Eidgenössische Technische Hochschule Zürich

a Bachelor Thesis

An introduction to the parametrised Picard-Vessiot Theory with a view towards a Theorem of Hölder

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Abstract

We will give an introduction to the parametrised Picard-Vessiot Theory and introduce the necessary prerequisites. As an application we will prove with it a Theorem by Hölder on the Gamma function.

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1 Introduction

In 1887, Hölder stated and proved the following theorem for the Gamma-function $\Gamma(x) := \int_0^{+\infty} t^{x-1}e^{-t}dt$:

Theorem 1.1 (Hölder) *There does not exist a polynomial $P \in \mathbb{C}[z, X_0, \ldots, X_n]$ for any $n \in \mathbb{Z}^{\geqslant 0}$ such that $P(z, \Gamma(z), \Gamma(z)', \ldots, \Gamma^{(n)}(z)) = 0$.*

He proved this by contradiction. For this, he used the lexicographic ordering $z > X_0 > \ldots > X_n$ on $\mathbb{C}[z, X_0, \ldots, X_n]$ and assumed the existence of a P as in the theorem which is minimal with respect to this ordering. Using the fact that the Gamma-function satisfies the following recursion formula $\Gamma(z+1) = z\Gamma(z)$, he constructed another $P' \in \mathbb{C}[z, X_0, \ldots, X_n]$ of smaller degree then P such that $P'(z, \Gamma(z), \Gamma(z)', \ldots, \Gamma^{(n)}(z)) = 0$. This is a contradiction to the minimality of P.

For an ring-automorphism σ, one calls an equation of the form $\sigma(y) = ay$ a difference equation. An example of this is the automorphism $\sigma : \mathbb{C}[x] \to \mathbb{C}[x]$ induced by the identity on $\mathbb{C}$ and $x \mapsto x+1$ and the difference equation $\sigma(y) = xy$. Note that the Gamma-function is a solution of this equation. This provides another angle to tackle questions like Hölder's Theorem. A theory has been developed to study differential equations which solutions of a difference equation satisfy which is known as *parametrised Picard-Vessiot Theory*. This Thesis follows the expository texts on the subject by C.Hardouin and M.F.Singer, [2] and [3]. For the purpose of this theory, one not only assumes a difference structure on K, but also a differential structure. A derivation on a ring is an additive morphism δ respecting the Leibniz rule. A differential ring is a ring with a specified derivation. An examples is $\mathbb{C}[x]$ with σ induced by the identity and $\sigma(x) = x + 1$ and δ the formal derivation. One also assumes that the derivation δ and the automorphism σ commute.

In the parametrised Picard-Vessiot Theory, one associates to a linear difference equation a differential ring, called the *parametrised Picard-Vessiot-ring* of this equation, which contains a non-trivial solution to the difference equation. This can be seen as an analogue of the splitting field in Galois Theory.

The study of differential rings began in the 1930's by the american mathematician Ritt. He showed that familiar Theorems from commutative algebra, for instance Hilbert's Base Theorem, sensibly adjusted, still hold. Ritt also initated together with his student Ellis Kolchin the field of Differential Algebraic Geometry. They tried to develop a theory akin to Algebraic Geometry. Algebraic Geometry at that time was still in its "classical form", hence the Differential Algebraic Geometry introduced by Ritt and Kolchin is similar to classical algebraic geometry. In a similar way that classical algebraic geometry needs an algebraically closed base field, the differential algebraic geometry developed by Ritt and Kolchin needed a differentially closed base field. We will work at first with them and then find a way to circumvent this restriction. In section 3 we introduce some properties of differential fields and in section 4 give an introduction of to differential algebraic geometry.

Section 5 will give a short review of rings with both a difference and differential structure on it.

Algebraic Geometry allows one to study algebraic groups, that is algebraic varieties with a group structure on it. Mirroring this approach, one can study differential algebraic groups, that is differential varieties with a group structure on it. Classical examples of this would be the vector group $\mathbb{G}_a$ or the torus $\mathbb{G}_m$. An interesting fact is that there exists a constant differential group morphism $\mathbb{G}_m \to \mathbb{G}_a$, whereas there does not exist such a morphism in the algebraic group context. Using this morphism, one can find a precise classification of the differential subgroups of $\mathbb{G}_a$ and $\mathbb{G}_m$. These classification results were first discovered by P.Cassidy in the 1970's. This material will be covered in section 6 & 7

A key fact for the parametrised Picard-Vessiot theory is that the group of $\sigma\delta$-automorphism of the parametrised Picard-Vessiot ring is a differential algebraic group. Furthermore, the differential dimension of this group corresponds to the differential transcendence degree of the solution to the difference equation one is studying. If the difference equation is scalar, the parametrised Galois-Group is a differential subgroup of a $\mathbb{G}_m$. This allows to use the classification results by Cassidy. In particular, this gives a theorem allowing us to study the differential transcendence degree of the solution by studying the coefficient: For $\mathcal{F}$ the field of one-periodic meromorphic functions on $\mathbb{C}$, define a derivation on $\mathcal{F}(x)$ by $\frac{\partial}{\partial x}$ and an automorphism $\sigma : f(x) \mapsto f(x+1)$. Then:

Proposition 1.2 *For $a \in \mathbb{C}(x)\backslash\{0\}$, consider a nonzero meromorphic function g in x over $\mathbb{C}$ such that*

$$\sigma(g) = ag.$$

Then g is differentially algebraic over $\mathcal{F}(x)$ if and only if there exists a non-zero linear homogeneous differential polynomial $\ell \in \mathbb{C}\{X\}$ and an element $f \in \mathbb{C}(x)$ such that $\ell(\frac{\delta(a)}{a}) = \sigma(f) - f$.

This provides us with the tools to give a simple proof to the theorem by Hölder.

I would like to thank my supervisor Professor Richard Pink for his effort, constructive comments and our interesting conversations throughout this thesis.

2 Differential Algebra

Definition 2.1 *A* derivation *on a commutative ring R is an additive map $\delta{:}R \to R$ which satisfies the Leibniz rule, i.e. for all $(a,b) \in R^2$ we have $\delta(ab) = \delta(a)b + a\delta(b)$.*

Definition 2.2 *A pair (R,δ) consisting of a commutative, unitary ring and a derivation on it is called a* differential ring*. If R is a field, then we call it a* differential field*.*

We will drop the derivation from the notation and sometimes call a differential ring a δ-ring for short.

Definition 2.3 *For any differential ring R, we call $R^\delta := \{r \in R \mid \delta(r) = 0\}$* the ring of δ-constants*.*

Note that integers are δ-constants in any differential ring.

Proposition 2.4 *The quotient rule holds, i.e. for any $a,b \in R$ where b is a unit, we have $\delta(\frac{a}{b}) = \frac{\delta(a)b - a\delta(b)}{b^2}$.*

Proof. We have $\delta(a) = \delta(\frac{a}{b}b) = \delta(\frac{a}{b})b + \frac{a}{b}\delta(b)$, which gives $\delta(a)b - a\delta(b)a = \delta(\frac{a}{b})b^2$ and since b is a unit we can conclude. □

Proposition 2.5 *For any differential ring (R,∂), any collection $\{X_i \mid i \in I\}$ of independent variables over R and any choice of $f_i \in R[X_i \mid i \in I]$, there exists a unique derivative δ on the polynomial ring $R[X_i \mid i \in I]$ extending δ such that $\delta X_i = f_i$ for all $i \in I$.*

Example 2.6 For any ring R, the ring $R[x]$ can be seen as a differential ring with the formal derivation $\partial = \frac{d}{dx}$. Another example of a derivation on $R[x]$ is obtained by multiplying the formal derivation with x, i.e. for all $f \in R[x]$ we set $\delta(f) := x\frac{\partial f}{\partial x}$.

Example 2.7 The ring of smooth functions $\mathcal{C}^\infty(\mathbb{R})$ in the indeterminate x together with derivation $\delta = \frac{\partial}{\partial x}$ is a differential ring.

Definition 2.8 *A morphism of rings $\varphi : R \to S$ for differential rings (R,δ) and (S,∂) which commutes with derivation, i.e. $\varphi\delta = \partial\varphi$, is called a* morphism of differential rings*. A* differential isomorphism *is a differential morphism with a two-sided inverse differential morphism.*

Proposition 2.9 *A differential morphism is an differential isomorphism if and only if it is bijective.*

Definition 2.10 *For differential rings R, S, where R is a subring of S, we call R a* differential subring *of S if the inclusion is a differential morphism.*

Definition 2.11 *For a differential ring R, we call an ideal $\mathfrak{a}$ a* differential ideal *if it is stable under derivation, i.e. $\delta(\mathfrak{a}) \subset \mathfrak{a}$.*

Proposition-Definition 2.12 *For any subset $A \subset R$ of a differential ring, there exists a smallest differential ideal $[A]$ containing A.*

Proof. The set A is contained in the differential ideal R. Furthermore, the intersection of a non-empty collection of differential ideals is again a differential ideal since it is already an ideal and for any $x \in \bigcap_{i \in \mathcal{I}} \mathfrak{a}_i$, we have for all $i \in \mathcal{I}$ that $\delta(x) \in \mathfrak{a}_i$ by the definition of differential ideals. If all $\mathfrak{a}_i$ contain A, then so does $\bigcap_{i \in \mathcal{I}} \mathfrak{a}_i$, hence $[A]$ is well defined as the intersection of all differential ideals containing A. $\square$

Example 2.13 For $x^2 \in \mathbb{Z}[x]$, the ideal $[x^2]$ has to contain $2 = \delta^2(x^2)$. Since $(x^2, 2x, 2)$ is stable under derivation we have $(x^2, 2) = [x^2]$.

Definition 2.14 *A* maximal differential ideal *is a proper differential ideal which is maximal among all proper differential ideals.*

Proposition 2.15 *For any proper differential ideal $\mathfrak{a}$, there exists a maximal differential ideal $\mathfrak{m}$ containing $\mathfrak{a}$. In particular, any nontrivial differential ring possesses a maximal differential ideal.*

Proof. This proof works exactly like the proof of Krull's Theorem in "classical" algebra: We consider the set M of proper differential ideal containing $\mathfrak{a}$. This set is non-empty as $\mathfrak{a}$ is an element. For any ascending chain of proper differential ideals containing $\mathfrak{a}$, there exists a upper bound given by the union of all elements of the chain. By Zorn's Lemma there exists a maximal element $\mathfrak{m} \in M$, which is a maximal differential ideal containing $\mathfrak{a}$. $\square$

Definition 2.16 *A differential ideal which is prime as an ideal is called a* prime differential ideal.

Proposition 2.17 *Every maximal differential ideal is prime.*

Proof. Assume that there were a differential ring R containing a maximal differential ideal $\mathfrak{m}$ which is not prime. Then there exist $a, b \in R \backslash \mathfrak{m}$ such that $ab \in \mathfrak{m}$. But then $[a, \mathfrak{m}] = [b, \mathfrak{m}] = R$ and thus $\mathfrak{m} = [ab, \mathfrak{m}] = R$, which is a contradiction. $\square$

Example 2.18 Maximal differential ideals need not be maximal ideals. For an example consider $\mathbb{Q}[x]$ with the formal derivation. If we derive any non-zero polynomial in $\mathbb{Q}[x]$ sufficiently often, we get an non-zero element in $\mathbb{Q}$ and thus a unit. Thus $(0) \subset \mathbb{Q}[x]$ is the maximal differential ideal, but plainly not a maximal ideal.

Definition 2.19 *For any subset $A \subset R$ we define the* radical of A *to be*

$$\operatorname{rad}(A) := \operatorname{rad}([A]) = \{a \in R \mid \exists n \geqslant 1 : a^n \in [A]\}.$$

We call a differential ideal $\mathfrak{a}$ radical *if* $\operatorname{rad}(\mathfrak{a}) = \mathfrak{a}$.

Unfortunately, in general this is not a differential ideal. For instance consider the differential ring $\mathbb{Z}[x]$ with the formal derivation $\partial = \frac{d}{dx}$. If $\mathrm{rad}(x^2)$ were a differential ideal, then $x \in \mathrm{rad}(x^2)$ and $\delta(x) = 1 \in \mathrm{rad}(x^2)$. However $1^n = 1 \notin [x^2] = (x^2, 2)$. The solution to this is to only consider so-called Keigher rings.

Definition 2.20 *A differential ring such that the radical of any differential ideal is a differential ideal is called* Keigher*.*

Proposition 2.21 *If $\mathbb{Q}$ is a subring of a differential ring R, then R is a Keigher ring.*

Proof. Consider a differential ring R such that $\mathbb{Q}$ is a subring. We fix a differential ideal $\mathfrak{a} \subset R$ and $a \in \mathrm{rad}(\mathfrak{a})$. Then there exists $n \in \mathbb{Z}^{\geqslant 0}$ such that $a^n \in \mathfrak{a}$. By differentiating we get $\delta(a^n) = n\delta(a)a^{n-1} \in \mathfrak{a}$ and since $\mathbb{Q}$ is a subring of R, we get $\delta(a)a^{n-1} \in \mathfrak{a}$. We define $a_k = \delta(a)^{2k+1}a^{n-k-1}$ and show by induction that for all $k \leqslant n-1$ we have $a_k \in \mathfrak{a}$. We have just dealt with the base case $k = 0$. Assume that for $0 \leqslant k \leqslant n-2$ we have $a_k \in \mathfrak{a}$. Then

$$\begin{aligned}\mathfrak{a} \ni & \delta(a_k)\delta(a) - a_k(2k+1)\delta^2(a) = \\ =&\delta(a)(n-k-1)a^{n-k-2}\delta(a)^{2k+1}\delta(a) + \delta(\delta(a)^{2k+1})a^{n-k-1}\delta(a) - (2k+1)\delta(a)^{2k+1}a^{n-k-1}\delta^2(a) \\ =&(n-k-1)a^{n-k-2}\delta(a)^{2k+3} + \delta^2(a)(2k+1)\delta(a)^{2k}a^{n-k-1}\delta(a) - \delta^2(a)(2k+1)\delta(a)^{2k}a^{n-k-1}\delta(a) \\ =&(n-k-1)a^{n-k-2}\delta(a)^{2k+3} \\ =&(n-k-1)a_{k+1}\end{aligned}$$

and since $(n-k-1) \neq 0$ invertible in R we have $a_{k+1} \in \mathfrak{a}$. In particular for $k = n-1$ we have $\delta(a)^{2n-1} \in \mathfrak{a}$. Thus we have $\delta(\mathrm{rad}(\mathfrak{a})) \subset \mathrm{rad}(\mathfrak{a})$ and hence $\mathrm{rad}(\mathfrak{a})$ is a differential ideal. □

Remark 2.22 As a convention, we will from now on assume that $\mathbb{Q}$ is a subring of any differential ring. This allows us to assume by the previous proposition that the radical of any set is a differential ideal.

Proposition 2.23 *The radical of a set is a radical ideal. Hence we will call the radical of a set the* radical ideal of a set*.*

Proof. For any differential ring R, consider $A \subset R$. We have $\mathrm{rad}(\mathrm{rad}(A)) = \{r \in R \mid \exists n \in \mathbb{Z}^{\geqslant 1} \text{ st. } r^n \in \mathrm{rad}(A)\}$. By definition of $\mathrm{rad}(A)$, for any $r \in \mathrm{rad}(\mathrm{rad}(A))$ we have $r^m \in [A]$ for some large enough $m \in \mathbb{Z}^{\geqslant 1}$. Hence $\mathrm{rad}(\mathrm{rad}(A)) \subset \mathrm{rad}(A)$. Since the other inclusion is clear, we are done. □

Definition 2.24 *Let (R, ∂) be a differential ring and n a non-negative integer. For each $i \geqslant 0$ and $n \geqslant j \geqslant 0$ let $\delta^i(x_j)$ be a variable. We define a derivation on the polynomial ring with infinitely many variables $R[x_1, ..., x_n, \delta(x_1), ..., \delta(x_n), \delta^2(x_1), ...]$ by defining for all $i \geqslant 0$ and all $r \in R$*

$$\delta(\delta^i(x)) := \delta^{i+1}(x) \quad \text{and} \quad \delta(r) := \partial(r).$$

We call this the differential polynomial ring *and denote it by $R\{x_1, ..., x_n\}$.*

Proposition 2.25 *Let R be a differential ring and $S \subset R$ a multiplicative subset. Then by defining for all $\frac{a}{s} \in S^{-1}R$ a derivation $\delta(\frac{a}{b}) = \frac{\delta(a)b - a\delta(b)}{b^2}$ we get a differential ring structure such that the inclusion $\iota : R \to S^{-1}R$ is a differential ring morphism.*

Proof. This is a computation. $\square$

Example 2.26 The field of rational functions with complex coefficients $\mathbb{C}(x)$ together with $\delta = \frac{\partial}{\partial x}$ is a differential field.

