The Isogeny Conjecture for t-Motives Associated to Direct Sums of Drinfeld Modules

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Abstract

Let K be a finitely generated field of transcendence degree 1 over a finite field. Let M be a t-motive over K of characteristic $\mathfrak{p}_0$, which is semisimple up to isogeny. The isogeny conjecture for M says that there are only finitely many isomorphism classes of t-motives M' over K, for which there exists a separable isogeny $M' \to M$ of degree not divisible by $\mathfrak{p}_0$. For the t-motive associated to a Drinfeld module this was proved by Taguchi. In this article we prove it for the t-motive associated to any direct sum of Drinfeld modules of characteristic $\mathfrak{p}_0 \neq 0$.

1 Introduction

Let K be a finitely generated field of transcendence degree 1 over a finite field. The isogeny conjecture for t-motives is the following statement, formulated more generally for A-motives (compare Section 4).

Conjecture 1.1 (Isogeny conjecture) *For any A-motive M over K of characteristic $\mathfrak{p}_0$, which is semisimple up to isogeny, there are only finitely many isomorphism classes of A-motives M' over K, for which there exists a separable isogeny $M' \to M$ of degree not divisible by $\mathfrak{p}_0$.*

For the A-motive associated to a Drinfeld module this was proved by Taguchi. In this article we prove the following generalization in special characteristic.

Theorem 1.2 *Conjecture 1.1 is true for any A-motive over K which is a direct sum of A-motives associated to Drinfeld A-modules of characteristic $\mathfrak{p}_0 \neq 0$.*

The proof is based on the following results for Drinfeld modules ϕ over K. First, Taguchi has proved the isogeny conjecture for ϕ and the semisimplicity and the Tate conjecture for the Galois representation on the rational Tate module $V_\mathfrak{p}(\phi)$ for all $\mathfrak{p} \neq \mathfrak{p}_0$ (the latter was also proved by Tamagawa). Second, in an earlier paper [14] we have shown that the image of the group ring $A_\mathfrak{p}[\mathrm{Gal}(K^{\mathrm{sep}}/K)]$ in its action on $T_\mathfrak{p}(\phi)$ is maximal for almost all $\mathfrak{p}$, provided that $\mathfrak{p}_0 \neq 0$ (cf. Theorem 2.8). In the case $\mathrm{End}_K(\phi) = A$ this means essentially that the residual representation modulo $\mathfrak{p}$ is absolutely irreducible for almost all $\mathfrak{p}$. As a third ingredient we show (Theorem 3.1) that the $\mathfrak{p}$-adic Tate modules of non-isogenous Drinfeld modules over K have no isomorphic non-trivial finite $A_\mathfrak{p}[G_K]$-subquotients for almost all $\mathfrak{p}$.

These results are translated to the corresponding A-motives. From then on, the proof follows Faltings's method [6] for abelian varieties over number fields, which

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is based on a classification of isogenies by Galois invariant sublattices of the Tate modules.

The assumption $\mathfrak{p}_0 \neq 0$ is imposed by the fact that the result of [14] was proved only under this restriction. An analogous result in the case $\mathfrak{p}_0 = 0$, which we believe to be true, would imply Theorem 1.2 in general, because all other ingredients and arguments are valid without restriction on the characteristic.

By contrast, a proof of the isogeny conjecture for general A-motives will require a different approach. Our proof for the direct sum of A-motives corresponding to Drinfeld modules relies on the isogeny conjecture for the direct summands as an essential ingredient. Furthermore, it relies on special results [11], [12], [13], [14] for the Galois representations associated to Drinfeld modules, which cannot be obtained for A-motives with the same methods.

The material in this article and in [14] was part of the doctoral thesis of the second author [24].

2 Drinfeld modules and Galois representations

Throughout this article we use the following notation.

Let p be a prime number and q a power of p. Let $\mathcal{C}$ and $\mathcal{X}$ be two smooth, irreducible, projective curves over the finite field $\mathbb{F}_q$ with q elements. By F and K we denote the respective function fields. We fix a closed point ∞ on $\mathcal{C}$ and let A be the ring of functions in F which are regular outside ∞. We also fix a homomorphism $\iota\colon A \to K$ and let $\mathfrak{p}_0$ denote its kernel.

Let $K\{\tau\}$ be the twisted (noncommutative) polynomial ring in one variable, which satisfies the relation $\tau x = x^q\tau$ for all $x \in K$. Identifying τ with the endomorphism $x \mapsto x^q$, the ring $K\{\tau\}$ is isomorphic to the ring of $\mathbb{F}_q$-linear endomorphisms of the additive group scheme $\mathbb{G}_{a,K}$. Let $\phi : A \to K\{\tau\}$, $a \mapsto \phi_a$ be a Drinfeld A-module of rank r over K. We assume that its constant coefficient is given by ι; then $\mathfrak{p}_0$ is called the characteristic of ϕ. For the general theory of Drinfeld modules see Drinfeld [5] or Deligne-Husemöller [4].

The following theorem is due to Taguchi and appeared in [16] Theorem 0.2 for the case of special characteristic and in [20] for the case of generic characteristic. By the anti-equivalence 4.10 below it is equivalent to Conjecture 1.1 for the A-motive associated to ϕ:

Theorem 2.1 (Isogeny conjecture for Drinfeld modules) *There are only finitely many isomorphism classes of Drinfeld A-modules ϕ' over K, for which there exists a separable isogeny $\phi \to \phi'$ over K of degree not divisible by $\mathfrak{p}_0$.*

The isogeny conjecture is intimately related to Galois representations. Let $\overline{K}$ be an algebraic closure of K and K^{sep} the separable closure of K in $\overline{K}$. By $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ we denote the absolute Galois group of K. For all nonzero ideals $\mathfrak{a}$ in A, we let

$$\phi[\mathfrak{a}] := \bigl\{x \in \overline{K} \;\big|\; \forall a \in \mathfrak{a}\colon \phi_a(x) = 0\bigr\}$$

denote the module of $\mathfrak{a}$-torsion of ϕ. If $\mathfrak{p}_0 \nmid \mathfrak{a}$, its points are defined over K^{sep} and form a free $A/\mathfrak{a}$-module of rank r. For any prime $\mathfrak{p}$ of A, we let $A_\mathfrak{p} \subset F_\mathfrak{p}$ denote the completions of $A \subset F$ at $\mathfrak{p}$. For $\mathfrak{p} \neq \mathfrak{p}_0$ the $\mathfrak{p}$-adic Tate module $T_\mathfrak{p}(\phi) := \varprojlim \phi[\mathfrak{p}^n]$ of ϕ is a free $A_\mathfrak{p}$-module of rank r, and the rational $\mathfrak{p}$-adic Tate module $V_\mathfrak{p}(\phi) := T_\mathfrak{p}(\phi) \otimes_{A_\mathfrak{p}} F_\mathfrak{p}$ is an $F_\mathfrak{p}$-vector space of dimension r.

On all these modules there is a natural Galois action. In particular, for all $\mathfrak{p} \neq \mathfrak{p}_0$ we have a continuous representation

$$\rho_\mathfrak{p} :\; G_K \longrightarrow \mathrm{Aut}_{A_\mathfrak{p}}\bigl(T_\mathfrak{p}(\phi)\bigr) \cong \mathrm{GL}_r(A_\mathfrak{p}).$$

They form a compatible system of Galois representations in the following sense; see Goss [7] 4.12.12 (2). Let $\mathcal{U}$ be an open dense subscheme of $\mathcal{X}$ over which ϕ has good reduction.

Proposition 2.2 *For all closed points $x \in \mathcal{U}$, and all primes $\mathfrak{p} \neq \mathfrak{p}_0$ of A not below x, the representation $\rho_\mathfrak{p}$ is unramified at x, and the characteristic polynomial of $\rho_\mathfrak{p}(\mathrm{Frob}_x)$ has coefficients in A and is independent of $\mathfrak{p}$.*

Now we turn to representation theoretic properties. The following result can be deduced from Theorem 2.1, as Taguchi does it in special characteristic in [16] Theorem 0.1, but in generic characteristic he proved it before that in [17] Theorem 0.1:

Theorem 2.3 (Semisimplicity) *For all primes $\mathfrak{p} \neq \mathfrak{p}_0$ of A, the $F_\mathfrak{p}[G_K]$-module $V_\mathfrak{p}(\phi)$ is semisimple.*

Next a homomorphism $\phi \to \psi$ of Drinfeld A-modules over K is an element of

$$\mathrm{Hom}_K(\phi, \psi) := \bigl\{ u \in K\{\tau\} \bigm| \forall a \in A \colon \psi_a \circ u = u \circ \phi_a \bigr\}.$$

By construction every such homomorphism induces G_K-equivariant homomorphisms $\phi[\mathfrak{a}] \to \psi[\mathfrak{a}]$, $T_\mathfrak{p}(\phi) \to T_\mathfrak{p}(\psi)$, and $V_\mathfrak{p}(\phi) \to V_\mathfrak{p}(\psi)$. The following theorem was proved independently by Taguchi [18] and Tamagawa [21]; compare Remark 4.12 below.

