(G_2))$, and their difference as $G_1 \setminus G_2 = (V, E(G_1) \setminus E(G_2))$. Moreover, we let $G \setminus X$ be the induced subgraph $G[V \setminus X]$.

When there are several graphs under consideration, to avoid ambiguity, we use subscripts such as $N_G(X)$ to indicate the graph that we are currently interested in. We also use subscripts with asymptotic notations to indicate dependency. For example, O_ε will be used to indicate that the hidden constant depends on ε. To simplify the presentation, we often omit floor and ceiling signs whenever these are not crucial and make no attempts to optimize absolute constants involved. We also assume that the parameter k (which will denote the minimum degree of the graph under consideration) tends to infinity and therefore is sufficiently large whenever necessary. All logarithms will be in base $e \approx 2.718$.

2. PRELIMINARIES

2.1. Depth First Search Algorithm

Our argument will utilize repeatedly the notion of the depth first search algorithm. This is a well known graph exploration algorithm, and we briefly describe it in this section.

The DFS (standing for Depth First Search) algorithm is a graph search algorithm that visits all vertices of a graph $G = (V, E)$ as follows. It maintains three sets of vertices, where S is the set of vertices whose exploration is complete, T is the set of unvisited vertices, and $U = V \setminus (S \cup T)$. The vertices of U are kept in a stack (the last in, first out data structure). These three sets will be updated as the algorithm proceeds. We assume that some order σ on the vertices of G is fixed, and the algorithm prioritizes vertices according to σ. The algorithm starts with $S = U = \emptyset$ and $T = V$, and runs until $U \cup T = \emptyset$. At each round of the algorithm, if the set U is non-empty, the algorithm queries T for neighbors of the last vertex v that has been added to U, scanning T according to σ. If v has a neighbor u in T, the algorithm deletes u from T and inserts it into U. If v does not have a neighbor in T, then v is popped out of U and is moved to S. If U is empty, the algorithm chooses the first vertex of T according to σ, deletes it from T and pushes it into U.

Observe that at the time we reach $U \cup T = \emptyset$, we obtain a rooted spanning forest of our graph (the root of each tree is the first vertex added to it). At this stage, in order to complete the exploration of the graph, we make the algorithm to query all remaining pairs of vertices in $S = V$, not queried before, in an arbitrary fixed order.

The following properties of the DFS algorithm will be relevant to us:

- if $T \neq \emptyset$, then every positively answered query increases the size of $S \cup U$ by one (however, note that having h positive queries will only guarantee that $|S \cup U| \geq h$, not $|S \cup U| = h$, since $|S \cup U|$ can also increase at a step where the stack U is empty);
- the set U always spans a path (indeed, when a vertex u is added to U, it happens so because u is a neighbor of the last vertex v in U; thus, u augments the path spanned by U, of which v is the last vertex);
- at any stage, G has no edges between the current set S and the current set T;
- for every edge $\{v, w\}$ of the graph, there exists a tree component in the forest produced by the DFS algorithm, in which v lies on the path from w to the root of the component, or vice versa.

In this paper, we utilize the DFS algorithm on random graphs, and will expose an edge only at the moment at which the existence of it is queried by the algorithm. More precisely, given a graph G and a real $p \in [0, 1]$, fix an order σ to be an arbitrary permutation, and assume that there is an underlying sequence $\overline{X} = (X_i)_{i=1}^{e(G)}$ of i.i.d. Bernoulli random variables with parameter p, which we call as the *query sequence*. The DFS algorithm gets an answer to its i-th query, asking whether some edge of G exists in G_p or not, according to the value of X_i; thus the query is answered positively if $X_i = 1$, and is answered negatively otherwise. Hence, the edge in G whose existence will be examined on the i-th query, depends on the outcome of the randomized algorithm. Note that the obtained graph is distributed according to G_p. Recently, Krivelevich and Sudakov [16] successfully used this idea to give a simple proof that $p = \frac{1}{n}$ is a sharp threshold for the appearance of a giant component in a random graph.

2.2. Probabilistic Tools

We will repeatedly use the technique known as *sprinkling*. Suppose that for some probability p, we wish to establish the fact $G_p \in \mathcal{P}$. It is often more convenient to establish this fact indirectly by choosing p_1 and p_2 so that G_p and $G_{p_1} \cup G_{p_2}$ have the same distribution (G_{p_1} and G_{p_2} are independent). We then prove that $G_{p_1} \in \mathcal{P}_1$ and $G_{p_2} \in \mathcal{P}_2$ for some properties $\mathcal{P}_1$ and $\mathcal{P}_2$ which together will imply the fact that $G_{p_1} \cup G_{p_2} \in \mathcal{P}$. Of course, a similar argument can be applied when we split the graph into several independent copies of random subgraphs formed with probabilities $p_1, p_2, \ldots, p_k$. Suppose that we have reals $0 \le p, p_1, \ldots, p_k \le 1$ and $p_1, \ldots, p_k = o(1)$ satisfying $p = \sum_{i=1}^{k} p_i$. Then the probability of a fixed pair of vertices forming an edge in $G_{p_1} \cup \cdots \cup G_{p_k}$ is $1 - (1 - p_1) \cdots (1 - p_k) = (1 - o(1))p$. Therefore, in this case $G_{p_1} \cup \ldots \cup G_{p_k}$ has the same distribution as $G_{(1-o(1))p}$, and thus for convenience, we consider $G_{p_1} \cup \ldots \cup G_{p_k}$ instead of the graph G_p, even though the distribution is not exactly the same. Since we are only interested in monotone properties $\mathcal{P}$, if we have $G_{(1-o(1))p} \in \mathcal{P}$ a.a.s., then we also have $G_p \in \mathcal{P}$ a.a.s. Moreover, when we are given the values of $p_1, p_2, \ldots$ beforehand, it is useful to expose the graphs G_{p_i} one at a time. By saying that we *sprinkle* the next round of edges, we suppose that we consider the outcome of the graph G_{p_i} for the first index i for which G_{p_i} has not been exposed.

The following two concentration results are the main probabilistic tools of this paper (see, e.g., [20]). The first theorem is Chernoff's inequality.

Theorem 2.1.   *Let $\lambda \le np$ be a positive real. If X is a binomial random variable with parameters n and p, then*

$$\mathbb{P}(|X - np| \ge \lambda) \le 2e^{-\lambda^2/(3np)}.$$

We will also use the following concentration result proved by Hoeffding [20, Theorem 2.3].

Theorem 2.2.   *Let $X_1, \ldots, X_n$ be independent random variables, with $0 \le X_k \le 1$ for each k. Let $S_n = \sum_{k=1}^{n} X_k$ and $\mu = \mathbb{E}[S_n]$. Then for every positive ε,*

$$\mathbb{P}(S_n \ge (1 + \varepsilon)\mu) \le e^{-\frac{\varepsilon^2 \mu}{2(1+\varepsilon/3)}}.$$

3. LONG PATHS

In this section we prove Theorems 1.1 and 1.2. Our first theorem is a slightly stronger version of Theorem 1.1.

Theorem 3.1.   *Let $p = \frac{c}{k}$ for some $c = o(k)$, and let G be a graph of minimum degree at least k.*

  i. *G_p a.a.s. contains a path of length $(1 - 2c^{-1/2})k$.*
 ii. *If G is a bipartite graph, then G_p a.a.s. contains a path of length $(2 - 6c^{-1/2})k$.*
iii. *If c tends to infinity with k, then for a fixed vertex v, there a.a.s. exists a path of length $(1 - 2c^{-1/2})k$ in G_p which starts at vertex v.*

Proof.   If $c < 4$, then the conclusions are vacuously true. Thus we may assume $c \geq 4$, and then for $\varepsilon := c^{-1/2}$, we have $\varepsilon \leq \frac{1}{2}$.

We will apply the DFS algorithm to the random graph G_p, as described in Section 2. Given a vertex v, let σ be an arbitrary ordering of the vertices which has v as its first vertex. Also assume that we have an underlying query sequence $\overline{X}$.

i. By Chernoff's inequality, after examining the query sequence for $\frac{k}{p} = \frac{k^2}{c}$ rounds, the probability of receiving at least $(1 - \varepsilon)k$ positive answers is at least

$$1 - e^{-\varepsilon^2 k^2/(3k)} = 1 - e^{-\varepsilon^2 k/3} = 1 - o(1).$$

Condition on this event, and we have $|S \cup U| \geq (1 - \varepsilon)k$. Consider the time at which we reach $|S \cup U| = (1 - \varepsilon)k$ (since $S = V$ in the end and $|S|$ changes by at most one at each step, there necessarily exists such a moment). Note that we asked less than $\frac{k^2}{c}$ queries until this stage. Suppose that at this time our set U is of size at most $(1 - 2\varepsilon)k$. Then we have $|S| > \varepsilon k$. Moreover, since the given graph has minimum degree at least k, each vertex in S has at least $k - |S \cup U| \geq \varepsilon k$ neighbors in T in the graph G. All the edges between S and T must have been queried by the algorithm and given a negative answer. Therefore, in order to be in a situation as above, we must have at least $|S| \cdot \varepsilon k > \varepsilon^2 k^2$ negative answers in our query sequence. However, since we asked at most $\frac{k^2}{c} = \varepsilon^2 k^2$ queries in total, this cannot happen. Thus we conclude that $|U| \geq (1 - 2\varepsilon)k = (1 - 2c^{-1/2})k$, which implies that there exists a path of length at least $(1 - 2c^{-1/2})k$, since the vertices in U form a path.

ii. By Chernoff's inequality, after examining the query sequence for $\frac{2k}{p} = \frac{2k^2}{c}$ rounds, we a.a.s. have $|S \cup U| \geq (2 - 5\varepsilon)k$. Condition on this event, and consider the time at which we reach $|S| = \varepsilon k$. If $|U| > (2 - 6\varepsilon)k$ at that point, then the vertices in U form a path of length at least $(2 - 6\varepsilon)k$. Thus assume that $|U| \leq (2 - 6\varepsilon)k$. Since $|S \cup U| \leq (2 - 5\varepsilon)k$, we examined the entries X_i of the sequence $\overline{X}$, only for indices $i \leq \frac{2k^2}{c}$. Moreover, since the given graph is bipartite, has minimum degree at least k, and U is a path, each vertex in S has at least $k - |S| - \frac{1}{2}|U| \geq 2\varepsilon k$ neighbors in T in G. Therefore, we must have at least $|S| \cdot 2\varepsilon k > 2\varepsilon^2 k^2$ negative answers in our query sequence so far. However, since we asked at most $\frac{2k^2}{c} = 2\varepsilon^2 k^2$ queries, this cannot happen. Therefore, G_p a.a.s. contains a path of length $(2 - 6c^{-1/2})k$.

iii. Let B be the event that there are less than $(1 - \varepsilon)k$ positive answers among the first $\frac{k}{p}$ rounds of the query sequence. By Chernoff's inequality, we have $\mathbb{P}(B) \leq 2e^{-\varepsilon^2 k/3} = o(1)$. As we have seen in the proof of (i), if B does not hold, then there exists a path of length at

least $(1-2\varepsilon)k$. In order to compute the probability that there is a path of length $(1-2\varepsilon)k$ starting at v, we will bound the probability of the event that $U \neq \emptyset$ during all the steps involved in reaching $|S \cup U| = (1-\varepsilon)k$, since if this is the case, then the path of length $(1-2\varepsilon)k$ that we found above necessarily starts at v (recall that v is the first vertex in σ). Let A_i be the event that $U = \emptyset$ at the time we reach $|S \cup U| = i$, thus we necessarily have $|S| = i$ if this event occurs. Since $i \leq (1-\varepsilon)k$, each vertex in S has at least εk neighbors in T in G at that moment. Therefore, when A_i occurred, we received at most i positive answers and at least $i \cdot \varepsilon k$ negative answers to our queries. Thus we can bound the probability that A_i occurs from above by the probability of the event that there are at most i positive answers among the first $i \cdot \varepsilon k$ queries. Hence by Chernoff's inequality with $\lambda = i\varepsilon c/2 \geq i$, we have $\mathbb{P}(A_i) \leq 2e^{-(i\varepsilon c)^2/(12i\varepsilon c)} = 2e^{-ic^{1/2}/12}$.

