ℓ-ADIC ALGEBRAIC MONODROMY GROUPS, COCHARACTERS, AND THE MUMFORD-TATE CONJECTURE

by

Richard Pink

Fakultät für Mathematik und Informatik
Universität Mannheim
D-68131 Mannheim, Germany
e-mail: pink@math.uni-mannheim.de

May 14, 1997

Abstract:

We prove that the ℓ-adic algebraic monodromy groups associated to a motive over a number field are generated by certain one-parameter subgroups determined by Hodge numbers. In the special case of an abelian variety we obtain stronger statements saying roughly that the ℓ-adic algebraic monodromy groups look like a Mumford-Tate group of some (other?) abelian variety. When the endomorphism ring is $\mathbb{Z}$ and the dimension satisfies certain numerical conditions, we deduce the Mumford-Tate conjecture for this abelian variety. We also discuss the problem of finding places of ordinary reduction.

§0. Introduction

Galois representations arising from motives: Consider a smooth proper algebraic variety X over a number field K, an integer d, and a rational prime ℓ. Let ρ_ℓ denote the continuous representation of $\mathrm{Gal}(\bar{K}/K)$ on the ℓ-adic étale cohomology group

$$V_\ell := H^d(X \times_K \bar{K}, \mathbb{Q}_\ell) \ ,$$

where $\bar{K}$ denotes an algebraic closure of K. The main object of interest in this article is the associated global algebraic monodromy group G_ℓ, defined as the Zariski closure of the image of ρ_ℓ in the algebraic group $\underline{\mathrm{Aut}}_{\mathbb{Q}_\ell}(V_\ell)$. Unfortunately our methods cannot say anything about the unipotent part of this group. Therefore we replace ρ_ℓ by its semisimplification, after which G_ℓ is a reductive group. Note that this modification is unnecessary in the case $d = 1$, where ρ_ℓ is dual to the Galois representation on the Tate module of an abelian variety.

As ℓ varies, the different ρ_ℓ form a strictly compatible system of Galois representations in the sense of Serre [28]. This means the following. Consider a non-archimedean place v of K, say with residue characteristic p, where X has good reduction. If $\ell \neq p$, then ρ_ℓ is unramified at v and the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ has coefficients in $\mathbb{Q}$ and is independent of ℓ.

Frobenius tori: Serre systematically analyzed the group theoretic consequences of strict compatibility ([31], [33], see also [6]). One of his main tools is the Zariski closure of the subgroup generated by $\rho_\ell(\mathrm{Frob}_v)$, which gives rise to the so-called Frobenius torus T_v. The compatibility condition implies that this torus has a natural form over $\mathbb{Q}$ and can be conjugated into G_ℓ for each $\ell \neq p$. One of Serre's main results asserts that for many places v this yields a maximal torus of the identity component G_ℓ° (cf. Theorem 3.7). It follows that the rank and the formal character of the different groups G_ℓ are independent of ℓ.

Local algebraic monodromy groups and their cocharacters: The first main theme of the present article, expounded in Sections 1–3, is the relation between T_v and G_ℓ in the case $\ell = p$. The motivation arose from studying some unpublished ideas of William W. Barker, but our methods are different. Let $H_{V,v} \subset G_p$ denote the Zariski closure of the image of the local Galois group $\mathrm{Gal}(\bar{K}_v/K_v)$. As the local Galois representation is very ramified in general, this group is more difficult to describe than T_v. Nevertheless, one can get hold of some of its structure and combinatorics using the so-called "mysterious functor" relating V_p with the crystalline cohomology group

$$M_v := H^d(X_v/\mathcal{O}_v) \otimes_{\mathcal{O}_v} K_v \ .$$

(Again we replace this by its semisimplification.) The local Galois representation is determined by the filtered module structure of M_v (cf. Illusie [18]). This data involves two things: a crystalline Frobenius, and a Hodge decomposition (see Wintenberger [43] Th. 3.1.2). The first piece of information leads to a natural representation of the Frobenius torus T_v on M_v. Via the mysterious functor one obtains a unique conjugacy class of embeddings $T_{v,\bar{\mathbb{Q}}_p} \hookrightarrow H_{V,v,\bar{\mathbb{Q}}_p}$ (see 3.12). In particular, some form of T_v can be found inside G_p.

An important tool in Serre's study of the Frobenius torus was the (quasi)-cocharacter of T_v determined by the p-adic valuations of the Frobenius eigenvalues (cf. 3.4). Motivated by its relation with Newton polygons we call it the Newton cocharacter of T_v. In the case $\ell = p$ the Hodge decomposition of M_v also determines a cocharacter, which via the mysterious functor determines a unique conjugacy class of cocharacters of $H_{V,v,\bar{\mathbb{Q}}_p}$. (This conjugacy class can be characterized using the Hodge-Tate decomposition.) These cocharacters are called Hodge cocharacters because of their relation with Hodge polygons. The fundamental idea in Sections 1–3 is to systematically study and exploit the relations between Newton and Hodge cocharacters.

Geometric relations between cocharacters: The main ingredient from the theory of local Galois representations is the fact (a former conjecture of Katz) that the Newton polygon lies above the Hodge polygon for every algebraic representation of $H_{V,v}$. This combinatorial statement depends only on the algebraic group $H_{V,v}$ and its Newton and Hodge cocharacters. Thus it may be analyzed in an abstract setting. This is done in Section 1. We translate combinatorial statements on polygons such as the above into geometric relations between the cocharacters themselves. The abstract results, Theorems 1.3–5, apply to any linear algebraic group over a field together with two (quasi)-cocharacters defined over an algebraic closure.

In the case of a crystalline local Galois representation the result is formulated in Theorem 2.3. To state it in words, let us conjugate both the Newton and Hodge cocharacters into a fixed maximal torus $T \subset H_{V,v}$, so that they may be viewed as elements of the cocharacter space $Y := Y_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The resulting Hodge cocharacter is not unique, but determines a unique finite subset of Y. Let S_{μ_V} denote the union of all $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-conjugates of this set. Then the Newton cocharacter lies in the interior of the convex closure of S_{μ_V}. (A similar assertion is in Rapoport-Richartz [27] Theorem 4.2.)

Sections 1–2 contain a few other results on subgroups generated by cocharacters, and describing $H_{V,v}$. The case of ordinary local Galois representations is considered in 2.7–9.

Consequences for global algebraic monodromy groups: One central result in Serre's theory says that the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugates of the Frobenius cocharacter generate the cocharacter space of T_v (see Proposition 3.5). Since by Theorem 2.3 the Frobenius cocharacter is a linear combination of Hodge cocharacters, one can deduce a similar assertion for Hodge cocharacters, stated in Theorem 3.16. Here one must allow conjugates under both $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and the Weyl group of G_p, and the result is valid only for certain v.

Disregarding such fine points one can draw the following general conclusion for the groups G_ℓ. Let us call a cocharacter of $G_{\ell,\bar{\mathbb{Q}}_\ell}$ a weak Hodge cocharacter if and only if the multiplicity of each weight $i \in \mathbb{Z}$ on V_ℓ is equal to the corresponding Hodge number $h^{i,d-i}$ of X. Our result is:

Theorem (3.18). *For every rational prime ℓ the identity component of $G_{\ell,\bar{\mathbb{Q}}_\ell}$ is generated by the images of weak Hodge cocharacters.*

Abelian varieties and the Mumford-Tate conjecture: The remaining sections of this article are devoted to the special case $d = 1$. We may suppose without loss of generality that the algebraic variety X is equal to an abelian variety A over K, say of dimension g. Consider the singular cohomology group $V := H^1(A(\mathbb{C}), \mathbb{Q})$ with respect to some fixed embedding $K \subset \mathbb{C}$, and let $G_\infty \subset \underline{\mathrm{Aut}}_{\mathbb{Q}}(V)$ denote the Mumford-Tate group associated to the natural Hodge structure on V. By definition it is generated by the images of weak Hodge cocharacters, so the fact that the only non-zero Hodge numbers are $h^{1,0} = h^{0,1} = g$ imposes strong restrictions on the form of G_∞. The comparison isomorphism $V_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ makes it possible to compare the identity component G_ℓ° with $G_{\infty,\mathbb{Q}_\ell}$, and according to the Mumford-Tate conjecture these groups should coincide.

The results of Sections 1–3 imply that the Hodge cocharacters impose combinatorial restrictions on G_ℓ° similar to those for G_∞. Thus in some sense G_ℓ° looks like a Mumford-Tate group of an abelian variety. For a weak version of this statement, resulting from Theorem 3.18 cited above, see Theorem 5.10. Using classification results due to Serre [30] §3, explained and augmented in Section 4, one can deduce in particular the so-called minuscule weights conjecture (see Zarhin [47] 0.4):

Corollary (5.11). *Each simple factor of the root system of G_ℓ° has type A, B, C, or D, and its highest weights in the tautological representation are minuscule.*

Using the finer result of Theorem 3.16 it is possible to obtain stronger restrictions on G_ℓ°, up to proving the Mumford-Tate conjecture under suitable numerical assumptions on $g = \dim(A)$ and the endomorphism ring $\mathrm{End}(A_{\bar{K}})$. Here the main other ingredient is Faltings' theorem ([14] Theorems 3–4). For simplicity we restrict ourselves to the special case $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$ in this article, although the results could be generalized to some extent along standard lines.

Interpolation of ℓ-adic algebraic monodromy groups: First note that, if we disregard the Mumford-Tate group, the Mumford-Tate conjecture still implies that the groups G_ℓ° are "independent of ℓ" in that they all come from one and the same algebraic group over $\mathbb{Q}$. A version of this weaker statement was proved already under certain restrictions in Larsen-Pink [22]. Here we can go significantly beyond that result:

Theorem (5.13). *Assume that* $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$.

(a) There exists a connected reductive subgroup $G \subset \mathrm{GL}_{2g,\mathbb{Q}}$ such that G_ℓ° is conjugate to $G \times_{\mathbb{Q}} \mathbb{Q}_\ell$ under $\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$ for every ℓ in some set $\mathbb{L}$ of primes of Dirichlet density 1.

(b) The pair consisting of G together with its absolutely irreducible tautological representation is a strong Mumford-Tate pair of weights $\{0, 1\}$ over $F = \mathbb{Q}$ in the sense of Definition 4.1 (b).

(c) The derived group G^{der} is $\mathbb{Q}$-simple.

(d) If the root system of G is determined uniquely by its formal character, i.e. if G does not have an ambiguous factor (cf. Section 4), then in (a) we can take $\mathbb{L}$ to contain all but at most finitely many primes.

Parts (b–c) of this theorem form a stronger version of the statement that G_ℓ° looks like a Mumford-Tate group. The proof is given in Section 6. It avoids crystalline theory and is thus independent of Sections 1–2. It is based on two main principles: First, we find arithmetic information relating Frobenius tori with roots of G_ℓ, using arguments very similar to those used by Serre [28], Katz, Ogus [24], and others to detect places of ordinary reduction in some cases. Second, we exploit the fact that Frobenius tori impose relations between the $\mathbb{Q}_\ell$-structures of the different G_ℓ, using the method of Larsen-Pink [21], [22].

New instances of the Mumford-Tate conjecture: Under certain numerical conditions on $g = \dim(A)$ the restrictions on G_ℓ° given by Theorem 5.13 are sufficient to imply the Mumford-Tate conjecture itself. This yields a significant improvement of earlier results of Serre [28], [32], [33], as well as those of Tankeev ([40] et al., [35], [38]).

Theorem (5.14). *Assume that* $\mathrm{End}(A_{\bar K}) = \mathbb{Z}$. *Assume moreover that* $2g$ *is neither*

(a) a k^{th} *power for any odd* $k > 1$, *nor*

(b) of the form $\binom{2k}{k}$ *for any odd* $k > 1$.

Then we have $G_\infty = \mathrm{CSp}_{2g,\mathbb{Q}}$ *and* $G_\ell^\circ = \mathrm{CSp}_{2g,\mathbb{Q}_\ell}$ *for every* ℓ. *In particular the Mumford-Tate conjecture holds for* A.

To indicate the scope of this result observe that in the range ≤ 1000 the only excluded dimensions are $g = 4$, 10, 16, 32, 64, 108, 126, 256, 500, 512, 864. The number of excluded values $g \leq 10^6$ is only 82. Alas, in the smallest interesting dimension $g = 4$ the Mumford-Tate conjecture still remains open.

If the Mumford-Tate group is in some sense small, one can also prove the Mumford-Tate conjecture by showing that the classification does not allow G_ℓ° to be smaller:

Theorem (5.15). *Assume that* $\mathrm{End}(A_{\bar K}) = \mathbb{Z}$ *and that the root system of each simple factor of* $G_{\infty,\bar{\mathbb{Q}}}$ *has type* A_{2s-1} *with* $s \geq 1$ *or* B_r *with* $r \geq 1$ *(cf. Table 4.6). Then the Mumford-Tate conjecture holds for* A.

This result, like the others cited above, depends on the classification results collected in Section 4.

Places of Ordinary Reduction: The arguments in the proof of Theorem 5.13 can be used to obtain some new results on the frequency of places with given Newton polygon. The smaller the groups G_ℓ are, the better the method works. Thus when the G_ℓ are sufficiently special, one can show the existence of many places of ordinary reduction. In the following results K^{conn} is a certain finite extension of K determined as in Theorem 3.6:

Theorem (7.1). *Assume that* $\mathrm{End}(A_{\bar K}) = \mathbb{Z}$, *and let* G *be as in Theorem 5.13. Suppose that the root system of the simple factors of* $G_{\bar{\mathbb{Q}}}$ *does not have type* C_r *with* $r \geq 3$. *Then the abelian variety* $A_{K^{\mathrm{conn}}}$ *has ordinary reduction at a set of places of* K^{conn} *of Dirichlet density* 1.

When the Mumford-Tate group is already small, the same follows for the groups G_ℓ°. Thus we can deduce:

Corollary (7.2). *Assume that* $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$*. Suppose that the root system of the simple factors of* $G_{\infty,\bar{\mathbb{Q}}}$ *does not have type* C_r *with* $r \geq 3$*. Then the abelian variety* $A_{K^{\mathrm{conn}}}$ *has ordinary reduction at a set of places of* K^{conn} *of Dirichlet density* 1*.*

The proof of these results, given in Section 7, distinguishes cases according to the type of the root system of G. In two of three cases the assertion follows easily from the intermediate results of Section 6. In the remaining case we encounter a new problem, which is solved with the help of a theorem of Wintenberger [46] concerning the lift of a compatible system of ℓ-adic representations under an isogeny of algebraic groups.

For further explanations see the introductions to the individual sections.

§1. Algebraic Groups, Cocharacters, and Polygons

Consider a linear algebraic group H over a field F. The subject of this preparatory section is to study relations between different cocharacters of H. To a cocharacter μ and a representation V of H are associated the weights of μ in V together with their respective multiplicities. These data are encoded in a certain polygon, and the main objects of this section are

(a) to reformulate certain geometric relations between these polygons in terms of the cocharacters themselves, and

(b) to deduce group theoretic consequences from these properties.

In the following we fix H, F, and an algebraically closed overfield E of F.

Tannaka duality: Let $\mathbf{Rep}_H$ denote the category of all finite dimensional representations of H over F. This is a tannakian category in the sense of Deligne-Milne [12]. Let $\mathbf{Vec}_F$ denote the category of finite dimensional F-vector spaces, and ω the "forgetful" functor $\mathbf{Rep}_H \to \mathbf{Vec}_F$ which to a representation of H associates the underlying F-vector space. This is a fiber functor of tannakian categories, and we have a canonical isomorphism $H \cong \underline{\mathrm{Aut}}^{\otimes}(\omega)$ (cf. [12] §2).

Cocharacters and $\mathbb{Z}$-gradings: Let $\mathbb{G}_{m,E}$ denote the multiplicative group over E. A homomorphism of algebraic groups $\mu : \mathbb{G}_{m,E} \to H_E := H \times_F E$ is called a cocharacter of H. For any cocharacter μ and any finite dimensional representation V of H we have a natural $\mathbb{Z}$-grading $V_E := V \otimes_F E = \bigoplus_{i\in\mathbb{Z}} V_E^i$. Here V_E^i is the weight space of weight i under μ, that is, the subspace on which $\mu(x)$ acts by multiplication with x^i for every $x \in E^\times$. If μ is fixed, this grading is functorial in V and compatible with tensor products and duals. Conversely, suppose that for each V we are given a $\mathbb{Z}$-grading of V_E which is functorial in V and compatible with tensor products and duals. Then this data can be interpreted as an F-linear tensor functor $\mathbf{Rep}_H \to \mathbf{Rep}_{\mathbb{G}_{m,E}}$, so it comes from a unique cocharacter of H (compare [12] Example 2.30). In other words, the cocharacter and the associated grading determine each other.

Quasi-cocharacters and $\mathbb{Q}$-gradings: The following terminology allows for arbitrary rational weights instead of integral weights. Consider the following inverse system of linear algebraic groups G_n over E, indexed by positive integers n ordered by divisibility. For each n we set $G_n := \mathbb{G}_{m,E}$, and for any $n|n'$ the homomorphism $G_{n'} \to G_n$ is exponentiation by n'/n. Then $\hat{\mathbb{G}}_{m,E} := \varprojlim G_n$ is the affine group $\mathbf{Spec}\, E[X^r|_{r\in\mathbb{Q}}]$.

A homomorphism of algebraic groups $\mu : \hat{\mathbb{G}}_{m,E} \to H_E$ is called a quasi-cocharacter of H. Pulling back by the natural map $\hat{\mathbb{G}}_{m,E} \to \mathbb{G}_{m,E}$, any cocharacter can be viewed as a quasi-cocharacter. Conversely, any quasi-cocharacter factors through some G_n, so it can be viewed as an n^{th} root of a usual cocharacter.

Most properties of cocharacters extend naturally to quasi-cocharacters. For instance, every quasi-cocharacter factors through some torus in H_E. The quasi-cocharacters of a torus T form an abelian group which is canonically isomorphic to $Y_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $Y_*(T) := \mathrm{Hom}(\mathbb{G}_{m,E}, T \times_F E)$ is the usual cocharacter group of T. Furthermore the above correspondence between cocharacters and compatible systems of $\mathbb{Z}$-gradings extends in a natural way to a correspondence between quasi-cocharacters and compatible systems of $\mathbb{Q}$-gradings. Namely, if some positive power μ^n of a quasi-cocharacter μ is an honest cocharacter, the weight space of weight $i \in \mathbb{Q}$ for μ is just the weight space of weight ni for μ^n.

Gradings and polygons: To any quasi-cocharacter μ and any finite dimensional representation V of H is associated the following polygon. It is the graph in $\mathbb{R}^2$ of a piecewise linear convex function $[0, \dim_F(V)] \to \mathbb{R}$ which starts at $(0,0)$. All slopes of this function are rational numbers, and the length of the subinterval on which the function has a given slope $i \in \mathbb{Q}$ is the dimension of the corresponding weight space $\dim_E(V_E^i)$. The polygon thus constructed is denoted $P_\mu(V)$.

Conjugation into a fixed maximal torus: The polygon $P_\mu(V)$ does not change when μ is replaced by a conjugate under $H(E) \rtimes \mathrm{Aut}(E/F)$. For the following arguments it will be useful to conjugate all cocharacters into a given maximal torus. Let us fix a maximal torus $T_E \subset H_E$. Let $\bar{\Gamma}$ denote the image of $\mathrm{Aut}(E/F)$ in the outer automorphism group of H_E, and let Γ be the inverse image of $\bar{\Gamma}$ in the automorphism group of T_E. This is a finite group preserving the root system of H_E. For later use recall that we have a canonical perfect pairing

$$\langle\ ,\ \rangle:\ X^*(T_E) \times Y_*(T_E) \to \mathbb{Z},\ (\chi, \lambda) \mapsto \deg(\chi \circ \lambda),$$

where $X^*(T_E) := \mathrm{Hom}(T_E, \mathbb{G}_{m,E})$ is the character group of T_E. After tensoring with $\mathbb{R}$ the cocharacter space $Y := Y_*(T_E) \otimes_{\mathbb{Z}} \mathbb{R}$ and the character space $X := X^*(T_E) \otimes_{\mathbb{Z}} \mathbb{R}$ are also in perfect duality $X \times Y \to \mathbb{R}$. All of this is equivariant under the action of Γ.

Definition (1.1). *For any quasi-cocharacter μ of H_E we let $S_\mu \subset Y$ denote the set of all $H(E) \rtimes \mathrm{Aut}(E/F)$-conjugates of μ which factor through T_E.*

By construction S_μ is a single orbit under the action of Γ. In particular, it is a finite set and its convex closure $\mathrm{Conv}(S_\mu)$ is a bounded convex polytope. As the corners of this polytope form a Γ-invariant non-empty subset of S_μ, this subset must be equal to S_μ. The interior $\mathrm{Conv}(S_\mu)^\circ$ is defined as the interior of the polytope $\mathrm{Conv}(S_\mu)$ inside the smallest affine linear subspace containing it.

Polygons and polytopes: Now we consider two cocharacters μ, ν of H.

Definition (1.2). *Let V be a finite dimensional representation of H.*

(a) We say that $P_\nu(V)$ lies above $P_\mu(V)$ *if and only if the first polygon lies on or above the second one at every point of the interval of definition* $[0, \dim_F(V)]$.

(b) We say that $P_\nu(V)$ lies strictly above $P_\mu(V)$ *if and only if in addition to (a) the polygons meet at most at the starting point* $(0,0)$ *and the endpoint.*

The following results translate these geometric relations into geometric relations between the cocharacters themselves. In the case that F is algebraically closed the assertion of Theorem 1.3 is contained in Rapoport-Richartz [27] Section 2 (cf. also Atiyah-Bott [1] Section 12).

Theorem (1.3). *The following conditions are equivalent:*

(a) $S_\nu \subset \mathrm{Conv}(S_\mu)$.

(a′) *For every* $\nu' \in S_\nu$ *and every character* $\chi \in X^*(T_E)$ *there exists* $\mu' \in S_\mu$ *such that* $\langle \chi, \nu' \rangle \geq \langle \chi, \mu' \rangle$.