Theorem 2.4 (Tate conjecture for homomorphisms) *For all primes $\mathfrak{p} \neq \mathfrak{p}_0$ of A, the natural homomorphism*

$$\mathrm{Hom}_K(\phi, \psi) \otimes_A A_\mathfrak{p} \longrightarrow \mathrm{Hom}_{A_\mathfrak{p}[G_K]}\bigl(T_\mathfrak{p}(\phi), T_\mathfrak{p}(\psi)\bigr)$$

is an isomorphism.

In particular, for $\psi := \phi$ the Galois representation commutes with the natural action of the endomorphism ring $E := \mathrm{End}_K(\phi)$, and Theorem 2.4 becomes:

Theorem 2.5 (Tate conjecture for endomorphisms) *For all primes $\mathfrak{p} \neq \mathfrak{p}_0$ of A, the natural algebra homomorphism*

$$E_\mathfrak{p} \ := \ E \otimes_A A_\mathfrak{p} \ \longrightarrow \ \mathrm{End}_{A_\mathfrak{p}[G_K]}\bigl(T_\mathfrak{p}(\phi)\bigr)$$

is an isomorphism.

Moreover, in [14] Proposition 2.5 we deduced the following result from Taguchi's Theorem 2.1:

Theorem 2.6 *For almost all primes $\mathfrak{p}$ of A, every $A_\mathfrak{p}[G_K]$-submodule of $T_\mathfrak{p}(\phi)$ has the form $\alpha(T_\mathfrak{p}(\phi))$ for some $\alpha \in E_\mathfrak{p}$.*

For yet finer information we decompose everything under $E_\mathfrak{p}$, as in [14] §4.1. Let Z denote the center of E. Then E is an order in a finite dimensional central division algebra over the quotient field of Z. Write $c := [Z/A]$ and $e^2 = [E/Z]$. Then the rank of ϕ is $r = cde$ for an integer $d > 0$. Let $Z_\mathfrak{P}$ denote the completion of Z at a prime $\mathfrak{P}$. Standard properties of division algebras over global fields imply that for almost all primes $\mathfrak{p}$ of A, we have

$$Z_\mathfrak{p} \ := \ Z \otimes_A A_\mathfrak{p} \ = \ \bigoplus_{\mathfrak{P}|\mathfrak{p}} Z_\mathfrak{P}$$

and

$$E_\mathfrak{p} \ \cong \ \mathrm{Mat}_{e\times e}(Z_\mathfrak{p}) \ = \ \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathrm{Mat}_{e\times e}(Z_\mathfrak{P}).$$

For such $\mathfrak{P}|\mathfrak{p}$ let $E_\mathfrak{p}$ act on $Z_\mathfrak{P}^{\oplus e}$ in the obvious way through its direct summand $\mathrm{Mat}_{e\times e}(Z_\mathfrak{P})$. Then $W_\mathfrak{P} := \mathrm{Hom}_{E_\mathfrak{p}}(Z_\mathfrak{P}^{\oplus e}, T_\mathfrak{p}(\phi))$ is a free $Z_\mathfrak{P}$-module of rank d. For all $\mathfrak{p}$ as above the above decomposition and the well-known structure theory of modules over matrix rings yield a natural decomposition

$$T_\mathfrak{p}(\phi) \;\cong\; \bigoplus_{\mathfrak{P}|\mathfrak{p}} W_\mathfrak{P} \otimes_{Z_\mathfrak{P}} Z_\mathfrak{P}^{\oplus e}. \tag{2.7}$$

Letting G_K act trivially on $Z_\mathfrak{P}^{\oplus e}$, by functoriality we obtain a natural continuous $Z_\mathfrak{P}$-linear representation of G_K on $W_\mathfrak{P}$. By construction the above isomorphism is $E_\mathfrak{p}[G_K]$-equivariant. Let $B_\mathfrak{p}$ denote the image of the natural homomorphism

$$A_\mathfrak{p}[G_K] \longrightarrow \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\phi)\big).$$

By Theorem 2.5, its commutant is $E_\mathfrak{p}$ for all $\mathfrak{p} \neq \mathfrak{p}_0$. In [14] Theorem B we proved:

Theorem 2.8 *Assume that $\mathfrak{p}_0 \neq 0$. Then for almost all primes $\mathfrak{p}$ of A the rings $E_\mathfrak{p}$ and $B_\mathfrak{p}$ are commutants of each other in $\mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\phi)\big)$. More precisely, for almost all $\mathfrak{p}$ we have $E_\mathfrak{p} \cong \mathrm{Mat}_{e\times e}(Z_\mathfrak{p})$ and $B_\mathfrak{p} \cong \mathrm{Mat}_{d\times d}(Z_\mathfrak{p})$.*

As explained in [14], Theorem 2.8 is expected to hold in the case $\mathfrak{p}_0 = 0$ as well.

3 Comparison of two Drinfeld modules

In this section we compare the Galois representations for any two Drinfeld A-modules ϕ_1, ϕ_2 over K of characteristic $\mathfrak{p}_0$. There are two possible cases.

Suppose first that there exists an isogeny $\phi_1 \to \phi_2$. Then for all $\mathfrak{p} \neq \mathfrak{p}_0$ the isogeny induces an $A_\mathfrak{p}[G_K]$-equivariant injection $T_\mathfrak{p}(\phi_1) \hookrightarrow T_\mathfrak{p}(\phi_2)$. In particular, it induces an isomorphism of the rational Galois representations $V_\mathfrak{p}(\phi_1) \xrightarrow{\sim} V_\mathfrak{p}(\phi_2)$. Moreover, any simple finite $A_\mathfrak{p}[G_K]$-subquotient of $T_\mathfrak{p}(\phi_1)$ is isomorphic to a subquotient of $T_\mathfrak{p}(\phi_2)$. Since there also exists an isogeny in the other direction $\phi_2 \to \phi_1$, the same holds vice versa.

The aim of this section is to prove that the opposite happens when ϕ_1 and ϕ_2 are non-isogenous. Then the Tate conjecture, Theorem 2.4, implies that

$$\mathrm{Hom}_{A_\mathfrak{p}[G_K]}\big(T_\mathfrak{p}(\phi_1), T_\mathfrak{p}(\phi_2)\big) = 0$$

for all $\mathfrak{p} \neq \mathfrak{p}_0$. In view of the semisimplicity from Theorem 2.3, this implies that $V_\mathfrak{p}(\phi_1)$ and $V_\mathfrak{p}(\phi_2)$ possess no isomorphic non-trivial $F_\mathfrak{p}[G_K]$-subquotients. By contrast, isomorphic simple finite $A_\mathfrak{p}[G_K]$-subquotients cannot be ruled out completely, because G_K acts on them through finite quotients, and so accidental isomorphisms between them can exist without any special meaning. But we prove that this happens at most finitely often:

Theorem 3.1 *If ϕ_1 and ϕ_2 are non-isogenous, the set of primes $\mathfrak{p}$ of A for which $T_\mathfrak{p}(\phi_1)$ and $T_\mathfrak{p}(\phi_2)$ have isomorphic non-trivial finite $A_\mathfrak{p}[G_K]$-subquotients is finite.*

The rest of this section is devoted to proving Theorem 3.1. Let us first sketch the argument in the case $\mathrm{End}_K(\phi_1) = \mathrm{End}_K(\phi_2) = A$. Theorem 2.6 implies that in this case $\phi_1[\mathfrak{p}]$ and $\phi_2[\mathfrak{p}]$ are irreducible finite $A_\mathfrak{p}[G_K]$-modules for almost all $\mathfrak{p}$. Assume that they are isomorphic for infinitely many $\mathfrak{p}$. Then for these $\mathfrak{p}$, the characteristic polynomials on $T_\mathfrak{p}(\phi_1)$ and $T_\mathfrak{p}(\phi_2)$ of every sufficiently good Frobenius element $\mathrm{Frob}_x \in G_K$ are congruent to each other modulo $\mathfrak{p}$. As the representations form a compatible system by Proposition 2.2, it follows that the characteristic polynomials are in fact equal. We apply this knowledge to $V_\mathfrak{p}(\phi_1)$ and $V_\mathfrak{p}(\phi_2)$ for any fixed $\mathfrak{p} \neq \mathfrak{p}_0$. Since the Frobenius elements are dense in G_K, we deduce that these two $F_\mathfrak{p}[G_K]$-modules have the same character. As they are also absolutely irreducible by Theorems 2.3 and 2.5, they are therefore isomorphic. Finally, by Theorem 2.4 this implies that ϕ_1 and ϕ_2 are isogenous, as desired.

