Minmax Methods in the Calculus of Variations of Curves and Surfaces

Tristan Rivière*

Abstract : *The study of the variations of curvature functionals takes its origins in the works of Euler and Bernouilli from the XVIIIth century on the Elastica. Since these very early times, special curves and surfaces such as geodesics, minimal surfaces, elastica, Willmore surfaces...etc have become central objects in mathematics much beyond the field of geometry stricto sensu with applications in analysis, in applied mathematics, in theoretical physics and natural sciences in general. Despite its venerable age the calculus of variations of length, area or curvature functionals for curves and surfaces is still a very active field of research with important developments that took place in the last decades. In the proposed mini-course we shall concentrate on the various minmax constructions of these critical curves and surfaces in euclidian space or closed manifolds. We will start by recalling the origins of minmax methods for the length functional and present in particular the "curve shortening process" of Birkhoff. We will mention the generalization of Birkhoff's approach to surfaces and the "harmonic map replacement" method by Colding and Minicozzi. We will then recall some fundamental notions of Palais Smale deformation theory in infinite dimensional spaces and apply it to the construction of closed geodesics and Elastica.*

In the second part of the mini-course we will present a new method based on smoothing arguments combined with Palais Smale deformation theory for performing successful minmax procedures for surfaces. We will present various applications of this so called "viscosity method" such as the problem of computing the cost of the sphere eversion in 3 dimensional euclidian space.

Math. Class. 49Q05, 53A10, 53C22, 49Q10, 58E12, 58E20

*Department of Mathematics, ETH Zentrum, CH-8093 Zürich, Switzerland.

I Lecture 1

The Origin of Minmax, Birkhoff Curve Shortening Process.

All a long the lecture N^n will denote a closed sub-manifold of $\mathbb{R}^m$ (i.e. N^n is compact without boundary) and it will be assumed to be at least C^2. The embedding $N^n \hookrightarrow \mathbb{R}^m$ induces a metric on N^n that we denote by h. For any $z \in \mathbb{N}^n$ we shall also denote by $P_z^{N^n}$ or simply P_z^T the orthonormal projection in $\mathbb{R}^m$ onto the tangent plane T_zN^n. Under these notations the Levi-Civita covariant derivative on N^n denoted ∇^h is given by

$$\nabla^h_{\vec{Y}}\vec{X}(z) = P^T\left(d\vec{X}_z\vec{Y}\right)$$

Let $\vec{\gamma}$ be a map from S^1 into N^n. The length of $\vec{\gamma}(S^1)$ is given by

$$L(\vec{\gamma}) := \int_{S^1} dl_{\vec{\gamma}}$$

where $dl_{\vec{\gamma}}$ is the one form on S^1 given by $dl_{\vec{\gamma}} := |\partial_\theta\vec{\gamma}|\, d\theta$. For any smooth variation $\vec{\gamma}_s$ of an immersion $\vec{\gamma}$ such that $\vec{w} = \partial_s\vec{\gamma}|_{s=0}$ we have $\vec{w} \in T_{\vec{\gamma}}N^n$ and

$$\begin{aligned}\frac{d}{ds}\int_{S^1} dl_{\vec{\gamma}_s}\Big|_{s=0} &= \int_{S^1} \partial_s dl_{\vec{\gamma}_s} = \int_{S^1}\left\langle \nabla^h_{\partial_s\vec{\gamma}}\partial_\theta\vec{\gamma}, \frac{\partial_\theta\vec{\gamma}}{|\partial_\theta\vec{\gamma}|}\right\rangle_{\mathbb{R}^m} d\theta\\ &= \int_{S^1}\left\langle \nabla^h_{\partial_\theta\vec{\gamma}}\partial_s\vec{\gamma}, \frac{\partial_\theta\vec{\gamma}}{|\partial_\theta\vec{\gamma}|}\right\rangle_{\mathbb{R}^m} d\theta = \int_{S^1}\left\langle \nabla^h\vec{w}, d\vec{\gamma}\right\rangle_{g_{\vec{\gamma}}} dl_{\vec{\gamma}}\end{aligned}$$