(b) *For every finite dimensional representation* V *of* H *over* F *the polygon* $P_\nu(V)$ *lies above the polygon* $P_\mu(V)$.

(c) *In every non-trivial finite dimensional representation* V *of* H *over* F *the smallest weight of* ν *is greater than or equal to the smallest weight of* μ.

Theorem (1.4). *The following conditions are equivalent:*

(a) $S_\nu \subset \mathrm{Conv}(S_\mu)^\circ$.

(a′) *For every* $\nu' \in S_\nu$ *and every character* $\chi \in X^*(T_E)$ *either there exists* $\mu' \in S_\mu$ *with* $\langle \chi, \nu' \rangle > \langle \chi, \mu' \rangle$, *or we have* $\langle \chi, \nu' \rangle = \langle \chi, \mu' \rangle$ *for every* $\mu' \in S_\mu$.

(b) *For every irreducible finite dimensional representation* V *of* H *over* F *the polygon* $P_\nu(V)$ *lies above the polygon* $P_\mu(V)$ *and lies strictly above unless the latter is a straight line.*

Theorem (1.5). *The following conditions are equivalent:*

(a) $S_\nu = S_\mu$.

(b) *For every finite dimensional representation* V *of* H *over* F *the polygon* $P_\nu(V)$ *coincides with the polygon* $P_\mu(V)$.

(c) *For every finite dimensional representation* V *of* H *over* F *the polygon* $P_\nu(V)$ *lies above the polygon* $P_\mu(V)$, *and for some faithful representation these polygons coincide.*

One should note that all of these statements refer to properties relative to the ground field F. Thus on the one hand the representation V of H must be defined over F, while on the other hand all Galois conjugates are included in the set S_μ. In the following proofs we may without loss of generality assume $\mu \in S_\mu$ and $\nu \in S_\nu$.

Proof of Theorem 1.3: The implication (b)⇒(c) is obvious. For its converse it is enough to look at the polygons above any integral point $0 < i \leq \dim_F(V)$. For each of the polygons the value at i is just the smallest weight of the associated quasi-cocharacter on the exterior power $\bigwedge^i V_E$. Thus (b) follows from (c). Next, the equivalence (a)⇔(a′) is a well-known characterization of the convex closure, provided that χ in (a′) is allowed to run through all of X. Since S_μ consists of rational points of Y, it is enough to work with rational χ. Scaling each χ makes it integral, so it suffices to consider usual characters, as desired. It remains to prove (a′)⇔(c).

For the implication (a′)⇒(c) consider any finite dimensional representation V of H, and choose a weight $\chi \in X^*(T_E)$ of T_E on V_E such that $\langle \chi, \nu \rangle$ attains the smallest possible value. By (a′) there exists $\mu' \in S_\mu$ such that $\langle \chi, \nu \rangle \geq \langle \chi, \mu' \rangle$. Here the right hand side is $\geq$ the smallest weight of μ' in V_E, hence also the smallest weight of μ, proving (c).

To prove (c)⇒(a′) fix a character $\chi \in X^*(T_E)$ and choose an order on the roots of H_E with respect to T_E, such that χ is a dominant weight. Let V be an irreducible representation of H over F which has χ among its highest weights. Then all highest weights of V are Γ-conjugate to χ, and therefore all weights of V are in the convex closure of the Γ-orbit of χ. It follows that the smallest weight of μ on V_E is of the form $\langle \chi^\gamma, \mu \rangle$ for some $\gamma \in \Gamma$. This is equal to $\langle \chi, \mu' \rangle$, where $\mu' := \mu^{\gamma^{-1}}$ is another element of S_μ. Now (c) implies $\langle \chi, \nu' \rangle \geq$ (the smallest weight of ν' on V_E) $=$ (the smallest weight of ν on V_E) $\geq \langle \chi, \mu' \rangle$, proving (a′). $\square$

Proof of Theorem 1.4: The equivalence (a)$\Leftrightarrow$(a′) is easy and left to the reader. To prove (b)$\Rightarrow$(a′) fix $\nu' \in S_\nu$ and a character $\chi \in X^*(T_E)$, and consider an irreducible representation V with highest weight χ. As in the preceding proof we find an element $\mu' \in S_\mu$ such that $\langle \chi, \mu' \rangle$ is the smallest weight of μ in V_E. When $P_\nu(V)$ lies strictly above $P_\mu(V)$ at the point 1, we have $\langle \chi, \nu \rangle \geq$ (the smallest weight of ν' on V_E) $=$ (the smallest weight of ν on V_E) $> \langle \chi, \mu' \rangle$. Assuming (b), this yields the desired assertion unless $P_\mu(V)$ is a straight line. In that case we compare the two polygons for V and its dual $V^\vee$. Since $P_\nu(V)$ lies above $P_\mu(V)$, and $P_\nu(V^\vee)$ above $P_\mu(V^\vee)$, one easily shows that the respective polygons must be equal. Then $\langle \chi, \nu' \rangle$ is some slope of $P_\nu(V)$, hence it is equal to every slope of $P_\mu(V)$, and therefore to $\langle \chi', \mu' \rangle$ for every weight χ' of T_E in V_E. In particular we have $\langle \chi, \nu' \rangle = \langle \chi^{\gamma^{-1}}, \mu' \rangle = \langle \chi, \mu'^\gamma \rangle$ for every $\gamma \in \Gamma$. Thus $\langle \chi, \nu' \rangle = \langle \chi, \mu' \rangle$ for every $\mu' \in S_\mu$, proving (a′).

It remains to prove (b) under the assumption (a′). Consider an irreducible representation V for which $P_\mu(V)$ is not a straight line but meets $P_\nu(V)$ at a point in the interior of its interval of definition. In view of Theorem 1.3 it suffices to derive a contradiction in this case. It is easy to see that the polygons must meet at a break point of $P_\nu(V)$, that is, a point where its slope changes. Suppose this point has coordinates (i, r) with $0 < i < \dim_F(V)$. Then r is the unique smallest weight of ν in the representation $\bigwedge^i V_E$. More precisely, if $V'_E \subset V_E$ denotes the F-subspace of dimension i corresponding to the slopes of $P_\nu(V)$ to the left of (i, r), then $\bigwedge^i V'_E$ is the unique line in $\bigwedge^i V_E$ on which ν has the smallest possible weight r.

By assumption the smallest weight of μ on $\bigwedge^i V_E$ is also equal to r. If χ denotes the weight of T_E on $\bigwedge^i V'_E$, it follows that we must be in the second case of condition (a′), that is, we have $\langle \chi, \nu' \rangle = \langle \chi, \mu' \rangle$ for all $\nu' \in S_\nu$ and $\mu' \in S_\mu$. Thus $\bigwedge^i V'_E$ is the unique line in $\bigwedge^i V_E$ on which the quasi-cocharacter $\nu_0 := \sum_{\nu' \in S_\nu} \nu'$ has the smallest possible weight $r \cdot \mathrm{card}(S_\nu)$. By construction ν_0 is a quasi-cocharacter in the center of H which is defined over F (i.e., any power of ν_0 which is an honest cocharacter is defined over F). Therefore the subspace $\bigwedge^i V'_E \subset \bigwedge^i V_E$ is H-invariant and defined over F. At last, this implies that V'_E comes from an H-invariant subspace $V' \subset V$, contradicting the assumption that V is irreducible. This proves (b), as desired. $\square$

Proof of Theorem 1.5: The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) are obvious. To prove (b)$\Rightarrow$(a) we first note that the situation is completely symmetric in μ and ν. Applying Theorem 1.3 twice, the second time with μ and ν interchanged, condition (b) implies $\mathrm{Conv}(S_\mu) = \mathrm{Conv}(S_\nu)$. Taking corners of these polytopes, the condition (a) follows.

It remains to prove the implication (c)⇒(b). Fix a faithful representation V for which the two polygons coincide. We must prove the same for any other representation V_1. If the polygons are equal for all irreducible subquotients of V_1, clearly the same follows for V_1 itself. Thus we may assume that V_1 is irreducible. Then V_1 is isomorphic to a subquotient of the tensor space $V^{m,n} := V^m \otimes (V^\vee)^n$ for suitable non-negative integers m, n. Clearly the two polygons coincide for $V^{m,n}$. Let V_2 be the direct sum of the Jordan-Hölder factors of $V^{m,n}$ other than V_1. Then $P_\nu(V^{m,n})$ is obtained by joining $P_\nu(V_1)$ and $P_\nu(V_2)$ and rearranging all edges in the order of increasing slopes. The same holds with μ in place of ν. As $P_\nu(V^{m,n}) = P_\mu(V^{m,n})$ and, by assumption, $P_\nu(V_i)$ is above $P_\mu(V_i)$ for both $i = 1, 2$, by an easy comparison of multiplicities it follows that the respective polygons coincide. This proves (b), as desired. □

Comparing subgroups generated by cocharacters: For any quasi-cocharacter μ of H let $H_\mu \subset H$ be the smallest normal algebraic subgroup, defined over F, such that μ factors through $H_{\mu,E}$. Equivalently, this subgroup can be characterized by the fact that $H_{\mu,E}$ is generated by the images of all $H(E) \rtimes \mathrm{Aut}(E/F)$-conjugates of μ. The geometric relations between cocharacters listed in the preceding results have the following consequences.

Proposition (1.6). *Suppose the equivalent conditions in Theorem 1.3 are satisfied. Then we have $H_\nu \subset H_\mu$.*

Proof: Since H_μ is a normal subgroup of H, the intersection $T_{\mu,E} := H_{\mu,E} \cap T_E$ is a maximal torus of $H_{\mu,E}$. The above characterization of $H_{\mu,E}$ implies that the cocharacter space $Y_*(T_{\mu,E}) \otimes_{\mathbb{Z}} \mathbb{R}$ is just the $\mathbb{R}$-subspace of Y generated by S_μ. The same statements hold with ν in place of μ. Now condition (a) of Theorem 1.3 implies that $\mathbb{R} \cdot S_\nu \subset \mathbb{R} \cdot S_\mu$. Therefore a maximal torus of H_μ contains a maximal torus of H_ν. As these groups normalize each other, we deduce that the factor group $H_\nu/(H_\nu \cap H_\mu)$ has rank zero. Since by construction it is also generated by the images of cocharacters, it must be trivial. Thus we have $H_\nu \subset H_\mu$, as desired. □

Proposition (1.7). *Suppose the equivalent conditions in both Theorem 1.4 and Theorem 1.5 are satisfied. Then we have $H_\nu = H_\mu$, this group is solvable, and its toric part is either trivial or isomorphic to $\mathbb{G}_{m,F}$.*

Proof: The conditions (a) of Theorems 1.4–5 together say that $S_\nu = S_\mu \subset \mathrm{Conv}(S_\mu)^\circ$. The first equality, combined with Proposition 1.6, already shows $H_\nu = H_\mu$. The latter inclusion implies that S_μ consists of a single element, say μ_0. By the remarks in the preceding proof μ_0 generates the cocharacter space of a maximal torus of $H_{\mu,E}$. As μ_0 is fixed under Γ, both the Weyl group of H_E and the automorphism group $\mathrm{Aut}(E/F)$ act trivially on this cocharacter space. Thus on the one hand the Weyl group of $H_{\mu,E}$ must be trivial, hence H_μ is solvable. On the other hand H_μ modulo its unipotent radical must be a split torus of dimension at most 1. □

§2. Crystalline Local Galois Representations

Fix a rational prime p and a finite extension $K/\mathbb{Q}_p$, which for simplicity we assume unramified. The Frobenius substitution of K over $\mathbb{Q}_p$ is denoted σ. Embed K into a fixed algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $\mathcal{D} := \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ denote the "decomposition" group. In this section we study crystalline Galois representations of $\mathcal{D}$ over $\mathbb{Q}_p$. For such representations it is well-known that "The Newton polygon lies above the Hodge polygon". Our main aim is to deduce from this certain consequences for the associated algebraic monodromy groups. We begin by reviewing some known facts concerning crystalline representations, filtered modules, and their associated algebraic monodromy groups.

The local algebraic monodromy group: Consider a continuous representation of $\mathcal{D}$ on a finite dimensional $\mathbb{Q}_p$-vector space V. The associated algebraic monodromy group H_V is the Zariski closure of the image of $\mathcal{D}$ in the general linear group $\underline{\mathrm{Aut}}_{\mathbb{Q}_p}(V)$. This group has a tannakian description, as follows.

Let $\mathbf{Rep}_{\mathcal{D}}$ denote the category of all finite dimensional continuous representations of $\mathcal{D}$ over $\mathbb{Q}_p$. This is a tannakian category in the sense of Deligne-Milne [12]. For V as above we let $((V))$ denote the full tannakian subcategory of $\mathbf{Rep}_{\mathcal{D}}$ which is generated by V, i.e. the smallest abelian full subcategory containing V which is stable by taking subquotients, tensor product, and duals. Let $\mathbf{Vec}_{\mathbb{Q}_p}$ denote the category of finite dimensional $\mathbb{Q}_p$-vector spaces, and ω_V the "forgetful" functor $((V)) \to \mathbf{Vec}_{\mathbb{Q}_p}$ which to a representation of $\mathcal{D}$ associates the underlying $\mathbb{Q}_p$-vector space. This is a fiber functor of tannakian categories, and we have a canonical isomorphism $H_V \cong \underline{\mathrm{Aut}}^{\otimes}(\omega_V)$. The category $((V))$ is then canonically equivalent to the category of representations of H_V (cf. [12] §2).

Filtered modules: Following Fontaine [15] 1.2, [16] 5.1 a "filtered module" over K consists of a finite dimensional K-vector space M together with

(a) a descending, exhaustive, separated filtration by K-subspaces $\mathrm{Fil}^i M$ ($i \in \mathbb{Z}$), and

(b) a σ-linear automorphism $f_M : M \xrightarrow{\sim} M$, i.e. an automorphism of additive groups satisfying $f_M(xm) = \sigma(x) f_M(m)$ for all $x \in K$ and $m \in M$.

Let $\mathbf{MF}_K$ denote the category of filtered modules over K. This is a $\mathbb{Q}_p$-linear category with tensor products and duals, but it is not abelian. Fontaine ([15] §4, [17] 1.3) defines a full subcategory $\mathbf{MF}_K^f$ of "weakly admissible filtered modules", which is abelian and tannakian. Its identity object consists of the vector space K with $0 = \mathrm{Fil}^1 K \subsetneq \mathrm{Fil}^0 K = K$ and $f_K = \sigma$.

For any object M of $\mathbf{MF}_K^f$ we let $((M))$ denote the full tannakian subcategory of $\mathbf{MF}_K^f$ which is generated by M. Then the functor $\omega_M : ((M)) \to \mathbf{Vec}_K$ which to each filtered module associates its underlying K-vector space is a fiber functor of tannakian categories. Its automorphism group $H_M := \underline{\mathrm{Aut}}^{\otimes}(\omega_M)$ is a certain algebraic subgroup of the general linear group $\underline{\mathrm{Aut}}_K(M)$, defined over K.

The "mysterious functor": Fontaine ([16] Th.5.2) defines:

(a) a full tannakian subcategory $\mathbf{Rep}_{\mathcal{D}}^{\mathrm{cris}}$ of $\mathbf{Rep}_{\mathcal{D}}$, stable under taking subquotients. Objects of $\mathbf{Rep}_{\mathcal{D}}^{\mathrm{cris}}$ are called "crystalline representations".

(b) a full tannakian subcategory $\mathbf{MF}_K^a$ of $\mathbf{MF}_K^f$, stable under taking subquotients. Objects of $\mathbf{MF}_K^a$ are called "admissible filtered modules".

(c) a (covariant) equivalence of tensor categories $\mathbf{D} : \mathbf{Rep}_{\mathcal{D}} \xrightarrow{\sim} \mathbf{MF}_K$.

In the following we fix a crystalline representation V and let $M := \mathbf{D}(V)$ denote the associated admissible filtered module. Then the functor $\mathbf{D}$ induces an equivalence of tannakian categories $((V)) \xrightarrow{\sim} ((M))$. By tannakian theory ([12] Th.3.2) it follows that the algebraic groups $H_{V,K} := H_V \times_{\mathbb{Q}_p} K$ and H_M are in a canonical way inner forms of each other.

Hodge decomposition and the Hodge cocharacter: Following Wintenberger ([43] 4.2.1, Th.3.1.2) any weakly admissible filtered module M possesses a canonical splitting of the filtration $\mathrm{Fil}^\bullet M$, i.e. a grading $M = \bigoplus_{i\in\mathbb{Z}} M^i$ by K-subspaces such that $\mathrm{Fil}^i M = \bigoplus_{i'\geq i} M^i$ for each $i \in \mathbb{Z}$. This grading is functorial in M and compatible with tensor products and duals. As explained in Section 1, this data corresponds to a unique cocharacter $\mu_M : \mathbb{G}_{m,K} \to H_M$, characterized by the fact that $x \in K^\times$ acts by multiplication with x^i on each M^i. This cocharacter is defined over K. By analogy with usual Hodge structures the grading of M may be called Hodge decomposition and μ_M the associated Hodge cocharacter.

Via the inner twist the conjugacy class of μ_M corresponds to a unique conjugacy class of cocharacters of $H_{V,\bar{\mathbb{Q}}_p} := H_V \times_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p$. We fix a representative μ_V. By construction we have:

Fact (2.1). *The $H_V(\bar{\mathbb{Q}}_p)$-conjugacy class of μ_V is defined over K.*

It is possible to characterize this conjugacy class using the Hodge-Tate decomposition associated to V (cf. [16], [44] §4, [18]). We have chosen the above construction via H_M because it will also apply to the Newton cocharacter below.

Frobenius: Let us set $m := [K/\mathbb{Q}_p]$. Then f_M^m is a K-linear automorphism of M. As its formation is functorial in M and compatible with tensor products and duals, this defines an element of $H_M(K) \subset \mathrm{Aut}_K(M)$. Via the inner twist its conjugacy class corresponds to a unique conjugacy class in $H_V(\bar{\mathbb{Q}}_p)$, for which we fix a representative Φ_V.

Proposition (2.2). *The $H_V(\bar{\mathbb{Q}}_p)$-conjugacy class of Φ_V is defined over $\mathbb{Q}_p$.*

Proof: Since f_M is a σ-linear automorphism of M, the map $\psi : h \mapsto f_M \circ h \circ f_M^{-1}$ defines an isomorphism $\sigma^* H_M \xrightarrow{\sim} H_M$. Via the inner twist we thus obtain an isomorphism $(\sigma^* H_{V,K}) \times_K \bar{\mathbb{Q}}_p \xrightarrow{\sim} H_{V,K} \times_K \bar{\mathbb{Q}}_p$ which is unique up to an inner automorphism. Now H_V is defined over $\mathbb{Q}_p$, so this isomorphism amounts to a $\bar{\mathbb{Q}}_p$-valued automorphism of H_V. As its construction was intrinsic, i.e. functorial in M and compatible with tensor constructions, it is already an inner automorphism.

Obviously f_M^m is a K-valued element of H_M, and by definition we have $\psi(f_M^m) = f_M^m$. As ψ corresponds to an inner automorphism of $H_{V,\bar{\mathbb{Q}}_p}$, this shows that the conjugacy class of Φ_V is defined over $\mathbb{Q}_p$, as desired. □

The Newton cocharacter: Next let $\mathrm{ord}_p : \bar{\mathbb{Q}}_p \to \mathbb{Q} \cup \{\infty\}$ be the normalized valuation with $\mathrm{ord}_p(p) = 1$. Then there is a unique f_M^m-invariant $\mathbb{Q}$-grading $M = \bigoplus_{i\in\mathbb{Q}} M_i$ of K-vector spaces such that all eigenvalues of f_M^m on M_i have normalized valuation mi. This decomposition can be obtained, for instance, from the eigenspace decomposition of

$M \otimes_K \bar{\mathbb{Q}}_p$ under the semisimple part of f_M^m. The grading is functorial in M and compatible with tensor products and duals, hence corresponds to a unique quasi-cocharacter $\nu_M : \hat{\mathbb{G}}_{m,K} \to H_M$.

In the same way Φ_V gives rise to a quasi-cocharacter ν_V of H_V, defined over $\bar{\mathbb{Q}}_p$. The conjugacy classes of ν_V and ν_M correspond to each other via the inner twist between H_V and H_M. From Proposition 2.2 it follows that the conjugacy class of ν_V is defined over $\mathbb{Q}_p$. For ease of terminology we call ν_M and ν_V Newton cocharacters, even when they are only quasi-cocharacters. The reason for the name "Newton" is the following relation with the Newton polygon.

Hodge and Newton polygons: In Section 1 we have associated a polygon to any cocharacter and any representation of an algebraic group. Consider an object W of $((V))$, corresponding to the filtered module $N := \mathbf{D}(W)$ in $((M))$. From the respective Hodge cocharacters we then obtain the Hodge polygon $P_{\mu_M}(N) = P_{\mu_V}(W)$ of N and V. The respective Newton cocharacters give rise to the Newton polygon $P_{\nu_M}(M) = P_{\nu_V}(V)$.

Geometric location of the Newton cocharacter: It is known that for crystalline representations the Newton polygon lies above the Hodge polygon. We shall express this information in intrinsic group theoretic terms, as follows. Putting $F := \mathbb{Q}_p$, $E := \bar{\mathbb{Q}}_p$, and $H := H_V$, we are in the situation of Section 1. Again we fix a maximal torus $T_{\bar{\mathbb{Q}}_p}$ of $H_{V,\bar{\mathbb{Q}}_p}$ and work inside its cocharacter space $Y := Y_*(T_{\bar{\mathbb{Q}}_p}) \otimes_{\mathbb{Z}} \mathbb{R}$. As in Definition 1.1 the set of all $H_V(\bar{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-conjugates of μ_V, resp. of ν_V, which factor through $T_{\bar{\mathbb{Q}}_p}$ is denoted S_{μ_V}, resp. S_{ν_V}. Note that in the special case $K = \mathbb{Q}_p$ Fact 2.1 and Proposition 2.2 imply that these sets do not change if only the $H_V(\bar{\mathbb{Q}}_p)$-conjugates are taken.