In the general case we first establish the necessary machinery for each of the Drinfeld modules ϕ_i separately. Set $E_i := \mathrm{End}_K(\phi_i)$, let Z_i be its center, and write $e_i^2 = [E_i/Z_i]$. Let ψ_i denote the tautological extension of ϕ to a Drinfeld Z_i-module. Then for almost all primes $\mathfrak{p}$ of A, the decomposition (2.7) yields an isomorphism

$$T_\mathfrak{p}(\phi_i) \;=\; \bigoplus_{\mathfrak{P}_i|\mathfrak{p}} T_{\mathfrak{P}_i}(\psi_i) \;\cong\; \bigoplus_{\mathfrak{P}_i|\mathfrak{p}} (W_{\mathfrak{P}_i})^{\oplus e_i}, \tag{3.2}$$

where $\mathfrak{P}_i|\mathfrak{p}$ runs through primes of Z_i. By Proposition 2.2 the representation of G_K on $T_{\mathfrak{P}_i}(\psi_i)$ is unramified at all closed points $x \in \mathcal{U}$ not above $\mathfrak{P}_i$, and the characteristic polynomial

$$f_{i,x}(t) \;:=\; \det\nolimits_{Z_{i,\mathfrak{P}_i}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, T_{\mathfrak{P}_i}(\psi_i)\big)$$

has coefficients in Z_i and is independent of $\mathfrak{P}_i$. The corresponding characteristic polynomial over $A_\mathfrak{p}$ is

$$\det\nolimits_{A_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, T_{\mathfrak{P}_i}(\psi_i)\big) \;=\; \mathrm{Nm}_{Z_{i,\mathfrak{P}_i}/A_\mathfrak{p}}\big(f_{i,x}(t)\big). \tag{3.3}$$

This uses the norm for the local extension $Z_{i,\mathfrak{P}_i}/A_\mathfrak{p}$, but the fact that $f_{i,x}$ has coefficients in the global ring Z_i can be exploited as follows.

Fix a finite normal field extension $\tilde{F}$ of F into which Z_i can be embedded, and let $\tilde{A}$ be the normalization of A in $\tilde{F}$. For any primes $\mathfrak{P}_i$ of Z_i and $\tilde{\mathfrak{p}}$ of $\tilde{A}$ above the same prime $\mathfrak{p}$ of A, we observe that

$$\begin{aligned} \Sigma_{i,\mathfrak{P}_i,\tilde{\mathfrak{p}}} \;&:=\; \big\{\sigma \in \mathrm{Hom}_A(Z_i,\tilde{A}) \;\big|\; \mathfrak{P}_i = \sigma^{-1}(\tilde{\mathfrak{p}})\big\} \\ &\cong\; \mathrm{Hom}_{A_\mathfrak{p}}(Z_{i,\mathfrak{P}_i}, \tilde{A}_{\tilde{\mathfrak{p}}}). \end{aligned}$$

Let m_i denote the inseparability degree of Z_i over A. This is also the inseparability degree of $Z_{i,\mathfrak{P}_i}$ over $A_\mathfrak{p}$. Thus the local norm can be calculated within $\tilde{A}_{\tilde{\mathfrak{p}}}$ as

$$\mathrm{Nm}_{Z_{i,\mathfrak{P}_i}/A_\mathfrak{p}}\big(f_{i,x}(t)\big) \;=\; \prod_{\sigma\in\Sigma_{i,\mathfrak{P}_i,\tilde{\mathfrak{p}}}} \sigma(f_{i,x}(t))^{m_i}. \tag{3.4}$$

Note that the right hand side has coefficients in $\tilde{A}$ and depends only on i, x, and the subset $\Sigma_{i,\mathfrak{P}_i,\tilde{\mathfrak{p}}} \subset \mathrm{Hom}_A(Z_i,\tilde{A})$.

On the other hand let $k_\mathfrak{p}$ denote the residue field at $\mathfrak{p}$, and consider the quotient $\overline{W}_{\mathfrak{P}_i} := W_{\mathfrak{P}_i}/\mathfrak{P}_i W_{\mathfrak{P}_i}$. For almost all $\mathfrak{P}_i|\mathfrak{p}$ the ramification degree is m_i; hence the $k_\mathfrak{p}[G_K]$-module $W_{\mathfrak{P}_i}/\mathfrak{p}W_{\mathfrak{P}_i}$ is a successive extension of m_i copies of $\overline{W}_{\mathfrak{P}_i}$. By combining the results obtained so far we can therefore deduce that

$$\begin{aligned} \det\nolimits_{k_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, \overline{W}_{\mathfrak{P}_i}\big)^{m_i e_i} \;&=\; \det\nolimits_{k_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, W_{\mathfrak{P}_i}/\mathfrak{p}W_{\mathfrak{P}_i}\big)^{e_i} \\ &\overset{(3.2)}{=}\; \det\nolimits_{k_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, T_{\mathfrak{P}_i}(\psi_i)/\mathfrak{p}T_{\mathfrak{P}_i}(\psi_i)\big) \\ &=\; \det\nolimits_{A_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, T_{\mathfrak{P}_i}(\psi_i)\big) \bmod \mathfrak{p} \\ &\overset{(3.3)}{=}\; \mathrm{Nm}_{Z_{i,\mathfrak{P}_i}/A_\mathfrak{p}}\big(f_{i,x}(t)\big) \bmod \mathfrak{p} \\ &\overset{(3.4)}{=}\; \prod_{\sigma\in\Sigma_{i,\mathfrak{P}_i,\tilde{\mathfrak{p}}}} \sigma(f_{i,x}(t))^{m_i} \bmod \tilde{\mathfrak{p}}. \end{aligned} \tag{3.5}$$

Note also that for almost all $\mathfrak{P}_i|\mathfrak{p}$, Theorem 2.6 and the decomposition (3.2) together imply that $\overline{W}_{\mathfrak{P}_i}$ is an irreducible $k_\mathfrak{p}[G_K]$-module and that every irreducible $A_\mathfrak{p}[G_K]$-subquotient of $T_\mathfrak{p}(\phi_i)$ is isomorphic to some $\overline{W}_{\mathfrak{P}_i}$.

Proof of Theorem 3.1. We assume that $T_\mathfrak{p}(\phi_1)$ and $T_\mathfrak{p}(\phi_2)$ possess isomorphic non-trivial finite $A_\mathfrak{p}[G_K]$-subquotients for infinitely many $\mathfrak{p}$. We must then show that ϕ_1 and ϕ_2 are isogenous.

For the infinitely many $\mathfrak{p}$, there must exist primes $\mathfrak{P}_i|\mathfrak{p}$ of Z_i such that $\overline{W}_{\mathfrak{P}_1} \cong \overline{W}_{\mathfrak{P}_2}$ as $k_\mathfrak{p}[G_K]$-modules. Thus the characteristic polynomials on these representations must coincide. In view of the calculation (3.5) this implies that for all $x \in \mathcal{U}$ not above $\mathfrak{P}_1$ or $\mathfrak{P}_2$, and for any choice of $\tilde{\mathfrak{p}}$, we have

$$(3.6) \qquad \prod_{\sigma\in\Sigma_{1,\mathfrak{P}_1,\tilde{\mathfrak{p}}}} \sigma(f_{1,x}(t))^{m_1m_2e_2} \equiv \prod_{\sigma\in\Sigma_{2,\mathfrak{P}_2,\tilde{\mathfrak{p}}}} \sigma(f_{2,x}(t))^{m_2m_1e_1} \mod \tilde{\mathfrak{p}}.$$

By assumption this happens for infinitely many quadruples $(\mathfrak{p}, \mathfrak{P}_1, \mathfrak{P}_2, \tilde{\mathfrak{p}})$. Since there are only finitely many possibilities for the subsets $\Sigma_{i,\mathfrak{P}_i,\tilde{\mathfrak{p}}} \subset \mathrm{Hom}_A(Z_i, \tilde{A})$, it must happen infinitely often with $\Sigma_{i,\mathfrak{P}_i,\tilde{\mathfrak{p}}}$ equal to some fixed Σ_i. For every $x \in \mathcal{U}$, the congruence (3.6) then concerns the same elements of $\tilde{A}$ modulo infinitely many $\tilde{\mathfrak{p}}$; hence it is an equality

$$\prod_{\sigma\in\Sigma_1} \sigma(f_{1,x}(t))^{m_1m_2e_2} = \prod_{\sigma\in\Sigma_2} \sigma(f_{2,x}(t))^{m_2m_1e_1}.$$

To translate this equality back to the Tate modules, we can fix any quadruple $(\mathfrak{p}, \mathfrak{P}_1, \mathfrak{P}_2, \tilde{\mathfrak{p}})$ as above with $\Sigma_{i,\mathfrak{P}_i,\tilde{\mathfrak{p}}} = \Sigma_i$. Then for every $x \in \mathcal{U}$ not above $\mathfrak{P}_1$ or $\mathfrak{P}_2$, the equations (3.3) and (3.4) imply that

$$\det\nolimits_{A_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, T_{\mathfrak{P}_1}(\psi_1)\big)^{m_2e_2} = \det\nolimits_{A_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, T_{\mathfrak{P}_2}(\psi_2)\big)^{m_1e_1}.$$

In other words, we have

$$\det\nolimits_{F_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, V_{\mathfrak{P}_1}(\psi_1)^{\oplus m_2e_2}\big) = \det\nolimits_{F_\mathfrak{p}}\big(t\cdot\mathrm{Id} - \mathrm{Frob}_x \,\big|\, V_{\mathfrak{P}_2}(\psi_2)^{\oplus m_1e_1}\big).$$

Since the Frobenius elements are dense in G_K, it follows that the characteristic polynomials over $F_\mathfrak{p}$ of any element of G_K on $V_{\mathfrak{P}_1}(\psi_1)^{\oplus m_2e_2}$ and on $V_{\mathfrak{P}_2}(\psi_2)^{\oplus m_1e_1}$ coincide. As these $F_\mathfrak{p}[G_K]$-modules are semisimple, by Proposition 3.8 below this implies that they are actually isomorphic.