Proposition 2.27 *Let R be a differential ring and $\mathfrak{a}$ a differential ideal. Then the factor ring $R/\mathfrak{a}$ inherits a differential ring structure by defining $\delta : R/\mathfrak{a} \to R/\mathfrak{a}$, $r + \mathfrak{a} \mapsto \delta(r) + \mathfrak{a}$. The projection $\iota : R \to R/\mathfrak{a}$ is a differential morphism.*

Proof. This map is well defined since $\delta(\mathfrak{a}) \subset \mathfrak{a}$. Furthermore by direct computation, one sees that the additivity and the Leibniz rule hold. $\square$

Proposition 2.28 *The differential ideals of the factor ring $R/\mathfrak{a}$ correspond to the differential ideals of R which contain $\mathfrak{a}$.*

Proposition 2.29 *For any differential morphism $\varphi : R \to S$, the kernel is a differential ideal and the image a differential ring. Furthermore φ induces a differential isomorphism $R/\ker(\varphi) \cong \operatorname{im}(\varphi)$.*

Proof. Since the morphism and derivation commute, the kernel is a differential ideal and the image a differential ring. We already know that there is an isomorphism of rings $\bar{\varphi} : R/\ker(\varphi) \to \operatorname{im}(\varphi)$, $r + \ker(\varphi) \mapsto \varphi(r)$. It is also a differential morphism by the following commutative diagram:

$$\begin{tikzcd}
x + \ker(\varphi) \arrow[r, "\varphi"] \arrow[d, "\delta"] & \varphi(x) \arrow[d, "\delta"] \\
\delta(x) + \ker(\varphi) \arrow[r, "\varphi"] & \varphi(\delta(x)) = \delta(\varphi(x))
\end{tikzcd}$$

Hence it is a differential isomorphism. $\square$

Definition 2.30 *We call a differential ring S with a differential morphism $R \longrightarrow S$ an* R-δ-algebra. *If there exist $n \in \mathbb{Z}^{\geqslant 0}$ and $s_i \in S$ for all $i \in \{1, \ldots, n\}$ such that the differential morphism of rings $R\{x_1, \ldots, x_n\} \longrightarrow S$ induced by $x_i \mapsto s_i$ is surjective, we call S a* differentially finitely generated R-δ-algebra *and write $S = R\{s_1, \ldots, s_n\}$.*

Note that since the kernel of a differential morphism is a differential ideal, we can also write $S \cong R\{x, \ldots, x_n\}/\mathfrak{a}$.

Caution: A differentially finitely generated R-δ-algebra is not necessarily a finitely generated R-algebra, for example $\mathbb{Z}\{x\} = \mathbb{Z}[x, \delta(x), \delta^2(x), \ldots]$ is a differentially finitely generated $\mathbb{Z}$-algebra, but not a finitely generated $\mathbb{Z}$-algebra.

Definition 2.31 *A* morphism of R-δ-algebras *is a differential morphism of rings* $\varphi : U \to V$ *such that* φ *commutes with the given morphism* $\iota_U : R \to U$*,* $\iota_V : R \to V$*:*

$$\begin{tikzcd} R \arrow[r, "\iota_U"] \arrow[dr, "\iota_V"'] & U \arrow[d, "\varphi"] \\ & V \end{tikzcd}$$

Differentially finitely generated differential rings need not be Noetherian. For instance, $\mathbb{Z}\{x\}$ is not Noetherian as $[x]$ is not generated as ideal by finitely many elements. One might want to call a differential ring R "differentially Noetherian" if for every differential ideal $\mathfrak{a}$ the exists a finite subset $\{a_1, \ldots, a_m\} \subset R$ such that $\mathfrak{a} = [a_1, \ldots, a_m]$. However, Ritt showed that this does not hold for $\mathfrak{a} = [x^2, \delta(x)^2, \ldots]$. This is remedied by the following definition.

Proposition-Definition 2.32 *For a differential ring R the following conditions are equivalent:*

1. *Any ascending chain of radical differential ideals becomes stationary.*
2. *For any radical differential ideal $\mathfrak{a}$ there exists a finite subset $B \subset \mathfrak{a}$ such that* $\mathrm{rad}(B) = \mathfrak{a}$*. Such a set B is called a* finite basis of $\mathfrak{a}$*.*
3. *Every non-empty set of radical differential ideals has a maximal element.*

A differential ring satisfying these conditions is called Rittian*.*

This proof is similar to the one that Noetherian is well-defined.

Proof. For 1. $\Rightarrow$ 2. we consider an arbitrary ascending chain $\mathrm{rad}(b_1) \subset \mathrm{rad}(b_1, b_2) \subset \ldots$ where all $b_i \in \mathfrak{a}$. If there were no finite basis, then the chain would not become stationary, and we would have a contradiction of 1. For 2. $\Rightarrow$ 1. pick a arbitrary sequence of $b_i \in \mathfrak{a}$ where for $i \neq j$ we have $b_i \neq b_j$. Then the ascending chain of radical ideals $\mathrm{rad}(b_1) \subset \mathrm{rad}(b_1, b_2) \subset \ldots$ becomes stationary at $\mathrm{rad}(b_1, \ldots, b_n)$ for some $n \in \mathbb{Z}^{\geqslant 0}$. However, if $\mathrm{rad}(b1, \ldots, b_n) \subsetneq \mathfrak{a}$. But then for any $b \in \mathfrak{a} \backslash \mathrm{rad}(b_1, \ldots, b_n)$, we would have $\mathrm{rad}(b_1, \ldots, b_n) \subsetneq \mathrm{rad}(b_1, \ldots, b_n, b)$. This is a contradiction and thus $\mathrm{rad}(b_1, \ldots, b_n) = \mathfrak{a}$ and thus we have found a finite basis of $\mathfrak{a}$. For 1. $\Rightarrow$ 3. consider a non-empty set of radical differential ideals S and pick any $\mathfrak{s}_1 \in S$. If there were not a maximal element, we could always pick a strictly larger $\mathfrak{s}_i$ and thus get a chain of radical differential ideals which would not become stationary. For 3. $\Rightarrow$ 1. we note that an ascending chain of radical differential ideals is a non-empty set. By 3. there exists a maximal element in this set and we are done. $\square$

As for Noetherian, Rittian remains invariant under certain constructions.

Proposition 2.33 *For any Rittian differential ring R and any differential ideal $\mathfrak{a} \subset R$, the factor ring $R/\mathfrak{a}$ is Rittian.*

Proof. We check the ascending chain condition. Note that the differential ideals of the quotient ring $R/\mathfrak{a}$ correspond to the differential ideals of R which contain $\mathfrak{a}$. Check that the radical differential ideals of $R/\mathfrak{a}$ correspond to the radical differential ideals of R which contain $\mathfrak{a}$. If there were an ascending chain of radical ideals in $R/\mathfrak{a}$ which does not become stationary, there would be a ascending chain of radical differential ideal in R which would not become stationary. □

To show the differential analogue to Hilbert's Basis Theorem we need some more differential algebra, namely division with remainder.

Definition 2.34 *The* order of a differential polynomial $p \in R\{x\} \backslash R$ *is the largest integer* n *such that* $\delta^n(x)$ *occurs in* p*. The order of an element* $p \in R$ *is* -1*. The* degree of a differential polynomial *of order* $n \geqslant 0$ *is the degree of* P *as polynomial in* $\delta^n(x)$*.*

Definition 2.35 *We define the* rank *of any differential polynomial* $p \in R\{x\} \backslash R$ *to be the pair* $\operatorname{rank}(p) := (\operatorname{ord}(p), \deg(p))$ *ordered with the lexicographic order, i.e.for* $q \in R\{x\}$ *we set*

$$p \ll q \Longleftrightarrow \big[\operatorname{ord}(p) < \operatorname{ord}(q) \text{ or } (\operatorname{ord}(p) = \operatorname{ord}(q) \text{ and } \deg(p) < \deg(q))\big].$$

We define that for all $r \in R$ *and* $p \in R\{x\}$ *we have* $r \ll p$*.*

Definition 2.36 *The* initiant I_p *of a differential polynomial* p *of order* $n \geqslant 0$ *is the leading coefficient of* p *as polynomial in* $\delta^n(x)$*. The initiant of a differential polynomial of order* -1 *is the differential polynomial itself. The* separant S_p *of a differential polynomial* p *of order* $n \geqslant 0$ *is* $\frac{\partial p}{\partial(\delta^n(x))}$*, i.e. the formal partial derivative of* p *seen as polynomial in* $R[x, \delta(x), \ldots, \delta^{n-1}(x)][\delta^n(x)]$*. The separant of a differential polynomial of order* -1 *is the differential polynomial itself.*

Lemma 2.37 *For* $p \in R\{x\}$ *and* $n := \operatorname{ord}(p)$*, we have* $\delta^k(p) = S_p\delta^{n+k}(x) + r$ *for some* $r \in R\{x\}$ *and* $\operatorname{ord}(r) \leqslant n + k - 1$ *for all* $k \geqslant 0$*.*

Proof. We prove this by induction. For the base case $k = 0$, the statement is that $p - S_p\delta^n(x) =: r$ is of smaller order. By partial derivation we get $\frac{\partial r}{\partial \delta^n(x)} = 0$. Hence $\delta^n(x)$ does not occur in r and we are done. Now assume that the statement is true for $k \geqslant 0$, i.e. there exists some $r \in R\{x\}$ of order less or equal then $n + k - 1$ such that $\delta^k(p) = S_p\delta^{n+k}(x) + r$. By derivation we get $\delta^{k+1}(p) = S_p\delta^{n+k+1}(x) + \delta(S_p)\delta^{n+k}(x) + \delta(r)$. Note that $\delta(S_p)\delta^{n+k}(x) + \delta(r)$ is of order $\leqslant n + k$ and thus we are done. □

Proposition 2.38 *For any differential ring* R*,* $q \in R\{x\} \backslash R$ *and* $p \in R\{x\}$*, there exists an element* $r \in R\{x\}$ *and integers* $u, v \in \mathbb{Z}^{\geqslant 0}$ *such that* $S_q^u I_q^v p - r \in [q]$ *and* $r \ll q$*.*

Proof. If p is of smaller rank then q, we are done. So without loss of generality we assume that $\operatorname{ord}(p) \geqslant \operatorname{ord}(q)$. We will first reduce this to the case where p and q are of the same order. Let $n := \operatorname{ord}(q)$ and $\operatorname{ord}(p) = n + k$ for some strictly positive integer. By our previous Lemma, we write $\delta^k(q) = S_q\delta^{n+k}(x) + r$ for some r of order $\leqslant n + k - 1$. Thus, there exists some a, b in $R\{x\}$ of order $\leqslant n + k - 1$ such that

$$S_q p - a\delta^{n+k}(x) = b.$$

Hence $S_q p - a\delta^k(q)$ is of order $\leqslant n + k - 1$. Hence by iterating this process we know that there exists some $u \in \mathbb{Z}^{\geqslant 0}$ and $f \in [q]$ such that $S_q^u p - f$ is of order $\leqslant n$. Hence we assume that $\mathrm{ord}(p) = \mathrm{ord}(q)$.

If the degree of p is smaller then the degree of q, we are done. If not, we consider p and q as polynomials $R[x, \delta(x), \ldots, \delta^{n-1}(x)][\delta^n(x)]$ where the leading coefficient of q is by definition I_q. By the "usual" polynomial division of p by q we get that there exist $v \in \mathbb{Z}^{\geqslant 0}$ and $r \in R[x, \delta(x), \ldots, \delta^{n-1}(x)][\delta^n(x)]$ of degree smaller than the degree of q such that

$$I_q^v p - sq = r.$$

Note that r is of smaller rank than q and we are done. □

Definition 2.39 *For any differential Ring R and any differential ideal $\mathfrak{a} \subset R$ and element $r \in R$, we define the saturation of $\mathfrak{a}$ with respect to r to be the ideal*

$$\mathfrak{a} : (r)^\infty := \{s \in R \mid \exists n \in \mathbb{Z}^{\geqslant 0} : r^n s \in \mathfrak{a}\}.$$

Proposition 2.40 *For a differential field K and a proper, non-zero differential prime ideal $\mathfrak{p} \subset K\{x\}$, and $p \in \mathfrak{p}\backslash\{0\}$ of minimal rank, we have*

$$\mathfrak{p} = [p] : (S_p I_p)^\infty.$$

Proof. For any $q \in \mathfrak{p}$, there exists an $r \in K\{x\}$ of smaller rank then p and integers $u, v \in \mathbb{Z}^{\geqslant 0}$ such that $S_p^u I_p^v q - r \in [p]$. Thus r is in $\mathfrak{p}$ and of smaller rank than p and thus in K. Since $\mathfrak{p}$ is a proper ideal, we have that $r = 0$. Hence we know that $\mathfrak{p} \subset ([p] : (S_p I_p)^\infty)$. Since S_p and I_p do not belong to the prime ideal $\mathfrak{p}$, we have $\mathfrak{p} = ([p] : (S_p I_p)^\infty)$. □

Altough it may be that for an irreducible element $p \in K\{x\}$ the ideal $[p]$ is not prime, it holds that:

Proposition 2.41 *For K a differential field and $p \in K[x]$ an irreducible polynomial, the differential ideal $([p] : (S_p I_p)^\infty)$ is a maximal differential ideal and thus prime.*

Proof. We first show that $([p] : (S_p I_p)^\infty)$ is a proper ideal. Assume that $1 \in ([p] : (S_p I_p)^\infty)$. Then $S_p I_p \in \mathrm{rad}([p])$. Since $I_p \in K^\times$ and $S_p \in K[x]\backslash\{0\}$ is of smaller degree than p, this cannot be, hence $([p] : (S_p I_p)^\infty)$ is proper.

Assume that there exists a differential ideal $\mathfrak{a}$ properly containing $([p] : (S_p I_p)^\infty)$. Then there exists a $q \in \mathfrak{a}\backslash([p] : (S_p I_p)^\infty)$. By division with remainder get the existence of $r \in K\{x\}$ of rank strictly smaller than p and $n \in \mathbb{Z}^{\geqslant 0}$ such that $(S_p I_p)^n p - r \in [p]$. Thus $r \in \mathfrak{a}$. Since r is of rank strictly smaller than p it is in $K[x]$ and of smaller degree than p. Since p is irreducible, we have $(p, r) = (1)$. Then $1 \in \mathfrak{a}$ and we have a contradiction. □

Theorem 2.42 (Ritt's Basis Theorem) *For any R Rittian, $R\{x_1, \ldots, x_n\}$ is Rittian.*

Proof. We will only prove this Theorem in the case that R is a field.

By induction it suffices to show that $R\{x\}$ is Rittian. Assume that there exists a radical differential ideal in $R\{x\}$ with no finite basis. Any ascending chain $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots$ of radical differential ideals without a finite basis has an upper bound in the form of the union $\bigcup_{i\geqslant 0} \mathfrak{a}_i$. This has no finite basis, because if it had a finite basis B then for some $i \geqslant 0$ we would have $B \subset \mathfrak{a}_i$, which would mean that B is a finite basis of $\mathfrak{a}_i$, which is a contradiction. Since by assumption the set of radical differential ideals without finite basis is not empty, we may apply Zorn's Lemma and get a radical differential ideal $\mathfrak{m}$ with no finite basis which is maximal among all radical differential ideals with no finite basis.

We now show that $\mathfrak{m}$ is a prime ideal. It is proper because $R\{x\} = [1]$ has a finite basis. Assume that $\mathfrak{m}$ is not prime, then there exist some $a, b \in R\{x\}\backslash\mathfrak{m}$ such that $ab \in \mathfrak{m}$. Both $\mathrm{rad}(a, \mathfrak{m})$ and $\mathrm{rad}(b, \mathfrak{m})$ have finite basis and we have $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\} \subset \mathfrak{m}$ such that $\mathrm{rad}(a, \mathfrak{m}) = \mathrm{rad}(a, a_1, \ldots, a_n)$ and $\mathrm{rad}(a, \mathfrak{m}) = \mathrm{rad}(b, b_1, \ldots, b_m)$. For any $m \in \mathfrak{m}$ we have

$$m^2 \in \mathfrak{m}^2 \subset \mathrm{rad}(a, a_1, \ldots, a_n)\mathrm{rad}(b, b_1, \ldots, b_m) \subset \mathrm{rad}(ab, ab_1, \ldots, ab_m, a_1b, a_1b_1, \ldots, a_nb_m).$$

Hence $m \in \mathrm{rad}(ab, ab_1, \ldots, ab_m, a_1b, a_1b_1, \ldots, a_nb_m)$ and since $ab \in \mathfrak{m}$ and all a_i's and b_i's are element of $\mathfrak{m}$ as well, this means that

$$\mathfrak{m} = \mathrm{rad}(ab, ab_1, \ldots, ab_m, a_1b, a_1b_1, \ldots, a_nb_m).$$

This is a contradiction and thus $\mathfrak{m}$ is prime.