By the union bound, we get

$$\mathbb{P}\left(B \cup \bigcup_{i=1}^{(1-\varepsilon)k} A_i\right) \leq \mathbb{P}(B) + \sum_{i=1}^{(1-\varepsilon)k} \mathbb{P}(A_i) = o(1) + \sum_{i=1}^{(1-\varepsilon)k} 2(e^{-c^{1/2}/12})^i$$

$$\leq \frac{2}{e^{c^{1/2}/12} - 1} + o(1) = o(1),$$

since c tends to infinity with k. ∎

The second and the third parts of our previous theorem turn out to be useful in proving Theorem 1.2.

Given a graph G, consider a path $P = (v_0, v_1, \ldots, v_\ell)$ of length ℓ in G. Suppose we wish to find a path longer than P in G. This could immediately be done if there exists an edge $\{v_\ell, x\}$ for some $x \notin V(P)$. Pósa noticed that an edge of the form $\{v_\ell, v_i\}$ can also be useful, since if such an edge is present in the graph, then we have a path $P' = (v_0, \ldots, v_i, v_\ell, v_{\ell-1}, \ldots, v_{i+1})$ of length ℓ in our graph. Therefore, now we also can find a path of length greater than ℓ if there exists an edge of the form $\{v_{i+1}, x\}$ for some $x \notin V(P)$. Pósa's rotation-extension technique is employed by repeatedly "rotating" the path until we can "extend" it.

We first state a special case of Theorem 1.2, which can be handled by using Pósa's rotation-extension technique. Since the proof is quite standard, we defer it to later.

Theorem 3.2. *There exists a positive real ε_0 such that following holds for every fixed positive real $\varepsilon \leq \varepsilon_0$. Let G be a graph on n vertices of minimum degree at least $(1-\varepsilon)k$, and assume that $n \leq (1+\varepsilon)k$. For $p \geq \frac{(1+4\varepsilon)\log k}{k}$, a random subgraph G_p is Hamiltonian a.a.s.*

Proof of Theorem 1.2. We may assume that ε is given so that $\varepsilon \leq \min\{\frac{\varepsilon_0}{48}, \frac{1}{67600}\}$ where ε_0 is given in Theorem 3.2, since the conclusion for larger values of ε follows immediately by monotonicity. Set $p_1 = p_2 = p_3 = \frac{\varepsilon \log k}{k}$, $p_4 = \frac{(1+52\varepsilon)\log k}{k}$, and $p = \frac{(1+55\varepsilon)\log k}{k}$. We will show that $G_{p_1} \cup G_{p_2} \cup G_{p_3} \cup G_{p_4}$ a.a.s. contains a path of length k. This will in turn imply that G_p a.a.s. contains a path of length k as discussed in Section 2.

Given a graph G of minimum degree at least k, by Theorem 1.1, we know that G_{p_1} a.a.s. contains a path P of length $\ell = (1-\varepsilon)k$. Let $X \subset V(G) \setminus V(P)$ be the set of vertices outside P which have at least $(1-10\varepsilon)|P|$ neighbors in $V(P)$. We consider two cases depending on the size of X.

Case 1. $|X| \geq 2\varepsilon k$.

Redefine X as an arbitrary subset of itself of size exactly $2\varepsilon|P| \leq 2\varepsilon k$. Partition the path P into $\frac{1}{2\varepsilon}$ intervals $P_1, \ldots, P_{1/2\varepsilon}$, each of length $2\varepsilon|P|$. By the averaging argument, one can see that there exists an interval P_i for which $e(X, P_i) \geq (1 - 10\varepsilon)|X||P_i|$. Consider the bipartite graph Γ induced by the two parts X and P_i, and note that the number of non-adjacent pairs is at most $10\varepsilon|X||P_i|$ (also note that $|X| = |P_i|$). Repeatedly remove vertices from Γ which have degree at most $(1 - 8\varepsilon^{1/2})|X|$. As long as the total number of removed vertices is at most $4\varepsilon^{1/2}|X|$, each such deletion accounts for at least $4\varepsilon^{1/2}|X|$ non-adjacent pairs of Γ. Thus if we continued the removal for at least $4\varepsilon^{1/2}|X|$ steps, then by counting the number of non-adjacent pairs in Γ in two ways, we have $4\varepsilon^{1/2}|X| \cdot 4\varepsilon^{1/2}|X| \leq 10\varepsilon|X||P_i|$, which is a contradiction. Hence, the deletion process stops at some step, and we obtain a subgraph Γ_1 of Γ of minimum degree at least $(1 - 8\varepsilon^{1/2})|X|$.

Let $P_{i,0}$ (and $P_{i,1}$) be the leftmost (and rightmost) $9\varepsilon^{1/2}|P_i|$ vertices of the interval P_i. Even after removing the vertices $P_{i,0}, P_{i,1}$ from Γ_1, we are left with a graph Γ_2 of minimum degree at least

$$(1 - 8\varepsilon^{1/2})|X| - 18\varepsilon^{1/2}|P_i| = (1 - 26\varepsilon^{1/2})|X| \geq \frac{9}{10}|X|.$$

By Theorem 3.1 (ii), since Γ_2 is a bipartite graph, in $(\Gamma_2)_{p_2} \subset G_{p_2}$, we can find a path of length at least $2(\frac{9}{10} - o(1))|X| \geq \frac{5}{3}|X|$. By removing at most two vertices, we may assume that the two endpoints x and y of this path are both in X. Since Γ_1 has minimum degree at least $(1 - 8\varepsilon^{1/2})|P_i|$, both of these endpoints have at least $\varepsilon^{1/2}|P_i| \geq \varepsilon^{3/2}k$ neighbors in the sets $P_{i,0}$ and $P_{i,1}$. By Chernoff's inequality, in the graph $(\Gamma_1)_{p_3} \subset G_{p_3}$, we a.a.s. can find edges of the form $\{x, x_0\}$ and $\{y, y_1\}$ for $x_0 \in P_{i,0}$ and $y_1 \in P_{i,1}$. Thus in the graph $G_{p_2} \cup G_{p_3}$, we found a path of length at least $\frac{5}{3}|X|$ which starts at x_0, ends at y_1, and uses only vertices from $X \cup (P_i \setminus (P_{i,0} \cup P_{i,1}))$ as internal vertices. Together with the path P, this gives a path of length at least

$$|P| - |P_i| + \frac{5}{3}|X| \geq |P| + \frac{2}{3}|X| = (1 + \frac{1}{3}\varepsilon)k,$$

in $G_{p_1} \cup G_{p_2} \cup G_{p_3}$.

Case 2. $|X| < 2\varepsilon k$.

Let $P = (v_0, v_1, \ldots, v_\ell)$. Let $A_0 = X \cup V(P)$ and $B_0 = V \setminus A_0$. Note that $|A_0| < (1 + \varepsilon)k$, and $G[B_0]$ has minimum degree at least $10\varepsilon|P| - |X| \geq 7\varepsilon k$.

If v_ℓ has at least $\frac{\varepsilon k}{2}$ neighbors in B_0 in G, then by Chernoff's inequality, in G_{p_2}, we a.a.s. can find an edge $\{v_\ell, w\}$ for some $w \in B_0$. Afterwards, by Theorem 3.1 (ii), we a.a.s. can find a path of length at least $5\varepsilon k$ in $G_{p_3}[B_0]$ starting at w. Together with P, these will form a path of length $\ell + 1 + 5\varepsilon k \geq (1 + 4\varepsilon)k$. Thus we may assume that v_ℓ has at least $(1 - \frac{\varepsilon}{2})k$ neighbors in A_0.

Let $Y \subset A_0$ be the set of vertices which have at most $(1 - 10\varepsilon)|P|$ neighbors in A_0 in G. Note that the vertices in Y have at least $k - (1 - 10\varepsilon)|P| \geq 10\varepsilon k$ neighbors in B_0. Moreover, by the definition of the set X, all vertices of Y belong to $V(P)$. Suppose that $|Y| \geq 2\varepsilon k$. Then since $|A_0| \leq (1 + \varepsilon)k$, there are at least

$$\left(1 - \frac{\varepsilon}{2}\right)k - (|A_0| - |Y|) \geq \frac{\varepsilon}{2}k$$

edges in G of the form $\{v_\ell, v_{i-1}\}$ where $v_i \in Y$, and a.a.s. in G_{p_2} we can find one such edge $\{v_\ell, v_{i-1}\}$. Afterwards, since v_i has at least $10\varepsilon k$ neighbors in B_0 in G, we a.a.s. can find an

edge $\{v_i, w\}$ in G_{p_2} for some $w \in B_0$. By Theorem 3.1 (iii), there a.a.s. exists a path P' of length at least $5\varepsilon k$ starting at w in G_{p_3}. The paths P and P' together with the edges $\{v_\ell, v_{i-1}\}$ and $\{v_i, w\}$ will give a path of length at least $(1 + 4\varepsilon)k$.

If $|Y| < 2\varepsilon k$, then let $A_1 = A_0 \setminus Y$ and let $B_1 = V \setminus A_1$. Note that $|A_1| \geq (1 - 3\varepsilon)k$, and $G[A_1]$ has minimum degree at least $(1 - 10\varepsilon)|P| - 2\varepsilon k \geq (1 - 13\varepsilon)k$. If $k + 1 \leq |A_1| < (1 + \varepsilon)k$, then we use Theorem 3.2 to find a.a.s. a path of length k inside A_1 in G_{p_4}. Finally, if $|A_1| \leq k$, then since G has minimum degree at least k, we have $e(A_1, B_1) \geq |A_1|(k + 1 - |A_1|) \geq k$. Thus in G_{p_2}, we can find a.a.s. an edge $\{v, w\}$ such that $v \in A_1$ and $w \in B_1$. If $w \in B_0$, then Theorem 3.2 a.a.s. gives a path of length at least $|A_1| - 1 \geq (1 - 3\varepsilon)k - 1 \geq (1 - 4\varepsilon)k$ in $G[A_1]_{p_4}$ starting at v, and Theorem 3.1 (iii) a.a.s. gives a path of length at least $6\varepsilon k$ in $G[B_0]_{p_3}$ starting at w. These two paths together with the edge $\{v, w\}$ will give a path of length at least $(1 + 2\varepsilon)k$. Finally, if $w \in B_1 \setminus B_0$, then w contains at least $10\varepsilon k - |Y| \geq 8\varepsilon k$ neighbors in the set B_0. Therefore in G_{p_2}, we a.a.s. can find an edge $\{w, w'\}$ such that $w' \in B_0$. Afterwards, we can proceed as in the previous case to finish the proof. ∎

We conclude the section with the proof of Theorem 3.2.

Proof of Theorem 3.2. Let $\varepsilon_0 \leq \frac{1}{20}$ be a small enough constant, and let $s = \frac{k}{(\log k)^{3/4}}$. Let $p = \frac{(1+3\varepsilon)\log k}{k}$ and $p_1 = \cdots = p_s = \frac{(\log k)^{5/4}}{k^2}$. We will prove that $G_p \cup G_{p_1} \cup \cdots \cup G_{p_s}$ is a.a.s. Hamiltonian. Since

$$p_1 + \cdots + p_s = s \cdot \frac{(\log k)^{5/4}}{k^2} = \frac{(\log k)^{1/2}}{k},$$

this will imply that $G_{(1+4\varepsilon)\log k/k}$ a.a.s. contains a Hamilton cycle.

We first claim that G_p a.a.s. satisfies the following properties.

1. for every subset A of vertices of size $|A| \leq \frac{n}{(\log k)^{3/2}}$, we have $|N_{G_p}(A)| \geq \varepsilon^3 |A| \cdot \log k$,
2. for every subset A of vertices of size $|A| \geq \frac{n}{(\log k)^{3/4}}$, we have $|N_{G_p}(A)| \geq (1 - 4\varepsilon)k$, and
3. G_p is connected.