Theorem (2.3). *We have $S_{\nu_V} \subset \mathrm{Conv}(S_{\mu_V})^\circ$.*

Proof: (A related assertion is in Rapoport-Richartz [27] Theorem 4.2.) Consider an irreducible representation W of H_V over $\mathbb{Q}_p$, corresponding to a simple filtered module N in $((M))$. Since N is a weakly admissible filtered module, by [15] Prop.4.3.3 we know already that its Newton polygon $P_{\nu_M}(N)$ lies above its Hodge polygon $P_{\mu_M}(N)$. Thus to apply Theorem 1.4 it suffices to prove that $P_{\nu_M}(N)$ lies strictly above $P_{\mu_M}(N)$ unless the latter is a straight line.

Assume that $P_{\mu_M}(N)$ is not a straight line but meets $P_{\nu_M}(N)$ above a point in the interior of its interval of definition. Then it is easy to see that the polygons must meet at a break point of the Newton polygon, that is, a point where its slope changes. Suppose this point lies at $0 < d' < \dim_K N$ and the greatest slope to the left of that point is s. Let $N' \subset N$ be the maximal f_N^m-invariant K-subspace on which the normalized valuation of every eigenvalue is $\leq ms$. Endowed with the induced filtration $\mathrm{Fil}^i N' := N' \cap \mathrm{Fil}^i N$ this is a subobject of N in the category $\mathbf{MF}_K$.

Let us show that N' is weakly admissible. The relevant polygons are sketched in Figure 2.4. By construction the Newton polygon $P_{\nu_M}(N')$ is just the initial segment of $P_{\nu_M}(N)$ above the interval $[0, d']$. For the Hodge polygon, on the other hand, the slopes of $P_{\mu_M}(N')$ are among the slopes of $P_{\mu_M}(N)$. Since the slopes are always arranged in ascending order, the endpoint of $P_{\mu_M}(N')$ lies on or above $P_{\mu_M}(N)$. Now one of the equivalent definitions of weak admissibility of N ([15] Def.4.1.4) states that the endpoint

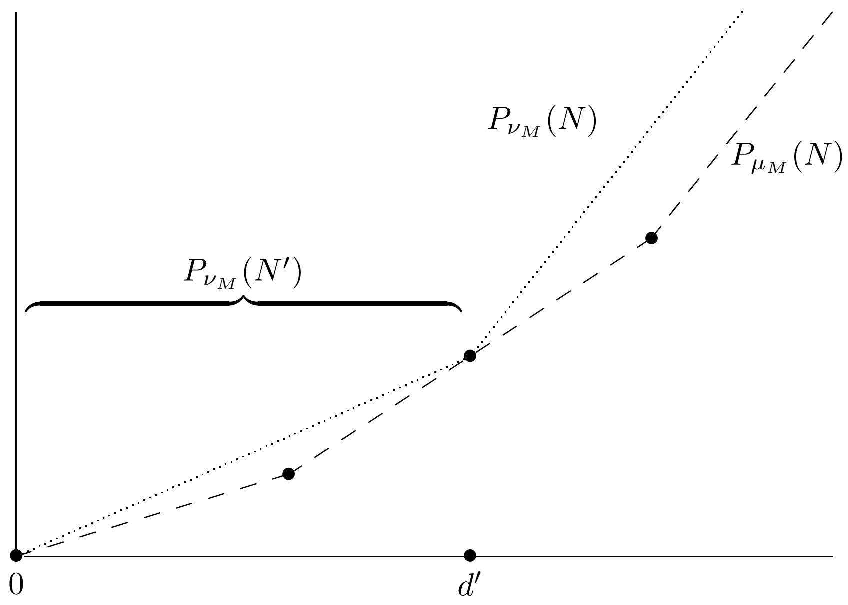

Figure 2.4

of $P_{\nu_M}(N')$ must lie on or above the endpoint of $P_{\mu_M}(N')$. Since $P_{\nu_M}(N)$ and $P_{\mu_M}(N)$ coincide at that point, it follows that the endpoints of $P_{\nu_M}(N')$ and $P_{\mu_M}(N')$ are equal. By [15] Prop.4.5.1 we now deduce that N' is weakly admissible, as desired.

Being weakly admissible, N' forms a non-zero proper subobject of N in the category $((M))$. This contradicts the assumption that N is simple. (For a related statement see Katz [19] Th.1.6.1.) □

It would be interesting to obtain further relations between the Hodge and Newton cocharacters. Later in this section we shall look at one of the possible extremes.

Description of H_M and H_V: In order to describe the group H_M we must take into account not only the Hodge cocharacter μ_M and the Frobenius element f_M^m, but also the various conjugates of μ_M under f_M. For any $i \in \mathbb{Z}$ there is a unique cocharacter ${}^{\psi^i}\mu_M : \mathbb{G}_{m,K} \to H_M$ characterized by

$$ {}^{\psi^i}\mu_M(\sigma^i(x)) = f_M^i \circ \mu_M(x) \circ f_M^{-i} $$

for all $x \in K^\times$. The following result is an analogue of Wintenberger [43] Prop. 4.2.3.

Proposition (2.5). *The subgroup of H_M which is generated by f_M^m and the images of ${}^{\psi^i}\mu_M$ for all $i \in \mathbb{Z}$ is Zariski dense in H_M.*

Proof: First we formalize the process of extension of scalars from $\mathbb{Q}_p$ to K, following general tannakian theory (see Deligne [11] §5). Let $((M)) \otimes_{\mathbb{Q}_p} K$ denote the category whose objects are objects $\tilde{N}$ of $((M))$ together with a homomorphism of $\mathbb{Q}_p$-algebras $K \to \mathrm{End}_{((M))}(\tilde{N})$. Since objects of $((M))$ are K-vector spaces (with extra structures), such $\tilde{N}$ is in particular a module over the ring $K \otimes_{\mathbb{Q}_p} K$, where the second factor refers to the additional, "external" K-action. The morphisms in $((M)) \otimes_{\mathbb{Q}_p} K$ are those morphisms in $((M))$ that commute with the additional K-action. With the tensor product over $K \otimes_{\mathbb{Q}_p} K$ we obtain a rigid abelian tensor category over K.

To get a closer look at the objects of this category consider the isomorphism

$$K \otimes_{\mathbb{Q}_p} K \longrightarrow \bigoplus_{i \bmod m} K, \quad x \otimes y \mapsto (\sigma^i(x) \cdot y)_i \ .$$

Any object has a corresponding decomposition $\tilde{N} = \bigoplus_{i \bmod m} N_i$. To analyze the filtered module structure on $\tilde{N}$, note first that the Hodge decomposition must consist of $K \otimes_{\mathbb{Q}_p} K$-submodules. Thus each N_i comes with its own Hodge decomposition. Furthermore, as the automorphism f_M is σ-linear in the first factor of $K \otimes_{\mathbb{Q}_p} K$, it permutes the N_i cyclically. It follows that $\tilde{N}$ is determined up to isomorphism by the K-vector space N_0 together with its automorphism induced by f_M^m and the pullback via f_M^i of the Hodge decomposition of N_i, for every $i \in \mathbb{Z}$. In other words, we have established an equivalence of categories between $((M)) \otimes_{\mathbb{Q}_p} K$ and the category of these N_0 with the indicated structures.

Now if G denotes the automorphism group of the fiber functor

$$\omega_{\tilde{M}}: \ ((M)) \otimes_{\mathbb{Q}_p} K \longrightarrow \mathbf{Vec}_K \ , \ \tilde{N} \mapsto N_0 \ ,$$

we have a natural equivalence of categories $((M)) \otimes_{\mathbb{Q}_p} K \longrightarrow \mathbf{Rep}_K$. Observe that $((M)) \otimes_{\mathbb{Q}_p} K$ is generated as a tensor category by the object $\tilde{M} = M \otimes_{\mathbb{Q}_p} K$, whose image $\omega_{\tilde{M}}(\tilde{M})$ is canonically isomorphic to M. Thus G is the Zariski closure of the subgroup of $\underline{\mathrm{Aut}}_K(M)$ that is generated by f_M^m and the images of ${}^{\psi^i}\mu_M$ for all $i \in \mathbb{Z}$.

On the other hand the composite of $\omega_{\tilde{M}}$ with the functor

$$((M)) \longrightarrow ((M)) \otimes_{\mathbb{Q}_p} K \ , \ N \mapsto N \otimes_{\mathbb{Q}_p} K$$

is just the original fiber functor ω_M. By universality of the construction of $((M)) \otimes_{\mathbb{Q}_p} K$ the induced functor $((M)) \longrightarrow \mathbf{Rep}_{H_M}$ factors through a natural tensor functor

$$((M)) \otimes_{\mathbb{Q}_p} K \longrightarrow \mathbf{Rep}_{H_M} \ .$$

This amounts to a functor $\mathbf{Rep}_G \to \mathbf{Rep}_{H_M}$ which is a right inverse of the restriction functor $\mathbf{Rep}_{H_M} \to \mathbf{Rep}_G$. It follows that $G = H_M$, as desired. $\square$

Coming to the group H_V, as in Section 1 we let H_{μ_V} denote the smallest normal algebraic subgroup, defined over $\mathbb{Q}_p$, such that μ_V factors through $H_{\mu_V,\bar{\mathbb{Q}}_p}$. In other words, the group $H_{\mu_V,\bar{\mathbb{Q}}_p} \subset H_{V,\bar{\mathbb{Q}}_p}$ is generated by the images of all $H_V(\bar{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-conjugates of μ_V.

Proposition (2.6). *The image of Φ_V in the factor group H_V/H_{μ_V} has coefficients in $\mathbb{Q}_p$ and generates a Zariski dense subgroup. In particular H_V/H_{μ_V} is commutative.*

Proof: Via the inner twist the group H_{μ_V} corresponds to the subgroup $H_{\mu_M} \subset H_M$ defined as in Section 1. By Proposition 2.5 the image of f_M^m in H_M/H_{μ_M} generates a Zariski dense subgroup. In particular this factor group is commutative. Being its inner twist the group H_V/H_{μ_V} is also commutative, and therefore the image of Φ_V in this group is the unique element of its conjugacy class. By Proposition 2.2 it is therefore defined over $\mathbb{Q}_p$. The rest follows again by inner twist. $\square$

Ordinary representations: The starting point is the following observation.

Proposition (2.7). *The following conditions are equivalent:*

(a) $S_{\nu_V} = S_{\mu_V}$.

(a′) S_{μ_V} *consists of one element.*

(b) The Hodge and Newton polygons for every object W of the category $((V))$ coincide.

(c) The Hodge and Newton polygons of V coincide.

Proof: From the proof of Theorem 2.3 we know already that for every W in $((V))$ the Newton polygon lies above the Hodge polygon. Thus the equivalences (a)⇔(b)⇔(c) follow directly from Theorem 1.5. The implication (a)⇒(a′) is a consequence of Theorem 2.3, and its converse is deduced as in the proof of Proposition 1.7. $\square$

Definition (2.8). *A crystalline representation V and its associated filtered module $\mathbf{D}(V)$ are called* ordinary *if and only if the equivalent conditions of Proposition 2.7 are met.*

This definition is equivalent to that in Wintenberger [43] §5.5. As a special case, when V is the Tate module of an abelian variety A over K with good reduction, property 2.7 (c) shows that V is ordinary if and only if the reduction of A is ordinary in the usual sense. The group theoretic consequences of the property "ordinary" are similar to those in that special case:

Proposition (2.9). *When V is ordinary, the group H_V is solvable.*

Proof: Using Theorem 2.3, Proposition 1.7 implies that H_{μ_V} is solvable. By Proposition 2.6 the factor group H_V/H_{μ_V} is commutative. Thus H_V is solvable, as desired. $\square$

“Quasi-ordinary” representations: To end with a bit of speculation, it might be useful to extend the concept of ordinary along the line suggested by Proposition 2.9.

Definition (2.10). *A crystalline representation V and its associated filtered module $\mathbf{D}(V)$ are called* quasi-ordinary *if and only if H_V is solvable.*

Motivated by the experience that group theoretic properties of algebraic monodromy groups correspond to arithmetic statements for the underlying motives, it would be interesting to find out more about quasi-ordinary representations. For instance, when V arises as part of the cohomology of a smooth projective algebraic variety X over a number field, is the local Galois representation quasi-ordinary for a set of places of Dirichlet density 1? This is true when X is an elliptic curve. Also, there might exist consequences for the local behavior of X.

Proposition (2.11). *If V is quasi-ordinary, then S_{ν_V} consists of the single element*

$$\frac{1}{\operatorname{card}(S_{\mu_V})} \cdot \sum_{\mu' \in S_{\mu_V}} \mu' \ .$$

Proof: The assertion does not change when the representation V is replaced by its semisimplification and hence H_V by its quotient by its unipotent radical. Thus without loss of generality we may assume that the identity component H_V° is a torus. Then the main point is to show that the quasi-cocharacter ν_V is centralized by H_V and defined over $\mathbb{Q}_p$.

For this first note that the algebraic group H_{μ_V} is connected, because over $\bar{\mathbb{Q}}_p$ it is generated by connected subgroups. Thus Proposition 2.6 implies that H_V is generated by H_V° together with Φ_V. Now by construction ν_V is centralized by Φ_V. It is also centralized by H_V° since the latter is a torus. Thus ν_V is centralized by all of H_V. By Proposition 2.2 the conjugacy class of ν_V is defined over $\mathbb{Q}_p$, hence ν_V itself is defined over $\mathbb{Q}_p$. In particular we must have $S_{\nu_V} = \{\nu_V\}$.

Now recall that S_{μ_V} and S_{ν_V} are Γ-orbits, where Γ is defined as in Section 1. Thus the unique element ν_V of the latter is itself Γ-invariant. By Theorem 2.3 it is an element of $\operatorname{Conv}(S_{\mu_V})$, and the unique Γ-invariant element is the one indicated. $\square$

§3. Global Algebraic Monodromy Groups and Generation by Cocharacters

In this section we shall consider a compatible system of representations of a global Galois group which arises from the ℓ-adic cohomology of an algebraic variety. As in the preceding section we are mainly interested in the associated algebraic monodromy groups. It was proved by Serre that these groups are generated essentially by the images of Newton cocharacters (combine Proposition 3.5 and Theorem 3.7 below). The main object of this section is to prove an analogous statement for Hodge cocharacters (see Theorem 3.18).

Galois representations arising from motives: In the following we fix a number field K and an algebraic closure $\bar{K}$. We also fix a smooth proper algebraic variety X over K and an integer d. Then for every rational prime ℓ the ℓ-adic cohomology group $H^d(X \times_K \bar{K}, \mathbb{Q}_\ell)$ is a finite dimensional $\mathbb{Q}_\ell$-vector space carrying a natural continuous action of $\mathrm{Gal}(\bar{K}/K)$. We shall concentrate on the semisimplification of this representation, denoted V_ℓ. It is known that $\dim_{\mathbb{Q}_\ell}(V_\ell)$ is independent of ℓ, say it is n. Once and for all we choose an identification $V_\ell \cong \mathbb{Q}_\ell^n$. Then the Galois action corresponds to a continuous homomorphism $\rho_\ell : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell)$, and our main object of interest is the associated global algebraic monodromy group:

Definition (3.1). *The Zariski closure in $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ of the image of ρ_ℓ is denoted G_ℓ.*

Since by construction ρ_ℓ is a semisimple representation, the algebraic group G_ℓ is reductive. All the results of this section are valid in some greater generality. For instance, one could work with the semisimplification of $\bigoplus_{d\in\mathbb{Z}} H^d(X \times_K \bar{K}, \mathbb{Q}_\ell)$ instead of a single cohomology group. Also, one could replace $H^d(X \times_K \bar{K}, \mathbb{Q}_\ell)$ by a direct factor which is cut out by a fixed algebraic cycle. Essentially we shall use only the fact that the representations ρ_ℓ form a strictly compatible system of ℓ-adic representations which are locally crystalline in equal residue characteristic at almost all places of K.

The local Galois representation: For any non-archimedean place v of K we let K_v denote the completion of K at v and k_v the residue field. We fix an extension $\bar{v}$ of v to $\bar{K}$ and let $\bar{k}_v$ denote the residue field at $\bar{v}$. Let $\mathcal{I}_{\bar{v}} \subset \mathcal{D}_{\bar{v}} \subset \mathrm{Gal}(\bar{K}/K)$ denote the inertia group and the decomposition group at $\bar{v}$. Then $\mathcal{D}_{\bar{v}}/\mathcal{I}_{\bar{v}} \cong \mathrm{Gal}(\bar{k}_v/k_v)$ is the free pro-finite group generated by Frobenius. We let $\mathrm{Frob}_v \in \mathcal{D}_{\bar{v}}$ denote any element that represents the geometric Frobenius in $\mathcal{D}_{\bar{v}}/\mathcal{I}_{\bar{v}}$.

In the following we shall discuss the restriction of the different representations ρ_ℓ to the decomposition group $\mathcal{D}_{\bar{v}}$. The behavior depends heavily on the relation between ℓ and the characteristic of k_v. Let us fix a finite set S of non-archimedean places of K such that X has good reduction outside S. For later use we assume that S contains all places where K is ramified over $\mathbb{Q}$. Throughout we shall restrict ourselves to places not in S.

Strict compatibility: Suppose that $v \notin S$ and $\ell \neq \mathrm{char}(k_v)$. Then the following fundamental facts are known (Deligne [8] Th. 1.6, [9] Cor. 3.3.9).

Theorem (3.2). *(a) ρ_ℓ is unramified at v, that is, its restriction to $\mathcal{I}_{\bar{v}}$ is trivial.*

(b) The characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$ has coefficients in $\mathbb{Z}$ and is independent of ℓ.

Frobenius eigenvalues: Let $\bar{\mathbb{Q}}$ denote a fixed algebraic closure of $\mathbb{Q}$. We choose an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and let $|\ |_\infty$ denote the associated complex absolute value. Likewise, for every rational prime ℓ we choose an embedding of $\bar{\mathbb{Q}}$ into the algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and let ord_ℓ denote the valuation on these which is normalized so that $\mathrm{ord}_\ell(\ell) = 1$.

By Theorem 3.2 the eigenvalues of $\rho_\ell(\mathrm{Frob}_v)$ are algebraic integers, independent of ℓ. The following can be said about their behavior at different places of $\bar{\mathbb{Q}}$.

Theorem (3.3). *Let $\xi \in \bar{\mathbb{Q}}$ be any eigenvalue of $\rho_\ell(\mathrm{Frob}_v)$ for $v \notin S$. Then we have*
(a) $|\xi|_\infty = \sqrt{\mathrm{card}(k_v)}$,
(b) $\mathrm{ord}_\ell(\xi) = 0$ *for any prime* $\ell \neq \mathrm{char}(k_v)$, *and*
(c) $0 \leq \mathrm{ord}_p(\xi) \leq d \cdot \mathrm{ord}_p(\mathrm{card}(k_v)) = d \cdot [k_v/\mathbb{F}_p]$ *for* $p = \mathrm{char}(k_v)$.

Here (b) follows from fundamental properties of ℓ-adic cohomology. The lower bound in (c) results from Theorem 3.2 (b), which by Poincaré duality implies the upper bound. Assertion (a) is Deligne's celebrated theorem concerning the analogue of the "Riemann Hypothesis" [9]. The valuations in (c) have been discussed in Section 2 and will play an important role below.

Frobenius tori: The information on the groups G_ℓ that results from properties of the Frobenius elements $\rho_\ell(\mathrm{Frob}_v)$ can be encoded neatly in terms of Frobenius tori, following Serre (cf. [31], or Chi [6]). For any $v \notin S$ choose a semisimple element $t_v \in \mathrm{GL}_n(\mathbb{Q})$ with the same characteristic polynomial as $\rho_\ell(\mathrm{Frob}_v)$. Let $T_v \subset \mathrm{GL}_{n,\mathbb{Q}}$ be the Zariski closure of the subgroup generated by t_v. By construction its identity component is a torus, called the Frobenius torus associated to v. Clearly t_v and T_v are determined uniquely up to conjugation.

By construction t_v is conjugate under $\mathrm{GL}_n(\mathbb{Q}_\ell)$ to the semisimple part of $\rho_\ell(\mathrm{Frob}_v)$, for every $\ell \neq \mathrm{char}(k_v)$. As any linear algebraic group contains the semisimple part of any of its elements, it follows that t_v is conjugate to an element of G_ℓ. Therefore $T_{v,\mathbb{Q}_\ell} := T_v \times_{\mathbb{Q}} \mathbb{Q}_\ell$ is conjugate under $\mathrm{GL}_n(\mathbb{Q}_\ell)$ to an algebraic subgroup of G_ℓ. The observation that this construction provides us with many subtori "common to all G_ℓ" has been exploited by Serre, with the following results (among others).

The Newton cocharacter: The cocharacter group and the character group

$$Y_*(T_v) := \mathrm{Hom}(\mathbb{G}_{m,\bar{\mathbb{Q}}}, T_v \times_{\mathbb{Q}} \bar{\mathbb{Q}}) \ ,$$
$$X^*(T_v) := \mathrm{Hom}(T_v \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{G}_{m,\bar{\mathbb{Q}}})$$

of T_v are in canonical perfect duality

$$\langle\ ,\ \rangle :\ X^*(T_v) \times Y_*(T_v) \to \mathbb{Z},\ \ (\chi, \lambda) \mapsto \deg(\chi \circ \lambda).$$

Thus giving a quasi-cocharacter of T_v, that is, an element of $Y_*(T_v) \otimes_{\mathbb{Z}} \mathbb{Q}$, is equivalent to giving a homomorphism $X^*(T_v) \to \mathbb{Q}$. In particular there is a unique quasi-cocharacter ν_v of T_v such that

$$\langle \chi, \nu_v \rangle = \frac{\mathrm{ord}_p\big(\chi(t_v)\big)}{[k_v/\mathbb{F}_p]} \tag{3.4}$$

for all $\chi \in X^*(T_v)$, where $p = \mathrm{char}(k_v)$. We call ν_v the Newton cocharacter of T_v, even when it is only a quasi-cocharacter.