By the previous proposition we can write $\mathfrak{m} = ([P] : (S_PI_P)^\infty)$ for some $P \in \mathfrak{m}$ of minimal rank. Since $\mathfrak{m}$ is a prime ideal, this means that $S_PI_P \notin \mathfrak{m}$. We claim that $S_PI_P\mathfrak{m} \subset \mathrm{rad}(P)$. Indeed, for any $m \in \mathfrak{m}$, there exists $n \in \mathbb{Z}^{\geqslant 0}$ such that $(S_PI_P)^n m \in [P]$, hence $(S_PI_P)^n m^n \in [P]$ and thus $S_PI_Pm \in \mathrm{rad}(P)$. Since $\mathrm{rad}(S_PI_P, \mathfrak{m}) \supsetneq \mathfrak{m}$, by maximality there exist $\{P_1, \ldots, P_r\} \subset \mathfrak{m}$ such that $\mathrm{rad}(S_PI_P, P_1, \ldots, P_r) = \mathrm{rad}(S_PI_P, \mathfrak{m})$. Hence

$$\mathfrak{m}^2 \subset \mathfrak{m}\mathrm{rad}(S_PI_P, \mathfrak{m}) \subset \mathrm{rad}(S_PI_P\mathfrak{m}, P_1\mathfrak{m}, \ldots, P_r\mathfrak{m}) \subset \mathrm{rad}(P, P_1\mathfrak{m}, \ldots, P_r\mathfrak{m}).$$

Furthermore, since $[P, P_1\mathfrak{m}, \ldots, P_r\mathfrak{m}] \subset [P, P_1, \ldots, P_r]$, we have $\mathrm{rad}(P, P_1\mathfrak{m}, \ldots, P_r\mathfrak{m}) \subset \mathrm{rad}(P, P_1, \ldots, P_r)$. Thus, since $\{P, P_1, \ldots, P_r\} \subset \mathfrak{m}$, we have $\mathfrak{m} = \mathrm{rad}(P, P_1, \ldots, P_r)$. Hence $\mathfrak{m}$ has a finite basis, which is a contradiction and we have shown that all radical differential ideals of $R\{x\}$ have a finite basis. □

Proposition 2.43 *Let R be a Rittian differential ring. Then any differentially finitely generated R-δ-algebra is Rittian.*

Proof. We can write any finitely generated R-δ-algebra as $R\{x_1, \ldots, x_n\}/\mathfrak{a}$. By Ritt's Basis Theorem, $R\{x_1, \ldots, x_n\}$ is Rittian, and as the quotient ring of a Rittian ring is Rittian, so is $R\{x_1, \ldots, x_n\}/\mathfrak{a}$. $\square$

Proposition 2.44 *For any Rittian differential ring R, any radical differential ideal $\mathfrak{a} \subset R$ can be written as the finite intersection of prime differential ideals not containing each other. This decomposition is unique up to permutation.*

Proof. Assume to the contrary that the set of radical differential ideals which are not the finite intersection of prime differential ideals is not empty. Then since R is Rittian, there exists a maximal element $\mathfrak{m}$ in that set. Since it is not prime, there exist $a, b \in R \backslash \mathfrak{m}$ such that $ab \in \mathfrak{m}$. Then $\mathrm{rad}(a, \mathfrak{m})$ and $\mathrm{rad}(b, \mathfrak{m})$ properly contain $\mathfrak{m}$ and can thus be written as the finite intersection of prime differential ideals. For any $c \in \mathrm{rad}(a, \mathfrak{m}) \cap \mathrm{rad}(b, \mathfrak{m})$, $c^2 \in \mathrm{rad}(a, \mathfrak{m})\mathrm{rad}(b, \mathfrak{m}) \subset \mathrm{rad}(ab, \mathfrak{m}) = \mathfrak{m}$, hence $\mathrm{rad}(a, \mathfrak{m}) \cap \mathrm{rad}(b, \mathfrak{m}) = \mathfrak{m}$. This is a contradiction. $\square$

3 Differential Fields

Definition 3.1 *For two differential fields K, L where L is a field extension of K, we call L a* differential field extension *of K if the inclusion is a differential morphism. We will denote this by L/K.*

Definition 3.2 *For a differential field extension L/K and $A \subset L$ we denote by $K\langle A \rangle \subset L$ the smallest differential subfield of L containing K and A.*

This is well defined since the intersection of any differential subfields containing both K and A is a differential subfield containing K and A.

Definition 3.3 *Let L/K be a differential field extension. We say that an element $a \in L$ is* differentially algebraic *over K if there exists a differential polynomial $f \in K\{x\} \backslash \{0\}$ such that $f(a) = 0$. If an element is not differentially algebraic over K, we say that it is* differentially transcendent *over K.*

Definition 3.4 *Let L/K be a differential field extension. We say that elements $a_1, \cdots, a_n$ are* differentially dependent over *K if there exists an $f \in K\{X_1, \cdots, X_n\} \backslash \{0\}$ such that $f(a_1, \cdots, a_n) = 0$. If there does not exist such an f, we say that $a_1, \cdots, a_n$ are* differentially independent over *K.*

Definition 3.5 *We call a differential field extension L/K* differentially algebraic *if every element of $x \in L$ is differentially algebraic over K.*

Note that a differentially algebraic field extension need not be algebraic. For instance $\mathbb{C}\langle e^x \rangle / \mathbb{C}$ is differentially algebraic, as $\delta(e^x) - e^x = 0$, but e^x is transcendent over $\mathbb{C}$. We will use this fact to define an order of differential field extensions.

Definition 3.6 *The* order of a differential field extension L/K *is the transcendence degree of the field extension* L/K.

Proposition 3.7 *For a differential field extension* L/K*, an element* $x \in L$ *is differentially algebraic over* K *if and only if the order of* $K\langle x\rangle/K$ *is finite.*

Proof. If x is differentially algebraic over K, there exists a $f \in K\{z\}\backslash\{0\}$ such that $f(x) = 0$. Then $f \notin K$, hence there exists $n \in \mathbb{Z}^{\geqslant 0}$ such that $\delta^n(x)$ is the highest order of derivation involved in f. Hence $\delta^n(x)$ is algebraic over $K(x, \delta(x), \ldots, \delta^{n-1}(x))$. We now show that $\delta^{n+i}(x)$ is algebraic over $K(x, \delta(x), \ldots, \delta^{n-1}(x))$ for all $i \geqslant 0$. We have just done the base case $i = 0$. Hence fix an $j \geqslant 0$ and assume that $\delta^{n+j}(x)$ is algebraic over $K(x, \delta(x), \ldots, \delta^{n-1}(x))$. There exists an $f \in K(x, \delta(x), \ldots, \delta^{n-1}(x))(X)$ such that $f(\delta^{n+j}(x)) = 0$. Then $\delta(f(\delta^{n+j}(x))) = 0$. This is a differential polynomial in $K\{x, \ldots, \delta^n(x), \delta^{n+j}(x)\}\{\delta^{n+j+1}(x)\}$ and thus $\delta^{n+j+1}(x)$ is differentially algebraic over $K\{x, \ldots, \delta^{n-1}(x)\}$.

If $K\langle x\rangle/K$ has finite order, then by definition there exists $n \geqslant 0$ such that $x, \delta(x), \ldots, \delta^n(x)$ are algebraically dependent over K. Thus x is differentially algebraic over K. □

Proposition 3.8 *A differential field extension* L/K *has order zero if and only if* L/K *is algebraic.*

Proposition-Definition 3.9 *A* differential transcendence basis of a differential field extension L/K *is a subset* B *of* L *such that* $L/K\langle B\rangle$ *is differentially algebraic and* B *is differentially independent over* K*. There exists such a set. The cardinality of such subsets is unique and we call it the differential transcendence degree of* L/K *and denote it by* $\delta\text{-trdeg}(L/K)$.

Proposition 3.10 *For differential field extensions* L/M *and* M/K *the differential transcendence degree is additive, i.e.* $\delta\text{-trdeg}(L/K) = \delta\text{-trdeg}(L/M) + \delta\text{-trdeg}(M/K)$.

We will now introduce differential analogues of an algebraically closed field and the algebraic closure of a field. However, a straightforward adaption doesn't work here. One may be tempted to say that a differential field is closed if any differentially algebraic extension is trivial. But for any differential field K and a variable x we can define a non-trivial differentially algebraic differential field extension $K(x)$ by defining $\delta(x) = 1$, thus it would be nonsensical to define differentially closed in this way as there aren't any fields satisfying this condition.

Definition 3.11 *A differential field* K *is called* differentially closed *if for any* $n, m \in \mathbb{Z}^{\geqslant 0}$ *and* $f_1, \ldots, f_n, g \in K\{x_1, \ldots, x_m\}$*, the existence of a joint solution of the equations* $f_i(x_1, \ldots, x_m) = 0$ *and the inequality* $g(x_1, \ldots, x_m) \neq 0$ *in some differential field extension implies the existence of a joint solution in* K.

Interestingly, the definition of a differentially closed field changed over time. The definition of a differentially closed field which we use was introduced by A.Robinson in 1959. In the 1960's, Kolchin introduced another definition, which is however equivalent to the one introduced by Robinson. In 1968 L.Blum introduced yet another equivalent definition of a differentially closed field, which she used to show that any differential field possesses a differential closure. A differentially closed field is "an enormous field" and even the "differential closure of $\mathbb{Q}$ is a monstrous object" [2], so one might not want to work with them.

Proposition-Definition 3.12 *For any differential field K, there exists a differential field extension $\tilde{K}$ of K such that $\tilde{K}$ is differentially closed and for any differential field extension L/K such that L is differentially closed, there exists a injective differential morphism $\iota : \tilde{K} \to L$. We call such $\tilde{K}$ a* differential closure *of K.*

Proposition 3.13 *For any two differential closures of K, there exists a differential isomorphism which restricts to the identity on K.*

Proposition 3.14 *The field of δ-constants of a differentially closed field is algebraically closed.*

Proof. Pick an arbitrary $f \in k^\delta[x]$ of degree $\geqslant 1$. Then there exists a root z of f in an algebraic closure of k^δ. If $z \in k$, then we are done. If not, we can endow the field extension $k(z)$ with derivation induced by $\delta(z) = 0$. Then $k(z)$ is a differential field extension of k which contains a solution to

$$f(y) = 0 \text{ and } \delta(y) = 0.$$

Since k is differentially algebraically closed, this means there exists a solution s which is a δ-constant. Hence k^δ is algebraically closed. $\square$

Proposition-Definition 3.15 *A differential field extension L of K is called* semiuniversal *if any differentially finitely generated extension of K can be embedded into L. Such an extension always exists.*

Theorem 3.16 *Consider a differential field K and an integral, differentially finitely generated K-δ-algebra R and a differential ring R_0 such that $R \supset R_0 \supset K$. There exists a nonzero element $u_0 \in R_0$ such that every K-δ-homomorphism φ into a semiuniversal extension L of K with $\varphi(u_0) \neq 0$ can be extended to a differential K-δ-homomorphism $\varphi' : R \to L$.*

For reference see Theorem 3 in [6].

4 $\sigma\delta$-Algebra

Definition 4.1 *A triple (R, δ, σ) consisting of a ring R, a derivation δ on R and a ring automorphism σ of R such that (R, δ) is a differential ring and σ and δ commute is called a* $\sigma\delta$-ring. *If (R, δ) is a differential field, then we call it* $\sigma\delta$-field.

In particular this means that σ is a differential morphism.

Remark 4.2 Any differential ring R is a $\sigma\delta$-ring for $\sigma = \mathrm{id}$.

Definition 4.3 *For a $\sigma\delta$-ring R we call $R^\sigma := \{r \in R \mid \sigma(r) = r\}$ the ring of* σ-constants of R. *For a $\sigma\delta$-field K this is a differential field.*

Definition 4.4 *A* $\sigma\delta$-morphism of $\sigma\delta$-rings *is a morphism of rings which commutes with σ and δ.*

Definition 4.5 *An ideal $\mathfrak{a}$ of a $\sigma\delta$-ring R is called a* $\sigma\delta$-ideal *if it is stable under σ and δ, i.e. if $\sigma(\mathfrak{a}) \subset \mathfrak{a}$ and $\delta(\mathfrak{a}) \subset \mathfrak{a}$.*

Definition 4.6 *A $\sigma\delta$-ideal $\mathfrak{m}$ of a $\sigma\delta$-ring R which is maximal among all $\sigma\delta$-ideals of R with respect to inclusion is called a* maximal $\sigma\delta$-ideal.

Proposition 4.7 *For any non-empty $\sigma\delta$-ring R there exists a maximal $\sigma\delta$-ideal $\mathfrak{m}$.*

Example 4.8 We endow the differential polynomial ring $R\{x\}$ with the automorphism $\sigma : R\{x\} \to R\{x\}$ induced by $x \mapsto -x$. The differential ideal $\mathrm{rad}(x^2 - 1)$ is a $\sigma\delta$-ideal. The only proper differential ideals containing it are $[x - 1]$ and $[x + 1]$. However neither $[x - 1]$ nor $[x + 1]$ are σ-stable. Hence $\mathrm{rad}(x^2 - 1)$ is the maximal $\sigma\delta$-ideal. Note however that $\mathrm{rad}(x^2 - 1)$ is not prime. This shows that a maximal $\sigma\delta$-ideal may not even be prime.

Definition 4.9 *A $\sigma\delta$-ring is called* $\sigma\delta$-simple *if it is non-zero and the only $\sigma\delta$-ideals are (0) and R.*

Proposition 4.10 *Let R be a $\sigma\delta$-ring and $\mathfrak{m} \subset R$ a maximal $\sigma\delta$-ideal. The factor ring $R/\mathfrak{a}$ inherits a $\sigma\delta$-ring structure by defining $\sigma : R/\mathfrak{m} \to R/\mathfrak{m}, r + \mathfrak{m} \mapsto r + \mathfrak{m}$. In particular $\iota : R \to R/\mathfrak{m}$ is a $\sigma\delta$-morphism.*

Proof. The ring homomorphism $\pi : R \to R/\mathfrak{m}$ commutes with σ and δ. We just need to show that $\sigma : R/\mathfrak{m} \to R/\mathfrak{m}$ is an automorphism. Surjectivity is clear. The kernel of π is a $\sigma\delta$-ideal of R, which contains $\mathfrak{m}$ and thus by maximality, $\ker(\pi) = \mathfrak{m}$. Hence σ is injective. $\square$

Proposition 4.11 *For any $\sigma\delta$-ring R and maximal $\sigma\delta$-ideal $\mathfrak{m}$, the factor ring $R/\mathfrak{m}$ is $\sigma\delta$-simple.*

Definition 4.12 *We call a $\sigma\delta$-ring S together with a $\sigma\delta$-morphism of $\sigma\delta$-rings $R \longrightarrow S$ a R-$\sigma\delta$-*algebra*.*

Definition 4.13 *For R-$\sigma\delta$-algebras U, V we call a $\sigma\delta$-morphism of $\sigma\delta$-rings $\varphi : U \to V$ a R-$\sigma\delta$-*morphism *if φ commutes with the given $\iota_U : R \to U$ and $\iota_V : R \to V$, i.e.*

$$\begin{tikzcd} R \arrow[r, "\iota_U"] \arrow[dr, "\iota_V"'] & U \arrow[d, "\varphi"] \\ & V \end{tikzcd}$$

Proposition 4.14 *For a $\sigma\delta$-ring S and two S-$\sigma\delta$-algebras R_1, R_2, define*

$$\begin{aligned} \sigma : R_1 \otimes_S R_2 \to R_1 \otimes_S R_2, \quad & \textit{induced by } a \otimes b \mapsto \sigma(a) \otimes \sigma(b) \\ \delta : R_1 \otimes_S R_2 \to R_1 \otimes_s R_2, \quad & \textit{induced by } a \otimes b \mapsto \delta(a) \otimes b + a \otimes \delta(b). \end{aligned}$$

This gives $R_1 \otimes_S R_2$ the structure of S-$\sigma\delta$-algebra such that $\iota_1 : R_1 \to R_1 \otimes_S R_2, r \mapsto r \otimes 1$ and $\iota_2 : R_2 \to R_1 \otimes_S R_2, r \mapsto 1 \otimes r$ are S-$\sigma\delta$-algebra morphisms.