We prove these claims through proving that G_p a.a.s. has the following properties:

a. minimum degree is at least $\varepsilon^2 \log k$,
b. for all pairs of sets A and B of sizes $|A| \leq \frac{n}{(\log k)^{3/2}}$ and $|B| \leq \varepsilon^3 |A| \cdot \log k$, have $e_{G_p}(A, B) < \frac{\varepsilon^2}{2}|A| \cdot \log k$, and
c. for all pairs of sets A and B of sizes $|A| \geq \frac{n}{(\log k)^{3/4}}$ and $|B| \geq 3\varepsilon k$, have $e_{G_p}(A, B) > 0$.

For a fixed vertex v, the probability v has degree less than $\varepsilon^2 \log k$ in the graph G_p is

$$\sum_{i=0}^{\varepsilon^2 \log k} \binom{\deg_G(v)}{i} p^i (1-p)^{\deg_G(v)-i} \leq \sum_{i=0}^{\varepsilon^2 \log k} \binom{(1-\varepsilon)k}{i} p^i (1-p)^{(1-\varepsilon)k-i}$$

$$\leq \sum_{i=0}^{\varepsilon^2 \log k} \left(\frac{e(1-\varepsilon)k}{i} \cdot \frac{p}{1-p} \right)^i \cdot (1-p)^{(1-\varepsilon)k}$$

$$\leq \sum_{i=0}^{\varepsilon^2 \log k} \left(\frac{e(1+3\varepsilon)\log k}{i} \right)^i \cdot e^{-(1+\varepsilon)\log k}$$

$$\leq (\varepsilon^2 \log k + 1) \cdot \left(\frac{e(1+3\varepsilon)}{\varepsilon^2} \right)^{\varepsilon^2 \log k} e^{-(1+\varepsilon)\log k},$$

which is $o(k^{-1})$ given that ε is small enough. By taking the union bound over all $n \leq (1+\varepsilon)k$ vertices, we can deduce (a). For (b), let A be a set of size $t \leq \frac{n}{(\log k)^{3/2}}$ and let B be a set of size $\varepsilon^3 t \log k$. If $e_{G_p}(A, B) \geq \frac{\varepsilon^2}{2} t \log k$ and A and B are not disjoint, then let $A' = A \setminus B$, $B' = B \setminus A$, and add the vertices in $A \cap B$, independently and uniformly at random to A' or B'. By linearity of expectation, we have

$$\mathbb{E}[e_{G_p}(A', B')] \geq \frac{1}{4} e_{G_p}(A, B) \geq \frac{\varepsilon^2}{8} t \log k.$$

Hence there exists a choice of disjoint sets $A' \subset A$ and $B' \subset N(A)$ such that $e_{G_p}(A', B') \geq \frac{\varepsilon^2}{8} t \log k$. Therefore it suffices to show that for every pair of disjoint sets A and B satisfying the bound on the sizes given in (b), we have $e_{G_p}(A, B) < \frac{\varepsilon^2}{8} t \log k$. The probability that $e_{G_p}(A, B) \geq \frac{\varepsilon^2}{8} t \log k$ is at most

$$\binom{|A||B|}{\varepsilon^2 t \log k / 8} \cdot p^{\varepsilon^2 t \log k / 8} \leq \left(\frac{e \varepsilon^3 t^2 \log k}{\varepsilon^2 t \log k / 8} \right)^{\varepsilon^2 t \log k / 8} \cdot p^{\varepsilon^2 t \log k / 8} = (8 e \varepsilon t p)^{\varepsilon^2 t \log k / 8}.$$

By taking the union bound over all possible sets, we see that the probability of having a pair of sets violating (b) is at most

$$\sum_{t=1}^{n/(\log k)^{3/2}} \binom{n}{\varepsilon^3 t \log k}^2 \cdot (8 e \varepsilon t p)^{\varepsilon^2 t \log k / 8} \leq \sum_{t=1}^{n/(\log k)^{3/2}} \left(\left(\frac{en}{\varepsilon^3 t \log k} \right)^{16\varepsilon} \cdot \frac{16 e \varepsilon t \log k}{k} \right)^{\varepsilon^2 t \log k / 8}.$$

By considering all the variables other than t as constant, the logarithm of the summand on the right hand side can be expressed as $at \log t + bt$ for some reals $a > 0$ and b (given that $8\varepsilon < 1$). Since the second derivative of this function is positive, the maximum of the summand occurs either at $t = 1$ or $t = n/(\log k)^{3/2}$. From this, one can deduce that the summand is always $o(n^{-1})$ and the sum is $o(1)$. For (c), first consider a fixed pair of sets A and B of sizes $|A| \geq \frac{n}{(\log k)^{3/4}}$ and $|B| \geq 3\varepsilon k$. Since the number of vertices of the graph is $n \leq (1+\varepsilon)k$ and minimum degree is at least $(1-\varepsilon)k$, we have $e_G(A, B) \geq \frac{1}{2}|A| \cdot \varepsilon k$. Therefore, $\mathbb{E}[e_{G_p}(A, B)] \geq \frac{\varepsilon}{2} n (\log k)^{1/4}$, and by Chernoff's inequality, the probability that $e_{G_p}(A, B) = 0$ is at most $e^{-\Omega(\varepsilon n (\log k)^{1/4})}$. Since the number of pairs of sets (A, B) is at most 2^{2n}, we can take the union bound over all choices of A and B to see that (c) hold.

Condition on the event that (a), (b), and (c) holds. Then for a set A of size at most $|A| \leq \frac{n}{(\log k)^{3/2}}$, note that by (a) we have $e_{G_p}(A, N_{G_p}(A)) \geq |A| \varepsilon^2 \log k$. Then by (b), we have $|N_{G_p}(A)| > \varepsilon^3 |A| \log k$. Therefore we have Property 1. For Property 2, let A be a set of size at least $\frac{n}{(\log k)^{3/4}}$. If $|N_{G_p}(A)| < (1-4\varepsilon)k$, then there are no edges between A and $B = V \setminus N_{G_p}(A)$ (recall that A and $N_{G_p}(A)$ are not necessarily disjoint), where $|B| \geq n - (1 - 4\varepsilon)k \geq 3\varepsilon k$. This contradicts (c) and cannot happen. Thus we have Property 2 as well. Property 3 follows

from Properties 1 and 2, since they imply that all connected components are of order at least $(1 - 4\varepsilon)k > \frac{1}{2}n$.

Condition on the event that G_p satisfies Properties 1, 2, and 3 given above. We claim that G_p contains a path of length at least $n - \frac{k}{(\log k)^{3/4}}$, and for all $i \leq \frac{k}{(\log k)^{3/4}}$, conditioned on the event that a longest path in $G_p \cup G_{p_1} \cup \cdots \cup G_{p_{i-1}}$ is of length ℓ_i, $G_p \cup G_{p_1} \cup \cdots \cup G_{p_i}$ contains a cycle of length $\ell_i + 1$ with probability at least $1 - o(k^{-1})$. Since G_p is connected, this will imply that as long as the graph does not contain a Hamilton path, the length of a longest path increases by at least one in every round of sprinkling. Since we start with a cycle of length at least $n - \frac{k}{(\log k)^{3/4}}$, this will prove that the final graph is a.a.s. Hamiltonian.

Let $P = (v_0, \ldots, v_{\ell_i})$ be a longest path in the graph $G_i = G_p \cup G_{p_1} \cup \cdots \cup G_{p_{i-1}}$ for some $i \geq 1$. For a set $X = \{v_{a_1}, \ldots, v_{a_k}\}$, we let $X^- = \{v_{a_1-1}, \ldots, v_{a_k-1}\}$ and $X^+ = \{v_{a_1+1}, \ldots, v_{a_k+1}\}$ (if the index becomes either 0 or $\ell_i + 1$, then we remove the corresponding vertex from the set). Note that for all sets X, we have $X \cup X^+ \cup X^- \subset V(P)$. Let $X_0 = \{v_0\}$.

We will iteratively construct sets X_t for $t = 0, 1, \ldots$ of size $|X_t| \geq (\frac{\varepsilon^3 \log k}{4})^t$, as long as $|X_t| \leq \frac{n}{(\log n)^{3/2}}$, such that $X_t \supset X_{t-1}$, and for every vertex $v \in X_t$, there exists a path of length ℓ_i over the vertex set $V(P)$ which starts at v and ends in v_{ℓ_i}. Given a set X_{t-1}, if $N_{G_i}(X_{t-1}) \not\subset V(P)$, then we can find a path of length at least $\ell_i + 1$ in G_i, which contradicts the assumption on maximality of P. Therefore, $N_{G_i}(X_{t-1}) \subset V(P)$, and each vertex in $N_{G_i}(X_{t-1}) \setminus (X_{t-1} \cup X_{t-1}^- \cup X_{t-1}^+)$ gives rise to an "endpoint" from which there exists a path of length ℓ_i, and at most two such vertices can give rise to the same endpoint (see the discussion before the statement of Theorem 3.2). Let X_t be the union of X_{t-1} with the set of endpoints obtained as above. We have,

$$|X_t| \geq |X_{t-1}| + \frac{1}{2}|N_{G_i}(X_{t-1}) \setminus (X_{t-1} \cup X_{t-1}^- \cup X_{t-1}^+)|$$

$$\geq \frac{1}{2}|N_{G_i}(X_{t-1})| - \frac{1}{2}|X_{t-1}| \geq \frac{1}{2}|X_{t-1}| \cdot (\varepsilon^3 \log k - 1) \geq \left(\frac{\varepsilon^3 \log k}{4}\right)^t$$

(where we used Property 1 in the second to last inequality). Repeat the argument until the first time we reach a set of size $|X_t| > \frac{n}{(\log k)^{3/2}}$, and redefine X_t as a subset of size exactly $\frac{n}{(\log k)^{3/2}}$ which contains X_{t-1}. By repeating the argument above, we can find a set X_{t+1} of size at least $\frac{\varepsilon^3 \log k}{4} \cdot \frac{n}{(\log k)^{3/2}} > \frac{n}{(\log k)^{3/4}}$. By repeating the argument once more (now using Property 2 instead of Property 1), we can find a set X_{t+2} of size at least $\frac{1}{2}(1 - 5\varepsilon)k$.

For each vertex $v \in X_{t+2}$, there exists a path of length ℓ_i which starts at v and uses only vertices from $V(P)$. Thus if there exists an edge between X_{t+2} and $V \setminus V(P)$ in G_i, then we can find a path of length at least $\ell_i + 1$ and contradict maximality of P. If $|V \setminus V(P)| \geq \frac{n}{(\log k)^{3/4}}$, then Property 2 implies the existence of such edge (since $\frac{1}{2}(1 - 5\varepsilon)k + (1 - 4\varepsilon)k > n$). This shows that we have $\ell_i \geq n - \frac{n}{(\log k)^{3/4}}$ for all $i \geq 1$. In particular, G_p contains a path of length at least $n - \frac{n}{(\log k)^{3/4}}$. For each vertex $v \in X_{r+2}$, by applying the argument of the previous paragraph to the other endpoint of the path starting at v, we can find a set Y_v of size at least $\frac{1}{2}(1 - 5\varepsilon)k$ such that for every $w \in Y_v$, there exists a path of length ℓ_i which starts at v and ends at w. Since $n \leq (1 + \varepsilon)k$ and the minimum degree of G is at least $(1 - \varepsilon)k$, there are at least $\frac{1}{2} \cdot (\frac{1}{2}(1 - 5\varepsilon)k - 2\varepsilon k)^2 \geq \frac{1}{16}k^2$ pairs such that if some pair appears in G_{p_i}, then G_{p_i} contains a cycle of length $\ell_i + 1$. Since $p_i = \frac{(\log k)^{5/4}}{k^2}$, by Chernoff's inequality, the probability that such edge appears in G_{p_i} is at least $1 - e^{-\Omega((\log k)^{5/4})} = 1 - o(k^{-1})$. This proves our claim.   ∎

4. CYCLES OF LENGTH $(1 - o(1))k$

In this section we prove Theorem 1.3.