where $g_{\vec{\gamma}}$ is the metric induced by the immersion $\vec{\gamma}$ on S^1 and where we have used that ∇^h is torsion free hence $\nabla^h_{\partial_\theta\vec{\gamma}}\partial_s\vec{\gamma} = \nabla^h_{\partial_s\vec{\gamma}}\partial_\theta\vec{\gamma} + [\partial_\theta\vec{\gamma}, \partial_s\vec{\gamma}]$ and $[\partial_\theta\vec{\gamma}, \partial_s\vec{\gamma}] = \vec{\gamma}_*[\partial_\theta, \partial_s] = 0$.

In **normal parametrization** (i.e. $|\partial_\theta\vec{\gamma}| \equiv Cte$), the immersion $\vec{\gamma}$ is a critical point of the length if and only if

$$\forall\ \vec{w} \in T_{\vec{\gamma}}N^n \qquad \int_{S^1} \partial_\theta\vec{w}\cdot\partial_\theta\vec{\gamma}\, d\theta = 0$$

which is equivalent to

$$P^T_{\vec{\gamma}}\left(\partial^2_{\theta^2}\vec{\gamma}\right) = 0 \quad \Longleftrightarrow \quad \nabla^h\partial_\theta\vec{\gamma} = 0 \tag{I.1}$$

This **geodesic equation** in **normal parametrization** is also equivalent to the **harmonic map** equation

$$-\ \partial^2_{\theta^2}\vec{\gamma} + \partial_\theta(P^T_{\vec{\gamma}})\partial_\theta\vec{\gamma} = 0 \tag{I.2}$$

Regarding the existence of geodesics on closed manifold we first state the following result.

Theorem I.1. [Cartan, 1927] *Assume $\pi_1(N^n) \neq 0$ and let $\alpha \in \pi_1(N^n)$ with $\alpha \neq 0$ then α is realized by a closed geodesic.* □

Proof. One minimizes

$$E(\vec{\gamma}) := \int_{S^1} \left|\frac{\partial \vec{\gamma}}{\partial \theta}\right|^2 \, d\theta$$

among all $\vec{\gamma}$ realizing α. The Dirichlet energy and the length are closely related to each other. Indeed we first have the following inequality

$$L^2(\vec{\gamma}) \leq 2\pi \, E(\vec{\gamma})$$

with equality if and only if the parametrization is of constant speed or *normal*. Moreover critical points to E satisfy the harmonic map equation (I.2) and are in normal parametrization.

The homotopy class prescription is moreover sub-critical in the sense that since we have the continuous embedding

$$W^{1,2}(S^1) \hookrightarrow C^{0,1/2}(S^1) \tag{I.3}$$

any minimizing sequence $\vec{\gamma}_k$ is pre-compact in C^0 i.e. there exists k' such that

$$\vec{\gamma}_{k'} \to \vec{\gamma}_\infty \quad \text{strongly in } C^0$$

Let $\delta > 0$ be such that all geodesic ball $B_\delta^{N^n}(z)$ in N^n are strictly convex. Hence for k' large enough $\vec{\gamma}_{k'}(\theta) \in B_\delta^{N^n}(\vec{\gamma}(\theta))$ for any θ in S^1. We can consider the deformation of $\vec{\gamma}_{k'}$ to $\vec{\gamma}_\infty$ given by the unique shortest geodesic in $B_\delta^{N^n}(\vec{\gamma}(\theta))$ connecting $\vec{\gamma}_{k'}(\theta)$ and $\vec{\gamma}(\theta))$ for all θ. This deformation realizes an homotopy equivalence between $\vec{\gamma}_{k'}(S^1)$ and $\vec{\gamma}_\infty(S^1))$ and hence $\vec{\gamma}_\infty \in \alpha$. It is also straightforward to check that $\vec{\gamma}_\infty$ satisfies (I.2) and minimizes both E and L in the class given by α. □

What about the existence of a closed geodesic when $\pi_1(N^n) = 0$?