Proof. To show that σ and δ commute it suffices to check on elements of the form $a \otimes b$,

$$\sigma(\delta(a \otimes b)) = \sigma(\delta(a) \otimes b + a \otimes \delta(b)) = \delta(\sigma(a)) \otimes \sigma(b) + \sigma(a) \otimes \delta(\sigma(b)) = \sigma\delta(a \otimes b).$$

Since for any $a, b \in R_1, c, d \in R_2$ we have

$$\begin{aligned} \delta((a \otimes c)(b \otimes d)) = \delta(ab) \otimes cd + ab \otimes \delta(cd) &= \delta(a)b \otimes cd + a\delta(b) \otimes cd + ab \otimes \delta(c)d + ab \otimes c\delta(d) \\ &= (\delta(a) \otimes c + a \otimes \delta(c))(b \otimes d) + (a \otimes c)(\delta(b) \otimes d + b \otimes \delta(d)) \\ &= \delta(a \otimes c)(b \otimes d) + (a \otimes c)\delta(b \otimes d), \end{aligned}$$

δ is a derivation. By right-exactness of the tensor product we get that $\sigma \otimes \mathrm{id}_{R_2}$ resp. $\mathrm{id}_{R_1} \otimes \sigma$ is an automorphism. Hence $R_1 \otimes_S R_2$ is a S-$\sigma\delta$-algebra. $\square$

Proposition 4.15 *Let any $\sigma\delta$-ring R and a multiplicative subset S which is stable under σ. Then by defining for all $\frac{a}{s} \in S^{-1}R$ that $\sigma(\frac{a}{s}) := \frac{\sigma(a)}{\sigma(s)}$ we get an automorphism and using the usual derivation on $S^{-1}R$, we get a $\sigma\delta$-ring structure on $S^{-1}R$ such that the map $\iota : R \to S^{-1}R$ is a $\sigma\delta$-morphism.*

When we say that we localize a $\sigma\delta$-ring R at an element s, we mean that we localize at $S = \{\sigma^i(s) \mid i \in \mathbb{Z}^{\geqslant 0}\}$.

5 Differential Algebraic Geometry

We will now need some differential algebraic geometry. For that we will use an adaption of classical algebraic geometry. This will necessitate the use of a differentially closed field. We will deal with this restriction later on. For the moment we will assume that k is differentially closed.

Definition 5.1 *For any set $I \subset k\{x_1, \dots, x_n\}$ we denote*

$$\mathbb{V}(I) = \{(k_1, \dots, k_n) \in k^n \mid f(k_1, \dots, k_n) = 0 \textit{ for all } f \in I\}.$$

Proposition-Definition 5.2 *We have $\mathbb{V}(\varnothing) = k^n$ and $\mathbb{V}(\{1\}) = \varnothing$. Furthermore $\bigcap_{i \in I} \mathbb{V}(i) = \mathbb{V}(\bigcup_{i \in I}\{i\})$ and $\mathbb{V}(i) \cup \mathbb{V}(j) = \mathbb{V}([i][j])$. Thus the $\mathbb{V}(S)$ form the closed sets of a topology on k^n, which we call the* Kolchin topology*. We will call Kolchin closed sets k-δ-*varieties*.*

Note that we restrict ourself to affine k-δ-varieties. Kolchin and Cassidy developped the theory in greater generality, but this won't be necessary for our purposes.

Any Zariski closed set is Kolchin closed. In general, the Kolchin topology is finer than the Zariski topology. For instance, for any polynomial $p \in k[x]$, the vanishing set $V(p)$ is a finite set of points whereas

$$\mathbb{V}(\delta(x)) = \{c \mid c \in k^\delta\}$$

is a vector space.

Definition 5.3 *For any $X \subset k^n$, we call*

$$\mathfrak{J}(X) := \{f \in k\{x_1, \dots, x_n\} \mid \textit{for all } x \in X : f(x) = 0\}$$

the defining ideal of X.

The defining ideal is a radical differential ideal. Furthermore $Y \subset X$ implies $\mathfrak{J}(X) \subset \mathfrak{J}(Y)$.

Proposition 5.4 *The Kolchin topology is noetherian.*

Proof. Consider a descending chain $X_1 \supset X_2 \supset \dots$ of Kolchin closed sets. This corresponds to an ascending chain of radical differential ideals $\mathfrak{J}(X_1) \subset \mathfrak{J}(X_2) \subset \dots$ in $k\{x_1, \dots, x_n\}$. By Ritt's basis theorem, we know that $k\{x_1, \dots, x_n\}$ is Rittian and thus the chain of radical differential ideals stabilizes. Thus the descending chain of closed sets stabilizes too and thus the topology is noetherian. □

Remark 5.5 In particular, any non-empty closed set can be written as the finite union of irreducible components.

Definition 5.6 *For any Kolchin closed subset, the ring $k\{X\} := k\{x_1, \dots, x_n\}/\mathfrak{J}(X)$ is called the* differential coordinate ring of X*. The localization of the differential coordinate ring of X with respect to all non-zero divisors is called the* total differential coordinate ring of X *and denoted by $k\langle X\rangle$.*

Proposition 5.7 (Differential Analogue to Hilbert's Nullstellensatz) *For a differentially closed field k the maps*

$$\{\textit{radical differential ideals of } k\{x_1,\ldots,x_n\}\} \leftrightarrow \{k\text{-}\delta\textit{-subvarieties of } k^n\}$$

$$\mathfrak{a} \mapsto \mathbb{V}(\mathfrak{a})$$

$$\mathfrak{J}(X) \mapsfrom X$$

are inverse to each other and bijections.

Proposition 5.8 *A subset $X \subset k^n$ is irreducible for the Kolchin topology if and only if $\mathfrak{J}(X)$ is a prime ideal and then $K\langle X\rangle$ is a field. Furthermore, the irreducible components correspond to the minimal prime ideals of $k\{X\}$.*

Proposition 5.9 *Let X be a Kolchin closed subset and $X_1,\ldots,X_n$ its irreducible components. Then $k\langle X\rangle \cong \prod_{i=1}^n k\langle X_i\rangle$.*

Definition 5.10 *For k-δ-varieties $X \subset k^n, Y \subset k^m$ we call a map $\chi : X \to Y$ a* morphism of differential varieties *or a k-δ-*morphism *if there exist $\chi_1,\ldots,\chi_m \in k\{X\}$ such that $\chi(x) = (\chi_1(x),\ldots,\chi_m(x))$. We call a k-δ-morphism an isomorphism if there exists a two-sided inverse k-δ-morphism.*

Definition 5.11 *The k-δ-*dimension $\delta\text{-}\dim_k(X)$ *of an irreducible k-δ-variety X is defined as* $\delta\text{-trdeg}(k\langle X\rangle \mid k)$. *If X is reducible, then we define its k-δ-dimension as the maximum of the k-δ-dimensions of its irreducible components.*

Note that k-δ-morphisms are continuous in the Kolchin topology. Furthermore this gives us a category **affine k-δ-varieties**, whose objects are k-δ-varieties as defined above and whose morphisms are k-δ-morphisms.

Proposition 5.12 *Given two affine varieties $V = \mathbb{V}(f_1,\ldots,f_r) \subset k^n$ for $\{f_1,\ldots,f_r\} \subset k\{x_1,\ldots,x_r\}$ and $W = \mathbb{V}(g_1,\ldots,g_s) \subset k^m$ for $\{g_1,\ldots,g_r\} \subset k\{y_1,\ldots,y_s\}$, define*

$$V \times_k W := \mathbb{V}(f_1,\ldots,f_r,g_1,\ldots,g_s) \subset k^{n+m}$$

*where the f_i's and g_j's are seen as elements in $k\{x_1,\ldots,x_r,y_1,\ldots,y_s\}$. This defines a product in the category of **affine k-δ-varieties**. Hence all finite products exist in this category.*

Proposition 5.13 *A singleton is an affine k-δ-variety and an terminal object in the category of affine k-δ-varieties.*

6 Linear Differential Algebraic Groups

Definition 6.1 *A* linear k-δ-differential group *is a group object in the category* ***affine k-δ-varieties****, i.e. it is a quadruple* $(G, e, \circ, \text{ inv})$ *consisting of an object G and $e \in G$ (or equivalently a morphism $e : 1 \to G$) and morphisms $\circ : G \times G \to G$ and* $\text{inv} : G \to G$*, where m is associative,* inv *is a two-sided inverse of $\circ$ and e is a two-sided unit of m.*

There exists a more general definition of a differential algebraic group, using a more general definition of a k-δ-morphism [7].

Example 6.2 Consider $G = k^n$ with $g_1 \circ g_2 := g_1 + g_2$ and $\text{inv}(g) = -g$. This is sometimes called the *vector group* and denoted by $\mathbb{G}_a^n$. Since $\mathfrak{I}(\mathbb{G}_a^n) = [0]$, we have $k\{\mathbb{G}_a^n\} = k\{x_1, \ldots, x_n\}$ and thus $\delta\text{-}\dim_k(\mathbb{G}_a^n) = n$.

Example 6.3 The set $\mathrm{GL}_n(k) = \{M \in k^{n^2} \mid \det(M) \neq 0\}$ is Kolchin open, hence it is not a k-δ-group. However, we can identify it with $\{(M, d) \in k^{n^2+1} \mid \det(M)d = 1\}$, which is Zariski closed and thus Kolchin closed in k^{n^2+1} and thus a linear k-δ-group.

Example 6.4 The torus $\mathbb{G}_m^n = \{(k^\times)^n\}$ with componentwise multiplication as group law is a k-δ-subgroup of $\mathrm{SL}_{n+1}(k)$. Any element of $(x_1, \ldots, x_n) \in \mathbb{G}_m^n$ can be seen as

$$\begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \frac{1}{\prod_{i=1}^n x_i} \end{pmatrix} \in \mathrm{SL}_{n+1}(k)$$

and thus $\mathbb{G}_m^n = \mathbb{V}\left(z_{i,j} \text{ for } i \neq j \ , \prod_{i=1}^{n+1} z_{i,i}\right)$, hence the coordinate ring is

$$k\{\mathbb{G}_m^n\} = k\{x_1, \ldots, x_{n+1}\}/\mathrm{rad}(\prod_{i=1}^n x_i - 1) \cong k\{x_1, \ldots, x_n, \frac{1}{x_1 \cdots x_n}\}.$$

In particular $\delta\text{-}\dim_k(\mathbb{G}_m^n) = n$. For ease of notation, we will denote elements $(x_1, \ldots, x_n, \frac{1}{x_1 \cdots x_n})$ with $(x_1, \ldots, x_n)$.

Definition 6.5 *A group morphism between two k-δ-groups G, H which is a k-δ-morphism of varieties is called a k-δ-*group morphism. *A k-δ-group isomorphism is a k-δ-group morphism with a both-sided inverse.*

Remark 6.6 The kernel of a k-δ-group morphism is a k-δ-subgroup of the source. This because it is a subgroup and Kolchin closed, since it is defined as the solution to elements of $k\{x_1, \ldots, x_n\}$. The image is a k-δ-subgroup of the target.

Proposition 6.7 *For any k-δ-group G and any normal k-δ-subgroup N of G, there exists a k-δ-group G/N together with a morphism $\pi : G \to G/N$ such that it satisfies the universal property of the quotient group, i.e. for any k-δ-morphism $\varphi : G \to G'$ such that $N \subset \ker(\varphi)$, there exists a unique k-δ-morphism $\tilde{\varphi} : G/N \to G'$ making the following diagram commute:*

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & G/N \arrow[d, "\exists!\tilde{\varphi}"] \\ & G' \end{tikzcd}$$

Proposition 6.8 *Any k-δ-morphism $\varphi : G \to G'$ induces an isomorphism $G/\ker(\varphi) \cong \mathrm{im}(G)$.*

Proposition 6.9 *For any k-δ-group G and a normal k-δ-subgroup N, the k-δ-subgroups of G/N correspond bijectively to the k-δ-subgroups of G containing N.*

Example 6.10 An important example of a k-δ-group morphism is the *logarithmic derivative* $\mathrm{dlog} : \mathbb{G}_m \to \mathbb{G}_a, g \mapsto \frac{\delta(g)}{g}$. For any $x, y \in \mathbb{G}_m$, we have $\mathrm{dlog}(xy) = \frac{\delta(xy)}{xy} = \frac{\delta(x)}{x} + \frac{\delta(y)}{y} = \mathrm{dlog}(x) + \mathrm{dlog}(y)$, thus this is a morphism of groups. In particular, since $\frac{\delta(x)}{x} \in k\{\mathbb{G}_m\} = k\{x, \frac{1}{x}\}$ we know that dlog is a k-δ-group morphism. However, it is not a morphism of algebraic groups.

This morphism is surjective. Indeed, pick an arbitrary element g of $\mathbb{G}_a$. Finding an element of the pre-image of g in $\mathbb{G}_m$ corresponds to finding a solution to the equations $\delta(z) = gz$ and $z \neq 0$. Since k is differentially closed, there exists such an element in k. The kernel of dlog is $\mathbb{G}_m(k^\delta)$. We get the following exact sequence:

$$1 \to \mathbb{G}_m(k^\delta) \to \mathbb{G}_m \to \mathbb{G}_a \to 1.$$

In particular, we have have the following differential isomorphism $\mathbb{G}_a \cong \mathbb{G}_m/\mathbb{G}_m(k^\delta)$.

Example 6.11 The k-δ-group $\mathbb{G}_m(k^\delta)$ is not an algebraic group over k, because it is not Zariski closed in k. However, it is Zariski closed in k^δ and thus an algebraic group over k^δ.

Proposition 6.12 *For any k-δ-group G there a exists a unique connected component containing the identity. This is a normal k-δ-subgroup G^0 of finite index of G.*

Proof. Since G^0 is Kolchin closed, to show that it is a k-δ-subgroup it suffices to show that it is a group. Pick an arbitrary element $g \in G^0$. Then $g^{-1}G^0 \cap G^0 \neq \varnothing$ and thus $g^{-1}G^0 = G^0$ and $g^{-1} \in G^0$. Hence we have for any $g \in G^0$ that $gG^0 \cap G^0 \neq \varnothing$ and thus $gG^0 = G^0$. Hence G^0 is a k-δ-subgroup.

For an arbitrary element $g \in G$, we have that an k-δ-isomorphism $y \mapsto gyg^{-1}$. Thus gG^0g^{-1} is closed in G and has a non-empty intersection with G^0. Thus we have shown that for all $g \in G$ we have $gG^0g^{-1} = G^0$ and thus G^0 is a normal k-δ-subgroup.

Furthermore G possesses a finite decompostion into connected components G_i. Since for any $g \in G_i$ we have $gG_0 \cap G_i \neq \varnothing$, we know that $G_i = gG_0$. Hence G^0 is of finite index in G. $\square$

Proposition 6.13 *The irreducible and connected components in the Kolchin topology of any k-δ-group coincide.*

In particular this means that the identity component of any k-δ-group is irreducible.

7 k-δ-subgroups of $\mathbb{G}_a$ and $\mathbb{G}_m$

Theorem 7.1 *A subset of k is a proper k-δ-subgroup of $\mathbb{G}_a$ if and only if it is the zero set of a non-zero linear homogeneous differential polynomial in $k\{x\}$.*

Proof. The zero set of a non-zero linear homogeneous differential polynomial ℓ is a proper Kolchin-closed subset of k. Since ℓ is homogeneous, $0 \in \mathbb{V}(\ell)$ and it is a subgroup as the sum of two solutions of ℓ is again a solution. Thus it is a proper k-δ-subgroup of $\mathbb{G}_a$.

Conversely, fix a proper k-δ-subgroup G of $\mathbb{G}_a$. We first show that G is a k^δ-vector space. Let t be a new variable, and extend δ from k to $k[t]$ by defining $\delta(t) := 0$. Pick an element $h \in \mathbb{G}\backslash\{0\}$ and an $f \in \mathfrak{J}(G)$. Then $f(th) \in k[t]$. Since $nh \in G$ for all $n \in \mathbb{Z}$, we have that $f(nh) = 0$. Since $\mathbb{Z}$ is Zariski dense in k^δ, we have $f(lh) = 0$ for all $l \in k^\delta$ and thus, since f was chosen arbitrarily in $\mathfrak{J}(G)$, we have $lh \in G$. Hence G is a k^δ-vector space.