An equivalent characterization of ν_v can be given as in Section 2. Namely, for every $i \in \mathbb{Q}$ let $V_i \subset \bar{\mathbb{Q}}^n$ be the weight space of weight i under ν_v. This is just the sum of the eigenspaces of t_v for all eigenvalues with p-adic valuation $m_v i$, and the decomposition $\bar{\mathbb{Q}}^n = \bigoplus_{i\in\mathbb{Q}} V_i$ determines ν_v uniquely.

Abundance of the Newton cocharacter: As T_v is defined over $\mathbb{Q}$ there is a natural action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $Y_*(T_v)$. The following result plays a central role in Serre's theory (cf. [31] p.10, also Chi [6] Th.3.4).

Proposition (3.5). *The cocharacter space $Y_v := Y_*(T_v) \otimes_{\mathbb{Z}} \mathbb{R}$ of T_v is generated over $\mathbb{R}$ by the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-orbit of ν_v.*

Proof: Suppose not. Then there exists a character $\chi \in X^*(T_v)$ of infinite order with

$$0 = \langle \chi, \nu_v^\sigma \rangle = \langle \chi^{\sigma^{-1}}, \nu_v \rangle \overset{(3.4)}{=} \frac{\mathrm{ord}_p\big(\chi^{\sigma^{-1}}(t_v)\big)}{[k_v/\mathbb{F}_p]} = \frac{\mathrm{ord}_p\big(\chi(t_v)^{\sigma^{-1}}\big)}{[k_v/\mathbb{F}_p]}.$$

for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, where $p = \mathrm{char}(k_v)$. This means that the algebraic number $\chi(t_v)$ is a unit at all primes above p. On the other hand Theorem 3.3 implies that $\chi(t_v)$ is a unit at all other finite primes and that its archimedean norms are equal. Thus the product formula shows that the archimedean norms are all equal to 1. In other words $\chi(t_v)$ is an algebraic number whose local norms are all 1. Any such number is a root of unity! Thus we have $\chi^m(t_v) = 1$ for some positive integer m. As t_v generates a Zariski dense subgroup of T_v, it follows that χ^m is the trivial character. Thus χ has finite order, contrary to the assumption. □

Connectedness and maximal tori: Let G_ℓ° denote the identity component of G_ℓ. Then the open subgroup $\rho_\ell^{-1}(G_\ell^\circ(\mathbb{Q}_\ell)) \subset \mathrm{Gal}(\bar{K}/K)$ corresponds to a unique finite Galois extension of K.

Theorem (3.6). *(Serre [31] p.17, [33] 2.2.3) This extension is independent of ℓ.*

We denote this extension by K^{conn}. Note that replacing K by K^{conn} has the effect of replacing each G_ℓ by its identity component, and afterwards we have $K = K^{\mathrm{conn}}$.

Theorem (3.7). *(Serre [31] p.13, [33] 2.2.4, cf. also Chi [6] Cor.3.8) Suppose that $K = K^{\mathrm{conn}}$. Then there is a set $V_{\max}$ of non-archimedean places of K, of Dirichlet density 1, such that for all $v \in V_{\max}$ we have*

(a) $v \notin S$,

(b) the group T_v is connected, hence a torus, and

(c) for every $\ell \neq \mathrm{char}(k_v)$ the torus $T_{v,\mathbb{Q}_\ell}$ is conjugate under $\mathrm{GL}_n(\mathbb{Q}_\ell)$ to a maximal torus of G_ℓ.

Corollary (3.8). *(Serre) The rank and the formal character of G_ℓ are independent of ℓ.*

Proof: Without loss of generality we may assume that $K = K^{\text{conn}}$. With any fixed $v \in V_{\max}$, Theorem 3.7 implies the assertion for all $\ell \neq \text{char}(k_v)$. Repeating the argument with a place v of different residue characteristic finishes the proof. □

For further consequences we look at the local Galois representation in equal residue characteristic, as follows.

Relation with crystalline cohomology: As before let us consider a non-archimedean place $v \notin S$ of K of residue characteristic p. By our assumption on S the local field extension $K_v/\mathbb{Q}_p$ is unramified. We now consider the case $\ell = p$ and shall apply the concepts and results of Section 2 to the restriction of ρ_p to the decomposition group $\mathcal{D}_{\bar{v}}$. Mostly we shall use the notations of Section 2 with an additional index $_v$.

Let $\mathcal{O}_v$ denote the ring of v-adic integers in K_v, and let X_v denote the closed fiber of a smooth proper model of X over $\mathcal{O}_v$. Then the crystalline cohomology group

$$H^d(X_v/\mathcal{O}_v) \otimes_{\mathcal{O}_v} K_v$$

possesses a natural structure of filtered module. Here the action of Frobenius is intrinsically defined, and the Hodge filtration results from comparison with the De Rham cohomology of X. Moreover by the C_{cris}-conjecture, proved by Fontaine-Messing and Faltings (cf. [18] Th. 3.2.3) the representation of $\mathcal{D}_v$ on the ℓ-adic cohomology group $H^d(X \times_K \bar{K}, \mathbb{Q}_p)$ is crystalline and its associated filtered module is canonically isomorphic to the above crystalline cohomology group.

Recall that V_p was defined as the semisimplification of $H^d(X \times_K \bar{K}, \mathbb{Q}_p)$ under the global Galois group $\text{Gal}(\bar{K}/K)$. Thus it is a partial semisimplification of the associated local Galois representation and hence again crystalline. Moreover its associated filtered module $M_v := \mathbf{D}(V_p)$ is a partial semisimplification of the above crystalline cohomology group.

The local algebraic monodromy groups: As in Section 2 we can now compare the following two algebraic groups.

Definition (3.9). *(a) The Zariski closure in* $\text{GL}_{n,\mathbb{Q}_p}$ *of* $\rho_p(\mathcal{D}_{\bar{v}})$ *is denoted* $H_{V,v}$.

(b) We set $H_{M,v} := \underline{\text{Aut}}^{\otimes}(\omega_{M_v})$*, where* ω_{M_v} *is the natural fiber functor* $((M_v)) \to \mathbf{Vec}_{K_v}$.

Clearly $H_{V,v}$ is contained in the global algebraic monodromy group G_p. Although G_p was forced to be reductive, this is not at all so for $H_{V,v}$. By Section 2 the algebraic groups $H_{V,v} \times_{\mathbb{Q}_p} K_v$ and $H_{M,v}$ are inner forms of each other. The reason for passing back and forth between these two groups is that in equal residue characteristic a Frobenius can be found naturally only on the crystalline side, that is, in $H_{M,v}$.

Crystalline Frobenius: Let f_{M_v} denote the σ-linear automorphism of M_v which is part of the structure of filtered module, and put $m_v := [K_v/\mathbb{Q}_p]$. Consider another rational prime $\ell \neq p$. The following statement is a consequence of the fact that crystalline cohomology is a "Weil cohomology".

Theorem (3.10). *(Katz-Messing, cf. [18] (1.3.5)) The characteristic polynomial of* $f_{M_v}^{m_v}$ *on* M_v *has coefficients in* $\mathbb{Z}$ *and is equal to that of* $\rho_\ell(\mathrm{Frob}_v)$*.*

From Section 2 recall that $f_{M_v}^{m_v}$ is an element of $H_{M,v}(K_v)$. Fix a representative $\Phi_{v,p}$ of the corresponding conjugacy class in $H_{V,v}(\bar{\mathbb{Q}}_p)$. Theorem 3.10 implies that there exists $g_v \in \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ such that $g_v t_v g_v^{-1}$ is equal to the semisimple part of $\Phi_{v,p}$. Note that, although g_v is not unique, the double coset

$$H_{V,v}(\bar{\mathbb{Q}}_p) \cdot g_v \cdot \mathrm{Cent}_{\mathrm{GL}_n(\bar{\mathbb{Q}}_p)}(T_v) \tag{3.11}$$

is independent of choices. As in the global case we obtain an element of $H_{V,v}(\bar{\mathbb{Q}}_p)$ and thus embeddings

$$g_v T_{v,\bar{\mathbb{Q}}_p} g_v^{-1} \subset H_{V,v,\bar{\mathbb{Q}}_p} \subset G_{p,\bar{\mathbb{Q}}_p}. \tag{3.12}$$

Thus in every local algebraic monodromy group some form of T_v can be found. Let us note the following consequence.

Proposition (3.13). *Suppose that* $K = K^{\mathrm{conn}}$ *and let* $V_{\max}$ *be as in Theorem 3.7. Consider* $v \in V_{\max}$*, say with residue characteristic* p*. Then all the groups* T_v*,* $H_{M,v}$*,* $H_{V,v}$*, and* G_p *are connected of the same rank, independent of* v *and* p*, and* $g_v T_{v,\bar{\mathbb{Q}}_p} g_v^{-1}$ *is a maximal torus of both* $H_{V,v,\bar{\mathbb{Q}}_p}$ *and* $G_{p,\bar{\mathbb{Q}}_p}$*.*

Proof: By Theorem 3.7 and Corollary 3.8 we already know that T_v and G_p are connected of equal rank, independent of v and p. Since $H_{V,v,\bar{\mathbb{Q}}_p}$ is pinched between forms of these two groups by 3.12, it also has the same rank. By Proposition 2.6 it is generated by a connected subgroup together with $g_v T_{v,\bar{\mathbb{Q}}_p} g_v^{-1}$. As the latter is connected, the assertions pertaining to $H_{V,v}$ are proved. Finally, the assertions for $H_{M,v}$ follow from the fact that this group is an inner form of $H_{V,v}$. □

Newton and Hodge cocharacters: Consider the Newton cocharacter ν_v of T_v, as defined above. Conjugating it into $H_{V,v,\bar{\mathbb{Q}}_p}$ as in 3.12 we find precisely the conjugacy class of Newton cocharacters from Section 2. Similarly we would like to find some conjugate of the Hodge cocharacter inside T_v. This is possible if T_v is sufficiently big.

So let us assume that $K = K^{\mathrm{conn}}$ and $v \in V_{\max}$. Let V_1 denote the set of places v of absolute degree 1, i.e. with $K_v \cong \mathbb{Q}_p$ if p denotes the residue characteristic of v. In the following we shall also assume $v \in V_1$. This restriction does not disturb since V_1 has Dirichlet density 1.

Consider the conjugacy class of Hodge cocharacters of $H_{V,v}$ defined in Section 2. By Proposition 3.13 we can find representatives in the maximal torus $g_v T_{v,\bar{\mathbb{Q}}_p} g_v^{-1}$ and conjugate them into T_v via g_v^{-1}.

Definition (3.14). *(a) The resulting cocharacters of T_v are called* strong Hodge cocharacters of T_v. *The set of all strong Hodge cocharacters of T_v is denoted $S_{\mu,v}$.*

(b) A cocharacter of T_v which is conjugate under $\mathrm{GL}_n(\bar{\mathbb{Q}})$ to a cocharacter in (a) is called a weak Hodge cocharacter of T_v.

The uniqueness of the double coset 3.11 implies that part (a) of this definition is independent of the choice of g_v. Next observe that the GL_n-conjugacy class of any cocharacter is determined uniquely by the system of multiplicities for all weights. As the Hodge filtration on M_v comes from the natural filtration on the De Rham cohomology group $\mathbb{H}^d(X, \Omega_X^\bullet)$, for Hodge cocharacters these multiplicities are just the Hodge numbers $h^{i,d-i}$ of X. Let $\mathcal{C}$ denote the GL_n-conjugacy class of cocharacters $\mathbb{G}_m \to \mathrm{GL}_n$ determined by these Hodge numbers. Then the weak Hodge cocharacters are precisely those cocharacters of T_v which also lie in $\mathcal{C}$.

Abundance of Hodge cocharacters: In the rest of this section we shall combine the main results obtained so far. First we restate Theorem 2.3 in terms of T_v. Recall that Y_v denotes the cocharacter space of T_v.

Theorem (3.15). *Assume $K = K^{\mathrm{conn}}$ and $v \in V_{\max} \cap V_1$. Then we have $\nu_v \in \mathrm{Conv}(S_{\mu,v})^\circ$.*

Proof: The assertion of Theorem 2.3 concerns a fixed but arbitrary maximal torus of $H_{V,v,\bar{\mathbb{Q}}_p}$. By Proposition 3.13 we may take the torus $g_v T_{v,\bar{\mathbb{Q}}_p} g_v^{-1}$. As in Definition 3.14 consider the conjugacy class of Hodge cocharacters of $H_{V,v,\bar{\mathbb{Q}}_p}$ defined in Section 2. Since $v \in V_1$, by Fact 2.1 this conjugacy class is defined over $\mathbb{Q}_p$. Thus the set $S_{\mu,v}$ in Definition 3.14 corresponds to the set S_{μ_V} in Theorem 2.3 via conjugation with g_v. The theorem is thus a restatement of Theorem 2.3. □

Combining Theorem 3.15 with Proposition 3.5 we deduce:

Theorem (3.16). *Assume that $K = K^{\mathrm{conn}}$ and $v \in V_{\max}$. Then $T_{v,\bar{\mathbb{Q}}}$ is generated by the images of all $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugates of all strong Hodge cocharacters. In particular, it is generated by the images of all weak Hodge cocharacters.*

At last we come back to the global algebraic monodromy group G_p for an arbitrary rational prime p. Recall that $\mathcal{C}$ denotes the GL_n-conjugacy class of cocharacters of GL_n determined by the Hodge numbers of X, and that $\mathcal{C}$ contains all the Hodge cocharacters under discussion.

Definition (3.17). *(a) Suppose that X has good reduction at all places v above p. Then a cocharacter μ of G_p is called a* strong Hodge cocharacter of G_p *if and only if there exists a non-archimedean place $v \notin S$ of K with residue characteristic p, such that μ is $G_p(\bar{\mathbb{Q}}_p)$-conjugate to a Hodge character of $H_{V,v}$ as in Section 2.*

(b) A cocharacter μ of G_p is called a weak Hodge cocharacter of G_p *if and only if $\mu \in \mathcal{C}$.*

Of course, any strong Hodge cocharacter is also a weak Hodge cocharacter. It is conjectured that the identity component $G^\circ_{p,\bar{\mathbb{Q}}_p}$ is generated by the images of strong Hodge cocharacters, provided that X has good reduction at all places v above p (compare, for instance, [45] Conjecture R). Our approach gives a slightly different result, involving a single conjugacy class of Hodge cocharacters and its transforms under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In particular, we can prove:

Theorem (3.18). *For every rational prime ℓ the identity component of $G_{\ell,\bar{\mathbb{Q}}_\ell}$ is generated by the images of weak Hodge cocharacters.*

Proof: Without loss of generality we may assume $K = K^{\mathrm{conn}}$. Choose any $v \in V_{\max}$, say with residue characteristic p. By Theorem 3.16 the associated Frobenius torus $T_{v,\bar{\mathbb{Q}}}$ is generated by the images of weak Hodge cocharacters. Moreover some $\mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$-conjugate of this torus is a maximal torus of $G_{\ell,\bar{\mathbb{Q}}_\ell}$. Indeed, this follows from Theorem 3.7 and Corollary 3.8 if $\ell \neq p$, respectively from Proposition 3.13 if $\ell = p$. Thus some maximal torus, and hence every maximal torus of $G_{\ell,\bar{\mathbb{Q}}_\ell}$ is generated by the images of weak Hodge cocharacters. As $G_{\ell,\bar{\mathbb{Q}}_\ell}$ is a reductive group, the desired assertion follows. □

§4. Classification of Certain Algebraic Groups of Mumford-Tate Type

In this section we collect some mostly known results on connected reductive groups which look like the Mumford-Tate group of an abelian variety. The main classification was obtained by Serre [30] §3. All the arguments are based on the classification and representation theory of reductive groups. We work over a given field F of characteristic zero with algebraic closure $\bar{F}$.

Mumford-Tate pairs: Consider a reductive algebraic group G over F and a faithful finite dimensional representation ρ of G. We are interested in cocharacters of G with special numerical properties vis-à-vis ρ. We shall use a slight variant of the definition in Serre [30] 3.2 (cf. also Wintenberger [44]).

Definition (4.1). *(a) The pair (G, ρ) is called* a weak Mumford-Tate pair of weights $\{0, 1\}$ *if and only if there exist cocharacters $\mu_i : \mathbb{G}_{m,\bar{F}} \to G_{\bar{F}}$ $(1 \leq i \leq k)$ such that*

(i) $G_{\bar{F}}$ is generated by the images of all $G(\bar{F})$-conjugates of all μ_i, and

(ii) the weights of each $\rho \circ \mu_i$ are in $\{0, 1\}$.

(b) The pair (G, ρ) is called a strong Mumford-Tate pair of weights $\{0, 1\}$ *if and only if the conditions in (a) hold and*

(iii) the μ_i are conjugate under $\mathrm{Gal}(\bar{F}/F)$.

Condition (i) implies that G must be connected. The possibilities for (G, ρ) and μ_i were determined by Serre [30] §3. We shall list his results, augmented by information on all possible autodualities and inclusions.

Reduction to the irreducible case: First we analyze weak Mumford-Tate pairs (G, ρ) of weights $\{0, 1\}$. By definition this condition is invariant under base extension, so to study it we may without loss of generality assume $F = \bar{F}$. Next it is clear that (G, ρ) satisfies the desired conditions if and only if $(\rho'(G), \rho')$ does so for every irreducible direct summand ρ' of ρ. Therefore we now assume that ρ is irreducible.

Tensor decomposition: Let Z denote the identity component of the center of G. When $G = Z$ we must have $\dim(\rho) = 1$ and there are the following two possibilities. If all μ_i are trivial, then $G = Z = 1$: we shall disregard this case. Otherwise we have $G = Z \cong \mathbb{G}_{m,F}$, and ρ is the standard representation $\mathbb{G}_{m,F} \xrightarrow{\sim} \mathrm{GL}_{1,F}$.

Let us now assume that $G \neq Z$ and let $G_1, \ldots, G_s$ denote the pairwise distinct almost simple factors of the derived group G^{der}. Then we have an almost direct product $G = Z \cdot G_1 \cdots G_s$. Correspondingly there are irreducible representations ρ_0 of Z and ρ_i of each G_i such that ρ decomposes as exterior tensor product $\rho \cong \rho_0 \boxtimes \ldots \boxtimes \rho_s$.

Each μ_i in Definition 4.1 (a) can be written uniquely as a product of quasi-cocharacters of Z and $G_1, \ldots, G_s$. If it has a non-trivial component in more than one G_j, one easily shows that $\rho \circ \mu$ has at least three distinct weights. This possibility is forbidden. On the other hand, for each G_j there must be at least one μ_i which has a non-trivial component in G_j. Then in particular $\rho \circ \mu_i$ is not central, so it must have both weights 0 and 1. It follows that $\det \circ \rho \circ \mu_i$ has weight > 0 and μ_i has a non-trivial component in Z. Therefore we have $Z \cong \mathbb{G}_{m,F}$, and its representation ρ_0 is, of course, the standard representation $\mathbb{G}_{m,F} \xrightarrow{\sim} \mathrm{GL}_{1,F}$. To summarize note that μ_i factors through the subgroup $Z \cdot G_j \subset G$.

Thus all in all we deduce that (G, ρ) is a Mumford-Tate pair of weights $\{0, 1\}$ if and only if the same is true for $(Z \cdot G_j, \rho_0 \boxtimes \rho_j)$ for every $1 \le j \le s$.

The simple case: Now we are reduced to the case $G = \mathbb{G}_{m,F} \cdot G^{\mathrm{der}}$ with G^{der} almost simple. Then the conjugates of a single cocharacter $\mu : \mathbb{G}_{m,F} \to G$ are enough to generate G. The triple (G, ρ, μ) is determined up to isomorphy by its "semisimple part" $(G^{\mathrm{der}}, \rho', \mu')$, where $\rho' := \rho|_{G^{\mathrm{der}}}$ and μ' denotes the component of μ in G^{der}. A list of all possibilities for this triple is given in Table 4.2.

root system	A_r	C_r	D_r	A_r	B_r	D_r
representation	standard	standard	standard	$\bigwedge^s$ (standard)	Spin	Spin^+
highest weight	ω_1	ω_1	ω_1	ω_s	ω_r	ω_r
$\dim(\rho)$	$r{+}1$	$2r$	$2r$	$\binom{r+1}{s}$	2^r	2^{r-1}
cocharacter	$\omega_s^\vee$	$\omega_r^\vee$	$\omega_r^\vee$	$\omega_1^\vee$	$\omega_1^\vee$	$\omega_1^\vee$
multiplicities	$s, r{+}1{-}s$	r, r	r, r	$\binom{r}{s-1}, \binom{r}{s}$	$2^{r-1}, 2^{r-1}$	$2^{r-2}, 2^{r-2}$
numerical conditions	$r{\ge}s{\ge}1$	$r{\ge}1$	$r{\ge}3$	$r{\ge}s{\ge}1$	$r{\ge}1$	$r{\ge}3$
autoduality	$-$ if $r{=}1$ no if $r{\neq}1$	$-$	$+$	$(-1)^s$ if $r{=}2s{-}1$ no if $r{\neq}2s{-}1$	$+$ if $r{\equiv}0{,}3\,(4)$ $-$ if $r{\equiv}1{,}2\,(4)$	$+$ if $r{\equiv}0\,(4)$ $-$ if $r{\equiv}2\,(4)$ no if $r{\equiv}1\,(2)$

Table 4.2: simple Mumford-Tate pairs of weights $\{0, 1\}$

Most of the information is from [30] §3. In each case ρ' is a fundamental representation with minuscule highest weight. Likewise the cocharacter μ' is a minuscule fundamental weight of the dual root system. The fundamental weights and co-weights are indexed as in Bourbaki [5] Planches. The last row contains the sign of the autoduality of ρ' if this representation is autodual, otherwise it contains the word "no". This information was taken from Dynkin [13] Ch.1, §3, Remark C. Note that some isomorphy classes of triples are listed more than once in Table 4.2 because of the exceptional isomorphisms between simple root systems of small rank.