4.1. High Connectivity and Long Cycles

We start with a simple lemma based on the DFS algorithm which allows us to claim the a.a.s. existence of a cycle of length linear in the average degree of the graph.

Lemma 4.1. *Let α be a fixed positive real. Let G be a graph of average degree αk, and let $p = \frac{\omega}{k}$ for some function $\omega = \omega(k) \ll k$ that tends to infinity as k does. Then G_p a.a.s. contains a cycle of length at least $(\frac{1}{2} - \frac{5}{\sqrt{\omega}})\alpha k$. Moreover, if G is a bipartite graph, then G_p a.a.s. contains a cycle of length at least $(1 - \frac{10}{\sqrt{\omega}})\alpha k$.*

Proof. Let n be the number of vertices of G and set $p_1 = p_2 = \frac{\omega}{2k} \ll 1$. We will show that $G_{p_1} \cup G_{p_2}$ a.a.s. contains a cycle of length at least $(\frac{1}{2} - \frac{5}{\sqrt{\omega}})\alpha k$.

Consider the DFS algorithm applied to the graph G_{p_1} starting from an arbitrary vertex. By Chernoff's inequality, after examining the query sequence for $\frac{2n}{p_1}$ steps, we a.a.s. receive at least n positive answers. Condition on this event. At the time at which T becomes an empty set (therefore when we complete exploring the component structure), we know that since $|S \cup U| = n$, the length of the query sequence is at most $\frac{2n}{p_1}$. The rooted spanning forest we found induces a partial order on the vertices of the graph, where for two vertices x, y, we have $x < y$ if and only if x is a predecessor of y in one of the rooted trees in the spanning forest.

In the DFS algorithm, every edge $\{x, y\}$ of G for which x and y are incomparable in the partial order, must have been queried and answered negatively. Therefore, there can only be at most $\frac{2n}{p_1}$ such edges. Since the average degree of the graph is αk, this means that there are at least $\frac{\alpha n k}{2} - \frac{2n}{p_1} \geq (\frac{1}{2} - \frac{4}{\alpha \omega})n \cdot \alpha k$ edges $\{x, y\}$ of G for which $x < y$ or $y < x$. Hence, there exists a vertex v incident to at least $(\frac{1}{2} - \frac{4}{\alpha \omega})\alpha k$ edges $\{w, v\}$ of G for which $w < v$. By definition, the other endpoint of all these edges lie on the path from v to the root of the tree that v belongs to. The probability that one of the edges among the $\frac{\alpha k}{\sqrt{\omega}}$ farthest reaching edges appear is

$$1 - (1 - p)^{\alpha k/\sqrt{\omega}} \geq 1 - e^{\alpha \sqrt{\omega}} = 1 - o(1).$$

Thus there a.a.s. exists at least one edge among the farthest reaching $\frac{\alpha k}{\sqrt{\omega}}$ edges. This edge gives a cycle of length at least $(\frac{1}{2} - \frac{5}{\sqrt{\omega}})\alpha k$.

Moreover, if G is a bipartite graph, then this edge gives a cycle of length $(1 - \frac{10}{\sqrt{\omega}})\alpha k$, since the vertex v can only be adjacent to every other vertex in the path from v to the root of the tree that contains v. ∎

Let t be a positive integer. A graph G is *t-vertex-connected* (or *t-connected* in short) if for every set S of at most $t - 1$ vertices, the graph $G \setminus S$ is connected. Here we state some facts about highly-connected graphs without proof. The fourth part is a result of Mader [18], and the fifth part is a result of Menger [19]. We refer readers to Diestel's graph theory book [7] for more information on highly-connected graphs.

Lemma 4.2. *Let t be a positive integer, and let G, G′ be t-connected graphs.*

   i. *G remains connected even after removing a combination of $t - 1$ edges and vertices.*
   ii. *If $v \notin V(G)$ has at least t neighbors in G, then $G \cup \{v\}$ is also t-connected.*
   iii. *If $|V(G) \cap V(G')| \geq t$, then $G \cup G'$ is also t-connected.*
   iv. *Every graph of average degree at least $4t$ contains a t-connected subgraph.*
   v. *For every pair of subsets A and B of $V(G)$, there are $\min\{t, |A|, |B|\}$ vertex-disjoint paths in G that connect A and B.*

The main strategy we use in proving Theorem 1.3 is to find in the random subgraph a highly connected subgraph that contains many vertex disjoint cycles. Lemma 4.1 will be used to find vertex disjoint cycles. Afterwards the connectivity condition will allow us to "patch" the cycles into a long cycle of desired length. This is similar in spirit to a theorem of Locke [17] which asserts that a 3-connected graph with a path of length ℓ contains a cycle of length at least $\frac{2}{5}\ell$.

Lemma 4.3. *Let α be a fixed positive real, t be a fixed positive integer, and let $p = \frac{\omega}{k}$ for some function $\omega = \omega(k) \ll k$ that tends to infinity as k does. Let G_1 and G_2 be graphs defined over the same set of n vertices. Suppose that at least $(1 - \frac{1}{t})n$ vertices of G_1 have degree at least αk, and that G_2 is t-connected. Then the graph $(G_1)_p \cup G_2$ a.a.s. contains a cycle of length at least $(1 - \frac{10}{t})\alpha k$.*

Proof. Let $p_1 = p_2 = \cdots = p_t = \frac{\omega}{tk}$. Suppose that we have found a cycle of length $\ell \leq (1 - \frac{10}{t})\alpha k$ after $i - 1$ rounds of sprinkling. We claim that we can then find a.a.s. a cycle of length at least $(1 - \frac{1}{t})\ell + \frac{2\alpha k}{t}$, by sprinkling one more round with probability p_i. If this is the case, then since

$$\left((1 - \frac{1}{t})\ell + \frac{2\alpha k}{t}\right) - \ell = \frac{2\alpha k}{t} - \frac{\ell}{t} \geq \left(\frac{2}{t} - \frac{1}{t}\right)\alpha k = \frac{1}{t}\alpha k,$$

after sprinkling at most t rounds, we will a.a.s. find a cycle of length at least $(1 - \frac{10}{t})\alpha k$.

To prove the claim, suppose that we are given a cycle C of length $\ell \leq (1 - \frac{10}{t})\alpha k$. Let $V' = V \setminus V(C)$. Since G_1 has $(1 - \frac{1}{t})n$ vertices of degree at least αk, the graph $G[V']$ has at least $(1 - \frac{1}{t})n - \ell$ vertices which have degree at least $\alpha k - \ell \geq \frac{10}{t}\alpha k$. Therefore the average degree of $G[V']$ is at least

$$\frac{\left((1 - \frac{1}{t})n - \ell\right) \cdot \frac{10}{t}\alpha k}{n - \ell} = \left(1 - \frac{1}{t} - \frac{\ell}{t(n - \ell)}\right) \cdot \frac{10}{t}\alpha k = \left(1 - \frac{1}{t} - \frac{1}{t(n/\ell - 1)}\right) \cdot \frac{10}{t}\alpha k,$$

which is minimized when ℓ is maximized. Since $\ell \leq (1 - \frac{10}{t})\alpha k \leq (1 - \frac{10}{t})n$, the average degree of $G[V']$ is at least

$$\left(1 - \frac{1}{t} - \frac{(1 - \frac{10}{t})n}{t \cdot \frac{10}{t}n}\right) \cdot \frac{10}{t}\alpha k = \frac{9}{t}\alpha k.$$

Thus by Lemma 4.1, after sprinkling one more round, we a.a.s. can find a cycle C' in $G[V']$ of length at least $(\frac{1}{2} - o(1))\frac{9\alpha k}{t} \geq \frac{4\alpha k}{t}$. Since G_2 is a t-connected graph, there exist t vertex disjoint paths that connect C to C' (see Lemma 4.2 (v)). Among these paths, consider

the two whose intersection point with C are closest to each other (along the distance induced by C). Using these two paths to merge C and C', we a.a.s. can find a cycle of length at least

$$\left(1 - \frac{1}{t}\right)|V(C)| + \frac{1}{2}|V(C')| \geq \left(1 - \frac{1}{t}\right)\ell + \frac{2\alpha k}{t},$$

as claimed. ∎

Our next lemma is similar to the lemma above, but will be applied under slightly different circumstances.

Lemma 4.4. *Suppose that ℓ and t are given integers satisfying $\ell \geq t$. Let G be a t-vertex-connected graph that contains s vertex-disjoint cycles of lengths at least ℓ each. Then G contains a cycle of length at least*

$$\left(1 - \frac{s}{t}\right)^{s-1}\ell + \sum_{i=0}^{s-2}\left(1 - \frac{s}{t}\right)^i \cdot \frac{\ell}{2}.$$

Thus if t is large enough depending on s, then G contains a cycle of length at least $\frac{s}{2}\ell$.

Proof. We will find a cycle of desired length by an iterative process; for $h = 1, 2, \ldots, s$, after the h-th step, we find a cycle of length at least $\ell_h = (1 - \frac{s}{t})^{h-1}\ell + \sum_{i=0}^{h-2}(1 - \frac{s}{t})^i \cdot \frac{\ell}{2}$, and $s - h$ other cycles of length at least ℓ each which are all vertex-disjoint. Note that the statement is vacuously true for $h = 1$ by the given condition. The statement for $h = s$ corresponds to the statment of the lemma.

Given a cycle C_h of length at least ℓ_h and $s - h$ other vertex disjoint cycles of length at least ℓ each, let $X = V(C_h)$ and Y be the union of the set of vertices of the cycles of length at least ℓ. By the t-connectivity of our graph, Lemma 4.2 (v), and the fact

$$\min\{|X|, |Y|\} \geq \ell \geq t,$$

we see that there are t vertex-disjoint paths that connect X to Y. By the pigeonhole principle, at least $\frac{t}{s-h}$ of these paths connect the cycle of length at least ℓ_h to one fixed cycle of length at least ℓ. Among these paths, consider the two whose intersection points with C_h are closest to each other. Using these two paths, we can find a cycle of length at least

$$\left(1 - \frac{s-h}{t}\right)\ell_h + \frac{\ell}{2} \geq \left(1 - \frac{s}{t}\right)\ell_h + \frac{\ell}{2} = \ell_{h+1}.$$

Moreover, note that we still have at least $s - h - 1$ vertex-disjoint cycles which are also disjoint to the new cycle we found. Therefore in the end, after the s-th step, we will find our desired cycle. For the second part, note that

$$\ell_s = \left(1 - \frac{s}{t}\right)^{s-1}\ell + \sum_{i=0}^{s-2}\left(1 - \frac{s}{t}\right)^i \cdot \frac{\ell}{2} = \left(1 - \frac{s}{t}\right)^{s-1}\frac{\ell}{2} + \sum_{i=0}^{s-1}\left(1 - \frac{s}{t}\right)^i \cdot \frac{\ell}{2}$$

$$= \left(1 - \frac{s}{t}\right)^{s-1}\frac{\ell}{2} + \frac{t}{s}\left(1 - \left(1 - \frac{s}{t}\right)^s\right) \cdot \frac{\ell}{2}.$$

If t is large enough depending on s, we have $(1 - \frac{s}{t})^{s-1} = 1 - o_t(1)$ and $1 - (1 - \frac{s}{t})^s = \frac{s^2}{t} - O_t(\frac{s^4}{t^2})$. Therefore in this case,

$$\ell_s \geq (1 - o_t(1)) \frac{\ell}{2} + \frac{t}{s} \left(\frac{s^2}{t} - O_t \left(\frac{s^4}{t^2} \right) \right) \frac{\ell}{2} \geq \frac{s}{2} \ell.$$

∎

4.2. Finding Long Cycles

In this subsection we prove Theorem 1.3.

We first state a structural lemma, which a.a.s. finds almost vertex-disjoint highly connected subgraphs in the random subgraph of our given graph. Afterwards, we will use the lemmas developed in the previous subsection in order to find a long cycle in various situations.