Since vector spaces do not possesses proper subgroups of finite index, G is irreducible and we know that $\mathfrak{J}(G)$ is prime. By 2.40, for $P \in \mathfrak{J}(G)\backslash\{0\}$ of minimal rank in $\mathfrak{J}(G)$ we have

$$\mathfrak{J}(G) = [P] : (I_P S_P)^\infty.$$

Because P is of minimal rank, and S_P and I_P are of smaller rank then P, we know that S_P and I_P are not in $\mathfrak{J}(G)$. Hence $S_P I_P \notin \mathfrak{J}(G)$ since $\mathfrak{J}(G)$ is prime. Thus there exists a $g \in G$ such that $S_P(g)I_P(g) \neq 0$. Since G is an k^δ-vector space, we have for of all $h \in G$ and all $t \in k^\delta$ that $P(g + th) = 0$. We introduce a new variable u and extend δ to $k[u]$ by setting $\delta(u) := 0$. Using the Taylor extension of $P(g + hu) \in k[u]$ at $u = 0$, we can write

$$P(g + hu) = \sum{}' P_k(g, h)u^k$$

where $P_k(g, h) \in k\{g, h\}$. This means that for all $k \geqslant 0$ we have that $P_k(g, h) = 0$ and thus in particular $P_1(g, h) = 0$. By definition of the Taylor extension, $P_1(g, h)$ is the partial derivative of $P(g + hu)$ with respect to u evaluated at $u = 0$. We recall that if we denote $\underline{x} = (\delta^i(x))_{\geqslant 0}$, we get $k\{x\} = k[x_0, x_1, \ldots]$. Furthermore, we denote by $\underline{g} := (\delta^i(g))_{i\geqslant 0}$ and

$\underline{h} := (\delta^i(h))_{i\geqslant 0}$. We then consider $P(\underline{x})$ as an element of $k[x_0, x_1, \ldots]$. Using the chain rule we get

$$\big[\frac{\partial}{\partial u}P(\underline{g}+\underline{h}u)\big]_{u=0} = \sum_{k\geqslant 0}\frac{\partial P}{\partial x_k}(\underline{g})h_k.$$

Thus, if we define

$$P_1(\underline{g},\underline{x}) := \sum_{k\in\mathbb{Z}^{\geqslant 0}}\frac{\partial P}{\partial x_k}(\underline{g})x_k \in k[\underline{x}],$$

we them have for all $g' \in G$ that $P_1(\underline{g},\underline{g'}) = 0$. Thus $P_1(\underline{g},\underline{x}) \in \mathfrak{J}(G)$. We will now show that $\mathfrak{J}(G) = [P_1(\underline{g},\underline{x})]$. By the linearity of $P_1(\underline{g},\underline{x})$, its rank is smaller or equal to the rank of P. Thus $P_1(\underline{g},\underline{x})$ is of minimal rank. Again by 2.40 $\mathfrak{J}(G) = [P_1(g,x)] : (I_{P_1(g,x)}S_{P_1(g,x)})^\infty$. However, since $I_{P_1(g,x)} = S_{P_1(g,x)} \in k^\times$, this means that

$$\mathfrak{J}(G) = [P_1(g,x)].$$

Since $P_1(g,x)$ is a non-zero linear homogeneous differential polynomial, we are done. $\square$

Theorem 7.2 *A subset of $k^\times$ is a proper k-δ-subgroup of $\mathbb{G}_m$ if and only if it is the zero set of an equation of the form*

- $x^n - 1$ *for $n \in \mathbb{Z}^{\geqslant 0}$ or*
- $\ell(\frac{\delta(x)}{x})$ *for some linear homogeneous differential polynomial in $\ell \in k\{x\}$.*

Proof. Consider a proper k-δ-subgroup H of $\mathbb{G}_m$ and its k-δ-subgroup of its k^δ-rational points $H(k^\delta)$. Note that $H(k^\delta)$ is a k-δ-subgroup of $\mathbb{G}_m(k^\delta)$ and thus either μ_n for some $n \in \mathbb{Z}^{\geqslant 1}$ or $\mathbb{G}_m(k^\delta)$.

Since $H^0(k^\delta) = H^0 \cap \mathbb{G}_m(k^\delta)$ we know by the differential analogue of Hilbert's Nullstellensatz that $\mathfrak{J}(H^0(k^\delta)) = \mathrm{rad}(\mathfrak{J}(H^0) + [\delta(x)])$. We also have that

$$k\{x, \tfrac{1}{x}\}/(\mathfrak{J}(H^0) + [\delta(x)]) \cong k[x, \tfrac{1}{x}]/(\mathfrak{J}(H^0) \cap k[x, \tfrac{1}{x}]).$$

Since H^0 is irreducible, $\mathfrak{J}(H^0) \cap k[x\frac{1}{x}]$ is prime and thus the ring above is an integral domain. Hence $\mathfrak{J}(H^0) + [\delta(x)]$ is radical. If $H^0(k^\delta)$ is finite and cyclic, we know that $\mathfrak{J}(H^0) \cap k[x\frac{1}{x}] \supset (x^n - 1)$ and thus H^0 is finite and cyclic. But since H^0 is a normal subgroup of finite index of H, we know that H is finite and cyclic.

We now assume that $H(k^\delta) = \mathbb{G}_m(k^\delta)$ and thus $H > \mathbb{G}_m(k^\delta)$. For this we recall form Ex. 6.10 the following differential isomorphism $\mathbb{G}_a \cong \mathbb{G}_m/\mathbb{G}_m(k^\delta)$. Thus, there exists a proper k-δ-subgroup H' of $\mathbb{G}_a$ such that dlog induces the isomorphism $H/\mathbb{G}_m(k^\delta) \cong H' \lneq \mathbb{G}_a$. In particular the image of H under dlog is a proper subgroup. Thus H is the zero set of an equations of the form $\ell(\frac{\delta(x)}{x})$ for some linear homogeneous differential polynomial $\ell \in k\{x\}$. Thus H is the zero-set of some equations of the form $\ell(\frac{\delta(x)}{x})$ or $x^n - 1$. $\square$

Note that any proper k-δ-subgroup G of $\mathbb{G}_m$ is contained in the zero set of a non-zero linear homogeneous differential polynomial. If G is the zero set of an equation of the form $x^n - 1$, then we know that for any $h \in H$ we have $h^n = 1$ and thus $\frac{\delta(h^n)}{h^n} = n\frac{\delta(h)}{h} = 0$. The second case is trivial.

One can generalize our results to k-δ-subgroups of $\mathbb{G}_a^n$ and $\mathbb{G}_m^n$:

Theorem 7.3 *A subset of k^n is a proper k-δ-subgroup of $\mathbb{G}_a^n$ if and only if it is the zero set of a finite number of non-zero linear homogeneous differential polynomials in $k\{x_1, \dots, x_n\}$.*

Theorem 7.4 *A subset of $(k^\times)^n$ is a proper k-δ-subgroup of $\mathbb{G}_m^n$ if and only if it is the zero set of a finite number of equation of the form*

- $x_1^{m_1} \cdots x_n^{m_n} - 1$ *for* $(m_1, \dots, m_n) \in \mathbb{Z}^n$ *or*
- $\ell(\frac{\delta(x_1)}{x_1}, \dots, \frac{\delta(x_n)}{x_n})$ *for some linear homogeneous differential polynomial* $\ell \in k\{X_1, \dots, X_n\}$.

Furthermore for any proper subgroup G of $\mathbb{G}_m^n$ there exists a non-zero linear homogeneous differential polynomial $L \in k\{X_1, \dots, X_n\}$ such that $G \subset \mathbb{V}\left(L\left(\frac{\delta(x_1)}{x_1}, \dots, \frac{\delta(x_n)}{x_n}\right)\right)$.

8 Parametrised Picard-Vessiot-Theory

8.1 $\sigma\delta$-Picard-Vessiot Ring

For ease of notation, when $Z = (Z_{i,j})_{i,j}$ an element of $\mathrm{GL}_n(R)$, we will denote $K\{(Z_{i,j})_{i,j=1,\dots,n}, \frac{1}{\det(Z)}\}$ by $K\{Z, \frac{1}{\det(Z)}\}$.

Definition 8.1 *For a $\sigma\delta$-field K and $A \in \mathrm{GL}_n(K)$, a* parametrised Picard-Vessiot ring *for the equation $\sigma(Y) = AY$ is a $\sigma\delta$-simple K-$\sigma\delta$-algebra R such that there exists $Z \in \mathrm{GL}_n(R)$ with $\sigma(Z) = AZ$ whose entries differentially generate R, i.e. $R = K\{Z, \frac{1}{\det(Z)}\}$, which we call a* fundamental solution. *We will sometimes abbreviate the name "parametrised Picard-Vessiot ring" to "$\sigma\delta$-Picard-Vessiot ring" or simply "$\sigma\delta$-PV-ring".*

Proposition 8.2 *For any $\sigma\delta$-field K and $A \in \mathrm{GL}_n(K)$, there exists a $\sigma\delta$-Picard-Vessiot ring for $\sigma(Y) = AY$.*

Proof. We give an explicit construction. Let X be an $n \times n$-matrix whose entries are variables. We first endow the differential ring $K\{(X_{i,j}), \frac{1}{\det(X)}\}$ with the structure of a K-$\sigma\delta$-algebra by extending σ. We do this by defining $\sigma(X_{i,j}) := (AX)_{i,j}$ for all $i, j \in \{1, \dots, n\}$ and $\sigma(\frac{1}{\det(X)}) := \frac{1}{\det(A)\det(X)}$. Since $K\{(X_{i,j}), \frac{1}{\det(X)}\}$ is generated by these elements and σ and δ have to commute, we can extend σ to $K\{(X_{i,j}), \frac{1}{\det(X)}\}$. Since this $\sigma\delta$-ring is non-zero, there exists a maximal $\sigma\delta$-ideal $\mathfrak{m}$. Then the quotient ring $R := K\{(X_{i,j}), \frac{1}{\det(X)}\}/\mathfrak{m}$ is $\sigma\delta$-simple. Furthermore, R is generated as a K-$\sigma\delta$-algebra by the images of $X_{i,j}$'s and $\frac{1}{\det(X)}$ into R, i.e. $R = K\{(\bar{X}_{i,j}), \frac{1}{\det(\bar{X})}\}$. Thus R is a $\sigma\delta$-Picard-Vessiot ring for $\sigma(Y) = AY$. $\square$

This $\sigma\delta$-Picard-Vessiot ring of an equation is in general not unique. However, we will later show that under the assumption that K^σ is differentially closed, two $\sigma\delta$-Picard-Vessiot rings are isomorphic as $\sigma\delta$-algebras.

Proposition 8.3 *For a $\sigma\delta$-field K and a differentially finitely generated $\sigma\delta$-simple K-$\sigma\delta$-algebra R , there exist $n \in \mathbb{Z}^{\geqslant 0}$ and $e_0, \dots, e_{n-1} \in R$ such that*

1. $e_0 + \ldots + e_{n-1} = 1$, $e_i^2 = e_i$, $e_i e_j = 0$ *for all* $i \neq j$
2. $R = Re_0 \oplus \ldots \oplus Re_{n-1}$ *and* $\sigma(e_i) = e_{i+1 \mod n}$
3. *the Re_i are integral δ-rings and $\sigma^n\delta$-simple $\sigma^n\delta$-rings*

Proof. Since R is $\sigma\delta$-simple, the ideal $[0]$ is the maximal $\sigma\delta$-ideal. In particular, it is radical. Since R is differentially finitely generated over a field, it is Rittian. Thus we can write $[0] = \bigcap_{i=0}^{n-1} \mathfrak{p}_\mathfrak{i}$ for prime differential ideals $\mathfrak{p}_\mathfrak{i}$ which do not contain one another. The $R/\mathfrak{p}_i$'s are integral domains. Since the $\mathfrak{p}_i$'s are unique up to permutation, the image of any one these prime ideals $\mathfrak{p}_i$ under σ is one of the prime ideals $\mathfrak{p}_j$. We now show that the action of σ on the $\mathfrak{p}_\mathfrak{i}$ is transitive. If not, there would exist a non-empty proper subset $\mathcal{I} \subsetneq \{0, \dots, n-1\}$ such that $\sigma(\bigcap_{i\in\mathcal{I}} \mathfrak{p}_\mathfrak{i}) = \bigcap_{i\in\mathcal{I}} \mathfrak{p}_\mathfrak{i} \supsetneq \{0\}$. In particular $\bigcap_{i\in\mathcal{I}} \mathfrak{p}_\mathfrak{i}$ is a $\sigma\delta$-ideal and non-trivial. But this contradicts the fact that R is $\sigma\delta$-simple. Thus, after possible renumbering, we can write $\sigma(\mathfrak{p}_\mathfrak{i}) = \mathfrak{p}_{i+1 \mod n}$. Thus, the $\mathfrak{p}_\mathfrak{i}$ are prime $\sigma^n\delta$-ideals and $\sigma(R/\mathfrak{p}_i) = R/\mathfrak{p}_{i+1 \mod n}$.

We also want to show that the $R/\mathfrak{p}_\mathfrak{i}$'s are $\sigma^n\delta$-simple rings. This is equivalent to the non-existence of proper $\sigma^n\delta$-ideals of R properly containing $\mathfrak{p}_\mathfrak{i}$. Assume that there exists such a differential ideal $\mathfrak{q}$. Then $\bigcap_{k=0}^{n-1} \sigma^k(\mathfrak{q})$ is stable under σ, since $\mathfrak{q}$ is a $\sigma^n\delta$-ideal. Hence $\bigcap_{k=0}^{n-1} \sigma^k(\mathfrak{q})$ is a $\sigma\delta$-ideal and since R is $\sigma\delta$-simple we know $\bigcap_{k=0}^{n-1} \sigma^k(\mathfrak{q}) = [0]$. In particular $\bigcap_{k=0}^{n-1} \sigma^k(\mathfrak{q}) \subset \mathfrak{p}_\mathfrak{i}$ and thus there exists a $j \in \{0, \dots, n-1\}$ such that $\sigma^j(\mathfrak{q}) \subset \mathfrak{p}_i$. Furthermore, since $\mathfrak{p}_i \subset \mathfrak{q}$, we have $\sigma^j(\mathfrak{p}_i) \subset \sigma^j(\mathfrak{q}) \subset \mathfrak{p}_i$. But since $\sigma^j(\mathfrak{p}_i) = \mathfrak{p}_{i'}$ for some i' and no $\mathfrak{p}_i$ are contained in one another, we have $\sigma^j(\mathfrak{p}_i) = \mathfrak{p}_i$ and thus $\mathfrak{p}_i \subset \sigma^j(\mathfrak{q}) \subset \mathfrak{p}_i$. Hence we have $\mathfrak{p}_i = \sigma^{-j}(\mathfrak{p}_i) = \mathfrak{q}$, which is a contradiction. Hence the $R/\mathfrak{p}_i$ are $\sigma^n\delta$-simple rings.

Furthermore, for all $i \neq j$, the sum $\mathfrak{p}_i + \mathfrak{p}_j$ is a $\sigma^n\delta$-ideal which properly contains $\mathfrak{p}_i$. Thus by the previous argument it has to be R, hence the $\mathfrak{p}_i$'s are coprime. In particular, this allows us to apply the Chinese Remainder Theorem and we know that $\pi : R \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} R/\mathfrak{p}_i$, $r \mapsto (\bar{r}, \dots, \bar{r})$ is an isomorphism. Since it is a differential ring morphism, this is also a differential isomorphism. We set $e_i := \pi^{-1}(0, \dots, 1_{R/\mathfrak{p}_i}, 0 \dots, 0)$ for all i. In particular, for all i we have $e_i^2 = \pi^{-1}(0, \dots, 1_{R/\mathfrak{p}_i}^2, 0 \dots, 0) = e_i$, for all $i \neq j$ $e_j e_i = \pi^{-1}(0, \dots, 0) = 0$ and $e_0 + \ldots e_n = \pi^{-1}(1_{R/\mathfrak{p}_0}, \dots, 1_{R/\mathfrak{p}_{n-1}}) = 1$. Furthermore $Re_i = \pi^{-1}(R/\mathfrak{p}_i)$. Since π is a differential isomorphism, the Re_i's are integral domains and δ-simple. □

This allows us to define the differential transcendence degree of a $\sigma\delta$-Picard-Vessiot Ring.