Inclusions between irreducible Mumford-Tate pairs: Consider a weak Mumford-Tate pair (G, ρ) of weights $\{0, 1\}$ with ρ absolutely irreducible. We want to determine all types of subgroups $G' \subset G$ such that $\rho' := \rho|_{G'}$ remains absolutely irreducible and (G', ρ') is also a weak Mumford-Tate pair of weights $\{0, 1\}$. Again we may assume $F = \bar{F}$.

Comparing the tensor decompositions of ρ with respect to G and G' we find that each simple factor of $(G')^{\mathrm{der}}$ must lie in a unique simple factor of G^{der}. Thus the problem reduces at once to the case that $G^{\rm der}$ is almost simple. However, it is not necessary that $(G')^{\rm der}$ is almost simple.

In the first three cases of Table 4.2 the group G is a classical group in its standard representation, so for these the possibilities are easily determined. In the first case we have $G = {\rm GL}_{r+1,F}$, hence any weak Mumford-Tate pair (G', ρ') of weights $\{0, 1\}$ with $\dim(\rho') = r + 1$ can occur inside (G, ρ). In the next two cases we have $G = {\rm CSp}_{2r,F}$, respectively $G = {\rm GSO}_{2r,F}$, so (G', ρ') occurs inside (G, ρ) if and only if ρ' has dimension $2r$ and possesses an autoduality of sign -1, resp. $+1$. Note that this means that each almost simple factor of (G', ρ') must be autodual, and that the signs of these individual autodualities must multiply up to the correct sign. The remaining three cases are covered by the following result.

Proposition (4.3). *Consider a weak Mumford-Tate pair (G, ρ) of weights $\{0, 1\}$ over $F = \bar{F}$. Assume that ρ is irreducible, that $G^{\rm der}$ is almost simple, and the type of (G, ρ) is one of the last three in Table 4.2. Consider a subgroup $G' \subset G$ such that $\rho' := \rho|_{G'}$ is irreducible and (G', ρ') is another weak Mumford-Tate pair of weights $\{0, 1\}$. Then we have either*

(a) $G' = G$, or

(b) there exist integers $r_1, r_2 \geq 0$ with $r := r_1 + r_2 + 1 \geq 3$ such that G has root system D_r and ρ is the Spin representation (i.e. the type of (G, ρ) is the last one in Table 4.2), and G' has root system $B_{r_1} + B_{r_2}$ and ρ' is the tensor product of the respective Spin representations of B_{r_1} and B_{r_2} (i.e. any simple factor of (G', ρ') has the fifth type in Table 4.2). Note that the cases $r_1 = 0$ or $r_2 = 0$ are included: here the root system of G' is B_{r-1}.

Proof: The possible triples (H, H', σ) consisting of a connected almost simple algebraic group H, a connected subgroup $H' \subset H$, and a representation σ of H whose restriction to H' is irreducible have been determined by Dynkin [13]. The proposition follows by comparing his list with ours. Assume that $G' \neq G$, and look at the type of (G, ρ) in Table 4.2. The fourth type is excluded by [13] Th. 4.7 and Table 6. The fifth type is impossible by [13] Th. 6.8. For the sixth type the result follows from [13] Th. 6.9 and Tables 16–17, provided that $r \geq 4$. The case $r = 3$ is finished by inspection of Table 4.2. □

Irreducible Mumford-Tate pairs and formal characters: As before consider a weak Mumford-Tate pair (G, ρ) of weights $\{0, 1\}$ over $F = \bar{F}$ with ρ irreducible. We shall determine to which extent (G, ρ) is determined by its formal character. This problem differs from that in the preceding paragraph because two irreducible connected subgroups of ${\rm GL}_{n,F}$ with the same formal character need not be contained in each other. (The problem was discussed in greater generality in Larsen-Pink [20] §4.)

Consider a maximal torus $T \subset G$ and let $X^*(T) := {\rm Hom}(T, \mathbb{G}_{m,F})$ denote its character group. The formal character of ρ is the formal sum

$$\mathrm{ch}_\rho := \sum_{\chi \in X^*(T)} \mathrm{mult}_\rho(\chi) \cdot \chi \;\in\; \mathbb{Z}[X^*(T)],$$

where $\mathrm{mult}_\rho(\chi) \in \mathbb{Z}$ denotes the multiplicity of χ as weight of $\rho|_T$. Since ρ was assumed to be faithful, the formal character determines the pair $(T, \rho|_T)$ up to isomorphism. Thus we may suppose that T and $\rho|_T$ are fixed. To determine the pair (G, ρ) up to isomorphism it then remains to determine the root system $\Phi \subset X^*(T)$ of G.

Let $\Phi^\circ \subset \Phi$ be the subset of roots which are short in their respective simple factor of Φ. By Larsen-Pink [20] §4 this set is determined uniquely by the formal character. Note that Φ° itself is a root system of the same rank as Φ, though in general it is not a closed root subsystem of Φ. It is known that the short roots of any simple root system form an isotypic root system, i.e. all simple factors are of the same type. Thus the isotypic decomposition of Φ° comes from some decomposition of Φ. If the formal character is taken into account, we obtain an isotypic decomposition of the pair $(\Phi^\circ, \mathrm{ch}_\rho)$ which corresponds to a certain tensor decomposition of (G, ρ). For the rest of the analysis it suffices to consider a single isotypic component, i.e. we may suppose that $(\Phi^\circ, \mathrm{ch}_\rho)$ is isotypic.

Any remaining ambiguity must now originate in one of the cases of Table 4.2 where the root system possesses roots of different lengths. In the case of type C_r the short roots form a simple root system of type D_r, and the standard representation of $\mathrm{Sp}_{2r,F}$ corresponds to the standard representation of $\mathrm{SO}_{2r,F}$. Here we may restrict attention to the case $r \geq 3$, since the case $r = 2$ will be included next. In the case B_r $(r \geq 1)$ the short roots form a reducible root system of type rA_1, and the Spin representation of $\mathrm{Spin}_{2r+1,F}$ corresponds to the exterior tensor product of the respective standard representations of $\mathrm{SL}_{2,F}$. If we have the first type of ambiguity, the simple factors of Φ and Φ° correspond to each other, and for each simple tensor factor of (G, ρ) we have exactly the choice between the second and the third type of Table 4.2, with $r \geq 3$ fixed. In the second type of ambiguity all simple tensor factors of (G, ρ) must be of the fifth type in Table 4.2, but now r may vary. The only other information determined by the formal character is the sum over the respective values of r.

Irreducible strong Mumford-Tate pairs: Now we return to an arbitrary field F of characteristic zero and consider strong Mumford-Tate pairs according to Definition 4.1 (b). We restrict ourselves to the irreducible case. The following result goes back to Borovoi [4]:

Proposition (4.4). *Consider a strong Mumford-Tate pair (G, ρ) of weights $\{0, 1\}$ over F, such that ρ is absolutely irreducible. Then G^{der} is almost simple over F or it is trivial. In particular, all simple tensor factors of (G, ρ) over $\bar{F}$ have the same type in Table 4.2 (with the same r).*

Proof: Suppose that $G^{\mathrm{der}} \neq 1$ and let G_1 be one of the almost simple factors of $G^{\mathrm{der}}_{\bar{F}}$. Recall that some μ_i must factor through $\mathbb{G}_{m,\bar{F}} \cdot G_1 \subset G_{\bar{F}}$ and have a non-trivial component in both the center and in G_1. As G is connected, the images of the $G(\bar{F})$-conjugates of μ_1 generate the subgroup $\mathbb{G}_{m,\bar{F}} \cdot G_1$. By conditions (i) and (iii) of Definition 4.1 the $\mathrm{Gal}(\bar{F}/F)$-conjugates of this subgroup must generate $G_{\bar{F}}$. Thus the almost simple factors of $G^{\mathrm{der}}_{\bar{F}}$ are permuted transitively by the Galois group. □

Proposition (4.5). *Consider a strong Mumford-Tate pair (G, ρ) of weights $\{0, 1\}$ over F, such that $\rho|_{G^{\mathrm{der}}}$ is absolutely irreducible and symplectic. Then all simple tensor factors of (G, ρ) over $\bar{F}$ are symplectic and their number is odd.*

Proof: As $\rho|_{G^{\mathrm{der}}}$ is symplectic, each simple tensor factor must be self-dual and the number of symplectic factors is odd. In particular there is at least one symplectic factor. As all the factors have the same type, they are all symplectic. This in turn implies that the number of factors is odd. $\square$

For later use we extract from Table 4.2 a list of all possibilities for the simple factors in the symplectic case, given in Table 4.6. The watchful reader will note that some unnecessary duplication was purged, but not all.

root system	C_r	A_{2s-1}	B_r	D_r
representation dim(ρ)	standard $2r$	$\bigwedge^s$ (standard) $\binom{2s}{s}$	Spin 2^r	Spin^+ 2^{r-1}
cocharacter	$\omega_r^\vee$	$\omega_1^\vee$	$\omega_1^\vee$	$\omega_1^\vee$
numerical conditions	$r \geq 1$	$s \geq 1$ $s \equiv 1\ (2)$	$r \geq 1$ $r \equiv 1, 2\ (4)$	$r \geq 6$ $r \equiv 2\ (4)$

Table 4.6: symplectic simple Mumford-Tate pairs of weights $\{0, 1\}$

Proposition (4.7). *Consider a strong Mumford-Tate pair (G, ρ) of weights $\{0, 1\}$ over F, such that $\rho|_{G^{\mathrm{der}}}$ is absolutely irreducible and symplectic. Assume that $n := \dim(\rho)$ is greater than 1 and neither*

(a) a k^{th} power for any odd $k > 1$, nor

(b) of the form $\binom{2k}{k}$ for any odd $k > 1$.

Then we have $G = \mathrm{CSp}_{n,\mathbb{Q}}$.

Proof: Let m denote the number of simple factors of $G_{\bar{F}}$, and n_1 the common dimension of the representation of any simple tensor factor. Then we have $n = n_1^m$. Since m is odd by Proposition 4.5, condition (a) implies $m = 1$. We must show that the unique simple factor has the first type in Table 4.6. The second type is forbidden by condition (b), unless $s = 1$ which coincides with the first case with $r = 1$. Next the third case with $r \leq 2$ is subsumed by the first case. In the third case with $r > 2$ as well as in the fourth case n is an odd power. This is again excluded by condition (a). $\square$

§5. Abelian Varieties and the Mumford-Tate Conjecture

Recall that Theorem 3.18 asserts that the identity component G_ℓ° of the ℓ-adic algebraic monodromy group is generated by certain Hodge cocharacters. In the present section we apply this result to the case of an abelian variety A over a number field K. It follows that G_ℓ° looks like the Mumford-Tate group of an abelian variety. A weak version of this statement is obtained in Theorem 5.10. Under the assumption $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$ we are able formulate a much stronger version in Theorem 5.13. The proof will be given in Section 6; it avoids crystalline theory and is thus independent of Sections 1–2. According to the Mumford-Tate conjecture G_ℓ° should be equal to the Mumford-Tate group of A. Using the classification results of Section 4, we can prove this in many cases where $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$.

In the following we fix an abelian variety A of dimension g over a number field K. We fix an embedding $K \subset \mathbb{C}$ and let $\bar{K}$ be the algebraic closure of K in $\mathbb{C}$. To keep the notations of the earlier sections we work with cohomology instead of homology.

The Mumford-Tate group: The singular cohomology group $V := H^1(A(\mathbb{C}), \mathbb{Q})$ is a vector space of dimension $2g$ over $\mathbb{Q}$. It is endowed with a natural Hodge structure of type $\{(1,0),(0,1)\}$, that is, a decomposition of $\mathbb{C}$-vector spaces $V \otimes_{\mathbb{Q}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$ such that $V^{0,1} = \overline{V^{1,0}}$. Once and for all we choose an identification $V \cong \mathbb{Q}^{2g}$. Let $\mu_\infty : \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}_{2g,\mathbb{C}}$ be the cocharacter through which any $z \in \mathbb{C}^\times$ acts by multiplication with z on $V^{1,0}$ and trivially on $V^{0,1}$.

Definition (5.1). *The Mumford-Tate group of $A(\mathbb{C})$ is the unique smallest algebraic subgroup $G_\infty \subset \mathrm{GL}_{2g,\mathbb{Q}}$, defined over $\mathbb{Q}$, such that μ_∞ factors through $G_\infty \times_{\mathbb{Q}} \mathbb{C}$.*

As the image of μ_∞ is connected, this definition implies that G_∞ is connected.

Relation with endomorphisms: The endomorphism algebra $\mathrm{End}(A(\mathbb{C}))$ acts naturally on the homology group $H_1(A(\mathbb{C}), \mathbb{Q}) \cong V^\vee$. Thus its opposite algebra acts on V. This action preserves the Hodge decomposition, so it commutes with μ_∞ and hence with G_∞. In fact, we have a natural isomorphism

$$\mathrm{End}(A(\mathbb{C}))^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{End}_{G_\infty}(V)\ . \tag{5.2}$$

All endomorphisms of $A(\mathbb{C})$ are algebraic over $\bar{K}$, that is, we have a canonical isomorphism $\mathrm{End}(A_{\bar{K}}) \cong \mathrm{End}(A(\mathbb{C}))$.

Relation with polarizations: Any polarization on A induces a non-degenerate alternating form $\Lambda : V \times V \to \mathbb{Q}(-1)$, where $\mathbb{Q}(-1)$ is a $\mathbb{Q}$-vector space of dimension 1, distinguished from $\mathbb{Q}$ in being viewed as a pure Hodge structure of type $\{(1,1)\}$. If $\mathrm{CSp}_{2g,\mathbb{Q}}$ denotes the group of symplectic similitudes with respect to Λ, it follows that μ_∞ factors through $\mathrm{CSp}_{2g,\mathbb{C}}$. Thus the definition of G_∞ implies

$$G_\infty \subset \mathrm{CSp}_{2g,\mathbb{Q}}\ . \tag{5.3}$$

The ℓ-adic algebraic monodromy group: For any rational prime ℓ the étale cohomology group $V_\ell := H^1(A \times_K \bar{K}, \mathbb{Q}_\ell)$ is canonically isomorphic to the dual of the ℓ-adic Tate module of A. The Galois action can thus be described by the action on the torsion points of $A(\bar{K})$. On the other hand, there is a canonical isomorphism $V_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, so the chosen isomorphism $V \cong \mathbb{Q}^{2g}$ induces an identification $V_\ell \cong \mathbb{Q}_\ell^{2g}$. The Galois action thus corresponds to a continuous homomorphism

$$\rho_\ell : \operatorname{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}_{2g}(\mathbb{Q}_\ell).$$

As in Section 3 the main object of our interest is the associated global algebraic monodromy group:

Definition (5.4). *The Zariski closure in* $\mathrm{GL}_{2g,\mathbb{Q}_\ell}$ *of the image of* ρ_ℓ *is denoted* G_ℓ.

The Mumford-Tate conjecture: Let G_ℓ° denote the identity component of G_ℓ. The Hodge and Tate conjectures for general algebraic cycles would imply the following conjecture.

Conjecture (5.5). *(Mumford-Tate, cf. [29] C.3.1) For any rational prime* ℓ *we have*

$$G_\ell^\circ = G_\infty \times_{\mathbb{Q}} \mathbb{Q}_\ell.$$

Certain parts of this conjecture have been proved. Most notably, the inclusion "⊂" was proved by Piatetskii-Shapiro [25], Deligne [10] I Prop. 6.2, Borovoi [2]:

Theorem (5.6). *For any rational prime* ℓ *we have*

$$G_\ell^\circ \subset G_\infty \times_{\mathbb{Q}} \mathbb{Q}_\ell.$$

If G_∞ is known, this result provides an upper bound on G_ℓ. As for lower bounds, the only known general result is the following theorem of Faltings ([14] Theorems 3–4).

Theorem (5.7). *The representation* ρ_ℓ *is semisimple, and the natural homomorphism*

$$\operatorname{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow \operatorname{End}_{\mathbb{Q}_\ell[\operatorname{Gal}(\bar{K}/K)]}\big(V_\ell\big)$$

is an isomorphism.

Corollary (5.8). *The group* G_ℓ° *is reductive, and* $\operatorname{End}(A_{\bar{K}})^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ *maps isomorphically to the commutant of* G_ℓ° *in* $M_{2g}(\mathbb{Q}_\ell)$.

Proof: After replacing K by a suitable finite extension we have $\operatorname{End}(A) = \operatorname{End}(A_{\bar{K}})$ and $G_\ell = G_\ell^\circ$ (compare Theorem 3.6). In this case the desired assertion is equivalent to that of Theorem 5.7. □

Using the results mentioned above and his theory of Frobenius tori Serre [28], [32], [33] was able to prove the Mumford-Tate conjecture for any odd dimensional abelian variety with $\operatorname{End}(A_{\bar{K}}) = \mathbb{Z}$. These methods were adapted by Tankeev to abelian varieties of dimension rp with p prime and $r \leq 9$ ([34], [36], [37], [39], [40]) and in other special cases

([35], [38], cf. also [42]). In the present paper we significantly extend these results (while relying heavily on Serre's ideas). Serre's theorem has been generalized to certain abelian varieties with larger endomorphism rings by Chi [6], [7], and Tankeev (loc. cit.). Our results could be generalized to some extent along the same lines.

The form of G_ℓ° in the general case: By Faltings' theorem (Theorem 5.7 above) the representation ρ_ℓ is semisimple. Therefore we may apply the general qualitative results of Section 3 to the present situation. By construction the Hodge cocharacter μ_∞ defined above has weights 0 and 1 in the given representation on V. Thus the following is obvious from the definition of the Mumford-Tate group.

Fact (5.9). *The pair consisting of G_∞ together with its tautological representation is a strong Mumford-Tate pair of weights $\{0,1\}$ over $F = \mathbb{Q}$ in the sense of Definition 4.1 (b).*

Any weak Hodge cocharacter of G_ℓ as defined in 3.17 has the same weights as μ_∞. Thus Theorem 3.18 implies:

Theorem (5.10). *The pair consisting of G_ℓ° together with its tautological representation is a weak Mumford-Tate pair of weights $\{0,1\}$ over $F = \mathbb{Q}_\ell$ in the sense of Definition 4.1 (a).*

Using Table 4.2 we deduce in particular:

Corollary (5.11). *Each simple factor of the root system of G_ℓ° has type A, B, C, or D, and its highest weights in the tautological representation are minuscule.*

The irreducible case: In the rest of this section we consider the case $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$. The isomorphy 5.2 then implies that the tautological representation of the Mumford-Tate group is absolutely irreducible. Combining Fact 5.9 with Proposition 4.4 we deduce:

Proposition (5.12). *Assume that $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$. Then G_∞^{der} is $\mathbb{Q}$-simple. In particular $G_{\infty,\bar{\mathbb{Q}}}^{\mathrm{der}} \subset \mathrm{GL}_{2g,\bar{\mathbb{Q}}}$ is $\otimes$-isotypic.*

Similarly Corollary 5.8 implies that the tautological representation of G_ℓ° on V_ℓ is absolutely irreducible. We cannot immediately deduce an analogue of Proposition 5.12, because by construction the ℓ-adic algebraic monodromy groups are defined over varying fields $\mathbb{Q}_\ell$ and only loosely connected with each other via Frobenius tori. We do know that the formal character of G_ℓ° is independent of ℓ. However, we have seen in Section 4 that the formal character does not always determine the root system. Therefore we cannot rule out the possibility that the root system of G_ℓ varies to some extent with ℓ. Nevertheless with the methods of Larsen-Pink [21], [22], we can show that deviations may happen only for few ℓ. By the following theorem most G_ℓ° can be "interpolated" by an algebraic group defined over $\mathbb{Q}$. At the same time, the statement that G_ℓ° looks like a Mumford-Tate group is made more precise.

Theorem (5.13). *Assume that $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$.*

(a) There exists a connected reductive subgroup $G \subset \mathrm{GL}_{2g,\mathbb{Q}}$ such that G_ℓ° is conjugate to $G \times_{\mathbb{Q}} \mathbb{Q}_\ell$ under $\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$ for every ℓ in some set $\mathbb{L}$ of primes of Dirichlet density 1.

(b) *The pair consisting of G together with its absolutely irreducible tautological representation is a strong Mumford-Tate pair of weights $\{0,1\}$ over $F=\mathbb{Q}$ in the sense of Definition 4.1 (b).*

(c) *The derived group G^{der} is $\mathbb{Q}$-simple.*

(d) *If the root system of G is determined uniquely by its formal character, i.e. if G does not have an ambiguous factor (cf. Section 4), then in (a) we can take $\mathbb{L}$ to contain all but at most finitely many primes.*

The proof will be given in the following section. In the remainder of this section we discuss some consequences. Let us first note that Theorem 5.13 (a) and the fact that the representation of every $(G_\ell^\circ)^{\mathrm{der}}$ is symplectic together imply that the tautological representation of G^{der} is symplectic. Therefore by Proposition 4.5 the possible types of the simple factors of G^{der} are those in Table 4.6.

New instances of the Mumford-Tate conjecture: The assertions of Fact 5.9 and Theorem 5.13 (b) put the same strong restrictions on the groups G_∞ and G. Exploiting the classification results of Section 4 we can deduce that they must indeed look the same under certain dimension restrictions.

Theorem (5.14). *Assume that* $\mathrm{End}(A_{\bar{K}})=\mathbb{Z}$. *Assume moreover that $2g$ is neither*

(a) *a k^{th} power for any odd $k>1$, nor*

(b) *of the form $\binom{2k}{k}$ for any odd $k>1$.*

Then we have $G_\infty=\mathrm{CSp}_{2g,\mathbb{Q}}$ and $G_\ell^\circ=\mathrm{CSp}_{2g,\mathbb{Q}_\ell}$ for every ℓ. In particular the Mumford-Tate conjecture holds for A.