Finally, by the decomposition (3.2) this shows that $\mathrm{Hom}_{F_\mathfrak{p}[G_K]}\big(V_\mathfrak{p}(\phi_1), V_\mathfrak{p}(\phi_2)\big)$ is non-zero. By Theorem 2.4 this implies that ϕ_1 and ϕ_2 are isogenous, as desired. This finishes the proof of Theorem 3.1. **q.e.d.**

For lack of a suitable reference we include proofs of the following facts:

Proposition 3.7 *Two finite dimensional representations of a group G over a field L have the same Jordan-Hölder factors with the same multiplicities if and only if they do so over an algebraic closure of L.*

Proof. By induction on the dimension it suffices to prove that two finite dimensional representations V and V' over L possess a common Jordan-Hölder factor if and only if they do so over $\bar{L}$. So assume that $V \otimes_L \bar{L}$ and $V' \otimes_L \bar{L}$ possess a common Jordan-Hölder factor $\bar{U}$. After replacing V and V' by suitable irreducible subquotients, we may assume that both representations are irreducible. We must then prove that they are isomorphic.

Let E denote the center of $\mathrm{End}_{L[G]}(V)$ and F the maximal subfield of E that is separable over L. Then $F \otimes_L \bar{L}$ is a direct sum of copies of $\bar{L}$, indexed by

$$\Sigma := \mathrm{Hom}_L(F, \bar{L}) \cong \mathrm{Hom}_L(E, \bar{L}),$$

and this implies that

$$V \underset{L}{\otimes} \bar{L} \;\cong\; V \underset{E}{\otimes} \big(E \underset{L}{\otimes} \bar{L}\big) \;\cong\; \bigoplus_{\sigma\in\Sigma} V \underset{E}{\otimes} \big(E \underset{F,\sigma}{\otimes} \bar{L}\big).$$

Since E is totally inseparable over F, each summand here is successive extension of copies of the semisimple representation $V \otimes_{E,\sigma} \bar{L}$. Thus every Jordan-Hölder factor occurs both as a subrepresentation and as a quotient, and so $\bar{U}$ occurs both as a subrepresentation and as a quotient of $V \otimes_L \bar{L}$.

The same argument applies to V' in place of V. Therefore there exist equivariant $\bar{L}$-linear homomorphisms $V \otimes_L \bar{L} \twoheadrightarrow \bar{U} \hookrightarrow V' \otimes_L \bar{L}$. This shows that the space

$$\mathrm{Hom}_{\bar{L}[G]}\big(V \underset{L}{\otimes} \bar{L}, V' \underset{L}{\otimes} \bar{L}\big) \;\cong\; \mathrm{Hom}_{L[G]}(V, V') \underset{L}{\otimes} \bar{L}$$

is non-zero, and so there exists a non-zero equivariant homomorphism $V \to V'$. Since V and V' are both irreducible, this homomorphism must be an isomorphism. Thus V and V' are isomorphic, as desired. **q.e.d.**

Proposition 3.8 *Let V be a finite dimensional representation of a group G over a field L. Then the Jordan-Hölder factors of V and their multiplicities are determined uniquely by the associated characteristic polynomials, i.e., by the map*

$$G \longrightarrow L[t],\; g \mapsto \det_L(t\cdot \mathrm{Id} - g \,|\, V).$$

Proof. By Proposition 3.7 we may extend scalars to an algebraic closure of L; hence we may assume that L is algebraically closed. We may also replace V by its semisimplification. Let V' be another semisimple finite dimensional representation over L with the same characteristic polynomials as V. Then both $\dim V$ and $\dim V'$ are equal to the degree of these characteristic polynomials and thus equal to each other. We may assume that this common dimension is positive, since otherwise the assertion is obvious.

Suppose first that V and V' possess a common irreducible component U. Writing $V \cong U \oplus W$ and $V' \cong U \oplus W'$, the multiplicativity of characteristic polynomials implies that W and W' again have the same characteristic polynomials of G. Thus in this case the desired assertion follows by induction on $\dim V$.

Assume now that V and V' have no irreducible components in common. Choose representatives U_i for the isomorphism classes of irreducible components of $V \oplus V'$. Let $A \subset \mathrm{End}_L(V \oplus V')$ denote the image of the group ring $L[G]$. Since $V \oplus V'$ is semisimple and L is algebraically closed, this is the direct sum of the matrix rings $\mathrm{End}_L(U_i)$. Furthermore, the assumption implies that $A = B \oplus B'$ for subrings $B \subset \mathrm{End}_L(V)$ and $B' \subset \mathrm{End}_L(V')$. As the trace is one of the coefficients of the characteristic polynomial, we have $\mathrm{tr}_L(g \,|\, V) = \mathrm{tr}_L(g \,|\, V')$ for all $g \in G$. Since the trace of a matrix is a linear map, this implies that $\mathrm{tr}_L(a \,|\, V) = \mathrm{tr}_L(a \,|\, V')$ for all $a \in A$. For any $b \in B$ we may apply this to the element $a = (b, 0) \in A$, deducing that $\mathrm{tr}_L(b \,|\, V) = \mathrm{tr}_L(0 \,|\, V') = 0$. If m_i denotes the multiplicity of U_i in V, we find in particular that $m_i \cdot \mathrm{tr}_L(c) = 0$ for any $c \in \mathrm{End}_L(U_i)$. But since the trace map $\mathrm{End}_L(U_i) \to L$ is surjective, this means that $m_i \cdot 1 = 0$ in L. In other words m_i is a multiple of the characteristic p of L.

As V is non-zero by assumption, some m_i is positive, and so p must be positive. The above result thus shows that $V \cong W^{\oplus p}$ for another representation W. The same result holds for V' in place of V; hence $V' \cong W'^{\oplus p}$ for a representation W'. The multiplicativity of characteristic polynomials then implies that W and W' again have the same characteristic polynomials of G. Thus the desired assertion follows by induction on $\dim V$. **q.e.d.**

4 A-Motives

We give a brief introduction to the notions and the basic algebraic theory of A-motives. For a more comprehensive exposition we refer to Anderson's original article [1] and to Goss's textbook [7]. There only the case $A = \mathbb{F}_q[t]$ is considered under the name of *t-motives*. However, the generalization to arbitrary A is straightforward and will allow extension of coefficients, just as for Drinfeld modules.

We keep the notations of Section 2. As a preparation we recall a consequence of Lang's theorem for GL_n over finite fields (Lang [10] Corollary to Theorem 1).

Let $\mathrm{Vec}'_\tau K$ denote the category of finite dimensional K-vector spaces together with an additive endomorphism $\tau : V \to V$ satisfying $\tau(xv) = x^q\tau(v)$ for all $x \in K$ and $v \in V$, such that $K\tau(V) = V$. For any such V we abbreviate $V^{\mathrm{sep}} := V \otimes_K K^{\mathrm{sep}}$ and denote again by τ its additive endomorphism $\tau(v \otimes x) := \tau(v) \otimes x^q$. For any module with an action of τ we denote by $(\ )^\tau$ the submodule of τ-invariants.

On the other hand, let $\mathrm{Rep}_{\mathbb{F}_q} G_K$ denote the category of finite dimensional continuous representations of G_K over $\mathbb{F}_q$. For any such representation H we let G_K act on $H \otimes_{\mathbb{F}_q} K^{\mathrm{sep}}$ by $\sigma(h \otimes x) := \sigma(h) \otimes \sigma(x)$. For every representation of G_K we denote by $(\ )^{G_K}$ the subgroup of G_K-invariants.

Proposition 4.1 *The maps $V \mapsto T(V) := (V^{\mathrm{sep}})^\tau$ and $H \mapsto D(H) := (H \otimes_{\mathbb{F}_q} K^{\mathrm{sep}})^{G_K}$ define mutually quasi-inverse equivalences of categories between $\mathrm{Vec}'_\tau K$ and $\mathrm{Rep}_{\mathbb{F}_q} G_K$.*

Proof. By SGA7 [8] exp.XXII §1 the natural map $v \otimes x \mapsto vx$ induces an isomorphism

$$T(V) \otimes_{\mathbb{F}_q} K^{\mathrm{sep}} = (V^{\mathrm{sep}})^\tau \otimes_{\mathbb{F}_q} K^{\mathrm{sep}} \longrightarrow V^{\mathrm{sep}}.$$

Taking G_K-invariants we deduce an isomorphism $D(T(V)) \to (V^{\mathrm{sep}})^{G_K} \cong V$, which is τ-equivariant by construction. Conversely by Galois descent the map $h \otimes x \otimes y \mapsto h \otimes xy$ yields an isomorphism

$$D(H) \otimes_K K^{\mathrm{sep}} = (H \otimes_{\mathbb{F}_q} K^{\mathrm{sep}})^{G_K} \otimes_K K^{\mathrm{sep}} \longrightarrow H \otimes_{\mathbb{F}_q} K^{\mathrm{sep}}.$$

Taking τ-invariants we obtain an isomorphism $T(D(H)) \to (H \otimes_{\mathbb{F}_q} K^{\mathrm{sep}})^\tau \cong H$, which is G_K-equivariant by construction. Clearly everything is functorial in V and H. **q.e.d.**

In the following we abbreviate $A_K = A \otimes_{\mathbb{F}_q} K$ and let I denote the kernel of the homomorphism $A_K \to K$, $a \otimes x \mapsto \iota(a)x$.