Proposition-Definition 8.4 *For any $\sigma\delta$-field K and $A \in \mathrm{GL}_n(K)$, consider a $\sigma\delta$-Picard-Vessiot ring R for $\sigma(Y) = AY$. Then the total ring of fractions L of R is the product of finitely many $\sigma^n\delta$-fields L_i which are all differentially isomorphic. We define δ-*trdeg$(R/K) := \delta$-trdeg(L_i/K).

Proof. Since R is a $\sigma\delta$-Picard-Vessiot ring over K, it is differentially finitely generated K-$\sigma\delta$-algebra and $\sigma\delta$-simple. Thus we know, thanks to the previous Proposition, that $R = Re_0 \oplus \ldots \oplus Re_n$. Since the total ring of fractions of the product is the product of the respective total rings of fractions, we have $L = \bigoplus_{i=0}^{n} \mathrm{Quot}(Re_i)$. It remains to show that the $L_i := \mathrm{Quot}(Re_i)$ are isomorphic as differential fields. Note that σ is an automorphism of R and that $\sigma(Re_i) = Re_{i+1 \mod n}$. In particular since R is a $\sigma\delta$-algebra, σ is a differential morphism of rings. Hence we have $Re_i \cong Re_j$ for all i, j and $\mathrm{Quot}(Re_i) = \mathrm{Quot}(Re_j)$. Since the transcendence degree is invariant under differential isomorphism, δ-trdeg(R/K) is well-defined. ☐

Proposition 8.5 *For any $\sigma\delta$-field K such that $K^\sigma =: k$ is differentially closed and any $\sigma\delta$-simple K-$\sigma\delta$-algebra R which is differentially finitely generated over K, we have $R^\sigma = K^\sigma$.*

Proof. By our previous proposition we may write $R = Re_0 \oplus \ldots \oplus Re_{n-1}$. In particular for any $r \in R^\sigma$ and $i \in \{1, \ldots, n\}$ we have $\sigma^n(re_i) = re_i$. Pick an arbitrary $r \in R^\sigma$. Then $re_i \in (Re_i)^{\sigma^n}$. Note that Re_i is am integral $\sigma^n\delta$-simple K-$\sigma^n\delta$-algebra which is finitely generated over K. If we prove the proposition in this case, we then have $(Re_i)^{\sigma^n} = K^{\sigma^n}$. We first show that it suffices to show this special case. Pick an arbitrary element $l \in K^{\sigma^n} \backslash K^\sigma$ and consider the polynomial $\prod_{i=0}^{n-1} y - \sigma^i(l) \in K\{y\}$. Note that $\sigma(\prod_{i=0}^{n-1}(y - \sigma^i(l))) = \prod_{i=0}^{n-1}(y - \sigma^i(l))$ and thus $\prod_{i=0}^{n-1}(y - \sigma^i(l)) \in (K\{y\})^\sigma$. We claim that $(K\{y\})^\sigma = K^\sigma\{y\}$. We write $y^{\underline{\alpha}} = y^{\alpha_0}\delta(y)^{\alpha_1} \cdots$ and pick an arbitrary element $\sum f_{\underline{\alpha}} y^{\underline{\alpha}} \in (K\{y\})^\sigma$. We then have $\sigma(\sum f_{\underline{\alpha}} y^{\underline{\alpha}}) = \sum \sigma(f_{\underline{\alpha}}) y^{\underline{\alpha}}$ and by assumption $\sum f_{\underline{\alpha}} y^{\underline{\alpha}}$. Thus we have $\sigma(f_{\underline{\alpha}}) = f_{\underline{\alpha}}$ and in particular $(K\{y\})^\sigma \subset K^\sigma\{y\}$. The other inclusion is trivial. Hence $\prod_{i=0}^{n-1}(y - \sigma^i(l)) \in K^\sigma\{y\}$. Since K^σ is differentially and thus algebraically closed, we know that $K^{\sigma^n} = K^\sigma$. Hence for all i we have $re_i = k_i$ for some k_i in K^σ. Then $r = r(e_1 + \ldots + e_n) = re_1 + \ldots + re_n = k_1 + \ldots + k_n \in K^\sigma$.

It remains to show that if R is an integral $\sigma\delta$-simple K-$\sigma\delta$-algebra which is finitely generated over K, we have $R^\sigma = K^\sigma$. We argue by contradiction and assume the existence of an $c \in R^\sigma \backslash K^\sigma$. We will show the existence of an differential homomorphism from R to a differential field extension of K sending c into K^σ.

For that let $R_0 := K\{c\}$, where we note that since R is differentially finitely generated over K, it is so too over R_0. Let u_0 be the element given by the statement of Theorem 3.17. Then it suffices to find a morphism $\varphi' : R_0 \to L$ such that $\varphi'(u_0) \neq 0$ and $\varphi'(c) \in K^\sigma$, as we can then extend the morphism to R thanks to Theorem 3.17. We write $u_0 = p(c)$ for some $p \in K\{y\}$, where y is a new variable. We choose a K^σ-vector space basis $(a_i)_{i \in \mathcal{I}}$ of K and write $p = \sum a_i V_i$ for some $V_i \in K^\sigma\{y\}$. Since $u_0 \neq 0$, there exists a k such that $V_k(c) \neq 0$.

We endow $K\{y\}$ with the structure of a $\sigma\delta$-ring by extending σ via $\sigma(y) = y$. Since $c \in R^\sigma$, the defining ideal $\mathfrak{J}(c) \subset K\{y\}$ is a $\sigma\delta$-ideal. We claim that $[\mathfrak{J}(c) \cap K^\sigma\{y\}] = \mathfrak{J}(c)$. Its clear that $[\mathfrak{J}(c) \cap K^\sigma\{y\}] \subset \mathfrak{J}(c)$. For the other inclusion we first define the length $\ell(f)$

of a differential polynomial $f \in K\{y\}$. We can write $f = \sum' f_{\underline{\alpha}} \underline{y^{\alpha}}$ where α is a multi-index, $f_{\underline{\alpha}} \in K$ and $\underline{y^{\alpha}} := y^{\alpha_0}\delta(y)^{\alpha_1}\cdots$. Then let the length be the number of non-zero $f_{\underline{\alpha}}$ involved. We now show that all $f \in \mathfrak{J}(c)$ are in $[\mathfrak{J}(c) \cap K^{\sigma}\{y\}]$ via induction on the length k. The base case $k = 0$ is true since 0 is an element of any ideal. Now assume that for $k \geqslant 0$ the claim is true. Let f be of length $k+1$. By normalizing, we may assume without loss of generality that there exists a $\underline{\beta}$ such that $f_{\underline{\beta}} = 1$. If all coefficients of f are in K^{σ}, we are done. Thus we assume that there is a $\underline{\alpha}$ such that $f_{\underline{\alpha}} \in K \backslash K^{\sigma}$. Consider the equation

$$\sigma(f_{\underline{\alpha}}^{-1} f) - f_{\underline{\alpha}}^{-1} f = \sigma(f_{\underline{\alpha}}^{-1})(\sigma(f) - f) + (\sigma(f_{\underline{\alpha}}^{-1}) - f_{\underline{\alpha}}^{-1}) f.$$

The left hand side has length k, since the $\underline{\alpha}$-monomial cancels itself. Thus the left hand side is, by induction-assumption, in $[\mathfrak{J}(c) \cap K^{\sigma}\{y\}]$. The first term on the right hand side has length k, since the term with $f_{\underline{\alpha}} = 1$ cancels itself. Again, by induction assumption, it is in $[\mathfrak{J}(c) \cap K^{\sigma}\{y\}]$. Thus we know that $(\sigma(f_{\underline{\alpha}}^{-1}) - f_{\underline{\alpha}}^{-1}) f \in [\mathfrak{J}(c) \cap K^{\sigma}\{y\}]$ and thus, since $(\sigma(f_{\underline{\alpha}}^{-1}) - f_{\underline{\alpha}}^{-1})$ is non-zero, we have $f \in [\mathfrak{J}(c) \cap K^{\sigma}\{y\}]$. Thus we have shown the claim.

By the claim and the fact that $K^{\sigma}\{y\}$ is Rittian, we know that $\mathfrak{J}(c) = [\mathrm{rad}(f_1, \ldots, f_r)]$ for $r \geqslant 0$ and $f_1, \ldots, f_r \in K^{\sigma}\{y\}$. Furthermore, since R is an integral domain, $\mathfrak{J}(c)$ is prime and in particular radical. Hence

$$\mathfrak{J}(c) = \mathrm{rad}(\mathfrak{J}(c)) = \mathrm{rad}([\mathrm{rad}(f_1, \ldots, f_r)]) = \mathrm{rad}(f_1, \ldots, f_r).$$

Thus the $f_1, \ldots, f_r$ and the inequality V_k form a system of equation in K^{σ} which possesses a joint solution in an extension of K, namely c. Since K^{σ} is differentially closed, there exists a joint solution $\tilde{c} \in K^{\sigma}$.

We induce a morphism $\varphi'' : K\{y\} \to K$ by letting φ'' be the identity on K and defining $\varphi''(x) := \tilde{c}$. Since the kernel of φ'' contains $f_1, \ldots, f_r$ and is a radical ideal, it also contains $\mathrm{rad}(f_1, \ldots, f_r)$. Thus, φ'' induces a differential morphism

$$\varphi' : K\{c\} \cong K\{y\}/\mathrm{rad}(f_1, \ldots, f_r) \to K$$

sending c to $\tilde{c} \in K^{\sigma}$. It remains to show that $\varphi(u_0) \neq 0$. We know that $V_k(\tilde{c}) \neq 0$. Thus, since all $V_i(\tilde{c}) \in K^{\sigma}\{\tilde{c}\} = K^{\sigma}$ and $V_k(\tilde{c}) \neq 0$, if $\varphi(u_0) = p(\tilde{c}) = \sum a_i V_i(\tilde{c}) = 0$, then the a_i's would be linearly dependent over K^{σ}. This is contradiction and thus $\varphi(u_0) \neq 0$. Thus there exists a differential morphism φ as we were looking for.

Hence $\varphi(c - \tilde{c}) = \varphi(c) - \tilde{c} = 0$. Since $c, \tilde{c} \in R^{\sigma}$ we have that $[c - \tilde{c}] \subset R$ is a $\sigma\delta$-ideal. But since R is $\sigma\delta$-simple, we know that $1 \in [c - \tilde{c}]$ and thus $\varphi(1) = 0$, which is a contradiction. Thus there cannot exist an $c \in R^{\sigma} \backslash K^{\sigma}$ and we are done.

$\square$

Remark 8.6 Note that the parametrised Picard-Vessiot theory did not make use of the condition that K^σ is differentially algebraically closed up until now. In everything that follows, the condition that K^σ be differentially algebraically closed is only used to apply this proposition, i.e. that the field σ-constants of K is the same as the field of σ-constants of the parametrised Picard-Vessiot ring. We will return to this observation later on.

Proposition 8.7 *For any $\sigma\delta$-field K such that K^σ is differentially closed and for any $\sigma\delta$-simple, differentially finitely generated K-algebra R and its total quotient ring* $\operatorname{Quot}(R)$, *we have* $\operatorname{Quot}(R)^\sigma = K^\sigma$.

Proof. Omitted, for reference see Cor 6.15 in [3]. □

Corollary 8.8 For any $\sigma\delta$-field K such that K^σ is differentially closed and $A \in \mathrm{GL}_n(K)$ and an $\sigma\delta$-Picard-Vessiot ring R for $\sigma(Y) = AY$ and two fundamental solutions $Z_1, Z_2 \in \mathrm{GL}_n(R)$, there exists a unique $C \in \mathrm{GL}_n(K^\sigma)$ such that $Z_2 = Z_1 C$.

Proof. We have $\sigma(Z_1^{-1}Z_2) = Z_1^{-1}A^{-1}AZ_2 = Z_1^{-1}Z_2$, hence $Z_1^{-1}Z_2 =: C \in \mathrm{GL}_n(R^\sigma) = \mathrm{GL}_n(K^\sigma)$ by 8.5 and thus $Z_2 = Z_1 C$. In particular this C is unique. □

Proposition 8.9 *For a $\sigma\delta$-field K such that K^σ is differentially closed and $A \in \mathrm{GL}_n(K)$, any two $\sigma\delta$-Picard-Vessiot rings for $\sigma(Y) = AY$ are isomorphic as K-$\sigma\delta$-algebras.*

Proof. Consider two such rings $R_1 = K\{U_{i,j}, \frac{1}{\det(U)}\}$ and $R_2 = K\{V_{i,j}, \frac{1}{\det(V)}\}$ and $(R_1 \otimes_K R_2)/\mathfrak{m}$ for a maximal $\sigma\delta$-ideal $\mathfrak{m} \subset R_1 \otimes_K R_2$. Since R_1, R_2 are $\sigma\delta$-simple, the canonical morphisms $\iota_1 : R_1 \to (R_1 \otimes_K R_2)/\mathfrak{m}, r \mapsto \overline{r \otimes 1}$ and $\iota_2 : R_2 \to (R_1 \otimes_K R_2)/\mathfrak{m}, r \mapsto \overline{1 \otimes r}$ are injective. The images of ι_1 and ι_2 are generated by $\iota_1(U_{i,j}), \iota_1(\frac{1}{\det(U)})$ and $\iota_2(V_{i,j}), \iota_2(\frac{1}{\det(V)})$ respectively. Since $(R_1 \otimes_K R_2)/\mathfrak{m}$ is a $\sigma\delta$-simple K-$\sigma\delta$-algebra which is differentially finitely generated over K we have $((R_1 \otimes_K R_2)/\mathfrak{m})^\sigma = K^\sigma$. Since $\sigma((1 \otimes V)^{-1}(U \otimes 1)) = (1 \otimes V)^{-1}A^{-}1A(U \otimes 1)$ we have $\iota_2(V) = \iota_1(U)C$ for some $C \in \mathrm{GL}_n(K^\sigma)$. Thus $\iota_1(R_1) = \iota_2(R_2)$ and thus $R_2 \cong R_1$ via $\iota_2^{-1}\iota_1$. □

8.2 Parametrised Galois Group

Definition 8.10 *A $\sigma\delta$-automorphism $\varphi : R \to R$ which restricts to the identity on K is called a $\sigma\delta$-*automorphism *over K.*

Definition 8.11 *For a $\sigma\delta$-field K let R be a $\sigma\delta$-Picard-Vessiot ring for $\sigma(Y) = AY$ where $A \in \mathrm{GL}_n(K)$. Then the $\sigma\delta$-*Galois group *or* parametrised Galois group $\sigma\delta\text{-}\operatorname{Gal}(R/K)$ *of R is the group of $\sigma\delta$-automorphisms φ of R over K.*

Notice that an element $\psi \in \sigma\delta\text{-}\operatorname{Gal}(R/K)$ gives a well defined K-$\sigma\delta$-morphism of the total ring of fraction L of R via $\psi(\frac{a}{b}) = \frac{\psi(a)}{\psi(b)}$. Since we have developed tools to study linear differential algebraic groups, we want to show that the parametrised Galois group is one.

Proposition 8.12 *Consider a $\sigma\delta$-field K such that $k := K^\sigma$ is differentially closed and a $\sigma\delta$-Picard-Vessiot ring R for $\sigma(Y) = AY$ where $A \in \mathrm{GL}_n(K)$. Then there exists a injective group morphism $\iota : \sigma\delta\text{-}\mathrm{Gal}(R/K) \to \mathrm{GL}_n(k)$ such that the image is a k-δ-subgroup of $\mathrm{GL}_n(k)$.*

Proof. We first define ι. Since R is a $\sigma\delta$-Picard-Vessiot ring, there exists a fundamental solution $Z \in \mathrm{GL}_n(R)$ such that $R = K\{(Z_{i,j}), \frac{1}{\det(Z)}\}$. Then for any $\psi \in \sigma\delta\text{-}\mathrm{Gal}(R/K)$, we have $\sigma(\psi(Z)) = \psi(\sigma(Z)) = \psi(AZ) = A\psi(Z)$ and hence $\psi(Z)$ is a solution of $\sigma(Y) = AY$. Furthermore, since $\psi(Z)$ is invertible in R, we have $\psi(Z) \in \mathrm{GL}_n(R)$. Hence $\psi(Z)$ is a fundamental solution of $\sigma(Y) = AY$. Therefore, by Cor. 8.8 there exists a unique $U_\psi \in \mathrm{GL}_n(k)$ such that $\psi(Z) = ZU_\psi$. Since for $\psi, \varphi \in \sigma\delta\text{-}\mathrm{Gal}(R/K)$ we have $\varphi(\psi(Z)) = \varphi(ZU_\psi) = ZU_\varphi U_\psi$, this defines a group morphism $\iota : \sigma\delta\text{-}\mathrm{Gal}(R/K) \to \mathrm{GL}_n(k)$, $\psi \mapsto U_\psi$.