We present below the resolution in the case[1] $n = 2$ - i.e. $N^n \simeq S^2$.

A **sweep-out** is a map $\vec{\sigma} \; : \; [0,1] \times S^1 \; \to \; N^2$ such that

i)
$$\vec{\sigma} \in C^0\left([0,1], W^{1,2}(S^1, N^2)\right)$$

ii) $\vec{\sigma}(0,\cdot)$ and $\vec{\sigma}(1,\cdot)$ are constant maps.

[1] The general case does not require more conceptual developments but it's presentation is a bit more tedious.

For any given sweep-out $\vec{\sigma}_0$ we define

$$\Omega_{\vec{\sigma}_0} := \{\vec{\sigma} \in \Omega \text{ such that } \vec{\sigma} \text{ and } \vec{\sigma}_0 \text{ are homotopic to eachother in } \Omega\}$$

For any $\vec{\sigma}_0 \in \Omega$ we define the **width** associated to $\vec{\sigma}_0$ to be the following number

$$W_{\vec{\sigma}_0} := \inf_{\vec{\sigma} \in \Omega_{\vec{\sigma}_0}} \max_{t \in [0,1]} E(\vec{\sigma}(t, \cdot)) \tag{I.4}$$

We have the following lemma

Lemma I.1. *For any closed two dimensional manifold N^2 and any sweep-out $\vec{\sigma}_0$ of N^2, $W_{\vec{\sigma}_0} > 0$ if and only if $\vec{\sigma}_0$ defines a non zero element of $\pi_2(N^2)$.* □

Proof of lemma I.1. One direction of the equivalence is straightforward, if $\vec{\sigma}_0$ is homotopic to a constant map then $W_{\vec{\sigma}_0} = 0$. Assume now $W_{\vec{\sigma}_0} = 0$ Let $\vec{\sigma}_k$ be a minimizing sequence. We then have

$$\lim_{k \to 0} \max_{t \in [0,1]} E(\vec{\sigma}_k(t, \cdot)) = 0$$

Hence, because of the continuous embedding (I.3), for k large enough and for each $t \in [0,1]$ the image $\vec{\sigma}_k(t, S^1)$ is included in a strictly convex ball $B_\delta^{N^n}(p_k(t))$ whose center $p_k(t)$ can be chosen to be continuous with respect to t. Similarly as in the proof of Cartan's theorem above, by taking the shortest geodesic map, one can then homotope $\vec{\sigma}_k(t, \cdot)$ to the continuous constant valued map $p_k(t)$. Hence $\vec{\sigma}_k$ is homotopic to such a map and has to be nul homotopic since $p_k([0,1])$ is contractible. □

From now on we assume $W_{\vec{\sigma}_0} > 0$ and ask

Does there exists a geodesic $\vec{\gamma}$ such that $L(\vec{\gamma}) = \sqrt{2\,\pi\, W_{\vec{\sigma}_0}}$?

The minimization procedure (I.4) is made complicated by the a-priori lack of coercivity. Indeed, for a minimizing sequence $\vec{\sigma}_k$ the only a-priori control is given by

$$\lim_{k \to 0} \max_{t \in [0,1]} E(\vec{\sigma}_k(t, \cdot)) = W_{\vec{\sigma}_0} \tag{I.5}$$

and the dependance with respect to t for instance is completely free a-priori. In order to give more coercivity to the minimization problem (I.4) one will restrict the minimization to a much smaller class (quasi "finite dimensional"). For any $Q \in \mathbb{N}^*$ we introduce the space of **piecewize**

linear maps with Q breaks

$$\Lambda^Q := \left\{ \begin{array}{c} \vec{\gamma} \in W^{1,2}(S^1,N^2) \quad \text{s.t. } L(\vec{\gamma}) \le Q\,\delta \\ \exists\ p_0 \le p_1 \le \cdots \le p_Q = p_0 \in S^1 \quad \text{ s.t. } L([p_i,p_{i+1}]) \le \delta \\ \nabla^h \partial_\theta \vec{\gamma} = 0 \quad \text{ and } \quad |\partial_\theta\vec{\gamma}|(\theta) \equiv Cte \quad \text{on } (p_i,p_{i+1}) \end{array} \right\}$$