Lemma 4.5. *Let $\varepsilon \leq \frac{1}{2}$ be a fixed positive real. Let G be a graph on n vertices with minimum degree at least k, and let $p = \frac{\omega}{k}$ for some function $\omega = \omega(k) \ll k$ that tends to infinity as k does. Suppose that G does not contain a bipartite subgraph of average degree at least $\frac{5}{4}k$. Then G_p a.a.s. admits a partition $V = X \cup Y$ of its vertex set, and contains a collection $\mathcal{C}$ of subgraphs of G_p satisfying:*

a. for every $C \in \mathcal{C}$, C is $(\log \omega)^{1/5}$-connected;
b. the sets $X \cap V(C)$ for $C \in \mathcal{C}$ form a partition of X, and $|Y| = o(n)$;
c. for every $C \in \mathcal{C}$, one of the following holds:
  i. the graph $G[V(C)]$ contains at least $(1-\varepsilon)|V(C)|$ vertices of degree at least $(1-\varepsilon)k$, or
  ii. $|Y \cap V(C)| = o(|V(C)|)$, the graph $G[X \cap V(C)]$ contains at least $(1 - \varepsilon)|V(C)|$ vertices of degree at least $\frac{k}{8}$, and there exists a bipartite graph $\Gamma_C \subseteq G$ with parts $X \cap V(C)$ and $Y \setminus V(C)$ which contains at least $\frac{|V(C)|\varepsilon^2 k}{4}$ edges and has maximum degree at most $\frac{8}{\varepsilon}k$.

We defer the proof of the structural lemma to later and first prove Theorem 1.3 using the structural lemma. Let G be a given graph of minimum degree at least k on n vertices, and let ε be a given positive real. For $p = \frac{\omega}{k}$, it suffices to prove the statement for $\omega \ll k$ since the conclusion for larger ω follows from monotonicity. Set $p_1 = p_2 = p_3 = \frac{\omega}{3k}$. Suppose that $\varepsilon \leq \frac{1}{50}$ is given (for larger values of ε, we may assume that $\varepsilon = \frac{1}{50}$).

Case 1. There exists a Bipartite Subgraph of G of Average Degree at Least $\frac{5}{4}k$.

We can apply Lemma 4.1 to the bipartite subgraph to a.a.s. find a cycle in G_{p_1} of length at least $(1 - o(1))\frac{5}{4}k \geq k + 1$.

If G does not contain such a bipartite subgraph, then we apply Lemma 4.5 to a.a.s. find a collection $\mathcal{C}$ of subgraphs which induce highly-connected subgraphs of G_{p_1}.

Case 2. There exists $C \in \mathcal{C}$ such that Property (c)-(i) holds.

By Lemma 4.5 (a), C is $\frac{1}{\varepsilon}$ connected, and we can apply Lemma 4.3 with $t = \frac{1}{\varepsilon}, \alpha = 1 - \varepsilon$, $G_1 = G[V(C)]$ and $G_2 = C$ to a.a.s. obtain a cycle of length at least $(1 - 10\varepsilon) \cdot (1 - \varepsilon)k \geq (1 - 11\varepsilon)k$ in $G_{p_2}[V(C)] \cup C \subset G_{p_1} \cup G_{p_2}$.

Case 3. Property (c)-(ii) holds for all $C \in \mathcal{C}$.

For each $C \in \mathcal{C}$, there exists a bipartite graph Γ_C with parts $X \cap V(C)$ and $Y \setminus V(C)$ which has at least $\frac{|V(C)| \varepsilon^2 k}{4}$ edges and maximum degree at most $\frac{8k}{\varepsilon}$. Expose the graph G_{p_2}, and for each $C \in \mathcal{C}$, let M_C be a maximum matching in $(\Gamma_C)_{p_2}$. Let $\mathcal{C}' = \{C \in \mathcal{C} \ : \ |M_C| \geq \frac{\varepsilon^3 |V(C)|}{128}\}$.

Lemma 4.6. *We a.a.s. have $\sum_{C \in \mathcal{C}'} |V(C)| \geq \frac{n}{2}$.*

Proof. For a graph $C \in \mathcal{C}$, we first estimate the probability that $C \notin \mathcal{C}'$. Let $X_C = X \cap V(C)$, $Y_C = Y \setminus V(C)$, and m_C be the number of edges of Γ_C (thus $\frac{|V(C)| \varepsilon^2 k}{4} \leq m_C \leq |V(C)| \cdot \frac{8k}{\varepsilon}$). Since the maximum degree is at most $\frac{8k}{\varepsilon}$, we know that for every collection of $t \leq \frac{\varepsilon^3 |V(C)|}{128}$ vertex-disjoint edges, there are at least $\frac{|V(C)| \varepsilon^2 k}{4} - 2t \cdot \frac{8k}{\varepsilon} \geq \frac{\varepsilon^2 |V(C)| k}{8}$ edges in Γ_C not intersecting any of the edges in the collection. Therefore the probability that $|M_C| = t$ is at most

$$\binom{m_C}{t} \cdot p_2^t \cdot (1 - p_2)^{\varepsilon^2 |V(C)| k / 8} \leq \left(\frac{e m_C p_2}{t} \right)^t (1 - p_2)^{\varepsilon^2 |V(C)| k / 8}$$

$$\leq \left(\frac{e}{t} \cdot \frac{8 |V(C)| k}{\varepsilon} \cdot \frac{\omega}{3k} \right)^t e^{-\varepsilon^2 |V(C)| k p_2 / 8}$$

$$= \left(\frac{8 e \omega |V(C)|}{3 t \varepsilon} \right)^t e^{-\varepsilon^2 |V(C)| \omega / 24}.$$

By parameterizing t as $t = \alpha |V(C)|$ ($\alpha \leq \frac{\varepsilon^3}{128}$), the right hand side becomes

$$\left(\frac{8 e \omega}{3 \alpha \varepsilon} \right)^{\alpha |V(C)|} e^{-\varepsilon^2 |V(C)| \omega / 24} = e^{\alpha \log(8 e \omega / (3 \alpha \varepsilon)) |V(C)|} e^{-\varepsilon^2 |V(C)| \omega / 24} \leq e^{-\varepsilon^2 |V(C)| \omega / 48}.$$

By taking the union bound over all values of t from 1 to $\frac{\varepsilon^3 |V(C)|}{128}$, we see that the probability of $|M_C| < \frac{\varepsilon^3 |V(C)|}{128}$, or equivalently $C \notin \mathcal{C}'$, is at most

$$\frac{\varepsilon^3 |V(C)|}{128} \cdot e^{-\varepsilon^2 |V(C)| \omega / 48} = o(1).$$

By Markov's inequality, it follows that $\sum_{C \notin \mathcal{C}'} |X \cap V(C)| < \frac{n}{4}$ a.a.s. If this event holds, then since $|X \cap V(C)| = (1 - o(1))|V(C)|$ for all $C \in \mathcal{C}$, we have

$$\sum_{C \in \mathcal{C}'} |V(C)| = (1 + o(1)) \sum_{C \in \mathcal{C}'} |X \cap V(C)|$$

$$= (1 + o(1)) \left(\sum_{C \in \mathcal{C}} |X \cap V(C)| - \sum_{C \notin \mathcal{C}'} |X \cap V(C)| \right)$$

$$= (1 + o(1)) \left((1 - o(1)) n - \frac{n}{4} \right) \geq \frac{n}{2}.$$

∎

Condition on the conclusion of Lemma 4.6. Consider an auxiliary bipartite graph Γ over a vertex set consisting of two parts $\mathcal{C}$ and Y (where Y is the set given by Lemma 4.5). A pair $\{C, y\}$ forms an edge if y is an endpoint of some edge in M_C. Since $|X \cap V(C)| \geq \frac{k}{8}$ and

$|X| = (1 - o(1))n$, the number of vertices of Γ is $|\mathcal{C}| + |Y| \leq \frac{n}{k/8} + o(n) = o(n)$ and the number of edges is $\sum_{C \in \mathcal{C}} |M_C| \geq \sum_{C \in \mathcal{C}'} \frac{\varepsilon^3}{128} |V(C)| \geq \frac{\varepsilon^3}{256} n$. Let $t \geq 100$ be a large enough constant. By Lemma 4.2 (iv), there exists a t-connected subgraph Γ' of Γ over the vertex set $C_1, \ldots, C_s, y_1, \ldots, y_{s'}$ of Γ. We claim that the induced subgraph H of $G_{p_1} \cup G_{p_2}$ on the vertex set $V(C_1) \cup \cdots \cup V(C_s) \cup \{y_1, \ldots, y_{s'}\}$ is t-connected (note that $s, s' \geq t$). Suppose that this is the case. Then since the sets $V(C_1) \cap X, \ldots, V(C_s) \cap X$ are vertex disjoint and each graph $G[V(C_i) \cap X]$ contains at least $(1 - \varepsilon)|V(C_i)|$ vertices of degree at least $\frac{k}{8}$, by Lemma 4.3 for each fixed i, $G_{p_3}[V(C_i) \cap X]$ a.a.s. contains a cycle of length at least $(1 - 10\varepsilon)\frac{k}{8} \geq \frac{k}{10}$. Thus in $H \cup G_{p_3}$ we a.a.s. have at least $(1 - o(1))s$ vertex disjoint cycles of length at least $\frac{k}{10}$ in the graph. Since H is t-connected, for large enough t, by Lemma 4.4 we can use 30 of the vertex disjoint cycles to a.a.s. find in $H \cup G_{p_3} \subseteq G_{p_1} \cup G_{p_2} \cup G_{p_3}$ a cycle of length at least $\frac{1}{2} \cdot 30 \cdot \frac{k}{10} > k$.

Therefore to conclude the proof of Theorem 1.3, it suffices to prove that H is t-connected. Let S be a subset of at most $t - 1$ vertices of $V(H)$. It suffices to prove that $H \setminus S$ is a connected graph. We do this by exploiting the t-connectivity of Γ'. Let v and w be two arbitrary vertices of $H \setminus S$. For a vertex $x \in S$, if $x \in Y$, then remove x from Γ'. Otherwise, if $x \in X$ and there is a matching edge belonging to some M_C incident to x, then remove the corresponding edge from Γ' (note that there is a one-to-one correspondence between such edges and edges of Γ').

First suppose that $v, w \in X$. Since we removed at most $t - 1$ vertices/edges from the graph Γ', without loss of generality, there still exists a path $C_1 z_1 C_2 \cdots z_{h-1} C_h$ in Γ' for $v \in V(C_1)$ and $w \in V(C_h)$. For each z_i, there exist vertices $z_i' \in V(C_i) \setminus S$ and $z_i'' \in V(C_{i+1}) \setminus S$ such that $\{z_i, z_i'\} \in M_{C_i}$ and $\{z_i, z_i''\} \in M_{C_{i+1}}$. Let $z_0'' = v$ and $z_h' = w$. Since each C_i is t-connected, for $i = 1, 2, \ldots, h$, we can find a path from z_{i-1}'' to z_i' in $C_{i+1} \setminus S$. By combining these paths with the edges $\{z_i', z_i\}$ and $\{z_i, z_i''\}$ for $i = 1, 2, \ldots, h - 1$, we obtain a path from $z_0'' = v$ to $z_h' = w$.

Second, suppose that $v \in X$ and $w \in Y$. Since we removed at most $t - 1$ edges from H, there exists an edge of H incident to w whose other endpoint w' is in X. By the case above, we see that there exists a path from v to w' which implies that there is a path from v to w. The last case when $v, w \in Y$ can be handled similarly.   ∎

4.3. Proof of the Structural Lemma

In this subsection, we prove the structural lemma, Lemma 4.5. The proof is quite technical so we begin this section by briefly explaining its idea.