Assume that $U_\psi = id_{\mathrm{GL}_n(k)}$. Then $\psi(Z) = Z$. Since R is differentially generated by the entries of Z and its determinant, we know that ψ is then the identity. Thus ι is injective. The image of ι is a subgroup of $\mathrm{GL}_n(k)$, but to show that it is a k-δ-subgroup, we need to show that it is Kolchin-closed. For that we will show that the image of ι is the zero-set of some differential polynomials. First we write $R = K\{(Y_{i,j}), \frac{1}{\det(Y)}\}/\mathfrak{J}$ where for $i, j = 1, \ldots, n$ the $Y_{i,j}$ are new variables and $\mathfrak{J}$ a radical differential ideal. The automorphism σ is then defined by $\sigma(Y_{i,j}) = (AY)_{i,j}$. Since $K\{(Y_{i,j}), \frac{1}{\det(Y)}\}$ is Rittian, we know that $\mathfrak{J}$ has a finite basis which we denote by $\{J_1, \ldots, J_s\}$. For any $M \in \mathrm{GL}_n(k)$ we consider a differential automorphism

$$\psi_M : K\{(Y_{i,j}), \frac{1}{\det(Y)}\} \to K\{(Y_{i,j}), \frac{1}{\det(Y)}\} \text{ defined by } Y_{i,j} \mapsto (YM)_{i,j}$$

and by the identity on K. Since M is a matrix of σ-constants, the differential automorphism ψ_M commutes with σ. Hence ψ_M is a $\sigma\delta$-automorphism which is the identity on K. M in the image of ι if and only if ψ_M fixes $\mathfrak{J}$, as then and only then ψ_M gives us a $\sigma\delta$-automorphism of R over K. Each of the finitely many J_i's can be written as a K-linear combination of finitely many $\delta^{k_{1,1}}(y_{1,1})^{l_{1,1}} \cdots \delta^{k_{1,2}}(y_{1,2})^{l_{1,2}} \cdots \delta^{k_{n,n}}(y_{n,n})^{l_{n,n}} \det(Y)^{-r}$ where all $l_{i,j}, k_{i,j}, r \in \mathbb{Z}^{\geqslant 0}$. Let $V \subset K\{Y, \frac{1}{\det(Y)}\}$ be the finite dimensional K-vector space generated by all monomials in $\delta^{k_{a,b}}(y_{a,b})^{l_{a,b}}$ and $\frac{1}{\det(Y)}$ which occur in this representation of $J_1, \ldots, J_s$. We can find a K-vector space basis $\{p_e\}_{e\in\mathcal{E}}$ of $V \cap \mathfrak{J}$ and extend it to a K-vector space basis $\{p_w\}_{w\in\mathcal{W}}$ of V. We now have for $w \in \mathcal{W}$

$$\psi_M(p_w) = p_w((YM)_{i,j}, (\det(Y)\det(M))^{-1}) = \sum_{v\in\mathcal{W}}{}' P_{v,w}\left((M_{i,j}), \frac{1}{\det(M)}\right) p_v$$

for some $P_{v,w}((M_{i,j}), \frac{1}{\det(M)})\bar{K}\{M, \frac{1}{det(M)}\}$. Thus $\mathfrak{J}$ is fixed by ψ_M if and only if $P_{v,w} = 0$ for all $v \in \mathcal{W}\backslash\mathcal{E}$, as only then $\psi_M(p_w)$ is a K-linear combination of p_v's for $v \in \mathcal{E}$. We now show that is equates to $((M_{i,j}), \frac{1}{\det(M)})$ being a solution to a set of differential polynomials with coefficient in k. Choose a basis $\{k_t\}_{t\in\mathcal{T}}$ of K as a k-vector space. Hence every differential polynomial $P_{v,w}((M_{i,j}), \frac{1}{\det(M)})$ can be written as $\sum'_{t\in\mathcal{T}} P_{v,w,t}((M_{i,j}), \frac{1}{\det(M)})k_t$, for finitely many $P_{v,w,t}((X_{i,j}), \frac{1}{\det(X)}) \in k\{(X_{i,j}), \frac{1}{\det(X)}\}$. Thus the image of ι is the zero set of the differential polynomials $\{P_{v,w,t}((X_{i,j}), \frac{1}{\det(X)}) \mid w \in \mathcal{E},\, v \in \mathcal{W}\backslash\mathcal{E},\, t \in \mathcal{T}\}$. Thus it is Kolchin-closed and the image of ι is a differential subgroup of $\mathrm{GL}_n(k)$. $\square$

Remark 8.13 Note that in the construction of ι in the proof of 8.11 we chose a fundamental solution Z. This choice is not unique and thus this identification of the parametrised Galois group is not unique. However, it is up to unique k-δ-group isomorphism. For two fundamental solutions Z_1, Z_2 there exists a unique $D \in \mathrm{GL}_n(k)$ such that $Z_1 = Z_2 D$. As in the proof, we have $\psi(Z_1) = Z_1 U_{\psi,1} = Z_2 D U_{\psi,1}$ and $\psi(Z_1) = \psi(Z_2 D) = Z_2 U_{\psi,2} D$. Hence $U_{\psi,1} = D^{-1} U_{\psi,2} D$, thus the two identification are conjugate.

Proposition 8.14 *For a a $\sigma\delta$-field K such that $k := K^\sigma$ is differentially closed, let R be a $\sigma\delta$-Picard-Vessiot ring. Then we have $\delta\text{-trdeg}(R/K) = \delta\text{-dim}_k\,(\sigma\delta\text{-}\mathrm{Gal}(R/K))$.*

A proof of this can be found in Prop 6.24 of [3]

Corollary 8.15 *The entries of a fundamental solution of a difference equation are differentially dependent if and only if $\delta\text{-dim}_k\,(\sigma\delta\text{-}\mathrm{Gal}(R/K)) = 0$.*

The next theorem shows that calling the parametrised Galois group a Galois group is justified. Indeed, there exists a Galois-correspondence, which we won't actually need.

Definition 8.16 *For any subset $F \subset L$ we define*

$$F^{\sigma\delta\text{-}\mathrm{Gal}(R/K)} := \{f \in F \mid \forall \varphi \in \sigma\delta\text{-}\mathrm{Gal}(R/K) \quad \varphi(f) = f\}.$$

For the next theorem we introduce for all $\sigma\delta$-rings $F \subset L$ the notation

$$\sigma\delta\text{-}\mathrm{Gal}(L/F) := \{\varphi \in \sigma\delta\text{-}\mathrm{Gal}(R/K) \mid \forall f \in F : \varphi(f) = f\}$$

Theorem 8.17 *Let K be a $\sigma\delta$-field such that $k := K^\sigma$ is differentially closed. For $A \in \mathrm{GL}_n(K)$, let R be a $\sigma\delta$-Picard-Vessiot ring of $\sigma(Y) = AY$ and L its total ring of fractions. Then we have $L^{\sigma\delta\text{-}\mathrm{Gal}(R/K)} = K$ and the following inclusion reversing bijective correspondence:*

$$\{F \mid F \text{ a } \sigma\delta\text{-ring, } K \subset F \subset L \text{ and } \mathrm{Quot}(F) = F\} \longleftrightarrow \{H \mid H \text{ a } \delta\text{-subgroup of } \sigma\delta\text{-}\mathrm{Gal}(R/K)\}$$

$$F \longrightarrow \sigma\delta\text{-}\mathrm{Gal}(L/F)$$

$$L^H \longleftarrow H$$

Proof. We only show that $L^{\sigma\delta\text{-}\operatorname{Gal}(R/K)} = K$, which is the only thing which we will need. Assume that there exists $\frac{a}{b} \in L^{\sigma\delta\text{-}\operatorname{Gal}(R/K)} \backslash K$. Then $0 \neq a \otimes b - b \otimes a \in R \otimes_K R$. Since $(R \otimes_K R)_{a\otimes b - b\otimes a}$ is a non-empty $\sigma\delta$-ring, we can pick a maximal $\sigma\delta$-ideal $\mathfrak{m} \subset (R \otimes_K R)_{a\otimes b-b\otimes a}$. Then the $\sigma\delta$-algebra morphisms $\psi_1 : R \to (R \otimes_K R)_{a\otimes b-b\otimes a}/\mathfrak{m}$, $r \mapsto \frac{r\otimes 1}{1} + \mathfrak{m}$ and $\psi_2 : R \to (R \otimes_K R)_{a\otimes b-b\otimes a}/\mathfrak{m}$, $r \mapsto \frac{1\otimes r}{1} + \mathfrak{m}$ are injective, since R is $\sigma\delta$-simple. Since $(R \otimes_K R)_{a\otimes b-b\otimes a}/\mathfrak{m}$ is a $\sigma\delta$-simple K-$\sigma\delta$-algebra which is differentially finitely generated over K we have $((R \otimes_K R)_{a\otimes b-b\otimes a}/\mathfrak{m})^\sigma = K^\sigma$ and since $\sigma((1\otimes Z)^{-1}(Z\otimes 1)) = (1\otimes Z)^{-1}A^{-1}A(Z\otimes 1)$ we have $\psi_2(Z) = \psi_1(Z)C$ for some $C \in \mathrm{GL}_n(K^\sigma)$. Thus $\psi_1(R) = \psi_2(R)$ and hence $\psi_2^{-1}\psi_1$ is a $\sigma\delta$-automorphism of R over K. The image of $a \otimes b - b \otimes a$ into $(R \otimes_K R)_{a\otimes b-b\otimes a}/\mathfrak{m}$ is equal to $\psi_1(a)\psi_2(b) - \psi_1(b)\psi_2(a) \neq 0$, since $a \otimes b - b \otimes a$ is a unit in $(R \otimes_K R)_{a\otimes b-b\otimes a}$ and thus not contained in the maximal ideal $\mathfrak{m}$. Thus $\psi_2^{-1}\psi_1(a)b - \psi_2^{-1}\psi_1(b)a \neq 0$, therefore we have found an element of $\sigma\delta\text{-}\operatorname{Gal}(R/K)$ such that $\frac{a}{b}$ isn't fixed by it. This is a contradiction and we are done. $\square$

9 Main Theorems

Theorem 9.1 *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is differentially closed. For $a \in K^\times$ let L be a K-$\sigma\delta$-algebra which contains z such that $\sigma(z) = az$ and $L^\sigma = k$. Then z is differentially algebraic over K if and only if there exists a non-zero homogenous linear differential polynomial $\ell \in k\{x\}$ and an element $f \in K^\times$ such that*

$$\ell\Big(\frac{\delta(a)}{a}\Big) = \sigma(f) - f.$$

If z is differentially algebraic, then $\ell(\frac{\delta z}{z}) - f \in k$.

Proof. Assume that there exist such ℓ and f as in the Theorem. Then we have

$$\begin{aligned}\sigma\Big(\ell\Big(\frac{\delta(z)}{z}\Big) - f\Big) =& \ell\Big(\frac{\delta(\sigma(z))}{\sigma(z)}\Big) - \sigma(f) = \ell\Big(\frac{\delta(az)}{az}\Big) - \sigma(f) \\ =& \ell\Big(\frac{\delta(a)}{a}\Big) + \ell\Big(\frac{\delta(x)}{x}\Big) - \sigma(f) = \ell\Big(\frac{\delta(z)}{z}\Big) + \ell\Big(\frac{\delta(a)}{a}\Big) - \sigma(f) = \ell\Big(\frac{\delta(z)}{z}\Big) - f\end{aligned}$$

and thus $\ell(\frac{\delta(z)}{z}) - f \in L^\sigma = k$. Hence z is differentially algebraic over K.

Conversely, assume that z is differentially algebraic over K. Since K is a field, $K\{z, \frac{1}{z}\}$ is non-zero and there exists a maximal $\sigma\delta$-ideal in $K\{z, \frac{1}{z}\}$. Hence $R := K\{z, \frac{1}{z}\}/\mathfrak{m} = K\{\bar{z}, \overline{\frac{1}{z}}\}$ is a $\sigma\delta$-simple ring. Thus R is a $\sigma\delta$-Picard-Vessiot ring. Consider any $\varphi \in \sigma\delta\text{-}\operatorname{Gal}(R/K)$. Then

$$\sigma\Big(\frac{\varphi(\bar{z})}{\bar{z}}\Big) = \frac{\varphi(\sigma(\bar{z}))}{\sigma(\bar{z})} = \frac{\bar{a}\varphi(\bar{z})}{\bar{a}\bar{z}} = \frac{\varphi(\bar{z})}{\bar{z}}$$

and thus $\frac{\varphi(\bar{z})}{\bar{z}} =: k_\varphi \in k$. And since

$$\frac{\varphi(\psi(\bar{z}))}{\bar{z}} = \frac{\varphi(\psi(\bar{z}))}{\psi(\bar{z})}\frac{\psi(\bar{z})}{\bar{z}} = k_\varphi k_\psi,$$

we can identify $\sigma\delta\text{-}\operatorname{Gal}(R/K)$ with a k-δ-subgroup of $\mathbb{G}_m$. Since z is differentially algebraic over K, by 8.13 we have

$$1 > \delta\text{-trdeg}(R/K) = \delta\text{-}\dim_k(\sigma\delta\text{-}\operatorname{Gal}(R/K)),$$

hence it is a proper subgroup. Thus we know that there exists a non-zero linear homogeneous polynomial $\ell \in k\{X\}$ such that $\sigma\delta\text{-}\operatorname{Gal}(R/K) \subset \mathbb{V}(\ell(\frac{\delta(X)}{X}))$. We now show that $\ell(\frac{\delta(\bar{z})}{\bar{z}}) \in K$. To show this we prove that $\ell(\frac{\delta\bar{z}}{\bar{z}})$ is invariant under $\sigma\delta\text{-}\operatorname{Gal}(R/K)$. For this pick any $\varphi \in \sigma\delta\text{-}\operatorname{Gal}(R/K)$. Then

$$\varphi\left(\ell\left(\frac{\delta(\bar{z})}{\bar{z}}\right)\right) = \ell\left(\frac{\delta(\varphi(\bar{z}))}{\varphi(\bar{z})}\right) = \ell\left(\frac{\delta(k_\varphi\bar{z})}{k_\varphi\bar{z}}\right)\ell\left(\frac{\delta(\bar{z})}{\bar{z}} + \frac{\delta(k_\varphi)}{k_\varphi}\right) = \ell\left(\frac{\delta(\bar{z})}{\bar{z}}\right)$$

and thus $f := \ell\left(\frac{\delta(\bar{z})}{\bar{z}}\right) \in K$. And since

$$\sigma(f) - f = \ell\left(\sigma\left(\frac{\delta(\bar{z})}{\bar{z}}\right) - \frac{\delta(\bar{z})}{\bar{z}}\right) = \ell\left(\frac{\delta(a)}{a}\right)$$

we have found both an ℓ and an f as in the statement and we are done. $\square$

It is somewhat unsatisfactory that we have to work with a differentially closed field. However, the results which we have suffice to prove some results regarding meromorphic functions. On the field $\mathcal{M}er(\mathbb{C})$ of meromorphic functions on $\mathbb{C}$ define $\sigma : \mathcal{M}er(\mathbb{C}) \to \mathcal{M}er(\mathbb{C})$, $f(x) \mapsto f(x+1)$ and $\delta = \frac{\partial}{\partial x}$. Let $\mathcal{F} = (\mathcal{M}er(\mathbb{C}))^\sigma$ and $\mathbb{C}(x)$ and $\mathcal{F}(x)$ be $\sigma\delta$-sub-fields of $\mathcal{M}er(\mathbb{C})$.

Proposition 9.2 *For $a \in \mathbb{C}(x)\backslash\{0\}$, consider a nonzero meromorphic function g in x over $\mathbb{C}$ such that*

$$\sigma(g) = ag.$$

Then g is differentially algebraic over $\mathcal{F}(x)$ if and only if there exists a non-zero linear homogeneous differential polynomial $\ell \in \mathbb{C}\{X\}$ and an element $f \in \mathbb{C}(x)$ such that $\ell(\frac{\delta(a)}{a}) = \sigma(f) - f$.