Proof: By Fact 5.9, the absolute irreducibility 5.2, and the inclusion (5.3), we may apply Proposition 4.7 to the Mumford-Tate group G_∞. It follows that $G_\infty=\mathrm{CSp}_{2g,\mathbb{Q}}$. Next consider the group G given by Theorem 5.13. Since the representation of G^{der} is symplectic, by Theorem 5.13 (b) we may apply Proposition 4.7 to G. Thus G is also a group of symplectic similitudes associated to some non-degenerate alternating form. By Theorem 5.13 (a) the same follows for G_ℓ°, as long as $\ell\in\mathbb{L}$. Thus the inclusion in Theorem 5.6 implies $G_\ell^\circ=\mathrm{CSp}_{2g,\mathbb{Q}_\ell}$ for these ℓ.

Since this last equality holds for one prime ℓ, it holds for every ℓ, by [22] Th. 4.3. To sketch the argument recall from Corollary 3.8 that the rank and the formal character of G_ℓ° are independent of ℓ. Thus every G_ℓ° is a subgroup of equal rank of $\mathrm{CSp}_{2g,\mathbb{Q}_\ell}$. It is also absolutely irreducible, hence the short roots of G_ℓ° are the same as those of $\mathrm{CSp}_{2g,\mathbb{Q}_\ell}$ (cf. Section 4). In characteristic zero it is well-known that the roots of any subgroup of equal rank form a closed root subsystem of the ambient root system. It follows that $G_\ell^\circ\subset\mathrm{CSp}_{2g,\mathbb{Q}_\ell}$ have the same roots, hence are equal, as desired. □

Theorem (5.15). *Assume that* $\mathrm{End}(A_{\bar{K}})=\mathbb{Z}$ *and that the root system of each simple factor of $G_{\infty,\bar{\mathbb{Q}}}$ has type A_{2s-1} with $s\geq 1$ or B_r with $r\geq 1$ (cf. Table 4.6). Then the Mumford-Tate conjecture holds for A.*

Proof: For every prime ℓ we know from Theorem 5.6, Corollary 5.8, and Theorem 5.10 that $G^\circ_{\ell,\bar{\mathbb{Q}}_\ell}$ is a subgroup of $G_{\infty,\bar{\mathbb{Q}}_\ell}$ whose tautological representation is irreducible and which forms a weak Mumford-Tate pair of weights $\{0,1\}$. It suffices to prove that $G_{\infty,\bar{\mathbb{Q}}_\ell}$ does not possess any proper subgroup with these properties. This assertion reduces to the same assertion for each simple tensor factor, where it is guaranteed by Proposition 4.3. $\square$

§6. Interpolation of ℓ-adic Algebraic Monodromy Groups

This section contains the proof of Theorem 5.13. The proof consists of two main parts. In the first half we show that the formal character is isotypic and its simple factors are permuted transitively by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Although the result is similar, this proof is independent of Sections 1–2. Here we concentrate more on the Newton cocharacter than the Hodge cocharacter. The arguments are very similar to those used to find places of ordinary reduction. For more explanations see Section 7. In the second half we show that the local data at the different primes ℓ fit together to a group defined over $\mathbb{Q}$. This group is constructed in some sense "around" a suitable Frobenius torus. The main ingredients here are the methods and results of Larsen-Pink [21], [22].

We keep the notations and assumptions of Section 5 and also assume $\mathrm{End}(A_{\bar{K}}) = \mathbb{Z}$. Then by Corollary 5.8 the tautological representation of G_ℓ° is absolutely irreducible. Since for Theorem 5.13 we may replace K by an arbitrary finite extension, we shall also assume $K = K^{\mathrm{conn}}$, so that all G_ℓ are connected. We fix a set $V_{\max}$ as in Theorem 3.7.

The setup: Recall from Theorem 3.7 that the Frobenius tori T_v for $v \in V_{\max}$ and the groups G_ℓ all have the same rank and the same formal character. Thus we may fix a split torus $T_0 \subset \mathrm{GL}_{2g,\mathbb{Q}}$ and conjugate it into each of these groups over the algebraic closure.

On the one hand we fix an element $f_\ell \in \mathrm{GL}_{2g}(\bar{\mathbb{Q}}_\ell)$ for every rational prime ℓ such that

$$T_{0,\bar{\mathbb{Q}}_\ell} \subset f_\ell^{-1} G_{\ell,\bar{\mathbb{Q}}_\ell} f_\ell \ . \tag{6.1}$$

This is a maximal torus. Let $\Gamma \subset \mathrm{Aut}(T_{0,\bar{\mathbb{Q}}})$ denote the stabilizer of the formal character of the tautological representation. Let $\Phi_\ell \subset X^*(T_0)$ denote the root system of $f_\ell^{-1} G_{\ell,\bar{\mathbb{Q}}_\ell} f_\ell$ and $W_\ell \subset \Gamma$ its Weyl group. Let $\Phi_\ell^\circ \subset \Phi_\ell$ be the subset of roots which are short in their respective simple factor of Φ_ℓ. By Larsen-Pink [20] §4 this set is determined uniquely by the formal character, and it is a root system of the same rank as Φ_ℓ, though not a closed root subsystem. In particular it is independent of ℓ, so we may abbreviate $\Phi^\circ := \Phi_\ell^\circ$. Note that Φ_ℓ is non-empty, because it is the root system of a connected reductive group with an irreducible representation of dimension $2g > 1$. Therefore Φ° is non-empty. Next we may suppose that the torus $f_\ell T_{0,\bar{\mathbb{Q}}_\ell} f_\ell^{-1}$ is defined over $\mathbb{Q}_\ell$. Its form over $\mathbb{Q}_\ell$ then corresponds to a homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \to \mathrm{Norm}_\Gamma(W_\ell)$. The composite with the projection map

$$\pi_\ell : \ \mathrm{Norm}_\Gamma(W_\ell) \twoheadrightarrow \mathrm{Norm}_\Gamma(W_\ell)/W_\ell \tag{6.2}$$

is a homomorphism

$$\bar{\varphi}_\ell : \ \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \longrightarrow \mathrm{Norm}_\Gamma(W_\ell)/W_\ell \tag{6.3}$$

which together with Φ_ℓ characterizes the form of G_ℓ up to inner twist.

On the other hand we choose an element $h_v \in \mathrm{GL}_{2g}(\bar{\mathbb{Q}})$ for every $v \in V_{\max}$ such that

$$T_{0,\bar{\mathbb{Q}}} = h_v^{-1} T_{v,\bar{\mathbb{Q}}} h_v \ . \tag{6.4}$$

The form of T_v over $\mathbb{Q}$ then corresponds to a homomorphism

$$\varphi_v : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \Gamma \ . \tag{6.5}$$

Roots and Frobenius eigenvalues: Let ν_v denote the Newton cocharacter of T_v defined in (3.4). Its conjugate by h_v is a cocharacter $\nu_{0,v}$ of T_0. By Theorem 3.3 (c) we have

$$0 \leq \langle \chi, \nu_{0,v} \rangle \leq 1 \tag{6.6}$$

for every weight $\chi \in X^*(T_0)$ which occurs in the given tautological representation. Any root $\alpha \in \Phi^\circ$ is a quotient of two such weights, hence

$$-1 \leq \langle \alpha, \nu_{0,v} \rangle \leq 1 \ . \tag{6.7}$$

We shall show that one of the inequalities in (6.7) is an equality sufficiently often. The following innocuous definition will provide us with the necessary arithmetic information. For any $v \in V_{\max}$ set

$$a_v := \sum_{\alpha \in \Phi^\circ} \alpha(h_v^{-1} t_v h_v) \ , \tag{6.8}$$

where $t_v \in T_v$ is as in Section 3. This is a rational number, since Φ° is invariant under the Galois action via φ_v. Recall that V_1 denotes the set of places of K of absolute degree one. Let V_{good} be the set of places $v \in V_{\max} \cap V_1$ satisfying

(6.9) $\qquad a_v$ is not a rational integer of absolute value $\leq \mathrm{card}(\Phi^\circ)$.

Proposition (6.10). *For any $v \in V_{\mathrm{good}}$ there exists $\alpha \in \Phi^\circ$ such that*

$$\langle \alpha, \nu_{0,v} \rangle = -1 \ .$$

Proof: Suppose $k_v \cong \mathbb{F}_p$, and consider a weight $\chi \in X^*(T_0)$ which occurs in the given representation. By Theorem 3.3 (b) the algebraic number $\chi(h_v^{-1} t_v h_v)$ is a unit at all finite primes not above p, and its archimedean valuations are independent of χ. For any $\alpha \in \Phi^\circ$ it follows that $\alpha(h_v^{-1} t_v h_v)$ is a unit at all finite primes not above p, and its archimedean norms are equal to 1. Thus a_v is always a rational number of absolute value $\leq \mathrm{card}(\Phi^\circ)$ which is integral outside p. By the hypothesis it cannot be an integer, so we have $\mathrm{ord}_p(a_v) \leq -1$. It follows that $\mathrm{ord}_p\big(\alpha(h_v^{-1} t_v h_v)\big) \leq -1$ for some $\alpha \in \Phi^\circ$. In view of the characterization (3.4) of ν_v this means that

$$\langle \alpha, \nu_{0,v} \rangle \leq \frac{-1}{[k_v/\mathbb{F}_p]} = -1.$$

The inequality (6.7) now shows that we must have equality, as desired. □

Proposition (6.11). *V_{good} has Dirichlet density* 1.

Proof: Fix any prime ℓ. The algebraic variety of semisimple conjugacy classes of $f_\ell^{-1}G_{\ell,\bar{\mathbb{Q}}_\ell}f_\ell$ is canonically isomorphic to $T_{0,\bar{\mathbb{Q}}_\ell}/W_\ell$. Thus there is a dominant morphism

$$G_{\ell,\bar{\mathbb{Q}}_\ell} \longrightarrow T_{0,\bar{\mathbb{Q}}_\ell}/\Gamma\ , \quad g \mapsto f_\ell^{-1}g_0 f_\ell \mod \Gamma\ ,$$

where g_0 denotes any $G_\ell(\bar{\mathbb{Q}}_\ell)$-conjugate of the semisimple part of g such that $f_\ell^{-1}g_0f_\ell \in T_0(\bar{\mathbb{Q}}_\ell)$. This morphism is already defined over $\mathbb{Q}_\ell$. Next the morphism

$$T_0 \to \mathbb{A}^1_{\mathbb{Q}}\ , \quad t \mapsto \sum_{\alpha\in\Phi^\circ} \alpha(t)$$

factors through T_0/Γ, since Φ° is Γ-invariant. Thus we can form the composite morphism

$$\psi :\ G_\ell \longrightarrow \mathbb{A}^1_{\mathbb{Q}_\ell}\ .$$

As Φ° is non-empty, it contains a non-zero weight, and the linear independence of characters implies that ψ is not constant. Let $X_\ell \subset G_\ell$ denote the set of points g for which $\psi(g)$ is a rational integer of absolute value $\leq \text{card}(\Phi^\circ)$. As only finitely many values are allowed, and G_ℓ is connected, this is a nowhere dense Zariski closed subset.

Now consider all places $v \in V_{\max} \cap V_1$ of residue characteristic different from ℓ. We already know that they form a set of Dirichlet density 1. For the condition (6.9) note that by construction we have $a_v = \psi(\rho_\ell(\text{Frob}_v))$. Thus condition (6.9) is satisfied if and only if $\rho_\ell(\text{Frob}_v) \notin X_\ell$. To analyze this property note that the image of Galois $\Gamma_\ell := \rho_\ell(\text{Gal}(\bar{K}/K))$ is a compact ℓ-adic analytic subgroup of $G_\ell(\mathbb{Q}_\ell)$ which is Zariski dense. Therefore $\Xi_\ell := X_\ell(\mathbb{Q}_\ell) \cap \Gamma_\ell$ is a nowhere dense closed analytic subset of Γ_ℓ. Let μ be the Haar measure on Γ_ℓ with total volume 1. One easily shows that $\text{vol}_\mu(U_\ell)$ goes to zero as U_ℓ runs through a cofinal system of open compact neighborhoods of Ξ_ℓ. Thus the desired assertion follows from the Čebotarev density theorem. □

Transitivity of the Galois action: Since the root system Φ° is determined uniquely by the formal character, it is invariant under the Galois action via φ_v. The following result is crucial.

Proposition (6.12). *For any $v \in V_{\text{good}}$ the action of* $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *via φ_v permutes the simple factors of Φ° transitively.*

We deduce immediately:

Corollary (6.13). *For every ℓ the formal character of $G^{\text{der}}_{\ell,\bar{\mathbb{Q}}_\ell} \subset \text{GL}_{2g,\bar{\mathbb{Q}}_\ell}$ is $\otimes$-isotypic.*

The mere fact that G_ℓ possesses a Hodge cocharacter has long been known as a consequence of Hodge-Tate theory. Together with Corollary 6.13 one thus obtains a different proof of Theorem 5.10 in this case, avoiding the general machinery of Sections 1–2. Conversely, Proposition 6.12 can be deduced almost, though not quite, from Theorems 3.15–16.

If the simple factors of Φ° have type A_1 it seems that an additional argument like in this section remains necessary.

Proof of Proposition 6.12: Let $\Phi^\circ = \Phi_1^\circ \oplus \ldots \oplus \Phi_k^\circ$ be the decomposition into simple factors. Consider the character space $X := X^*(T_0) \otimes \mathbb{R}$ and its corresponding decomposition $X = X_0 \oplus X_1 \oplus \ldots \oplus X_k$, where X_0 belongs to the central part. Let $Y := Y_*(T_0) \otimes \mathbb{R} = Y_0 \oplus \ldots \oplus Y_k$ be the analogous decomposition of the cocharacter space. For any $v \in V_{\text{good}}$ the Frobenius cocharacter $\nu_{0,v}$ is an element of Y. Let $\alpha \in \Phi^\circ$ be as in Proposition 6.10. Without loss of generality we may assume $\alpha \in \Phi_1^\circ$.

Lemma (6.14). $\nu_{0,v} \in Y_0 \oplus Y_1$.

Proof: By Proposition 6.10 we have $\langle \alpha, \nu_{0,v} \rangle = -1$. Assume that $\nu_{0,v}$ has a non-trivial part in the factor Y_i with $i > 1$. Then there exists a root $\beta \in \Phi_i^\circ$ with $\langle \beta, \nu_{0,v} \rangle < 0$. Now by Larsen-Pink [20] §4 the formal character of the given tautological representation factors as $\mathrm{ch} = \otimes_{j=0}^k \mathrm{ch}_j$ with $\mathrm{ch}_j \in \mathbb{Z}[X_j]$. For any $j > 0$ the factor ch_j is the formal character of a faithful representation of a simple Lie algebra with root system Φ_j°. For every $0 \leq j \leq k$ we shall choose a weight $\chi_j \in X_j$ which occurs in ch_j, as follows. Note that we may impose independent conditions on each χ_j. For $j \neq 1, i$ we make no additional assumption. For $j = 1$ we assume that $\chi_1 + \alpha$ also occurs in ch_1. Likewise for $j = i$ we assume that $\chi_i + \beta$ occurs in ch_i. Clearly such choices can be made. By construction both $\chi := \chi_0 + \ldots + \chi_k$ and $\chi' := \chi + \alpha + \beta$ are weights in the tautological representation which satisfy

$$\langle \chi', \nu_{0,v} \rangle - \langle \chi, \nu_{0,v} \rangle < -1 \ .$$

But the inequalities (6.6) imply that any such difference must be in the closed interval $[-1, 1]$. Thus we have a contradiction. □

To finish the proof of Proposition 6.12 note that by Proposition 3.5 the conjugates of $\nu_{0,v}$ under $\varphi_v(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$ generate the $\mathbb{R}$-vector space Y. As the factorization of Y is normalized by the Galois group, this action must permute $Y_1, \ldots, Y_k$ transitively. Thus the simple factors of Φ° are permuted transitively, as desired. □

The non-ambiguous cases: By Corollary 6.13 we may now distinguish cases according to the type of the formal character of each simple factor of Φ°. First we prove Theorem 5.13 in the non-ambiguous case, i.e. when each Φ_ℓ is determined by its formal character. Then for all ℓ we have $\Phi_\ell = \Phi^\circ$, and $W_\ell =: W^\circ$ is the Weyl group of Φ°. Note that this is a normal subgroup of Γ. Fix a place $v \in V_{\text{good}}$ and let

$$\bar{\psi} : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \Gamma / W^\circ$$

denote the composite of φ_v with the projection map

$$\Gamma \twoheadrightarrow \Gamma / W^\circ \ .$$

In the following a homomorphism emanating from $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ or $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ is called unramified (at ℓ) if and only if its restriction to the inertia group at ℓ is trivial. A connected reductive group over $\mathbb{Q}_\ell$ is called unramified if and only if it is quasi-split and splits over an unramified extension.

Proposition (6.15). *For all but at most finitely many ℓ we have*

(a) $\bar\psi$ *is unramified at* ℓ*,*

(b) G_ℓ *and hence* $\bar\varphi_\ell$ *is unramified, and*

(c) the homomorphisms $\bar\psi|_{\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ *and* $\bar\varphi_\ell$ *are conjugate to each other by some element of* Γ*.*

Proof: Assertion (a) is obvious, and (b) is Theorem 3.2 of Larsen-Pink [22]. Assertion (c) follows from the fact that $T_{v,\mathbb{Q}_\ell}$ is conjugate to a maximal torus of G_ℓ, whenever ℓ is different from the residue characteristic of v. □

The group G desired in Theorem 5.13 certainly exists over $\bar{\mathbb{Q}}$, say with maximal torus $T_{0,\bar{\mathbb{Q}}}$, and the problem is to choose a suitable model over $\mathbb{Q}$. Recall that the quasi-split forms over $\mathbb{Q}$ are classified by homomorphisms from $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to the outer automorphism group $\mathrm{Out}(\Phi^\circ)$. Thus $\bar\psi$ and the natural injection

$$\Gamma/W^\circ \hookrightarrow \mathrm{Out}(\Phi^\circ)$$

define a quasi-split connected reductive group G over $\mathbb{Q}$. By construction the formal character of the given irreducible representation is $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant with respect to this form of G. As G is quasi-split, this representation descends to an absolutely irreducible representation over $\mathbb{Q}$ (see Borel-Tits [3] Cor. 12.11). In particular we can realize G as a subgroup of $\mathrm{GL}_{2g,\mathbb{Q}}$.

For any ℓ as in Proposition 6.15 both G and G_ℓ are unramified at ℓ. Since $\bar\varphi_\ell$, respectively $\bar\psi$, determines G_ℓ, resp. $G_{\mathbb{Q}_\ell}$ up to inner twist, and both are quasi-split, they are isomorphic by Proposition 6.15 (c). As the given representations correspond, they are conjugate under $\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$, as desired. This proves Theorem 5.13 (a) and (d).

For the rest of Theorem 5.13 note that by construction G with its given representation forms a weak Mumford-Tate pair of weights $\{0, 1\}$ in the sense of Definition 4.1. On the other hand the simple factors of $G_{\bar{\mathbb{Q}}}$ are permuted transitively by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, by Proposition 6.12. This implies Theorem 5.13 (c) and that we have a strong Mumford-Tate pair, i.e. Theorem 5.13 (b). Thus Theorem 5.13 is proved in the non-ambiguous case. □

The ambiguous cases: In the remaining cases we need a technical result from [21] which relates the Galois action φ_v with the structure of the different G_ℓ via their Weyl groups W_ℓ. Again we fix a place $v \in V_{\mathrm{good}}$. Abbreviate $\Delta := \varphi_v(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}))$, consider a normal subgroup $\Delta_1 \lhd \Delta$, and let

$$\bar\varphi : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \Delta/\Delta_1$$

denote the composite of φ_v with the projection map

$$\pi : \ \Delta \twoheadrightarrow \Delta/\Delta_1 .$$

Let $\mathrm{Frob}_\ell \in \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ denote any element which acts as geometric Frobenius on unramified extensions. Finally, for any subset $X \subset \Gamma$ we let $[X]$ denote the set of elements of Γ that are conjugate to an element of X. The crucial point is the following.

Proposition (6.16). *(Larsen-Pink [21] Prop. 8.9) One can choose $v \in V_{\text{good}}$ and $\Delta_1 \triangleleft \Delta$ such that for all rational primes ℓ in a set $\mathbb{L}$ of Dirichlet density* 1 *we have:*

(a) $\bar{\varphi}$ is unramified at ℓ,

(b) G_ℓ and hence $\bar{\varphi}_\ell$ is unramified, and

(c) $[\pi_\ell^{-1}(\bar{\varphi}_\ell(\mathrm{Frob}_\ell))] = [\pi^{-1}(\bar{\varphi}(\mathrm{Frob}_\ell))]$.

Recall from Proposition 6.12 that Δ permutes the simple factors of Φ° transitively. This information was not available in [22]. It will allow us to identify Δ_1, essentially, as a Weyl group, and to strengthen Proposition 6.16 from an assertion on conjugacy classes to one on individual subgroups and their cosets. In the following we fix v, Δ_1, and $\mathbb{L}$ as in Proposition 6.16.

Let W° denote the Weyl group of Φ°. This is a normal subgroup of Γ. Put $\tilde{\Delta}_1 := W^\circ \cdot \Delta_1$ and $\tilde{\Delta} := W^\circ \cdot \Delta$, and consider the following commutative diagram.

(6.17)
$$\begin{array}{ccccc} \Delta & \xrightarrow{\;\pi\;} & \Delta/\Delta_1 & \xleftarrow{\;\overline{\varphi}\;} & \\ \big\downarrow & & \big\downarrow & & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \\ \tilde{\Delta} & \xrightarrow{\;\varpi\;} & \tilde{\Delta}/\tilde{\Delta}_1 & \xleftarrow{\;\overline{\psi}\;} & \end{array}$$

Note that $W^\circ \subset W_\ell$ for every ℓ. Thus Proposition 6.16 implies:

Proposition (6.18). *For all $\ell \in \mathbb{L}$ we have:*

(a) $\bar{\psi}$ is unramified at ℓ,

(b) G_ℓ and hence $\bar{\varphi}_\ell$ is unramified, and

(c) $[\pi_\ell^{-1}(\bar{\varphi}_\ell(\mathrm{Frob}_\ell))] = [\varpi^{-1}(\bar{\psi}(\mathrm{Frob}_\ell))]$.