Definition 4.2 (A-motives) *An* A-motive M over K of characteristic $\mathfrak{p}_0$ *is an A_K-module together with an additive endomorphism $\tau : M \to M$ satisfying*

$$\tau((a \otimes x)m) = (a \otimes x^q)\tau(m)$$

for all $a \in A$, $x \in K$ and $m \in M$, such that

(1) *M is finitely generated and projective over A_K,*

(2) *M is finitely generated over $K\{\tau\}$, and*

(3) *the A_K-module $M/A_K\tau(M)$ is annihilated by a power of I.*

The rank of M *is the rank of M as an A_K-module. A* homomorphism of A-motives *is a homomorphism of A_K-modules that commutes with τ.*

By Anderson [1] Proposition 1.8.3 we have:

Proposition-Definition 4.3 (Torsion and Tate modules) *Let M be an A-motive over K of rank r and characteristic $\mathfrak{p}_0$.*

(1) *For any ideal $\mathfrak{a} \subset A$ not divisible by $\mathfrak{p}_0$, the quotient $M/\mathfrak{a}M$ is an object in $\mathrm{Vec}'_\tau K$ and*
$$M[\mathfrak{a}] := T(M/\mathfrak{a}M)$$
is a free module of rank r over $A/\mathfrak{a}$, called the module of $\mathfrak{a}$-torsion of M.

(2) *For any prime $\mathfrak{p} \neq \mathfrak{p}_0$ of A, the* $\mathfrak{p}$-adic Tate module *and the* rational $\mathfrak{p}$-adic Tate module *of M are*
$$T_\mathfrak{p}(M) := \varprojlim_i M[\mathfrak{p}^i] \quad \text{and} \quad V_\mathfrak{p}(M) := T_\mathfrak{p}(M) \otimes_{A_\mathfrak{p}} F_\mathfrak{p}.$$
The former is a free module of rank r over $A_\mathfrak{p}$, and the latter is a vector space of dimension r over $F_\mathfrak{p}$.

By construction, we have continuous actions of the absolute Galois group G_K on $M[\mathfrak{a}]$, on $T_\mathfrak{p}(M)$ and on $V_\mathfrak{p}(M)$. Moreover, the definition is functorial in M, i.e., every homomorphism $\eta\colon N \to M$ of A-motives over K induces G_K-equivariant homomorphisms $N[\mathfrak{a}] \to M[\mathfrak{a}]$ and $T_\mathfrak{p}(\eta)\colon T_\mathfrak{p}(N) \to T_\mathfrak{p}(M)$. The following important theorem is the analog of Faltings's famous result and is independently due to Taguchi [18], [19] and Tamagawa [21], [22], [23].

Theorem 4.4 (Tate conjecture for A-motives) *For any A-motives N and M over K of characteristic $\mathfrak{p}_0$ and all primes $\mathfrak{p} \neq \mathfrak{p}_0$ of A, the natural map*
$$\mathrm{Hom}_K(N, M) \otimes_A A_\mathfrak{p} \longrightarrow \mathrm{Hom}_{A_\mathfrak{p}[G_K]}\big(T_\mathfrak{p}(N), T_\mathfrak{p}(M)\big)$$
is an isomorphism.

Definition 4.5 (Isogenies) *A homomorphism of A-motives η is called an* isogeny *if $\ker \eta = 0$ and $\operatorname{coker} \eta$ has finite dimension over K. An isogeny η is called* separable *if $A_K \tau(\operatorname{coker} \eta) = \operatorname{coker} \eta$.*

Consider a separable isogeny $\eta\colon N \to M$. Then $\operatorname{coker} \eta$ is an object of $\mathrm{Vec}'_\tau K$; hence by Proposition 4.1 it corresponds to the finite $\mathbb{F}_q[G_K]$-module $T(\operatorname{coker} \eta)$. By functoriality this is also an A-module and is therefore isomorphic to $\bigoplus_{i=1}^r A/\mathfrak{a}_i$ for suitable $r \geq 0$ and ideals $\mathfrak{a}_i \subset A$.

Definition 4.6 (Degree) *The* degree *of a separable isogeny η is the ideal $\deg \eta := \prod_{i=1}^r \mathfrak{a}_i \subset A$, where r and the $\mathfrak{a}_i$ are as above.*

In the following, by a *sublattice of* an $A_\mathfrak{p}$-module or an $F_\mathfrak{p}$-vector space we mean a finitely generated $A_\mathfrak{p}$-submodule of maximal rank.

Proposition 4.7 (Isogenies and lattices) *Let $\eta\colon N \to M$ be a separable isogeny of A-motives over K of characteristic $\mathfrak{p}_0$. Then $\mathrm{im}(T_\mathfrak{p}(\eta)) \subset T_\mathfrak{p}(M)$ is a G_K-invariant sublattice for all primes $\mathfrak{p} \neq \mathfrak{p}_0$ of A, with equality for all $\mathfrak{p} \nmid \deg \eta$.*

Proof. Since $\deg \eta$ annihilates $T(\operatorname{coker} \eta)$, Proposition 4.1 implies that it also annihilates $\operatorname{coker} \eta$. Thus for any non-zero element $a \in \deg \eta$ we have $aM \subset \eta(N) \subset M$, and so there exists an isogeny $\hat\eta\colon M \to N$ such that $\eta \circ \hat\eta = a \cdot \mathrm{Id}$. This implies that the image of $T_\mathfrak{p}(\eta)\colon T_\mathfrak{p}(N) \to T_\mathfrak{p}(M)$ contains $a \cdot T_\mathfrak{p}(M)$. In particular $\mathrm{im}(T_\mathfrak{p}(\eta))$ is a sublattice of $T_\mathfrak{p}(M)$ for all $\mathfrak{p}$, and is equal to $T_\mathfrak{p}(M)$ for all $\mathfrak{p} \nmid a$. Since for any $\mathfrak{p} \nmid \deg \eta$ we can choose $a \in (\deg \eta) \smallsetminus \mathfrak{p}$, we have equality for all $\mathfrak{p} \nmid \deg \eta$, as desired.
q.e.d.

In the following proposition we call two isogenies $\eta\colon N \to M$ and $\eta'\colon N' \to M$ isomorphic if there exists an isomorphism $\theta\colon N' \to N$ such that $\eta \circ \theta = \eta'$. This is equivalent to saying that the submodules $\eta(N)$ and $\eta'(N')$ of M coincide.

Proposition 4.8 (Classification of isogenies) *For any A-motive M over K of characteristic* $\mathfrak{p}_0$*, the map* $\eta \mapsto \big(\mathrm{im}(T_\mathfrak{p}(\eta))\big)_{\mathfrak{p}\neq\mathfrak{p}_0}$ *induces a bijection*

$$\left\{\begin{array}{l}\textit{isomorphism classes of}\\ \textit{separable isogenies } \eta\colon\\ N \to M \textit{ with } \mathfrak{p}_0 \nmid \deg\eta\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\textit{collections of } G_K\textit{-invariant}\\ \textit{sublattices } \Lambda_\mathfrak{p} \subset T_\mathfrak{p}(M)\\ \textit{for all } \mathfrak{p}\neq\mathfrak{p}_0 \textit{ such that } \Lambda_\mathfrak{p}\\ = T_\mathfrak{p}(M) \textit{ for almost all } \mathfrak{p}\end{array}\right\}$$

Proof. Clearly isomorphic isogenies yield the same lattices; hence the map is well-defined. To construct an inverse let $(\Lambda_\mathfrak{p})_{\mathfrak{p}\neq\mathfrak{p}_0}$ be a collection of sublattices as in the proposition. Let $\mathfrak{p}\neq\mathfrak{p}_0$ be a prime with $\Lambda_\mathfrak{p}\neq T_\mathfrak{p}(M)$. Then $\Lambda_\mathfrak{p}$ contains $\mathfrak{p}^m T_\mathfrak{p}(M)$ for some $m>0$, and so we have a natural surjection

$$M[\mathfrak{p}^m] \;\cong\; T_\mathfrak{p}(M)/\mathfrak{p}^m T_\mathfrak{p}(M) \;\twoheadrightarrow\; T_\mathfrak{p}(M)/\Lambda_\mathfrak{p}.$$

By applying the functor D from Proposition 4.1 we obtain surjections

$$M \;\twoheadrightarrow\; M/\mathfrak{p}^m M \;\cong\; D\big(M[\mathfrak{p}^m]\big) \;\twoheadrightarrow\; D\big(T_\mathfrak{p}(M)/\Lambda_\mathfrak{p}\big).$$

Let M' denote the kernel of the composite map. Then M' is an A-submotive of M such that the inclusion map $M'\hookrightarrow M$ is a separable isogeny of $\mathfrak{p}$-power degree with $T_\mathfrak{p}(M')=\Lambda_\mathfrak{p}$. We apply this construction recursively for every prime $\mathfrak{p}\neq\mathfrak{p}_0$ at which $\Lambda_\mathfrak{p}\neq T_\mathfrak{p}(M)$ and obtain an A-submotive N' such that the inclusion map $N'\hookrightarrow M$ is a separable isogeny with $T_\mathfrak{p}(N')=\Lambda_\mathfrak{p}$ for all $\mathfrak{p}\neq\mathfrak{p}_0$.