Proof. If ℓ and f exist as in the statement we note that $\ell(\frac{\delta(g)}{g}) - f$ is a linear combination of meromorphic functions, so it is meromorphic. If $\ell(\frac{\delta(a)}{a}) = \sigma(f) - f$ for some $f \in \mathbb{C}(x)$, then $\sigma(\ell(\frac{\delta(g)}{g}) - f) = \ell(\frac{\delta(g)}{g} + \frac{\delta(a)}{a}) - \sigma(f) = \ell(\frac{\delta(g)}{g}) - f$, hence $\ell(\frac{\delta(g)}{g}) - f$ is a 1-periodic meromorphic function and thus an element of $\mathcal{F}$. Thus g is differentially algebraic over $\mathcal{F}(x)$.

Conversely assume that g is differentially algebraic over $\mathcal{F}(x)$. We need to show the existence of an ℓ and f as specified in the proposition. We will first find an ℓ and f for the case that K^σ is differentially closed and then use these as an Ansatz to find a solution over $\mathbb{C}$. For this we choose a differential closure of $\mathcal{F}$, which we denote by $\tilde{\mathcal{F}}$, and $\mathcal{F}(x)\{g\}$ the differential ring extension of $\mathcal{F}(x)$ with g, which is a $\sigma\delta$-subring of the field of meromorphic functions. Endow $K := \tilde{\mathcal{F}}(x)$ with a $\sigma\delta$-structure by extending δ with $\delta(x) = 1$ and defining σ to be the automorphism induced by the identity on $\tilde{\mathcal{F}}$ and $\sigma(x) \mapsto x + 1$. Then $K\{g\}$ has a natural K-$\sigma\delta$-algebra structure. Pick a maximal $\sigma\delta$-ideal $\mathfrak{m} \subset K\{g\}$ and set $L := K\{g\}/\mathfrak{m}$, which is a K-$\sigma\delta$-algebra.

To apply Prop 9.1 we need to show that $L^\sigma = K^\sigma = \tilde{\mathcal{F}}$. To determine K^σ, we use Prop 8.7 to the $\tilde{\mathcal{F}}$-algebra $\tilde{\mathcal{F}}[x]$. To apply this, we first need to check that $\tilde{\mathcal{F}}[x]$ is differentially finitely generated over $\tilde{\mathcal{F}}$, $\sigma\delta$-simple and that $\tilde{\mathcal{F}}^\sigma$ is differentially closed. By definition of σ we have $\tilde{\mathcal{F}}^\sigma = \tilde{\mathcal{F}}$, which is differentially closed by choice. Since $\tilde{\mathcal{F}}[x]$ is already finitely generated as a $\tilde{\mathcal{F}}$-algebra, it is differentially finitely generated over $\tilde{\mathcal{F}}$. To check the $\sigma\delta$-simple condition, we pick an arbitrary $P \in \tilde{\mathcal{F}}[x]\backslash\{0\}$ and show that any σ-ideal containing P is the unit ideal. We do this by induction on the degree of P. The base case $\deg P = 0$ is clear. Assume that the claim is true for all P of degree $\leqslant n$. Then pick P of degree $n+1$. Since $\tilde{\mathcal{F}}[x]^\sigma = \tilde{\mathcal{F}}$, we know that $\sigma(P) \neq P$. Then $\deg(\sigma(P) - P) < \deg(P)$ and $\sigma(P) - P \neq 0$. Thus, by induction assumption we are done. Hence $\tilde{\mathcal{F}}[x]$ is σ-simple and in particular it is $\sigma\delta$-simple. By Prop.8.7 we then have $\mathrm{Quot}(\tilde{\mathcal{F}}[x])^\sigma = \tilde{\mathcal{F}}^\sigma = \tilde{\mathcal{F}}$.

To determine L^σ, we use Prop 8.5. To apply this, we need to check that $K\{g\}/\mathfrak{m}$ is $\sigma\delta$-simple, differentially finitely generated over K and that K^σ is differentially closed. We have just shown $K^\sigma = \tilde{\mathcal{F}}$. Furthermore, since $\mathfrak{m}$ is a maximal $\sigma\delta$-ideal, the $\sigma\delta$-simple condition is satisfied. The differentially finitely generated condition holds since L is generated by $g \in L$. Thus we have $L^\sigma = K^\sigma = \tilde{\mathcal{F}}$.

Notice that since g is differentially dependent over $\mathcal{F}(x)$, it is also differentially dependent over $\tilde{\mathcal{F}}(x)$. Since $g \in L$ is now a solution to a difference equation over K and K^σ is differentially closed, we can apply theorem 9.1. From this we get the existence of $\tilde{\ell} \in \tilde{\mathcal{F}}\{Y\}$ and $\tilde{f} \in \tilde{\mathcal{F}}(x)^\times$

$$\tilde{\ell}(Y) = \sum_{i=0}^{n} \tilde{l}_i \delta^i(Y) \text{ and } \tilde{f}(x) = \frac{\tilde{u}(x)}{\tilde{v}(x)}$$

for

$$\tilde{u}(x) = \sum_{i=0}^{m} \tilde{u}_i x^i \text{ and } \tilde{v}(x) = \sum_{i=0}^{n-1} \tilde{v}_i x^i + x^n$$

such that

$$\tilde{\ell}\left(\frac{\delta(a(x))}{a(x)}\right) - \frac{\tilde{u}}{\tilde{v}}(x+1) + \frac{\tilde{u}}{\tilde{v}}(x) = 0.$$

We make this into an Ansatz by introducing new variables $\underline{L}, \underline{U}, \underline{V}$:

$$\ell(\underline{L}, Y) = \sum_{i=0}^{n} L_i \delta^i(Y)$$

and

$$f(x) = \frac{u(\underline{U}, x)}{v(\underline{V}, x)}$$

for

$$u(\underline{U}, x) = \sum_{i=0}^{m} U_i x^i \text{ and } v(\underline{V}, x) = \sum_{i=0}^{n-1} V_i x^i + x^n.$$

We want to find some $\underline{l}, \underline{u}, \underline{v} \in \mathbb{C}$ such that after evaluating we have

$$\ell\left(\underline{l}, \frac{\delta(a(x))}{a(x)}\right) - \frac{u(\underline{u})}{v(\underline{v})}(x+1) + \frac{u(\underline{u})}{v(\underline{v})}(x) = 0.$$

After multiplying this equation with some power of the denominator d of a and $v(x)v(x+1)$, which is not zero since v is normalized by assumption, we get the following, equivalent equation

$$d^m v(\underline{V}, x) v(\underline{V}, x+1) \left(\ell\left(\underline{L}, \frac{\delta(a(x))}{a(x)}\right) - \sigma(\frac{u(\underline{U})}{v(\underline{V})}(x)) + \frac{u(\underline{U})}{v(\underline{V})}(x)\right) = 0 \in \mathbb{C}[\underline{L}, \underline{U}, \underline{V}, x].$$

This equation holds if and only if all coefficients of powers of x are zero. This gives us a system of polynomial equations in $\mathbb{C}[\underline{L}, \underline{U}, \underline{V}, x]$. However, by construction of our Ansatz, we know of the existence of a joint solution $\underline{\tilde{l}}, \underline{\tilde{u}}, \underline{\tilde{v}}$ in an extension of $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, there exists some solution $\underline{l}, \underline{u}, \underline{v}$ in $\mathbb{C}$. Evaluating ℓ, u, v at $\underline{l}, \underline{u}, \underline{v}$ we get ℓ and f as stated in the proposition. $\square$

Theorem 9.3 (Hölder) *The Gamma function is differentially transcendent over $\mathbb{C}(x)$.*

Proof. The Gamma function is a meromorphic solution to $\sigma(z) = xz$, hence by 9.2 it is differentially algebraic over $\mathcal{F}(x)$ only if there exists a non-zero linear homogeneous differential polynomial $\ell \in \mathbb{C}(X)$ and a $f \in \mathbb{C}(x)$ such that $\ell(\frac{\delta(x)}{x}) = \sigma(f) - f$. As $\delta(x) = 1$, for any $i \geqslant 0$ we have $\delta^i(\frac{\delta(x)}{x}) = \frac{(-1)^i i!}{x^{i+1}}$. Thus it is sufficient to show that for any $n \in \mathbb{Z}^{\geqslant 0}$, for $i = 1, \ldots, n$ and $a_i \in \mathbb{C}$ not all zero and $f(x) \in \mathbb{C}(x)$

$$\sum_{i=1}^{n} \frac{a_i}{x^i} = f(x+1) - f(x)$$

can not hold. We first note that $\sum_{i=1}^{n} \frac{a_i}{x^i}$ has only poles at the point 0. Assume that $f(x)$ has a pole at $u \in \mathbb{C}\backslash\{0\}$. Then in particular $u + \mathbb{Z}^{\geqslant 0} \not\ni 0$ or $u + \mathbb{Z}^{\leqslant 0} \not\ni 0$. If $0 \notin u + \mathbb{Z}^{\geqslant 0}$, then using $\sum_{i=1}^{n} \frac{a_i}{x^i} + f(x) = f(x+1)$, we see that $f(x)$ has a pole at $u+1$. Since $u+i \neq 0$ for all $i \geqslant 0$, we can apply this argument infinitely many times and see that $f(x)$ has infinitely many poles and thus is not an element of $\mathbb{C}(x)$. If $0 \notin u + \mathbb{Z}^{\leqslant 0}$, we can apply the analogous argument by using $\sum_{i=1}^{n} \frac{a_i}{x^i} - f(x+1) = -f(x)$.

Hence $f(x)$ can only have poles at zero and can thus be written $f(x) = \frac{g(x)}{x^m}$ for some $g(x) \in \mathbb{C}[x]$. Hence we have $(\sum_{i=1}^n \frac{a_i}{x^i} + \frac{g(x)}{x^m})x^{\max\{m,n\}} = \frac{g(x+1)x^{\max\{m,n\}}}{(x+1)^m}$. Note that the left hand side has no poles, thus right hand side does not either. In particular $f(x)$ has no poles.

If $f(x)$ does not have any poles, then $\sum_{i=1}^n \frac{a_i}{x^i}$ cannot have any poles either. This is only the case if all $a_i = 0$, which we excluded. Hence no such $f(x)$ and ℓ exist and thus the Gamma function is differentially transcendent over $\mathcal{F}(x)$ and thus in particular over $\mathbb{C}(x)$. □

We can generalize Theorem 9.1 to:

Theorem 9.4 *Let K be a $\sigma\delta$-field such that $k = K^\sigma$ is δ-closed. For $a_1, \dots, a_n \in K^\times$ let L be a K-$\sigma\delta$-algebra which contains $z_1, \dots, z_n$ such that for $i = 1, \dots, n$*

$$\sigma(z_i) = a_i z_i$$

and $L^\sigma = k$. Then $z_1, \dots, z_n$ are differentially dependent over K if and only if there exists a non-zero homogenous linear differential polynomial $\ell(x_1, \dots, x_n) \in k\{x_1, \dots, x_n\}$ and an element $f \in K^\times$ such that

$$\ell\left(\frac{\delta(a_1)}{a_1}, \dots, \frac{\delta(a_n)}{a_n}\right) = \sigma(f) - f$$

We do not prove this Theorem, but rather note that the proof works analogously to the one of Theorem 9.1. using Theorem 7.4 instead of Theorem 7.2. The "descent" argument given for Proposition 9.2 can also be generalized:

Proposition 9.5 *For $a_1, \dots, a_n \in \mathbb{C}(x)^\times$ consider $z_1, \dots, z_n$ meromorphic functions over $\mathbb{C}$ such that*

$$\sigma(z_i) = a_i z_i$$

for all $i = 1, \dots, n$. Then the z_i's are differentially dependent over $\mathcal{F}(x)$ if and only if there exist a non-zero linear homogeneous differential polynomial $\ell \in \mathbb{C}\{X_1, \dots, X_n\}$ and a $f \in \mathbb{C}(x)$ such that $\ell(\frac{\delta(a_1)}{a_1}, \dots, \frac{\delta(a_n)}{a_n}) = \sigma(f) - f$.

A proof of it can be found in Cor. 3.2 of [2]. We use this to prove the following:

Proposition 9.6 (Ex. 4.45 in [2]) *Let $a_1, a_2 \in \mathbb{C}$. Then $\Gamma(x + a_1)$ and $\Gamma(x + a_2)$ are differentially algebraically dependent over $\mathcal{F}(x)$ if and only if $a_1 - a_2 \in \mathbb{Z}$.*

Proof. Note that $\Gamma(x + a)$ satisfies the difference equation $\sigma(z) = (x + a)z$. Using Proposition 9.5 and the same calculation as in the proof of 9.3, we see that the differential algebraical dependence is equivalent to the existence of some $b_{i,j} \in \mathbb{C}$ not all zero and an element $f \in \mathbb{C}(x)$ such that

$$\sum_{i,j \in \mathbb{Z}^{\geqslant 0}} \frac{b_{i,j}}{(x + a_1)^i (x + a_2)^j} = f(x + 1) - f(x).$$

It suffices to show that if $a_2 - a_1 = 1$, then $\Gamma(x+a_1)$ and $\Gamma(x+a_2)$ are differentially algebraically dependent. If we set $f(x) = \frac{1}{x+a_1}$, we then get

$$f(x+1) - f(x) = \frac{1}{x+a_1+1} - \frac{1}{x+a_1} = \frac{-1}{(x+a_1+1)(x+a_1)} = \frac{-1}{(x+a_2)(x+a_1)}$$

hence $\Gamma(x+a_2)$ and $\Gamma(x+a_1)$ are differentially algebraically dependent.

Now assume that $a_2 - a_1 \notin \mathbb{Z}$ and that there were some $b_{i,j}$'s and an $f(x)$ as demanded . We first assume that $f(x)$ has a pole at $u \in \mathbb{C}\backslash\{-a_1, -a_2\}$. Then $\{-a_1, -a_2\} \nsubseteq u+\mathbb{Z}$. Assume that $-a_1, -a_2 \notin u+\mathbb{Z}^{\geqslant 0}$. Then, thanks to $\sum_{i,j\in\mathbb{Z}^{\geqslant 0}} \frac{b_{i,j}}{(x+a_1)^i(x+a_2)^j} + f(x) = f(x+1)$ we know that f has a pole a $u+1$. Since $-a_1, -a_2 \notin u+\mathbb{Z}^{\geqslant 0}$, we get that f has infinitely many poles and we get a contradiction. The analogous argument works for if $-a_1, -a_2 \notin u+\mathbb{Z}^{\leqslant 0}$.

If f has only poles at $-a_1$ and $-a_2$, then we can write $f(x) = \frac{g(x)}{(x+a_1)^n(x+a_2)^m}$ for some $g(x) \in \mathbb{C}[x]$ and $n, m \in \mathbb{Z}^{\geqslant 0}$. But then there exists a $N \in \mathbb{Z}^{\geqslant 0}$ such that

$$(x+a_1)^N(x+a_2)^N(\sum_{i,j\in\mathbb{Z}^{\geqslant 0}} \frac{b_{i,j}}{(x+a_1)^i(x+a_2)^j} + f(x))$$

has no poles. Thus $\frac{g(x)(x+a_1)^N(x+a_2)^N}{(x+a_1+)^n(x+a_2+1)^m}$ has no poles. Hence f does not have any poles.

If $f(x)$ does not have any poles, then $\sum_{i,j\in\mathbb{Z}^{\geqslant 0}} \frac{b_{i,j}}{(x+a_1)^i(x+a_2)^j}$ cannot have any poles either. This only the case if all $b_{i,j} = 0$, which we excluded. Thus no $b_{i,j}$'s and $f(x)$ exist and thus $\Gamma(x+a_1)$ and $\Gamma(x+a_2)$ are not differentially algebraically dependent over $\mathcal{F}(x)$ and in particular over $\mathbb{C}(x)$. $\square$

Remark 9.7 As already mentioned in Rem.8.6, the main "ingredient" of the the parametrised Picard-Vessiot Theory is the non-increase of the field of σ-constants. This is were we used the fact that the σ-constants were differentially closed. This gives hope that one can do without this condition. Di Vizio and Hardouin in [1] and Wibmer in [4] showed that if K^σ is algebraically closed one can construct a parametrised Picard-Vessiot ring R for $\sigma(Y) = AY$ where $A \in \mathrm{GL}_n(K)$ such that the σ-constants do not increase, i.e. $R^\sigma = K^\sigma$. Unfortunately, the Picard-Vessiot ring constructed this way does not have to be unique, so one then still needs to deal with that. This has for instance been done by Di Vizio and Hardouin in [1], where they consider linear differential groups schemes instead of differential groups. This functorial approach allows them to generalize several results of the parametrised Picard-Vessiot-Theory, in particular theorem 9.1 can be stated without the condition that the field of σ-constants need to be differentially closed.

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