Let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be a collection of edge-disjoint t-connected subgraphs of G_p (where t is some large integer), which covers the maximum number of edges of G_1 and has the minimum number of subgraphs in it. Note that if the number of vertices of G is $O(k)$, then the collection $\mathcal{C}$ likely consits of a single subgraph. However, we put no restriction on the number of vertices, and thus G_p might even consist of several connected components. Thus $\mathcal{C}$ is a non-trivial collection of subgraphs of G_p. This collection will have interesting properties which will eventually imply our lemma.

Note that two subgraphs C_i and C_j can only share at most $t - 1$ vertices since otherwise C_i and C_j can be combined into a single t-connected subgraph of G_p to contradict the minimality of the collection. Hence every pair subgraphs in $\mathcal{C}$ are "almost" disjoint. Moreover, if there are too many edges of the graph G not covered by any of the subgraphs in $\mathcal{C}$, then we will be able to find a t-connected subgraph of G_1 which is edge-disjoint to all subgraphs in $\mathcal{C}$

and thus contradicts the maximality of the collection. Thus most edges of G are covered by some subgraph in $\mathcal{C}$.

Afterwards, we find a subcollection $\mathcal{C}'$ of $\mathcal{C}$ for which the following holds: the number of vertices which are covered by at least two subgraphs in $\mathcal{C}'$ is small. Moreover, the collection $\mathcal{C}'$ will maintain the property that most edges of G are covered by some subgraph. Now let X be the set of vertices which are covered by exactly one subgraph in $\mathcal{C}'$, and Y be the rest of the vertices. We can see that most edges of G lie within subgraphs in the collection $\mathcal{C}'$, and that every pair of subgraphs in $\mathcal{C}'$ are only allowed to intersect in Y. This illustrates how the structure claimed in Lemma 4.5 arises from the collection $\mathcal{C}$.

We first prove the following lemma, which forms an intermediate step in proving Lemma 4.5.

Lemma 4.7. *Let G be a graph on n vertices with minimum degree at least k, and let $p = \frac{\omega}{k}$ for some function $\omega = \omega(k) \ll k$ that tends to infinity as k does. Suppose that G does not contain a bipartite subgraph of average degree at least $\frac{5}{4}k$. Then G_p a.a.s. admits a partition $V = X \cup Y$ of its vertex set, and contains a collection $\hat{\mathcal{C}}$ of subgraphs satisfying the following:*

  a. *every graph $C \in \mathcal{C}$ is $(\log \omega)^{1/5}$-connected;*
  b. *the sets $X \cap V(C)$ for $C \in \mathcal{C}$ form a partition of X, and $|Y| = o(n)$;*
  c. *for every $C \in \mathcal{C}$, $|Y \cap V(C)| = o(|V(C)|)$ and the induced subgraph $G[X \cap V(C)]$ contains at least $(1 - o(1))|V(C)|$ vertices of degree at least $k/8$;*
  d. *for every $C \in \mathcal{C}$ and every vertex $v \in X \cap V(C)$, there are at most $o(k)$ edges of G incident to v whose other endpoint lies in $X \setminus V(C)$.*

Proof. Let $t = (\log \omega)^{1/5}$ and $V = V(G)$. A straightforward application of Chernoff's inequality and of the union bound shows that G_p a.a.s. satisfies the following property: "for every pair of sets A and B that have $e_G(A, B) \geq \frac{nk}{\omega^{1/2}}$, we have $e_{G_p}(A, B) \geq \frac{1}{2}n\omega^{1/2}$." Expose G_p and condition on this event.

Let C be a graph defined over a subset of vertices of V. For an edge e of G, we say that e is *covered* by C if both of the endpoints of e belong to $V(C)$ (note that this does not necessarily imply that e is an edge of C). Let $\mathcal{C}_0$ be a collection of t-connected edge-disjoint subgraphs of G_p of order at least t^4 each, which maximizes the total number of edges covered and whose sum of orders is minimized. Note that for every pair of graphs $C, C' \in \mathcal{C}_0$, we have $|V(C) \cap V(C')| < t$ as otherwise by Lemma 4.2 (iii) we can replace the two graphs C and C' in $\mathcal{C}_0$ by a single graph $C \cup C'$ in order to find a collection of t-connected edge-disjoint subgraphs that contradicts the minimality of sum of orders of the collection $\mathcal{C}_0$. We will repeatedly apply this idea throughout this proof; the graphs in $\mathcal{C}_0$ cannot be combined to give another t-connected subgraph.

Step 1 : Initial Partition

Let X_0 be the set of vertices which are contained in at most t^4 graphs in $\mathcal{C}_0$, and let $Y_0 = V \setminus X_0$.

Claim 4.8. $|Y_0| \leq \frac{4n}{t^3}$.

Proof. Consider an auxiliary bipartite graph Γ_0 whose vertex set consists of two parts, where one part is V and the other part is $\mathcal{C}_0$. A pair (v, C) forms an edge in Γ_0 if $v \in V(C)$.

Suppose that Γ_0 contains a t-connected subgraph Γ_0', and let $C_1, C_2, \ldots, C_s$ be the graphs in $V(\Gamma_0') \cap \mathcal{C}_0$. We claim that the union $C_\sigma = C_1 \cup C_2 \cup \cdots \cup C_s$ forms a t-connected graph; thus deducing a contradiction to the minimality assumption of $\mathcal{C}_0$. Indeed, suppose that we removed a set S of $t-1$ vertices from C_σ, and let v, w be two vertices which have not been removed. Since Γ_0' is t-connected, after removing S from the V part of the graph Γ_0', we still have a connected graph, and thus without loss of generality we can find a path of the form $(C_1, y_1, C_2, y_2, \ldots, y_{h-1}, C_h)$ in the graph Γ_0' where $v \in C_1$ and $w \in C_h$. Let $y_0 = v$ and $y_h = w$. For each i, we have $y_{i-1}, y_i \in V(C_i)$, and since C_i is t-connected, there exists a path from y_{i-i} to y_i in the graph $C_i \setminus S$. By combining these paths, we can find a path from $y_0 = v$ to $y_h = w$ in $C_\sigma \setminus S$.

As mentioned above, this implies that we cannot have a t-connected subgraph of Γ_0. Since each vertex in Y_0 is contained in more than t^4 graphs in $\mathcal{C}_0$, and each graph in $\mathcal{C}_0$ is of order at least t^4, the number of edges of Γ_0 is at least $\frac{1}{2}(t^4|Y_0| + t^4|\mathcal{C}_0|)$, by Lemma 4.2 (iv) we have

$$\frac{1}{2}\left(t^4|Y_0| + t^4|\mathcal{C}_0|\right) \leq 2t(n + |\mathcal{C}_0|),$$

from which it follows that $|Y_0| \leq \frac{4n}{t^3}$.   ∎

Let X_0' be the subset of vertices $v \in X_0$ for which there are at least $\frac{k}{\omega^{1/4}}$ edges in $G[X_0]$ incident to v that are not covered by any of the graphs $C \in \mathcal{C}_0$.

Claim 4.9. $|X_0'| < \frac{n}{t^3}$.

Proof.   Suppose that $|X_0'| \geq \frac{n}{t^3}$. In this case, we claim that we can find a t^4-connected subgraph of G_p which is edge-disjoint from all the graphs in $\mathcal{C}_0$. Since a t^4-connected graph is necessarily a t-connected graph with at least t^4 vertices, this will contradict the maximality of the family $\mathcal{C}_0$.

Color each graph in $\mathcal{C}_0$ by either red or blue, uniformly and independently at random. Let A be the collection of vertices $v \in X_0'$ for which all the graphs in $\mathcal{C}_0$ that contain v are of color red, and similarly define B for blue graphs. Let $\{v, w\}$ be an edge in $G[X_0]$ which is not covered by any graph in $\mathcal{C}_0$. Then since there are no graphs in $\mathcal{C}_0$ containing both v and w, the probability that $\{v, w\}$ contributes towards $e_G(A, B)$ is exactly $2^{-d_v - d_w}$, where d_v and d_w are the numbers of graphs in $\mathcal{C}_0$ that contain v and w, respectively. By the definition of X_0', there are at least $\frac{1}{2}|X_0'| \cdot \frac{k}{\omega^{1/4}}$ edges which are not covered by any graph in $\mathcal{C}_0$. Since the vertices in X_0 are covered at most t^4 times, we have

$$\mathbb{E}[e_G(A, B)] \geq \frac{1}{2}|X_0'| \cdot \frac{k}{\omega^{1/4}} \cdot 2^{-2t^4} \geq \frac{n}{t^3} \cdot \frac{k}{\omega^{1/4}2^{2t^4 + 1}} \geq \frac{kn}{\omega^{1/2}},$$

where we used the fact that $t = (\log \omega)^{1/5}$. Therefore, there exists a choice of coloring of graphs in $\mathcal{C}_0$ such that $e_G(A, B) \geq \frac{kn}{\omega^{1/2}}$, and this implies that $e_{G_p}(A, B) \geq \frac{1}{2}\omega^{1/2}n$ (recall that we conditioned on this fact). By Lemma 4.2 (iv), there exists a t^4-connected subgraph of the bipartite subgraph of G_p induced by $A \cup B$. Furthermore, none of the edges of this t^4-connected subgraph could have been covered by a graph in $\mathcal{C}_0$. Indeed, such a graph should be colored by both red and blue, which is impossible. Therefore, we found a t^4-connected subgraph of G_p as claimed.   ∎

Let $\mathcal{C}_1 = \{C \in \mathcal{C}_0 : |C \cap X_0| \geq \frac{k}{t^5}\}$. Our next claim establishes a useful property regarding vertices not in X_0'.

Claim 4.10.   *For every vertex $v \in X_0 \setminus X_0'$, there are at most $o(k)$ edges of $G[X_0]$ incident to v not covered by any graph in $\mathcal{C}_1$.*

Proof.   Note that there are two possible circumstances in which an edge in $G[X_0]$ is not covered by some graph in $\mathcal{C}_1$. First is if it is not covered by any graph in $\mathcal{C}_0$, and second is if it is covered by some graph in $\mathcal{C}_0 \setminus \mathcal{C}_1$. For a fixed vertex $x \in X_0 \setminus X_0'$, by the definition of the set X_0', there are at most $\frac{k}{\omega^{1/4}} = o(k)$ edges incident to x of the first type. Also, since x is contained in at most t^4 graphs in $\mathcal{C}_0$ and each graph $C \in \mathcal{C}_0 \setminus \mathcal{C}_1$ satisfies $|X_0 \cap V(C)| \leq \frac{k}{t^5}$, there are at most $t^4 \cdot \frac{k}{t^5} = o(k)$ edges incident to x of the second type. Thus we establish our claim.   ■

Let X_0'' be the set of vertices $v \in X_0$ which are covered by at least two graphs in $\mathcal{C}_1$, or are not covered by any graph in $\mathcal{C}_1$. We defer the proof of the following claim, which is somewhat similar to that of Claim 4.8, to later.

Claim 4.11. $|X_0'' \setminus X_0'| < \frac{21n}{t^3}$.

Step 2 : Intermediate Partition

Let $X_1 = X_0 \setminus (X_0' \cup X_0'')$ and $Y_1 = V \setminus X_1 = Y_0 \cup (X_0' \cup X_0'')$. We first verify that the partition $V = X_1 \cup Y_1$ and the collection of graphs $\mathcal{C}_1$ satisfy the following list of properties from the statement of Lemma 4.7:

  a. every graph $C \in \mathcal{C}_1$ is $(\log \omega)^{1/5}$-connected;
  b. the sets $X_1 \cap V(C)$ for $C \in \mathcal{C}_1$ form a partition of X_1, and $|Y_1| < \frac{26n}{t^3} = o(n)$;
  d. for every $C \in \mathcal{C}_1$ and every vertex $v \in X_1 \cap V(C)$, there are at most $o(k)$ edges of G incident to v whose other endpoint lies in $X_1 \setminus V(C)$.

Property (a) follows from the definition of $\mathcal{C}_0$. Property (b) follows from the definition of X_0'' and Claims 4.8, 4.9 and 4.11 which imply that $|Y_1| < \frac{26n}{t^3}$. Property (d) follows from Claim 4.10, Property (b), and the fact that $X_1 \subset X_0 \setminus X_0'$.