Combinatorial arguments: First we note the following special case of Proposition 6.18.

Proposition (6.19). *There exists a prime $\ell \in \mathbb{L}$ such that $[W_\ell] = [\tilde{\Delta}_1]$.*

Proof: Since $\mathbb{L}$ has Dirichlet density 1 we may choose $\ell \in \mathbb{L}$ such that $\bar{\psi}(\mathrm{Frob}_\ell) = 1$. Then the right hand side in Proposition 6.18 (c) is equal to $[\varpi^{-1}(1)] = [\tilde{\Delta}_1]$. This subset contains the identity element of Γ, hence so does the left hand side in Proposition 6.18 (c). As the left hand side is the union of all conjugates of some W_ℓ-coset, and contains the identity, the coset must be equal to W_ℓ. Thus the assertion of Proposition 6.18 (c) reads $[W_\ell] = [\tilde{\Delta}_1]$, as desired. □

Proposition (6.20). *There exists a connected reductive group $G_0 \subset \mathrm{GL}_{2g,\bar{\mathbb{Q}}}$ with maximal torus $T_{0,\bar{\mathbb{Q}}}$ and root system $\Phi \subset X^*(T_0)$ such that*

(a) the set of short roots in Φ is Φ°, and

(b) the Weyl group of Φ is $\tilde{\Delta}_1$.

With some additional effort one can probably prove $W^\circ \subset \Delta_1$, so that $\Delta_1 = \tilde{\Delta}_1$ is itself the Weyl group of Φ. For our purposes this improvement is not necessary.

Proof: The problem is to select Φ among the different types with the same set of short roots allowed by Table 4.2. We distinguish cases according to the type of ambiguity.

Assume first that the simple factors of Φ_ℓ have type C_r or D_r with fixed $r \geq 3$, coming with the standard representation of dimension $2r$. Here the simple factors of Φ and Φ° correspond to each other. Note that the Weyl group of D_r has index 2 in the Weyl group of C_r. Let us identify the factor group with the additive group of the field with 2 elements $\mathbb{F}_2$. Then the factor group $\bar{\Gamma} := \Gamma/W^\circ$ is isomorphic to the wreath product $\mathbb{F}_2^k \rtimes S_k$, where k is the number of simple factors of Φ° and S_k denotes the symmetric group on k letters, acting on $\mathbb{F}_2^k$ by permuting the coefficients. Consider the subgroup $\bar{\Delta}_1 := \tilde{\Delta}_1/W^\circ \subset \bar{\Gamma}$.

Lemma (6.21). *Either $\bar{\Delta}_1 = 1$ or $\bar{\Delta}_1 = \mathbb{F}_2^k$.*

Proof: By Proposition 6.19 we have $[\tilde{\Delta}_1] = [W_\ell]$ for some $\ell \in \mathbb{L}$. Note that up to conjugation by Γ we have $W_\ell/W^\circ = \mathbb{F}_2^{k'} \times \{0\}^{k-k'}$, where k' is the number of C_r-factors of G_ℓ. Thus we can already deduce $\bar{\Delta}_1 \subset \mathbb{F}_2^k$.

Suppose that $\bar{\Delta}_1 \neq 1$. Then $W_\ell \neq W^\circ$, and hence $k' > 0$. It follows that W_ℓ/W°, and hence $\bar{\Delta}_1$, contains an element of $\mathbb{F}_2^k$ which has precisely one entry equal to 1. Now recall that $\bar{\Delta}_1$ is normalized by Δ whose image in S_k is transitive. Thus the Δ-conjugates of the element just found generate $\mathbb{F}_2^k$. Hence we have $\bar{\Delta}_1 = \mathbb{F}_2^k$, as desired. □

Depending on the case in Lemma 6.21 we choose Φ of type kD_r respectively kC_r. In both cases $\tilde{\Delta}_1$ is the Weyl group of Φ, hence Proposition 6.20 is proved for this type of ambiguity.

Now assume that each simple factor of Φ° has type A_1, coming with the standard representation. If the number of these factors is k, we can identify $\bar{\Gamma} := \Gamma/W^\circ$ with the symmetric group S_k. Consider the subgroup $\bar{\Delta}_1 := \tilde{\Delta}_1/W^\circ \subset \bar{\Gamma} \cong S_k$.

Lemma (6.22). *There exists $k'|k$ such that $\bar{\Delta}_1 = (S_{k'})^{k/k'} \subset S_k$ up to conjugation by S_k.*

Proof: By Proposition 6.19 we have $[\tilde{\Delta}_1] = [W_\ell]$ for some $\ell \in \mathbb{L}$. Since the simple factors of Φ_ℓ have root system B_{r_i} for certain positive integers r_i with $r_1 + \ldots + r_s = k$, the subgroup $W_\ell/W^\circ \subset S_k$ is equal to $S_{r_1} \times \ldots \times S_{r_s}$ up to conjugation by S_k.

If $r_i = 1$ for all $1 \leq i \leq s$, we have $\bar{\Delta}_1 = 1$, hence the desired assertion holds with $k' = 1$. So let us assume that some r_i is greater than 1. Then W_ℓ/W°, and therefore $\bar{\Delta}_1$, contains a transposition. Let $\bar{\Delta}_2 \subset \bar{\Delta}_1$ be the subgroup generated by all transpositions in $\bar{\Delta}_1$. Any subgroup of S_k which is generated by transpositions has the form $S_{k_1} \times \ldots \times S_{k_t}$ up to conjugation by S_k, where $k = k_1 + \ldots + k_t$ is a partition with $k_i > 0$. By construction $\bar{\Delta}_2$ is normalized by Δ whose image in S_k is transitive. Thus all k_i are equal, that is, we have $k_i = k'$ for some $k'|k$. Note that we must have $k' > 1$, since $\bar{\Delta}_2 \neq 1$. To prove the lemma it remains to show that $\bar{\Delta}_1 = \bar{\Delta}_2$.

Suppose that $\bar{\Delta}_1 \neq \bar{\Delta}_2$. Note that $\bar{\Delta}_1$ is contained in the normalizer of $\bar{\Delta}_2$, which is the wreath product $(S_{k'})^{k/k'} \rtimes S_{k/k'}$. Take any nontrivial element of the image of $\bar{\Delta}_1$

in $S_{k/k'}$. We can lift it to an element of $\bar{\Delta}_1$ which possesses a cycle of length $\geq 2k'$. By the formula $[W_\ell] = [\tilde{\Delta}_1]$ the same is true for some element of W_ℓ/W°. Thus we must have $r_i \geq 2k'$ for some $1 \leq i \leq s$. It follows that W_ℓ/W° contains a pure cycle of length $2k'-1$, and again the same is true for $\bar{\Delta}_1$. But the group $(S_{k'})^{k/k'} \rtimes S_{k/k'}$ cannot contain a pure cycle of length $2k'-1$. Indeed, consider any element of this group. If its image in $S_{k/k'}$ is non-trivial, the number of letters which are moved is $\geq 2k' > 2k'-1$. If the element lies in $(S_{k'})^{k/k'}$, all its cycles have length $\leq k'$, which is $< 2k'-1$ since $k' > 1$. Thus we have a contradiction, and Lemma 6.22 is proved. □

Lemma 6.22 implies that $\tilde{\Delta}_1$ is the Weyl group of the root system $(k/k') \cdot B_{k'}$, up to conjugation by Γ. Letting Φ be a suitable conjugate of $(k/k') \cdot B_{k'}$, Proposition 6.20 is proved in this case, and thus in general. □

Fix G_0 and Φ as in Proposition 6.20. We can now strengthen Proposition 6.18.

Proposition (6.23). *For each $\ell \in \mathbb{L}$ we have*
(a) $\bar{\psi}$ is unramified at ℓ,
(b) G_ℓ and hence $\bar{\varphi}_\ell$ is unramified, and
and there exists $\gamma_\ell \in \Gamma$ such that
(c) $\Phi_\ell = \gamma_\ell(\Phi)$, and
(d) $\pi_\ell^{-1}(\bar{\varphi}_\ell(\mathrm{Frob}_\ell)) = \gamma_\ell \cdot \varpi^{-1}(\bar{\psi}(\mathrm{Frob}_\ell)) \cdot \gamma_\ell^{-1}$.

Proof: Fix $\ell \in \mathbb{L}$. Parts (a) and (b) are repetitions from Proposition 6.18. For the remaining assertions let us write the respective cosets in the form

$$\pi_\ell^{-1}(\bar{\varphi}_\ell(\mathrm{Frob}_\ell)) = \sigma_\ell W_\ell$$
$$\varpi^{-1}(\bar{\psi}(\mathrm{Frob}_\ell)) = \tau_\ell \tilde{\Delta}_1$$

with suitable σ_ℓ, $\tau_\ell \in \Gamma$. Then Proposition 6.18 (c) reads

$$[\sigma_\ell W_\ell] = [\tau_\ell \tilde{\Delta}_1] \ . \tag{6.24}$$

The rest of the proof is essentially an exercise in group theory and could be left to the reader. The important combinatorial information here is that both W_ℓ and $\tilde{\Delta}_1$ are Weyl groups. To avoid explicit calculations in symmetric groups we shall use a general result from [21]. Let us identify Γ with a subgroup of the automorphism group of the cocharacter space $X := X^*(T_0) \otimes \mathbb{R}$. Then [21] Th. 2.1 implies that the triples $(X, W_\ell, \sigma_\ell W_\ell)$ and $(X, \tilde{\Delta}_1, \tau_\ell \tilde{\Delta}_1)$ are abstractly isomorphic. In other words, there exists an automorphism ι of X such that $W_\ell = \iota \tilde{\Delta}_1 \iota^{-1}$ and $\sigma_\ell W_\ell = \iota \tau_\ell \tilde{\Delta}_1 \iota^{-1}$. As the set of reflections at roots is intrinsic in a Weyl group, the first equality implies that Φ_ℓ and $\iota(\Phi)$ differ only in the lengths of their elements. In particular the isomorphy classes of Φ_ℓ and Φ differ at most by replacing certain factors of type B_r by C_r or vice versa, with $r \geq 3$. Since neither of the isotypic cases in Table 4.2 allows the occurrence of both B_r and C_r with $r \geq 3$, it follows that Φ_ℓ and Φ are abstractly isomorphic. One easily shows that ι can be chosen such that $\Phi_\ell = \iota(\Phi)$. Finally Table 4.2 shows that in the ambiguous cases under consideration the formal character is intrinsic to the root system. Thus ι preserves the formal character, hence comes from an element $\gamma_\ell \in \Gamma$. The desired assertions follow. $\square$

End of the proof of Theorem 5.13: To construct the group G desired in Theorem 5.13 we start with the group G_0 of Proposition 6.20 and choose a suitable model over $\mathbb{Q}$. In the same way as in the non-ambiguous cases we define G as the quasi-split form of G_0 associated to the composite of $\bar{\psi}$ of Diagram (6.17) with the natural injection

$$\tilde{\Delta}/\tilde{\Delta}_1 \hookrightarrow \mathrm{Out}(\Phi) .$$

By the same arguments as before we can realize G as a subgroup of $\mathrm{GL}_{2g,\mathbb{Q}}$.

To show that G satisfies condition (a) of Theorem 5.13, consider any $\ell \in \mathbb{L}$. By Proposition 6.23 (a–b) and by construction both $G_{\mathbb{Q}_\ell}$ and G_ℓ are unramified connected reductive groups over $\mathbb{Q}_\ell$. Their isomorphy classes are therefore uniquely determined by the respective root datum and the homomorphism from $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to the outer automorphism group. Moreover, the isomorphy class of the given representation is determined if the formal character is taken into account. By Proposition 6.23 (c–d) all these data coincide for the two groups, up to isomorphism. Hence the groups are isomorphic and their respective representations correspond. This means that $G_{\mathbb{Q}_\ell}$ and G_ℓ are conjugate under $\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$, as desired.

The rest of Theorem 5.13 is proved as in the non-ambiguous case. $\square$

§7. Places of Ordinary Reduction

Consider an abelian variety A over a number field K. It is conjectured that there exists a finite extension K' of K such that the set of places of K' where $A_{K'}$ possesses ordinary reduction has Dirichlet density 1. It is also conjectured that the extension $K' = K^{\rm conn}$ of Theorem 3.6 is enough. We prove this conjecture when the algebraic monodromy groups G_ℓ associated to A are sufficiently special.

Theorem (7.1). *Assume that* $\mathrm{End}(A_{\bar K}) = \mathbb{Z}$*, and let* G *be as in Theorem 5.13. Suppose that the root system of the simple factors of* $G_{\bar{\mathbb{Q}}}$ *does not have type* C_r *with* $r \geq 3$*. Then the abelian variety* $A_{K^{\rm conn}}$ *has ordinary reduction at a set of places of* $K^{\rm conn}$ *of Dirichlet density* 1*.*

The hypothesis means that G, and hence the groups G_ℓ°, are in some sense small (see Table 4.6). Thus it applies, for instance, when the Mumford-Tate group G_∞ of A is small. We can deduce:

Corollary (7.2). *Assume that* $\mathrm{End}(A_{\bar K}) = \mathbb{Z}$*. Suppose that the root system of the simple factors of* $G_{\infty,\bar{\mathbb{Q}}}$ *does not have type* C_r *with* $r \geq 3$*. Then the abelian variety* $A_{K^{\rm conn}}$ *has ordinary reduction at a set of places of* $K^{\rm conn}$ *of Dirichlet density* 1*.*

Proof: If the Mumford-Tate conjecture holds for A, the assertion follows directly from Theorem 7.1. Thus by Theorem 5.15, and a look at Table 4.6, it remains to consider the case that the simple factors of $G_{\infty,\bar{\mathbb{Q}}}$ have type D_r with $r \geq 6$ and the Mumford-Tate conjecture is false. Then Proposition 4.3 implies that the simple factors of $G_{\bar{\mathbb{Q}}}$ have type B_r in the Spin representation, for some $r \geq 1$. In this case, again Theorem 7.1 applies. □

Note that if the Mumford-Tate conjecture were to fail for A, then G_ℓ° would be smaller than G_∞, making it easier to prove the existence of places of ordinary reduction.

Summary of the proof: The rest of this section is devoted to proving Theorem 7.1. Throughout we assume $\mathrm{End}(A_{\bar K}) = \mathbb{Z}$ and $K = K^{\rm conn}$, and use the notations of Sections 5–6. Let G and $\mathbb{L}$ be as in Theorem 5.13.

We shall distinguish cases according to the isotypic type of the formal character of $G^{\rm der}$, following Table 4.6. Note that the case C_r with $r \leq 2$ is subsumed by the corresponding case B_r. Thus it suffices to look at the remaining three cases. In the second and the third case the theorem follows from Proposition 7.3 together with Proposition 6.11. The fourth case is treated in Proposition 7.4.

In each of these cases the proof follows the lines of a well-known argument going back to Serre [28], Katz, Ogus [24] 2.7–9. It is also found in Noot [23], Tankeev [41], and in the analogous case of Drinfeld modules in Pink [26].

Let us briefly sketch the main ideas. Choose a central function on the ℓ-adic algebraic monodromy group G_ℓ, for every ℓ, which is sufficiently intrinsic so that its values on Frobenius elements $\rho_\ell(\mathrm{Frob}_v)$ are rational numbers and independent of ℓ. For example this central function may be the trace of an intrinsically defined representation of G_ℓ, such as its given tautological representation, or its adjoint representation. In any case its value on $\rho_\ell(\mathrm{Frob}_v)$ can be written as a sum of certain multiplicative combinations of eigenvalues of Frobenius on the Tate module of A. Thus this value is subject to rather sharp bounds with respect to all valuations of $\mathbb{Q}$ not associated to the residue characteristic p of v. The bound at p is weaker; in fact, it is weakest when A has ordinary reduction at v. With luck for non-ordinary reduction the bound is strong enough to show that the value lies in a fixed finite set which is independent of v. Looking at the ℓ-adic representation for some fixed ℓ and using Čebotarev's density theorem one then proves that these places v have Dirichlet density zero, finishing the proof.

The cases A_{2s-1} and B_r: The following proposition generalizes Noot [23] Th. 2.2, its proof being similar. Most of the work was done already in the first half of Section 6. Let V_{good} be as defined in (6.9). Note that the statement applies to the case B_r as well, since its short roots are isotypic of type A_1.

Proposition (7.3). *Assume that the simple factors of Φ° have type A_{2s-1} with $s \geq 1$. Then the abelian variety A has ordinary reduction at all places $v \in V_{\mathrm{good}}$.*

Proof: We use the same notations as in the proof of Propositions 6.10 and 6.12. Take $v \in V_{\mathrm{good}}$ of residue characteristic p and let ν_i denote the component of the Newton cocharacter $\nu_{0,v}$ in Y_i, for each $0 \leq i \leq k$. By Proposition 6.10 and Lemma 6.14 we have $\nu_1 \neq 0$ and $\nu_i = 0$ for all $i > 1$.

Take a strong Hodge cocharacter of T_v as in Definition 3.14 and conjugate it via h_v into a cocharacter $\mu_{0,v}$ of T_v. By Theorem 3.15 $\nu_{0,v}$ lies in the convex closure of the orbit of $\mu_{0,v}$ under some Weyl group $W_v \subset \Gamma$. Recall from Section 4 that $\mu_{0,v} \in Y_0 \oplus Y_i$ for some $i \geq 1$. As $\nu_1 \neq 0$, some W_v-conjugate of $\mu_{0,v}$ must have a non-trivial component in Y_1. Thus without loss of generality we may assume $\mu_{0,v} \in Y_0 \oplus Y_1$. Let μ_i denote the component of $\mu_{0,v}$ in Y_i, for each $0 \leq i \leq k$. The convex closure theorem first of all implies $\nu_0 = \mu_0$. Next observe that the W_v-conjugates of $\mu_{0,v}$ in $Y_0 \oplus Y_1$ are precisely the conjugates under the Weyl group W_1 of Φ_1°. Thus ν_1 lies in the convex closure of the W_1-orbit of μ_1.

To calculate explicitly let us now identify Φ_1° with the subset

$$\left\{\, e_i - e_j \;\middle|\; 1 \leq i, j \leq 2s,\ i \neq j \,\right\} \;\subset\; \mathbb{R}^{2s}\,,$$

where $e_1, \ldots, e_{2s}$ denotes the standard basis of $\mathbb{R}^{2s}$. Then the character space X_1 is identified with the subspace

$$\left\{\, (x_1, \ldots, x_{2s}) \in \mathbb{R}^{2s} \;\middle|\; x_1 + \ldots + x_{2s} = 0 \,\right\},$$

and the cocharacter space Y_1 with the quotient space $\mathbb{R}^{2s}/\mathbb{R}\cdot(1,\ldots,1)$. We write elements of Y_1 in the form $[y_1, \ldots, y_{2s}]$, keeping in mind that different tuples may represent the same element. The Weyl group W_1 is identified with the symmetric group S_{2s}. By Table 4.6 we have $\mu_1 = [1, 0, \ldots, 0]$ up to conjugation by W_1. Since $\nu_1 \in \mathrm{Conv}(W_1 \cdot \mu_1)$, we can write $\nu_1 = [y_1, \ldots, y_{2s}]$ with $y_i \geq 0$ and $y_1 + \ldots + y_{2s} = 1$. Then we also have $y_i \leq 1$ for all i. After conjugation by W_1 we may without loss of generality assume that $1 \geq$

$y_1 \geq y_2 \geq \ldots \geq y_{2s} \geq 0$. Now by Proposition 6.10 there exist $1 \leq i, j \leq 2s$ such that $\langle e_i - e_j, \nu_1 \rangle = -1$. Therefore

$$1 = \langle e_j - e_i, \nu_1 \rangle = y_j - y_i \leq y_1 \leq 1 \,.$$

It follows that we have equality, and in particular $y_1 = 1$. The remaining y_i must then vanish, hence we have $\nu_1 = [1, 0, \ldots, 0] = \mu_1$.

All in all we have now shown that $\nu_{0,v} = \mu_{0,v}$ up to conjugation by Γ. Thus the Newton polygon and the Hodge polygon of the given local Galois representation on V_p coincide. Therefore the local Galois representation is ordinary, hence A has ordinary reduction at v. □

The case D_r: In this case the method sketched above does not quite work in the stated form. The problem is that there is no central function on G which leads to bounds that are tight enough at places of non-ordinary reduction. One can show that the only candidates for such a function come from the trace of the standard representation of SO_{2r}, but this representation is defined only over a central extension. If the ℓ-adic representations were to lift to a compatible system of representations into SO_{2r}, the previous method would be applicable there. Fortunately, using a theorem of Wintenberger we can perform enough of this lift so that essentially the same arguments succeed.

Proposition (7.4). *Assume that the simple factors of Φ° have type D_r with $r \geq 6$. Then the abelian variety A has ordinary reduction at all places v in some set $V_{\mathrm{ord}} \subset V_{\max} \cap V_1$ of Dirichlet density* 1.

Proof: We use the same notations as in the proof of Proposition 6.12. Recall that the root system of G corresponds to Φ°, whose Weyl group was denoted W°. To calculate explicitly we identify each simple factor Φ_i° with the subset

$$\left\{ \pm e_j \pm e_{j'} \;\middle|\; 1 \leq j < j' \leq r \right\} \;\subset\; \mathbb{R}^r \,,$$

where $e_1, \ldots, e_r$ denotes the standard basis of $\mathbb{R}^r$. Accordingly, both the character spaces X_i and the cocharacter spaces Y_i are identified with $\mathbb{R}^r$. Recall that any possible Hodge cocharacter has a non-trivial component in at most one Y_i with $i > 0$, and the type of this component can be read off from Table 4.6. In the above notation we find that this component is equal to $e_1 = (1, 0, \ldots, 0) \in Y_i$ up to conjugation by W°.