Thus to any collection $(\Lambda_\mathfrak{p})_{\mathfrak{p}\neq\mathfrak{p}_0}$ we have associated an isogeny which gives back the lattices $\Lambda_\mathfrak{p}$. It remains to show that for any separable isogeny $\eta: N\to M$ of degree not divisible by $\mathfrak{p}_0$, the above construction applied to the lattices $\Lambda_\mathfrak{p} := \mathrm{im}(T_\mathfrak{p}(\eta))$ yields an isogeny isomorphic to η. For any $\mathfrak{p}\neq\mathfrak{p}_0$ with $\Lambda_\mathfrak{p}\neq T_\mathfrak{p}(M)$ let $M'\subset M$ be as above. Then the construction together with the equivalence of categories 4.1 implies that η factors through a separable isogeny $N\to M'$ of degree prime to $\mathfrak{p}$. After repeating this with all $\mathfrak{p}|\deg\eta$ we obtain a factorization $N\to N'\hookrightarrow M$ of η, where N' is as above and $N\to N'$ is a separable isogeny of degree 1. This is the desired isomorphism. **q.e.d.**

Proposition 4.9 (Isomorphism classes in an isogeny class) *Let M be an A-motive over K of characteristic* $\mathfrak{p}_0$*. Set* $E := \mathrm{End}_K(M)$ *and* $E_{(\mathfrak{p}_0)} := E\otimes_A A_{(\mathfrak{p}_0)}$*, where* $A_{(\mathfrak{p}_0)}\subset F$ *denotes the localization of A at* $\mathfrak{p}_0$*. Then the multiplicative group* $E^*_{(\mathfrak{p}_0)}$ *acts naturally on the set of all sublattices of* $V_\mathfrak{p}(M)$*, and there exists a natural bijection*

$$\left\{\begin{array}{l}\textit{isomorphism classes of}\\ \textit{A-motives } N \textit{ over } K \textit{ such that}\\ \textit{there exists a separable isogeny}\\ \eta\colon N\to M \textit{ with } \mathfrak{p}_0\nmid\deg\eta\end{array}\right\} \longrightarrow E^*_{(\mathfrak{p}_0)}\left\backslash\left\{\begin{array}{l}\textit{collections of } G_K\textit{-invariant}\\ \textit{sublattices } \Lambda_\mathfrak{p}\subset V_\mathfrak{p}(M)\\ \textit{for all } \mathfrak{p}\neq\mathfrak{p}_0 \textit{ such that } \Lambda_\mathfrak{p}\\ = T_\mathfrak{p}(M) \textit{ for almost all } \mathfrak{p}\end{array}\right\}\right. .$$

Proof. The map is defined by choosing some η and setting $\Lambda_\mathfrak{p} := \mathrm{im}(T_\mathfrak{p}(\eta))$. To show that it is well-defined consider any two separable isogenies $\eta,\eta': N\to M$ of degree not divisible by $\mathfrak{p}_0$. Take any element $a\in(\deg\eta)\smallsetminus\mathfrak{p}_0$ and let $\hat\eta\colon M\to N$ be such that $\eta\circ\hat\eta = a\cdot\mathrm{Id}$, as in the proof of Proposition 4.7. Then the equality $\eta\circ\hat\eta\circ\eta = a\cdot\eta = \eta\circ(a\cdot\mathrm{Id})$ implies that $\hat\eta\circ\eta = a\cdot\mathrm{Id}$ on N. Similarly, we can find an element $a'\in(\deg\eta')\smallsetminus\mathfrak{p}_0$ and an isogeny $\hat\eta'\colon M\to N$ such that $\eta'\circ\hat\eta' = a'\cdot\mathrm{Id}$ and $\hat\eta'\circ\eta' = a'\cdot\mathrm{Id}$. The calculation

$$(\eta'\circ\hat\eta)\circ\eta = \eta'\circ(\hat\eta\circ\eta) = \eta'\circ(a\cdot\mathrm{Id}) = (a\cdot\mathrm{Id})\circ\eta'$$

then implies that

$$T_\mathfrak{p}(\eta'\circ\hat\eta)\big(\mathrm{im}(T_\mathfrak{p}(\eta))\big) = T_\mathfrak{p}(a\cdot\mathrm{Id})\big(\mathrm{im}(T_\mathfrak{p}(\eta'))\big)$$

for all $\mathfrak{p} \neq \mathfrak{p}_0$. By construction $a \cdot \mathrm{Id}$, $a' \cdot \mathrm{Id} \in E$ become invertible in $E_{(\mathfrak{p}_0)}$, and so the calculation

$$(\eta \circ \hat{\eta}') \circ (\eta' \circ \hat{\eta}) = \eta \circ (a' \cdot \mathrm{Id}) \circ \hat{\eta} = (a' \cdot \mathrm{Id}) \circ (\eta \circ \hat{\eta}) = (a' \cdot \mathrm{Id}) \circ (a \cdot \mathrm{Id})$$

implies that $\eta' \circ \hat{\eta}$ becomes invertible in $E_{(\mathfrak{p}_0)}$, too. Thus the two collections of lattices are equivalent by the element $a^{-1}(\eta' \circ \hat{\eta}) \in E^*_{(\mathfrak{p}_0)}$; hence the map is well-defined.

To show that it is injective consider two separable isogenies $\eta : N \to M$ and $\eta' : N' \to M$ of degree not divisible by $\mathfrak{p}_0$, such that the associated collections of lattices are equivalent under $E^*_{(\mathfrak{p}_0)}$. Then there exist a, $a' \in A \smallsetminus \mathfrak{p}_0$ such that

$$\mathrm{im}(T_\mathfrak{p}(a \cdot \eta)) = T_\mathfrak{p}(a \cdot \mathrm{Id})\big(\mathrm{im}(T_\mathfrak{p}(\eta))\big) = T_\mathfrak{p}(a' \cdot \mathrm{Id})\big(\mathrm{im}(T_\mathfrak{p}(\eta'))\big) = \mathrm{im}(T_\mathfrak{p}(a' \cdot \eta'))$$

for all $\mathfrak{p} \neq \mathfrak{p}_0$. Since $a \cdot \eta$ and $a' \cdot \eta'$ are again separable of degree not divisible by $\mathfrak{p}_0$, Proposition 4.8 implies that N and N' are isomorphic, as desired.

To show that the map is surjective let $(\Lambda_\mathfrak{p})_{\mathfrak{p} \neq \mathfrak{p}_0}$ be a collection of sublattices as in the proposition. Then there are at most finitely many $\mathfrak{p} \neq \mathfrak{p}_0$ with $\Lambda_\mathfrak{p} \not\subset T_\mathfrak{p}(M)$. Choose any element $a \in A \smallsetminus \mathfrak{p}_0$ such that $a\Lambda_\mathfrak{p} \subset T_\mathfrak{p}(M)$ for these $\mathfrak{p}$. Then we have $a\Lambda_\mathfrak{p} \subset T_\mathfrak{p}(M)$ for all $\mathfrak{p} \neq \mathfrak{p}_0$, with equality for almost all $\mathfrak{p}$. Thus Proposition 4.8 yields an A-motive mapping to the collection of lattices $(a\Lambda_\mathfrak{p})_{\mathfrak{p} \neq \mathfrak{p}_0}$. By construction this collection is equivalent to the collection $(\Lambda_\mathfrak{p})_{\mathfrak{p} \neq \mathfrak{p}_0}$, and the surjectivity follows.

q.e.d.

Finally we explain the relation with Drinfeld modules. For every Drinfeld A-module ϕ over K we set $M_\phi := K\{\tau\}$ with the action of $a \otimes x \in A_K$ by $(a \otimes x)m = xm\phi_a$ and of τ by left multiplication. One easily shows that this defines an A-motive and that the construction is functorial in ϕ. More precisely, we have (cf. Anderson [1] Theorem 1):

Proposition 4.10 *This construction defines a fully faithful contravariant functor from the category of Drinfeld A-modules over K of characteristic $\mathfrak{p}_0$ to the category of A-motives over K characteristic $\mathfrak{p}_0$. Its essential image consists of all A-motives which are free of rank one over $K\{\tau\}$.*

The contravariance of this functor is also reflected in a duality between the torsion modules of ϕ and of M_ϕ (cf. Anderson [1] Proposition 1.8.3). Let Ω_A denote the module of Kähler differentials of A.