In order to find a partition of the vertex set and a collection of graphs satisfying Property (c) as well, we will identify the graphs $C \in \mathcal{C}_1$ that do not satisfy Property (c), and will move the vertices of $X_1 \cap V(C)$ to Y_1. Note that this adjustment does not affect Properties (a) and (d). Our goal is to maintain Property (b) as well by keeping the total number of vertices that we move small enough.

Let X_1' be the subset of vertices of X_1 which have at least $\frac{3k}{4}$ neighbors in the set Y_1. If $|X_1'| \geq \frac{130n}{t^3}$, then the bipartite subgraph induced by $X_1' \cup Y_1$ has at most $\frac{6}{5}|X_1'|$ vertices and at least $\frac{3k}{4}|X_1'|$ edges. Thus the average degree of this graph is at least $2 \cdot \frac{3k}{4} \cdot \frac{5}{6} = \frac{5k}{4}$, which contradicts our assumption saying that G does not contain such a subgraph. Therefore we have

$$|X_1'| < \frac{130n}{t^3}. \tag{1}$$

Claim 4.12.   *For a vertex $x \in X_1 \setminus X_1'$ contained in $C_x \in \mathcal{C}_1$, x has degree at least $\frac{k}{8}$ in the subgraph of G induced by the vertex set $X_1 \cap V(C_x)$.*

Proof. For a vertex $x \in X_1 \setminus X_1'$, let $C_x \in \mathcal{C}_1$ be the graph containing x. Since $x \notin X_1'$, there are at least $\frac{k}{4}$ edges of G incident to x in $G[X_1]$. By Property (d), at most $o(k)$ edges among them are incident to a vertex not in C_x. Therefore, x has degree at least $\frac{k}{4} - o(k) \geq \frac{k}{8}$ in the subgraph of G induced by $X_1 \cap V(C_x)$. ∎

Let $\mathcal{C}_1' = \{C \in \mathcal{C}_1 : |V(C) \cap X_1'| \geq \frac{|V(C)|}{t}\}$ and $\mathcal{C}_1'' = \{C \in \mathcal{C}_1 : |V(C) \cap Y_1| \geq \frac{|V(C)|}{t}\}$.

Claim 4.13. $\sum_{C \in \mathcal{C}_1'} |V(C)| = o(n)$.

Proof. By the definition of $\mathcal{C}_1'$, we have $\sum_{C \in \mathcal{C}_1'} \frac{|V(C)|}{t} \leq \sum_{C \in \mathcal{C}_1'} |V(C) \cap X_1'| \leq |X_1'|$. By (1), this implies $\sum_{C \in \mathcal{C}_1'} \frac{|V(C)|}{t} \leq \frac{130n}{t^3}$, from which it follows that $\sum_{C \in \mathcal{C}_1'} |V(C)| \leq \frac{130n}{t^2}$. ∎

Claim 4.14. $\sum_{C \in \mathcal{C}_1''} |V(C)| = o(n)$.

The proof of Claim 4.14 will be given later.

Step 3 : Final Partition and the Collection of t-Connected Subgraphs

Let $\mathcal{C}_2 = \mathcal{C}_1 \setminus (\mathcal{C}_1' \cup \mathcal{C}_1'')$. Let X_2 be the subset of vertices of X_1 which are covered by some graph in $\mathcal{C}_2$, and let $Y_2 = V \setminus X_2$. We claim that the partition $V = X_2 \cup Y_2$ and the collection $\mathcal{C}_2$ satisfy the claims of the lemma. We recall the properties that we wish to establish.

a. every graph $C \in \mathcal{C}_2$ is $(\log \omega)^{1/5}$-connected;
b. the sets $X_2 \cap V(C)$ for $C \in \mathcal{C}_2$ form a partition of X_2, and $|Y_2| = o(n)$;
c. for every $C \in \mathcal{C}_2$, $|Y_2 \cap V(C)| = o(|V(C)|)$ and the induced subgraph $G[X_2 \cap V(C)]$ contains at least $(1 - o(1))|V(C)|$ vertices of degree at least $k/8$;
d. for every vertex $v \in X_2 \cap V(C)$, there are at most $o(k)$ edges of G incident to v whose other endpoint lies in $X_2 \setminus V(C)$.

As mentioned above, Properties (a) and (d) follow from the same properties for X_1, Y_1, and $\mathcal{C}_1$. Since $|X_2| \geq |X_1| - \sum_{C \in \mathcal{C}_1' \cup \mathcal{C}_1''} |V(C)| = (1 - o(1))n$, and $|Y_2| = o(n)$, Property (b) follows as well. Note that $X_1 \supseteq X_2$, and that the vertices in $X_1 \setminus X_2$ are covered exactly once by some graph in $\mathcal{C}_1$. Therefore, for all $C \in \mathcal{C}_2$, we have $V(C) \cap X_1 = V(C) \cap X_2$ and $V(C) \cap Y_1 = V(C) \cap Y_2$. Thus for $C \in \mathcal{C}_2$, since $C \notin \mathcal{C}_1''$, we have $|V(C) \cap Y_2| = |V(C) \cap Y_1| < \frac{|V(C)|}{t}$, and the first part of Property (c) holds. Also, by Claim 4.12, for $C \in \mathcal{C}_2$ the vertices in $V(C) \cap (X_2 \setminus X_1') = V(C) \cap (X_1 \setminus X_1')$ have degree at least $\frac{k}{8}$ in the subgraph of G induced by $V(C) \cap X_1 = V(C) \cap X_2$. By the fact $C \notin \mathcal{C}_1'$, we have

$$|V(C) \cap (X_1 \setminus X_1')| = (1 - o(1))|V(C) \cap X_1| = (1 - o(1))|V(C)|,$$

and this establishes the second part of Property (c). ∎

It remains to prove Claims 4.11 and 4.14.

Proof of Claim 4.11. Recall that $\mathcal{C}_1 = \{C \in \mathcal{C}_0 : |C \cap X_0| \geq \frac{k}{t^5}\}$ and X_0'' is the set of vertices $v \in X_0$ which are covered by at least two graphs in $\mathcal{C}_1$, or are not covered by any graph in $\mathcal{C}_1$. Let $X_{0,\geq 2}''$ be the vertices which are covered by at least two graphs in $\mathcal{C}_1$ and X_0''' be the vertices not covered by any graph in $\mathcal{C}_1$.

We first estimate the size of the set $X''_{0,\geq 2}$. Since the graphs in $\mathcal{C}_1$ intersect X_0 in at least $\frac{k}{t^5}$ vertices and each vertex in X_0 is covered at most t^4 times, we have $\frac{k}{t^5}|\mathcal{C}_1| \leq t^4|X_0|$, from which it follows that

$$|\mathcal{C}_1| \leq \frac{t^9 n}{k}. \tag{2}$$

Consider the following auxiliary graph Γ_1 over the vertex set $\mathcal{C}_1$, where two vertices $C, C' \in \mathcal{C}_1$ are connected by an edge if they share a vertex from $X''_{0,\geq 2}$ (we place only one edge for each vertex even it is contained in more than two graphs in $\mathcal{C}_1$). The number of vertices of Γ_1 is at most $\frac{t^9 n}{k}$. Since every two graphs in $\mathcal{C}_1$ intersect in less than t vertices, each edge of Γ_1 can account for less than t vertices of $X''_{0,\geq 2}$, and thus the number of edges of Γ_1 is at least $\frac{|X''_{0,\geq 2}|}{t}$.

Suppose that Γ_1 contains a t-connected subgraph over the vertices $C_1, C_2, \ldots, C_s$ of Γ_1. We claim that $C_\sigma = C_1 \cup \cdots \cup C_s$ is a t-connected subgraph and this will contradict the minimality of the family $\mathcal{C}_0$. It suffices to prove that C_σ is connected even after removing a set S of at most $t-1$ vertices. Let v and w be two vertices in $V(C_\sigma) \setminus S$. Each vertex in S corresponds to at most one edge in the auxiliary graph Γ_1, and thus even after removing the edges corresponding to vertices in S, without loss of generality there exists a path $(C_1, C_2, \ldots, C_h)$ of Γ_1 for which $v \in C_1$ and $w \in C_h$. By the definition of the graph Γ_1, for each i, there exists a vertex $v_i \in C_i \cap C_{i+1}$ such that $v_i \notin S$. Let $v_0 = v$ and $v_h = w$. Then for all $0 \leq i < h$, we can find a path from v_i to v_{i+1} in the graph $C_i \setminus S$ (recall that C_i is t-connected). This implies that there exists a path from v to w in $C_\sigma \setminus S$.

Thus Γ_1 cannot contain a t-connected subgraph. By Lemma 4.2 (iv), we then have

$$\frac{|X''_{0,\geq 2}|}{t} \leq 2t \cdot \frac{t^9 n}{k},$$

which implies $|X''_{0,\geq 2}| \leq \frac{2t^{11} n}{k} < \frac{n}{t^3}$ (note that $t = (\log \omega)^{1/5} \leq (\log k)^{1/5}$).

Now consider the set X'''_0. By Claim 4.10 and the definition of X'''_0, each vertex in $Z = X'''_0 \setminus X'_0$ has at least $k - o(k)$ neighbors in the set Y_0. Therefore, if $|Z| \geq 5|Y_0|$, then we obtain a bipartite subgraph of G with at least $|Z| \cdot (k - o(k))$ edges and at most $\frac{6}{5}|Z|$ vertices. Thus this bipartite graph has average degree at least $2 \cdot (k - o(k))\frac{5}{6} \geq \frac{5}{4}k$. However, this contradicts our assumption, and thus we have $|Z| < 5|Y_0| \leq \frac{20n}{t^3}$. Therefore, $|X'''_0 \setminus X'_0| \leq |X''_{0,\geq 2}| + |X'''_0 \setminus X'_0| \leq \frac{21n}{t^3}$. ∎

Proof of Claim 4.14. Recall that $\mathcal{C}''_1 = \{C \in \mathcal{C}_1 : |V(C) \cap Y_1| \geq \frac{|V(C)|}{t}\}$. Consider an auxiliary bipartite graph Γ_2 whose vertex set consists of two parts, where one part is the set Y_1 and the other part is $\mathcal{C}''_1$. A pair (v, C) forms an edge in Γ_2 if $v \in V(C)$. As we have seen in the proof of Claim 4.8, this graph cannot contain a t-connected subgraph (in fact, Γ_2 is a subgraph of Γ_0 defined in the proof of Claim 4.8). By Property (b) in Step 2 which follows from Claims 4.8, 4.9 and 4.11), we have $|Y_1| \leq \frac{26n}{t^3}$, and by (2) we have $|\mathcal{C}''_1| \leq |\mathcal{C}_1| \leq \frac{t^9 n}{k}$. Hence the number of vertices of Γ_2 is $|Y_1| + |\mathcal{C}''_1| \leq (\frac{26}{t^3} + \frac{t^9}{k})n$. The number of edges is at least $\sum_{C \in \mathcal{C}''_1} |V(C) \cap Y_1| \geq \sum_{C \in \mathcal{C}''_1} \frac{|V(C)|}{t}$. Therefore by Lemma 4.2 (iv), we have

$$\sum_{C \in \mathcal{C}''_1} \frac{|V(C)|}{t} \leq 2t \cdot \left(\frac{26}{t^3} + \frac{t^9}{k}\right)n,$$

which implies that $\sum_{C \in \mathcal{C}''_1} |V(C)| < \frac{53n}{t}$ (recall that $t = (\log w)^{1/5} \leq (\log k)^{1/5}$). ∎

One more round of sprinkling will give us our desired structural lemma, Lemma 4.5, which says the following. Let $0 < \varepsilon \leq \frac{1}{2}$ be fixed, G be a graph on n vertices with minimum degree at least k that does not contain a bipartite subgraph of average degree at least $\frac{5}{4}k$ and let $p = \frac{\omega}{k}$ for some function $\omega = \omega(k) \ll k$ that tends to infinity as k does. Then G_p a.a.s. admits a partition $V = X \cup Y$ of its vertex set, and contains a collection $\mathcal{C}$ of subgraphs of G_p satisfying:

  a. for every $C \in \mathcal{C}$, C is $(\log \omega)^{1/5}$-connected;
  b. the sets $X \cap V(C)$ for $C \in \mathcal{C}$ form a partition of X, and $|Y| = o(n)$;
  c. for every $C \in \mathcal{C}$, one of the following holds:
      i. the graph $G[V(C)]$ contains at least $(1-\varepsilon)|V(C)|$ vertices of degree at least $(1-\varepsilon)k$, or
      ii. $|Y \cap V(C)| = o(|V(C)|)$, the graph $G[X \cap V(C)]$ contains at least $(1 - \varepsilon)|V(C)|$ vertices of degree at least $\frac{k}{8}$, and there exists a bipartite graph $\Gamma_C \subseteq G$ with parts $X \cap V(C)$ and $Y \setminus V(C)$ which contains at least $\frac{|V(C)|\varepsilon^2 k}{4}$ edges and has maximum degree at most $\frac{8}{\varepsilon}k$.