Next recall that the lattices between the root lattice and the weight lattice of a root system correspond to the different isomorphy types of semisimple groups in a fixed isogeny class. In this case $\mathbb{Z}^r \subset X_i$ is the weight lattice of SO_{2r}, and $(1, 0, \ldots, 0)$ is an element of the dual lattice $(\mathbb{Z}^r)^\vee = \mathbb{Z}^r \subset Y_i$. In other words, the component of the Hodge cocharacter in any simple factor of $G^{\mathrm{ad}}_{\bar{\mathbb{Q}}}$ comes from a cocharacter of $\mathrm{SO}_{2r,\bar{\mathbb{Q}}}$. This observation plays a central role in the following argument.

Lifting the Galois representation: Let us now look at the structure of the adjoint group G^{ad} over $\mathbb{Q}$. By construction G^{ad} is isomorphic to the Weil restriction $\mathcal{R}_{F/\mathbb{Q}}\,\mathrm{PSO}_{2r,F}$, where $\mathrm{PSO}_{2r,F}$ denotes a quasi-split projective special orthogonal group over a number field F of degree $(F/\mathbb{Q}) = k$. The exterior tensor product of the standard representations of the different simple factors of $\mathcal{R}_{F/\mathbb{Q}}\mathrm{SO}_{2r,F}$ over $\bar{\mathbb{Q}}$ descends to a representation over $\mathbb{Q}$, since its formal character is invariant under all automorphisms of the root system. Let $\tilde{G}$ denote the image of $\mathcal{R}_{F/\mathbb{Q}}\mathrm{SO}_{2r,F}$ in this representation and σ the induced faithful representation of $\tilde{G}$. We have thus constructed a central extension of G^{ad} of degree 2, i.e. a short exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \tilde{G} \longrightarrow G^{\mathrm{ad}} \longrightarrow 1 \; . \tag{7.5}$$

Let us call a cocharacter of G^{ad} a Hodge cocharacter if and only if it comes from a Hodge cocharacter of G. By the remarks above any Hodge cocharacter of G^{ad} lifts to a cocharacter of $\mathcal{R}_{F/\mathbb{Q}}\mathrm{SO}_{2r,F}$ and hence to a cocharacter of $\tilde{G}$.

Now consider any prime $\ell \in \mathbb{L}$. By Theorem 5.13 (a) the given ℓ-adic representation ρ_ℓ can be conjugated from G_ℓ into G. Composing with the projection $G \to G^{\mathrm{ad}}$ we obtain a homomorphism

$$\bar{\rho}_\ell : \mathrm{Gal}(\bar{K}/K) \longrightarrow G^{\mathrm{ad}}(\mathbb{Q}_\ell) \; .$$

As in Section 3 let S denote a finite set of non-archimedean places of K such that A has good reduction outside S and $K/\mathbb{Q}$ is unramified outside S. Then $\bar{\rho}_\ell$ is unramified at all non-archimedean places $v \notin S$ with $v \nmid \ell$. For $v \notin S$ with $v|\ell$ the local representation $\bar{\rho}_\ell|_{\mathcal{D}_{\bar{v}}}$ is crystalline in the sense that, for instance, its composite with the adjoint representation of G^{ad} is crystalline. Note that the Hodge weights of this representation are in the interval $[-1, 1]$. As we have seen above, the associated Hodge cocharacters lift to cocharacters of $\tilde{G}$.

Using Wintenberger [46] Th. 2.1.4 one can now lift each $\bar{\rho}_\ell$ to $\tilde{G}$ on some fixed open subgroup of $\mathrm{Gal}(\bar{K}/K)$. For our purposes an elementwise lift will suffice. For this we shall quote an intermediate result of Wintenberger's proof. Consider the long exact sequence of $\mathbb{Q}_\ell$-valued points associated to the short exact sequence (7.5). It is the first row of the following diagram, in which $K_1 \subset \bar{K}$ denotes a Galois extension of K, to be chosen presently, and the dotted arrow is not yet defined.

$$\begin{array}{ccccccc} \cdots \longrightarrow & \tilde{G}(\mathbb{Q}_\ell) & \longrightarrow & G^{\mathrm{ad}}(\mathbb{Q}_\ell) & \xrightarrow{\ \delta\ } & H^1(\mathbb{Q}_\ell, \{\pm 1\}) \\ & & & \uparrow{\scriptstyle \bar{\rho}_\ell} & & \vdots\uparrow \\ & & & \mathrm{Gal}(\bar{K}/K) & \longrightarrow & \mathrm{Gal}(K_1/K) \; . \end{array} \tag{7.6}$$

The proof of [46] Lemme 2.3.2 shows:

Lemma (7.7). *There exists a finite abelian extension K_1/K of exponent 2 such that for each $\ell \in \mathbb{L}$ there exists a dotted arrow making Diagram (7.6) commutative.*

Compatibility of Frobenius conjugacy classes: Next we translate the compatibility condition for characteristic polynomials of Theorem 3.2 (b) into an ℓ-independence statement on conjugacy classes. The following result is enough for our purposes.

Lemma (7.8). *For any $\ell \in \mathbb{L}$ and any $v \in V_{\max}$ with $v \nmid \ell$ the* $\mathrm{Aut}(G^{\mathrm{ad}}_{\bar{\mathbb{Q}}})$*-conjugacy class of the semisimple part of $\bar\rho_\ell(\mathrm{Frob}_v)$ is defined over $\mathbb{Q}$ and independent of ℓ.*

Proof: To begin with, consider any torus $T \subset \mathrm{GL}_{2g,E}$, defined over an overfield $E/\mathbb{Q}$, which has the same dimension and the same formal character as T_0. Then T can be conjugated into T_0 over the algebraic closure $\bar{E}$ of E. By definition of Γ the resulting identification $T_{\bar{E}} \xrightarrow{\sim} T_{0,\bar{E}}$ is unique up to conjugation by Γ. It follows that the morphism $T \to T_0/\Gamma$ is unique and defined over E. We shall use this remark several times.

First suppose that T is a maximal torus of G defined over $\mathbb{Q}$. Let W denote the associated Weyl group. Then the algebraic variety $G^\natural$ of semisimple conjugacy classes of G is naturally isomorphic to the quotient variety T/W. On conjugating T into T_0 the Weyl group W is mapped to a subgroup of Γ. Hence we have natural morphisms, defined over $\mathbb{Q}$:

$$G \twoheadrightarrow G^\natural \cong T/W \twoheadrightarrow T_0/\Gamma\,. \tag{7.9}$$

Note that the composite map can be described as follows. Take an element $\gamma \in G$ and conjugate its semisimple part into T via G. Next conjugate the resulting element together with the whole torus T into T_0 under GL_{2g}, and take the image in T_0/Γ. In any case, the semisimple part of γ is conjugated into T_0 under GL_{2g} in some way, and the result is mapped to T_0/Γ. The problem is that this shorter description does not characterize the image uniquely for every γ.

Now consider $\ell \in \mathbb{L}$ and suppose that the conjugation $G_\ell \xrightarrow{\sim} G_{\mathbb{Q}_\ell}$ in Theorem 5.13 (a) was done with the element $u_\ell \in \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$. Take $v \in V_{\max}$ with $v \nmid \ell$. I claim that the ambiguity disappears for the element $u_\ell\rho_\ell(\mathrm{Frob}_v)u_\ell^{-1} \in G(\mathbb{Q}_\ell)$. Indeed, its semisimple part is conjugate to the abstract Frobenius element t_v. Recall that t_v generates a Zariski dense subgroup of T_v, and note that T_v is a torus of the right type, since $v \in V_{\max}$. Thus conjugating t_v is equivalent to conjugating T_v. It follows that the image of t_v in T_0/Γ is unique and equal to the image of $u_\ell\rho_\ell(\mathrm{Frob}_v)u_\ell^{-1}$ via the morphisms (7.9). It also follows that the image of t_v is defined over $\mathbb{Q}$.

To transfer these assertions to the group G^{ad} note that the morphisms (7.9) lie in the following commutative diagram.

$$\begin{array}{ccccccc}
G & \twoheadrightarrow & G^\natural & \cong & T/W & \twoheadrightarrow & T_0/\Gamma \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G^{\mathrm{ad}} & \twoheadrightarrow & (G^{\mathrm{ad}})^\natural & \cong & T^{\mathrm{ad}}/W & \twoheadrightarrow & T_0^{\mathrm{ad}}/\Gamma \\
 & & \downarrow & & & & \downarrow \\
 & & (G^{\mathrm{ad}})^\natural/\,\underline{\mathrm{Aut}}(G^{\mathrm{ad}}) & & \xrightarrow{\ \sim\ } & & T_0^{\mathrm{ad}}/\,\mathrm{Aut}(\Phi^\circ)
\end{array} \tag{7.10}$$

Here we abbreviate $T^{\mathrm{ad}} := T/\mathbb{G}_{m,\mathbb{Q}}$ and $T_0^{\mathrm{ad}} := T_0/\mathbb{G}_{m,\mathbb{Q}}$. The isomorphy in the last row results from the fact that T_0^{ad} conjugates into a maximal torus of G^{ad}, which identifies the root system of G^{ad} with Φ°. By construction $u_\ell \rho_\ell(\mathrm{Frob}_v) u_\ell^{-1}$ maps to $\bar\rho_\ell(\mathrm{Frob}_v)$ in G^{ad}. As the image in T_0/Γ is $\mathbb{Q}$-rational, so is the image in the last row, as desired. $\square$

Traces of Frobenius lifts and their arithmetic properties: Now we lift the Frobenius elements to $\tilde G$ and discuss their traces. First note the following fact.

Lemma (7.11). *The morphism*

$$\tilde G \longrightarrow \mathbb{A}^1_{\mathbb{Q}}, \quad \tilde\gamma \mapsto \mathrm{tr}(\sigma(\tilde\gamma))^2$$

factors through a non-constant morphism $\psi : G^{\mathrm{ad}} \longrightarrow \mathbb{A}^1_{\mathbb{Q}}$ which is invariant under all automorphisms of $G^{\mathrm{ad}}_{\bar{\mathbb{Q}}}$.

Proof: As σ is a non-trivial irreducible representation, the linear independence of characters implies that the morphism $\mathrm{tr}\circ\sigma$ is non-constant. Since the isomorphy class of σ is invariant under all automorphisms of the root system, the same is true for this morphism. Its square factors through G^{ad} because it is invariant under $\tilde\gamma \mapsto -\tilde\gamma$. $\square$

Now consider any $\ell \in \mathbb{L}$ and $v \in V_{\max}$ with $v \nmid \ell$. Choose an element $\widetilde{\mathrm{Frob}}_{v,\ell} \in \tilde G(\bar{\mathbb{Q}}_\ell)$ mapping to $\bar\rho_\ell(\mathrm{Frob}_v) \in G^{\mathrm{ad}}(\mathbb{Q}_\ell)$. It is unique up to sign. Therefore, the same is true for $a_{v,\ell} := \mathrm{tr}(\sigma(\widetilde{\mathrm{Frob}}_{v,\ell})) \in \bar{\mathbb{Q}}_\ell$.

Lemma (7.12). *The number $a_{v,\ell}^2$ is in $\mathbb{Q}$ and independent of ℓ.*

Proof: By construction we have $a_{v,\ell}^2 = \psi(\bar\rho_\ell(\mathrm{Frob}_v))$, with ψ as in Lemma 7.11. Thus Lemma 7.8 implies that this is a rational number which is independent of ℓ. $\square$

In particular, the $a_{v,\ell}$ are algebraic numbers which differ at most up to sign when v is fixed. After replacing certain $\widetilde{\mathrm{Frob}}_{v,\ell}$ by their negatives, if necessary, we may suppose without loss of generality that $a_{v,\ell}$ is independent of ℓ. Therefore we may now abbreviate $a_v := a_{v,\ell}$. By Lemma 7.12 this is an algebraic number of degree at most 2 over $\mathbb{Q}$. The following lemma says that these a_v generate only finitely many distinct extensions of $\mathbb{Q}$.

Lemma (7.13). *There exists a finite subset $M \subset V_{\max}$ such that for every $v \in V_{\max}$ we have $a_v \in \mathbb{Q}\cdot a_w$ for some $w \in M$.*

Proof: Let K_1 be as in Lemma 7.7. We first require that M contains all places $v \in V_{\max}$ which ramify in K_1. Next for each element $\tau \in \mathrm{Gal}(K_1/K)$ consider the places $v \in V_{\max}$ which are unramified in K_1 and whose associated Frobenius substitution is τ. By Čebotarev's density theorem we may assume that M contains such a place, say v_τ. If we have $a_v \neq 0$ for one of these places associated to τ, we also assume $a_{v_\tau} \neq 0$. Otherwise, any v_τ will do.

Let us show that such a set M meets our requirements. If $v \in V_{\max}$ is ramified in K_1 we may take $w = v$, so the assertion is obvious. It is also obvious if $a_v = 0$. Otherwise we can find $w \in M$ which is unramified in K_1, whose Frobenius substitution in $\mathrm{Gal}(K_1/K)$ is equal to that associated to v, and which satisfies $a_w \neq 0$. I claim that $a_v/a_w \in \mathbb{Q}$ for this choice of w.

To prove this consider any $\ell \in \mathbb{L}$ with $v, w \nmid \ell$ and look at the diagram (7.6). Recall that Kummer theory induces a natural isomorphism

$$H^1(\mathbb{Q}_\ell, \{\pm 1\}) \cong \mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^2 . \tag{7.14}$$

Since $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ induces the same sign change on $\widetilde{\mathrm{Frob}}_{v,\ell}$ as on a_v, the cohomology class $\delta(\bar\rho_\ell(\mathrm{Frob}_v))$ corresponds to the class of a_v^2 via (7.14). The same remarks apply to w in place of v, and by Lemma 7.7 the resulting cohomology classes are equal. It follows that $(a_v/a_w)^2 \in (\mathbb{Q}_\ell^\times)^2$. Therefore ℓ splits in the (at most quadratic) extension $\mathbb{Q}(a_v/a_w)$ of $\mathbb{Q}$. Since this holds for all primes ℓ in a set of Dirichlet density 1, Čebotarev's density theorem implies $\mathbb{Q}(a_v/a_w) = \mathbb{Q}$, as desired. □

Next we want to bound a_v with respect to every valuation on $\mathbb{Q}$. Consider any eigenvalue $\xi \in \bar{\mathbb{Q}}_\ell$ of $\sigma(\widetilde{\mathrm{Frob}}_{v,\ell})$. Then ξ^2 is an eigenvalue in the representation $\sigma^{\otimes 2}$. This factors through a representation of G^{ad}, where ξ^2 is an eigenvalue of $\bar\rho_\ell(\mathrm{Frob}_v)$. It is therefore an algebraic number, hence $\xi \in \bar{\mathbb{Q}}$. In the following, let p denote the residue characteristic of v.

Lemma (7.15). *Assume that $v \in V_1$, i.e. v has absolute degree one. Then we have*
(a) $|\xi|_\infty = 1$,
(b) $\mathrm{ord}_\ell(\xi) = 0$ *for any rational prime* $\ell \neq p$, *and*
(c) $|\mathrm{ord}_p(\xi)| \leq 1$.

Proof: The valuations of the eigenvalues of $\rho_\ell(\mathrm{Frob}_v)$ in the given representation are described in Theorem 3.3. Taking ratios of two such eigenvalues it follows that for any eigenvalue ξ' of $\bar\rho_\ell(\mathrm{Frob}_v)$ in the adjoint representation we have
(a) $|\xi'|_\infty = 1$,
(b) $\mathrm{ord}_\ell(\xi') = 0$ for any $\ell \neq p$, and
(c) $|\mathrm{ord}_p(\xi')| \leq [k_v/\mathbb{F}_p] = 1$.
The first two relations extend directly to any other representation of G^{ad}. Taking $\xi' = \xi^2$, this proves the first two assertions.

The valuations at p correspond to the slopes of the Newton polygon, which in turn is determined by the Newton cocharacter. So assertion (c) amounts to a bound on the weights of the Newton cocharacter in the representation σ. We may do the necessary calculation inside the character and cocharacter spaces of T_0. As in Section 6 the Newton cocharacter at v corresponds to a quasi-cocharacter $\nu_{0,v}$ of T_0. Let us write

$$\nu_{0,v} = \big(\nu_0, \nu_1, \ldots, \nu_k\big) \in Y = Y_0 \oplus Y_1 \oplus \ldots \oplus Y_k . \tag{7.16}$$

We have already seen that the Hodge cocharacter has the form

$$\mu_{0,v} = \big(\mu_0, e_1, 0, \ldots, 0\big) \tag{7.17}$$

up to conjugation by Γ. From Theorem 3.15 we know that $\nu_{0,v}$ lies in the convex closure of the orbit of $\mu_{0,v}$ under the Weyl group W°.

Now recall that σ was defined as the exterior tensor product of the respective standard representations for all simple factors of $G_{\bar{\mathbb{Q}}}$. Thus its weights correspond to the elements

$$\chi = \big(0, \pm e_{i_1}, \ldots, \pm e_{i_k}\big) \in X = X_0 \oplus X_1 \oplus \ldots \oplus X_k \; , \tag{7.18}$$

for all possible $1 \leq i_j \leq r$ and all signs. Therefore the possible values of $\langle \chi, \mu_{0,v}\rangle$ are ± 1 and 0. Since $\nu_{0,v} \in \mathrm{Conv}(W^\circ \cdot \mu_{0,v})$, it follows that $|\langle \chi, \nu_{0,v}\rangle| \leq 1$. This implies assertion (c). □

From Lemma 7.15 we immediately deduce the following bounds on a_v.

Lemma (7.19). *Assume that $v \in V_1$, i.e. v has absolute degree one. Then we have*

(a) $|a_v|_\infty \leq \dim(\sigma) = (2r)^k$,

(b) $\mathrm{ord}_\ell(a_v) \geq 0$ *for any rational prime $\ell \neq p$, and*

(c) $\mathrm{ord}_p(a_v) \geq -1$.

Proof of Proposition 7.4: First we give a sufficient condition for ordinary reduction in terms of the elements a_v. Later we show that this condition holds for a set of places of Dirichlet density 1.

Lemma (7.20). *Assume that $v \in V_{\max} \cap V_1$ with* $\mathrm{ord}_p(a_v) = -1$. *Then A has ordinary reduction at v.*

Proof: Let $\nu_{0,v}$ and $\mu_{0,v}$ be as in (7.16–17) and recall that $\nu_{0,v} \in \mathrm{Conv}(W^\circ \cdot \mu_{0,v})$. Note that the Weyl group W° stabilizes each summand of the decomposition $Y = Y_0 \oplus Y_1 \oplus \ldots \oplus Y_k$ and acts trivially on the central part Y_0. Thus we can deduce $\nu_0 = \mu_0$ and $\nu_i = 0$ for all $i > 1$. Next observe that the conjugates of e_1 under the Weyl group of D_r are precisely the elements $\pm e_i$ with $1 \leq i \leq r$. Therefore we must have $\nu_1 = (y_1, \ldots, y_r)$ with $|y_1| + \ldots + |y_r| \leq 1$. After conjugation by the Weyl group we may without loss of generality assume $y_1 \geq \ldots \geq y_{r-1} \geq |y_r|$.

Now by the proof of Lemma 7.15 the given assumptions imply that $\langle \chi, \nu_{0,v}\rangle = -1$ for some weight χ as in (7.18). In explicit terms we have

$$-1 = \langle \chi, \nu_{0,v}\rangle = \langle \pm e_{i_1}, \nu_1\rangle = \pm y_{i_1} \; .$$

Therefore

$$1 = |y_{i_1}| \leq |y_1| + \ldots + |y_r| \leq 1 \; .$$

Thus we must have equality, whence $y_i = 0$ for all $i \neq i_1$. Our normalization of ν_1 now implies $i_1 = 1$ and $y_1 = 1$. All in all we find $\nu_1 = (1, 0, \ldots, 0) = \mu_1$, and therefore $\nu_{0,v} = \mu_{0,v}$. This means that the local Galois representation is ordinary, hence A has ordinary reduction at v. □

Lemma (7.21). *For any $b \in \mathbb{Q}$ the set of places $v \in V_{\max} \cap V_1$ with $a_v^2 \neq b$ has Dirichlet density* 1*.*

Proof: Let us fix any prime $\ell \in \mathbb{L}$ and restrict attention to places $v \in V_{\max}$ with $v \nmid \ell$. By construction we have $a_v^2 = \psi(\bar{\rho}_\ell(\mathrm{Frob}_v))$, where ψ is the morphism from Lemma 7.11. Thus we are speaking of those places v for which $\bar{\rho}_\ell(\mathrm{Frob}_v)$ does not lie in the Zariski closed subscheme $\psi^{-1}(b) \subset G^{\mathrm{ad}}$. Since ψ is a non-constant morphism and G^{ad} is connected, this is a nowhere dense Zariski closed subscheme. The rest of the proof proceeds as in Proposition 6.11. $\square$

Lemma (7.22). *The set of places $v \in V_{\max} \cap V_1$ with $\mathrm{ord}_p(a_v) = -1$ has Dirichlet density* 1*.*

Proof: Consider a place $v \in V_{\max} \cap V_1$ with $\mathrm{ord}_p(a_v) > -1$. We shall show that the number a_v^2 attains one of only finitely many values. The desired assertion then follows from Lemma 7.21. If $a_v = 0$ there is nothing more to prove, so let us suppose $a_v \neq 0$.

Fix a finite set M as in Lemma 7.13. We may disregard the finite number of primes v at which some non-zero a_w for $w \in M$ is not a unit. Take $w \in M$ such that $a_v \in \mathbb{Q} \cdot a_w$. Then we must have $a_w \neq 0$, so we can look at the rational number a_v/a_w. By Lemma 7.19 (b) its denominator outside p is bounded by that of a_w. Next we have $\mathrm{ord}_p(a_v/a_w) = \mathrm{ord}_p(a_v) > -1$, by assumption. Since the number is rational, this valuation is in $\mathbb{Z}$, so a_v/a_w is integral at p. It follows that the denominator of a_v/a_w is bounded by that of a_w at all finite primes. On the other hand, by Lemma 7.19 (a) its archimedean absolute value is bounded by $(2r)^k/|a_w|$. Therefore a_v/a_w takes only finitely many values. We conclude that a_v takes only finitely many values, as desired. $\square$

Proposition 7.4 now results by combining Lemma 7.20 with Lemma 7.22. $\square$

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