Proposition 4.11 *Let ϕ be a Drinfeld A-module over K of characteristic $\mathfrak{p}_0$.*

(1) *For all ideals $\mathfrak{a}$ in A not divisible by $\mathfrak{p}_0$, there is a natural G_K-equivariant isomorphism*
$$M_\phi[\mathfrak{a}] \cong \mathrm{Hom}_A\big(\phi[\mathfrak{a}], \mathfrak{a}^{-1}\Omega_A/\Omega_A\big).$$

(2) *For all primes $\mathfrak{p} \neq \mathfrak{p}_0$, there is a natural G_K-equivariant isomorphism*
$$T_\mathfrak{p}(M_\phi) \cong \mathrm{Hom}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\phi), \Omega_A \otimes_A A_\mathfrak{p}\big).$$

Remark 4.12 For any two Drinfeld A-modules ϕ and ψ over K the above correspondences yield a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_K(\phi, \psi) \otimes_A A_\mathfrak{p} & \longrightarrow & \mathrm{Hom}_{A_\mathfrak{p}[G_K]}\big(T_\mathfrak{p}(\phi), T_\mathfrak{p}(\psi)\big) \\
\wr \big\| \, 4.10 & & \wr \big\| \, 4.11(2) \\
\mathrm{Hom}_K(M_\psi, M_\phi) \otimes_A A_\mathfrak{p} & \longrightarrow & \mathrm{Hom}_{A_\mathfrak{p}[G_K]}\big(T_\mathfrak{p}(M_\psi), T_\mathfrak{p}(M_\phi)\big).
\end{array}$$

Thus Theorem 2.4 becomes a special case of Theorem 4.4.

5 Proof of the main theorem

Throughout this section, we fix an A-motive M over K which is the direct sum of A-motives associated to Drinfeld A-modules of special characteristic $\mathfrak{p}_0$. The proof of Theorem 1.2 follows the argument of Deligne [3] Corollaire 2.8 for abelian varieties over number fields. An important step is the classification of isogenies by lattices from Propositions 4.8 and 4.9. Thus in this section we first study the Galois invariant sublattices of $V_\mathfrak{p}(M)$ for any fixed $\mathfrak{p} \neq \mathfrak{p}_0$. We prove that the action of $\big(\mathrm{End}_K(M) \otimes_A F_\mathfrak{p}\big)^*$ on the set of these sublattices is transitive for almost all $\mathfrak{p} \neq \mathfrak{p}_0$ and 'almost transitive' for all $\mathfrak{p} \neq \mathfrak{p}_0$. Working adèlically, the desired finiteness is then reduced to the finiteness of the class number.

First we group the direct summands of M by their isogeny classes. Thus we write

$$M = \bigoplus_{i=1}^{n} M_i \qquad \text{and} \qquad M_i = \bigoplus_{j=1}^{k_i} M_{\phi^{i,j}}$$

with Drinfeld A-modules $\phi^{i,j}$ such that $\phi^{i,j}$ and $\phi^{i',j'}$ are isogenous over K if and only if $i = i'$. Then the endomorphism ring of M decomposes accordingly as

$$E \ := \ \mathrm{End}_K(M) \ = \ \bigoplus_{i=1}^{n} \mathrm{End}_K(M_i).$$

In particular,

$$E \otimes_A F \ \cong \ \bigoplus_{i=1}^{n} \mathrm{Mat}_{k_i \times k_i}\big(\mathrm{End}_K(\phi^{i,1})^{\mathrm{op}} \otimes_A F\big)$$

is a finite dimensional semisimple F-algebra.

Next for every prime $\mathfrak{p} \neq \mathfrak{p}_0$, Proposition 4.11 (2) yields a natural isomorphism

$$(5.1) \qquad T_\mathfrak{p}(M) \ = \ \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{k_i} T_\mathfrak{p}(M_{\phi^{i,j}}) \ \cong \ \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{k_i} \mathrm{Hom}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\phi^{i,j}), \Omega_A \otimes_A A_\mathfrak{p}\big).$$

Since Ω_A is locally free of rank 1 over A, the representation theoretic properties of $T_\mathfrak{p}(M)$ can therefore be read off from those of $T_\mathfrak{p}(\phi^{i,j})$. In particular, the results of Sections 2 and 3 apply.

5.1 Galois invariant sublattices

In this subsection we investigate the G_K-invariant sublattices of $V_\mathfrak{p}(M)$ for $\mathfrak{p} \neq \mathfrak{p}_0$. For this we first analyze the image of the group ring $A_\mathfrak{p}[G_K]$ in

$$\mathrm{End}_{E \otimes_A F_\mathfrak{p}}\big(V_\mathfrak{p}(M)\big).$$

Proposition 5.2 *For all $\mathfrak{p} \neq \mathfrak{p}_0$, the ring $\mathrm{End}_{E \otimes_A F_\mathfrak{p}}\big(V_\mathfrak{p}(M)\big)$ is a semisimple $F_\mathfrak{p}$-algebra, and the image of $A_\mathfrak{p}[G_K]$ is an $A_\mathfrak{p}$-order in it.*

Proof. By Theorem 2.3 and the decomposition (5.1) the $F_\mathfrak{p}[G_K]$-module

$$V_\mathfrak{p}(M) = \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{k_i} V_\mathfrak{p}\big(M_{\phi^{i,j}}\big)$$

is semisimple. Thus the image of $F_\mathfrak{p}[G_K]$ in $\mathrm{End}_{F_\mathfrak{p}}\big(V_\mathfrak{p}(M)\big)$ is a semisimple subalgebra, and by Jacobson's density theorem it is equal to its bicommutant. But by the Tate conjecture, Theorem 4.4, its commutant is $E \otimes_A F_\mathfrak{p}$. Thus the image of $F_\mathfrak{p}[G_K]$ is the commutant of $E \otimes_A F_\mathfrak{p}$, i.e., equal to $\mathrm{End}_{E \otimes_A F_\mathfrak{p}}\big(V_\mathfrak{p}(M)\big)$. From these facts both assertions follow. **q.e.d.**

Proposition 5.3 *For all $\mathfrak{p} \neq \mathfrak{p}_0$, the number of orbits of $(E \otimes_A F_\mathfrak{p})^*$ in the set of G_K-invariant sublattices of $V_\mathfrak{p}(M)$ is finite.*

Proof. By the Jordan-Zassenhaus theorem (Reiner [15] Theorem 26.4) Proposition 5.2 implies that there are only finitely many isomorphism classes of G_K-invariant sublattices of $V_\mathfrak{p}(M)$. Every isomorphism between two G_K-invariant sublattices of $V_\mathfrak{p}(M)$ extends to a G_K-equivariant automorphism of $V_\mathfrak{p}(M)$. By Theorem 4.4 these automorphisms are precisely the elements of $(E \otimes_A F_\mathfrak{p})^*$. **q.e.d.**

Next we exploit Theorems 2.8 and 3.1.

Proposition 5.4 *There exists a finite set S_0 of primes of A, containing $\mathfrak{p}_0$, such that for all $\mathfrak{p}$ outside S_0, the image of the group ring $A_\mathfrak{p}[G_K]$ in $\mathrm{End}_{E\otimes_A F_\mathfrak{p}}\big(V_\mathfrak{p}(M)\big)$ is a finite direct sum of matrix rings over complete discrete valuations rings. In particular, this image is a maximal order in $\mathrm{End}_{E\otimes_A F_\mathfrak{p}}\big(V_\mathfrak{p}(M)\big)$.*

Proof. For any $i = 1, \ldots, n$ and all $\mathfrak{p} \neq \mathfrak{p}_0$ let $B_{i,\mathfrak{p}}$ denote the image of $A_\mathfrak{p}[G_K]$ in $\mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(M_i)\big)$. Since the direct summands $T_\mathfrak{p}(M_{\phi^{i,j}})$ of $T_\mathfrak{p}(M_i)$ become isomorphic over $F_\mathfrak{p}$, this is isomorphic to the image of $A_\mathfrak{p}[G_K]$ in $\mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(M_{\phi^{i,1}})\big)$. By Proposition 4.11 (2) it is therefore anti-isomorphic to the image of $A_\mathfrak{p}[G_K]$ in $\mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\phi^{i,1})\big)$. Thus Theorem 2.8 implies that for almost all $\mathfrak{p}$ we have

$$B_{i,\mathfrak{p}} \cong \mathrm{Mat}_{d_i\times d_i}(Z_i \otimes_A A_\mathfrak{p}),$$

where Z_i denotes the center of the endomorphism ring of $\phi^{i,1}$ and d_i is some positive integer. Moreover, Z_i is integrally closed above almost all primes $\mathfrak{p}$, and at all these primes $Z_i \otimes_A A_\mathfrak{p}$ is a finite direct sum of complete discrete valuations rings.