Proof of Lemma 4.5. Set $p_1 = p_2 = \frac{\omega}{2k}$ and $t = (\log \omega)^{1/5}$. Apply Lemma 4.7 to G_{p_1} to find a partition $V = X \cup Y$ and a collection $\mathcal{C}$ of subgraphs of G_{p_1} satisfying the following properties:

  a'. every graph $C \in \mathcal{C}$ is $(\log \omega)^{1/5}$-connected;
  b'. the sets $X \cap V(C)$ for $C \in \mathcal{C}$ form a partition of X, and $|Y| = o(n)$;
  c'. for every $C \in \mathcal{C}$, $|Y \cap V(C)| = o(|V(C)|)$ and the induced subgraph $G[X \cap V(C)]$ contains at least $(1 - o(1))|V(C)|$ vertices of degree at least $k/8$;
  d'. for every $C \in \mathcal{C}$ and every vertex $v \in X \cap V(C)$, there are at most $o(k)$ edges of G incident to v whose other endpoint lies in $X \setminus V(C)$.

(we denote the properties by (a'), (b'), (c'), and (d') in order to distinguish it from the properties (a), (b), and (c)).

For $C \in \mathcal{C}$, let $X_C = X \cap V(C)$, $Z_C = Y \setminus V(C)$. Define a bipartite subgraph Γ_C of G with bipartition $X_C \cup Z_C$ as follows: first take all the edges of G between X_C and Z_C, and for each vertex of X_C of degree at least k, retain k arbitrarily chosen edges incident with it. Let $Z'_C \subset Z_C$ be the vertices which have degree greater than $\frac{8k}{\varepsilon}$ in this bipartite subgraph, and let $Z''_C \subset Z_C$ be the vertices which have degree at most $\frac{8k}{\varepsilon}$.

Now expose the edges of G_{p_2}. If a vertex $z \in Z'_C$ has at least t neighbors in G_{p_2} in the set X_C, then we can add z to the graph C to obtain another t-connected subgraph (see Lemma 4.2 (ii)). In such a situation, we say that z is *absorbed* to C, and let the *enlarged* graph $\hat{C}$ be the union of C with the set of vertices which is absorbed by C. Note that even though the same holds for vertices in Z''_C, for technical reasons, we only absorb vertices from Z'_C to C. Further note that we allow a fixed vertex being absorbed to several graphs, and that this does not affect the property that $X \cap V(C)$ forms a partition of X, since each vertex being absorbed is a vertex in Y.

Let $\mathcal{C}'$ be the collection of graphs $C \in \mathcal{C}$ for which the number of edges of Γ_C incident to Z'_C which are not covered by the enlarged graph $\hat{C}$ is at least $\frac{\varepsilon^2 k}{8}|V_C|$.

Claim 4.15. *We a.a.s. have $\sum_{C \in \mathcal{C}'} |X \cap V(C)| = o(n)$.*

Proof.   Suppose that the vertices in Z'_C have degree $d_1, \ldots, d_s$ in Γ_C. Since we only consider at most k edges incident to each vertex of X_C, we have $\sum_i d_i \le k|V(C)|$ (also note that $d_i \le |V(C)|$ for all i). For a vertex $z \in Z'_C$, since z has degree $d_z \ge \frac{8k}{\varepsilon}$ in Γ_C, by Chernoff's inequality, the probability that z cannot be absorbed is at most $e^{-\Omega(\omega)}$. Let N be the random variable which counts the number of edges of Γ_C incident to non-absorbed vertices from Z'_C after exposing G_{p_2}. We have,

$$\mathbb{E}[N] = \sum_i d_i \cdot e^{-\Omega(\omega)} \le \left(\sum_i d_i \right) \cdot e^{-\Omega(\omega)} = o(|V(C)| \cdot k).$$

Let $\mathbf{1}_i$ be the indicator random variable of the event that the i-th vertex of Z'_C is absorbed to C. Note that the events $\mathbf{1}_i$ are independent since they depend on disjoint sets of edges, and that we have $N = \sum_{i=1}^s d_i \cdot \mathbf{1}_i$. Since $0 \le \frac{d_i}{|V(C)|} \le 1$, by applying Hoeffding's inequality to the random variables $\frac{d_i}{|V(C)|} \cdot \mathbf{1}_i$, we see that the probability of $\frac{N}{|V(C)|} \ge \frac{\varepsilon^2 k}{8}$, which is equivalent to $C \in \mathcal{C}'$, is at most $e^{-\Omega(k)}$. Then,

$$\mathbb{E}[\sum_{C \in \mathcal{C}'} |X \cap V(C)|] \le \sum_{C \in \mathcal{C}} |X \cap V(C)| \cdot e^{-\Omega(k)} = o(n).$$

Thus by Markov's inequality, it follows that $\sum_{C \in \mathcal{C}'} |X \cap V(C)| = o(n)$ a.a.s.   ∎

Condition on the conclusion of Claim 4.15. Let $\mathcal{C}_1 = \{\hat{C} : C \in \mathcal{C} \setminus \mathcal{C}'\}$ (recall that $\hat{C}$ is the enlarged graph obtained from C). Let X_1 be the subset of vertices of X covered by some graph in $\mathcal{C}_1$, and let $Y_1 = V \setminus X_1$. We claim that the partition $V = X_1 \cup Y_1$ and the collection of graphs $\mathcal{C}_1$ satisfy properties (a), (b), (c) of Lemma 4.5 (which we listed before this proof).

Property (a) immediately follows from how we constructed the enlarged graphs. Note that the difference between the sets X and X_1 consist of the vertices of $X \cap V(C)$ for $C \in \mathcal{C}'$, and that $\sum_{C \in \mathcal{C}'} |X \cap V(C)| = o(n)$. Since the difference between a graph $C \in \mathcal{C}$ and its enlarged graph $\hat{C}$ lie in $Y \subset Y_1$, Property (b) follows from Property (b'). We now focus on proving that (c) holds as well.

Take a graph $C \in \mathcal{C} \setminus \mathcal{C}'$. If $e_{\Gamma_C}(X_C, Z''_C) \ge \frac{\varepsilon^2 k}{4}|V(C)|$, then (ii) holds and there is nothing to prove (recall that the vertices in Z''_C are not added to the enlarged graph). Suppose that $e_{\Gamma_C}(X_C, Z''_C) < \frac{\varepsilon^2 k}{4}|V(C)|$. Since $C \notin \mathcal{C}'$, there are less than $\frac{\varepsilon^2 k}{8}|V(C)|$ edges of Γ_C incident to Z'_C that are not covered by $\hat{C}$. Therefore, the total number of edges in Γ_C not covered by $\hat{C}$ is at most $e_{\Gamma_C}(X_C, Z''_C) + \frac{\varepsilon^2 k}{8}|V(C)| \le \frac{3\varepsilon^2 k}{8}|V(C)|$.

We can count the number of such edges in another way. Let X'_C be the subset of vertices of X_C, whose degree in $G[V(\hat{C})]$ is less than $(1 - \varepsilon)k$. Since $X_1 \subset X$, by Property (d'), a vertex in X'_C can have at most $o(k)$ neighbors in $X_1 \setminus V(C)$. Therefore, the number of edges of Γ_C not covered by $\hat{C}$ is at least $|X'_C| \cdot \frac{\varepsilon}{2}k$. By combining this with the bound established above, we have

$$|X'_C| \cdot \frac{\varepsilon}{2}k \le \frac{3\varepsilon^2 k}{8}|V(C)|,$$

from which it follows that $|X'_C| \le \frac{3\varepsilon}{4}|V(C)|$. Recall that by Property (c'), we have $|X \cap V(C)| = (1 - o(1))|V(C)|$ for all $C \in \mathcal{C}$. Thus $G[V(\hat{C})]$ contains at least $|X \cap V(C)| - \frac{3\varepsilon}{4}|V(C)| \ge (1 - \frac{7\varepsilon}{8})|V(C)|$ vertices of degree at least $(1 - \varepsilon)k$. It then suffices to prove that

$|V(C)| \geq (1 - \frac{\varepsilon}{8})|V(\hat{C})|$. Since we only added the vertices of Z'_C to C in order to obtain $\hat{C}$, we have

$$|V(\hat{C}) \setminus V(C)| \leq |Z'_C| \leq \frac{e(\Gamma_C)}{(8/\varepsilon)k} \leq \frac{k|V(C)|}{(8/\varepsilon)k} = \frac{\varepsilon}{8}|V(C)|,$$

and it implies $|V(\hat{C})| \leq (1 + \frac{\varepsilon}{8})|V(C)| \leq \frac{1}{1-(\varepsilon/8)}|V(C)|.$ ∎

5. CONCLUDING REMARKS

In this paper, we studied random subgraphs of graphs with large minimum degree. Our goal was to extend classical results on random graphs to a more general setting, where we replace the host graph by a graph with large minimum degree. We proved that the results asserting the a.a.s. existence of long paths and cycles in $G(n, p)$ can in fact be extended to this setting. The problems we addressed in this paper are also closely related to our previous paper [15], where we studied random subgraphs of graphs on n vertices with minimum degree at least $\frac{n}{2}$, and proved that for every graph G of minimum degree at least $\frac{n}{2}$ and $p \gg \frac{\log n}{n}$, the random graph G_p a.a.s. is Hamiltonian.

Similarly to Theorem 1.2, it is natural to expect that for every graph G of minimum degree at least k and $p \geq \frac{(1+\varepsilon)\log k}{k}$, the graph G_p a.a.s. contains a cycle of length at least $k + 1$. While we are unable to settle this question at present, it seems that the techniques we developed in this paper can be useful in attacking this problem.

It is also known that a directed graph of minimum outdegree at least k contains a cycle of length at least $k+1$. However, it is no longer true that there exists a function $p_0 = p_0(k) < 1$ for which the following holds: if $p \geq p_0$, then for every directed graph D of minimum outdegree at least k, D_p a.a.s. contains a cycle of length k. Indeed, suppose that we are given a function p_0 depending only on k. Let N be a large enough integer depending on p_0, and consider a blow-up of a directed cycle of length N, where each vertex is replaced by an independent set of size k, and each edge is replaced by a complete bipartite graph, whose orientation of edges comes from that of the underlying edge in the directed cycle (call this directed graph D). A necessary condition for D_{p_0} to contain a cycle is that each complete bipartite graph contains at least one edge. The probability of this happening is exactly $(1 - (1 - p_0)^{k^2})^N$. However, this can be made arbitrarily small by choosing N to be large enough depending on p_0. Note that if the above event does not hold, then not only does D_{p_0} not contain a cycle of length k, but it also does not contain a cycle of any length. This gives a partial explanation to why the proof of Theorem 1.3 is unexpectedly challenging technically, as many "natural" approaches at one point reduce the problem to a problem of finding a cycle in some directed graph after taking a random subgraph of it.

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