Let $B_\mathfrak{p}$ denote the image of $A_\mathfrak{p}[G_K]$ in $\mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(M)\big)$. Then the projection maps induce an embedding

$$B_\mathfrak{p} \;\hookrightarrow\; \bigoplus_{i=1}^n B_{i,\mathfrak{p}} \;\subset\; \bigoplus_{i=1}^n \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(M_i)\big).$$

We will show that the inclusion on the left hand side is an equality for almost all $\mathfrak{p}$. To this end we look at these rings as left modules over $A_\mathfrak{p}[G_K]$. Let r_i denote the rank of the A-motive M_i. Then there is a (non-canonical) isomorphism of left $A_\mathfrak{p}[G_K]$-modules

$$\begin{aligned} N_{i,\mathfrak{p}} \;:=\; \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(M_i)\big) \quad &\cong \quad T_\mathfrak{p}(M_i)^{\oplus r_i} \;\cong\; \bigoplus_{j=1}^{k_i} T_\mathfrak{p}(M_{\phi^{i,j}})^{\oplus r_i} \\ &\overset{4.11}{\cong} \quad \bigoplus_{j=1}^{k_i} \mathrm{Hom}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\phi^{i,j}), \Omega_A \otimes_A A_\mathfrak{p}\big)^{\oplus r_i}. \end{aligned}$$

For any fixed $i \neq i'$, Theorem 3.1 implies that for almost all $\mathfrak{p}$, the modules $N_{i,\mathfrak{p}}$ and $N_{i',\mathfrak{p}}$ do not possess an isomorphic non-trivial finite $A_\mathfrak{p}[G_K]$-subquotient. Since there are only finitely many i and i', we deduce that for almost all $\mathfrak{p}$, no two direct summands of $\bigoplus_{i=1}^n N_{i,\mathfrak{p}}$ possess an isomorphic non-trivial finite $A_\mathfrak{p}[G_K]$-subquotient. Thus for these $\mathfrak{p}$, every $A_\mathfrak{p}[G_K]$-submodule of $\bigoplus_{i=1}^n N_{i,\mathfrak{p}}$ decomposes according to i. In particular $B_\mathfrak{p}$ decomposes, and since $B_{i,\mathfrak{p}}$ is its image in $N_{i,\mathfrak{p}}$, the inclusion $B_\mathfrak{p} \hookrightarrow \bigoplus_{i=1}^n B_{i,\mathfrak{p}}$ must be an equality.

Since $B_{i,\mathfrak{p}}$ is a finite direct sum of matrix rings over complete discrete valuations rings for almost all $\mathfrak{p}$, the same now follows for $B_\mathfrak{p}$, as desired. **q.e.d.**

Proposition 5.5 *Let S_0 be as in Proposition 5.4. Then for all primes $\mathfrak{p} \notin S_0$, the action of $(E \otimes_A F_\mathfrak{p})^*$ on the set of G_K-invariant sublattices of $V_\mathfrak{p}(M)$ is transitive.*

Proof. Since the image of $A_{\mathfrak{p}}[G_K]$ is a maximal order in $\mathrm{End}_{E\otimes_A F_{\mathfrak{p}}}\big(V_{\mathfrak{p}}(M)\big)$ by Proposition 5.4, it follows from Reiner [15] Theorem 18.10 that any two G_K-invariant sublattices of $V_{\mathfrak{p}}(M)$ are isomorphic $A_{\mathfrak{p}}[G_K]$-modules. As in the proof of Proposition 5.3 this implies that they are equivalent under $(E\otimes_A F_{\mathfrak{p}})^*$, as desired. **q.e.d.**

5.2 Adèlization

Set $S := \{\infty, \mathfrak{p}_0\}$ and let

$$\widehat{A}^S := \prod_{\mathfrak{p}\notin S} A_{\mathfrak{p}}$$

denote the profinite completion of A away from S. Let

$$\mathbb{A}_F^S := \widehat{A}^S \otimes_A F \cong \prod_{\mathfrak{p}\notin S} F_{\mathfrak{p}}$$

denote the ring of partial adèles of F away from S. Let $A_{(\mathfrak{p}_0)}$ be the localization of A at $\mathfrak{p}_0$.

Proposition 5.6 *For any open subgroup $\mathcal{K} \subset \big(E\otimes \mathbb{A}_F^S\big)^*$, the number of double cosets*

$$\big(E\otimes_A A_{(\mathfrak{p}_0)}\big)^* \setminus \big(E\otimes_A \mathbb{A}_F^S\big)^* \,/\, \mathcal{K}$$

is finite.

Proof. Since $\mathcal{K}$ is open, it contains a subgroup of finite index of the open compact subgroup $(E\otimes_A \widehat{A}^S)^*$. It therefore suffices to prove the proposition in the case $\mathcal{K} = (E\otimes_A \widehat{A}^S)^*$.

We can then translate the assertion into one about lattices, as follows. For every $\underline{e} \in \big(E\otimes_A \mathbb{A}_F^S\big)^*$ we define

$$\Lambda_{\underline{e}} := \big(E\otimes_A A_{(\mathfrak{p}_0)}\big) \cap \underline{e}\big(E\otimes_A \widehat{A}^S\big).$$

This is a right E-submodule of $E\otimes_A A_{(\mathfrak{p}_0)}$. Since $\underline{e}$ and $\underline{e}^{-1}$ have only finitely many poles, there exists an element $a \in A \smallsetminus \mathfrak{p}_0$ such that

$$a\big(E\otimes_A \widehat{A}^S\big) \subset \underline{e}\big(E\otimes_A \widehat{A}^S\big) \subset a^{-1}\big(E\otimes_A \widehat{A}^S\big).$$

It follows that

$$aE = \Lambda_a \subset \Lambda_{\underline{e}} \subset \Lambda_{a^{-1}} = a^{-1}E;$$

hence $\Lambda_{\underline{e}}$ is a finitely generated submodule satisfying

$$\Lambda_{\underline{e}} \otimes_A A_{(\mathfrak{p}_0)} = E\otimes_A A_{(\mathfrak{p}_0)}. \tag{5.7}$$

Moreover, approximation at the divisors of a shows that

$$\Lambda_{\underline{e}} \otimes_A \widehat{A}^S = \underline{e}\big(E\otimes_A \widehat{A}^S\big). \tag{5.8}$$

We claim that two such lattices $\Lambda_{\underline{e}}$ and $\Lambda_{\underline{e}'}$ are isomorphic as right E-modules if and only if $\underline{e}$ and $\underline{e}'$ lie in the same double coset. The 'if' part follows directly from the transformation rule $\Lambda_{\varepsilon\underline{e}\underline{k}} = \varepsilon\Lambda_{\underline{e}}$ for all $\varepsilon \in \big(E\otimes_A A_{(\mathfrak{p}_0)}\big)^*$ and $\underline{k} \in \big(E\otimes_A \widehat{A}^S\big)^*$. For the 'only if' part note that any isomorphism $\Lambda_{\underline{e}} \to \Lambda_{\underline{e}'}$ is induced by left multiplication with an element $\varepsilon \in E\otimes_A F$. The equation (5.7) for $\underline{e}$ and $\underline{e}'$ then implies that $\varepsilon \in (E\otimes_A A_{(\mathfrak{p}_0)})^*$. Moreover, the equation (5.8) for $\underline{e}$ and $\underline{e}'$ implies that $\varepsilon\underline{e}\big(E\otimes_A \widehat{A}^S\big) = \underline{e}'\big(E\otimes_A \widehat{A}^S\big)$. Thus $\varepsilon\underline{e}\underline{k} = \underline{e}'$ for some $\underline{k} \in \big(E\otimes_A \widehat{A}^S\big)^*$, and so the the double cosets of $\underline{e}$ and $\underline{e}'$ coincide, proving the claim.

Finally, since E is an A-order in the semisimple F-algebra $E \otimes_A F$, by the Jordan-Zassenhaus theorem (Reiner [15] Theorem 26.4) there are only finitely many isomorphism classes of finitely generated E-modules of any given rank. By the claim the proposition follows. **q.e.d.**

Proof of Theorem 1.2. By Proposition 4.9 the theorem is equivalent to saying that the set of equivalence classes under $(E \otimes_A A_{(\mathfrak{p}_0)})^*$ of collections of G_K-invariant sublattices $\Lambda_\mathfrak{p} \subset V_\mathfrak{p}(M)$ for all $\mathfrak{p} \neq \mathfrak{p}_0$, such that $\Lambda_\mathfrak{p} = T_\mathfrak{p}(M)$ for almost all $\mathfrak{p}$, is finite. The group $(E \otimes \mathbb{A}_F^S)^*$ acts on the set of all such collections $(\Lambda_\mathfrak{p})_{\mathfrak{p} \notin S}$, and Propositions 5.3 and 5.5 together imply that the number of orbits under this action is finite. Fix one of these orbits and let $\mathcal{K} \subset (E \otimes \mathbb{A}_F^S)^*$ be the stabilizer of an element. Then the set of isomorphism classes of A-motives corresponding to this orbit can be identified with the double quotient

$$\big(E \otimes_A A_{(\mathfrak{p}_0)}\big)^* \setminus \big(E \otimes_A \mathbb{A}_F^S\big)^* \,/\, \mathcal{K}\,.$$

Since $\mathcal{K}$ is an open compact subgroup of $\big(E \otimes \mathbb{A}_F^S\big)^*$, this double quotient is finite by Proposition 5.6, finishing the proof. **q.e.d.**

Remark. Instead of the Jordan-Zassenhaus theorem in the form of Proposition 5.6 one can use the general theory of reductive algebraic groups over global fields. By Behr [2] Satz 7 the class number of a connected reductive algebraic group over a global field is finite, and we know that $(E \otimes_A F)^*$ is reductive over the center of $E \otimes_A F$. Thanks to the reduction theory developed in Harder [9], the extra conditions (V) in Behr's paper are obsolete.

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