where we recall that δ is a positive number such that each geodesic ball $B^N_\delta(z)$ in N^n is strictly convex. In other words, maps in Λ^Q are made of a succession of Q geodesic arcs (possibly trivial) each contained in a geodesic ball $B^N_\delta(z)$ and each parametrized at a constant speed. Observe that any element in Λ^Q posses a constant speed reparametrization. This is obtained by merging the successive p_i in S^1 such that $|\partial_\theta\vec{\gamma}|(\theta) \equiv 0 \quad \text{on } (p_i,p_{i+1})$.

Taking an element $\vec{\sigma} \in \Omega_{\vec{\sigma}_0}$ such that $\max_{t\in[0,1]} L(\vec{\sigma}(t,\cdot)) \le L$ then we can reparametrize each $\vec{\gamma}(t,\cdot)$ so that $|\partial_\theta\vec{\gamma}(t,\cdot)| \equiv L(\vec{\sigma}(t,\cdot))/2\pi$ and we replace each portion of $\vec{\gamma}(t,\theta)$ on $(e^{i\,\theta_j}, e^{i\,\theta_{j+1}})$ where $\theta_j = 2\pi\, j/[L/\delta]$ by the shortest geodesic in the corresponding geodesic ball joining $\vec{\gamma}(e^{i\,\theta_j})$ and $\vec{\gamma}(e^{i\,\theta_{j+1}})$. It is not difficult that the map obtained is in $\Lambda^{[L/\delta]}$ and that it is homotopic to $\vec{\sigma}_0$. Hence, denoting $\Lambda := \Lambda^{[W/\delta]}$ we have proved[2] that

$$W_{\vec{\sigma}_0} = \inf_{\vec{\sigma}\in\Omega_{\vec{\sigma}_0}\cap\Lambda} \ \max_{t\in[0,1]} E(\vec{\sigma}(t,\cdot))$$

We shall simply write W for $W_{\vec{\sigma}_0}$ and we assume $W > 0$. Let

$$G := \Lambda \cap \{\text{immersed closed geodesics}\}$$

We shall now improve our minimizing sequence $\vec{\sigma}_k$ and deform it into a new minimizing sequence $\vec{\gamma}_k$ that we can take in $\Omega_{\vec{\sigma}_0}\cap\Lambda$, in order to impose for any $\varepsilon > 0$ the existence of $\eta > 0$ such that, for k large enough

$$(2\pi)^{-1}\, L^2(\vec{\gamma}_k(t,\cdot)) = E(\vec{\gamma}_k(t,\cdot)) \ge W_{\vec{\sigma}_0} - \eta \implies \operatorname{dist}(\vec{\gamma}_k(t,\cdot),G) \le \varepsilon \quad \text{(I.6)}$$

The main ingredient for improving the minimizing sequence is given by the following "pulling tight" morphism Ψ on Λ, known also as **curve shortening map**.

Theorem I.2. [Birkhoff 1918] *There exists a morphism $\Psi\ :\ \Lambda \longrightarrow \Lambda$ such that*

i) Ψ is continuous (Λ is equipped with the $W^{1,2}$ topology.)

[2] Observe that the previous unique geodesic replacement in convex balls is a continuous with respect to the $W^{1,2}-$norm

ii) for every $\vec{\sigma} \in \Lambda$, $\Psi(\vec{\sigma})$ is homotopic to $\vec{\sigma}$,

iii) for every $\vec{\sigma} \in \Lambda$,

$$L(\Psi(\vec{\sigma})) \leq L(\vec{\sigma}) \quad ,$$

iv) there exists a continuous function φ from $[0,\infty)$ into $[0,\infty)$ such that $\varphi(0) = 0$ and

$$dist^2\,(\vec{\sigma}, \Psi(\vec{\sigma})) \leq \varphi\left(\frac{L^2(\vec{\sigma}) - L^2(\Psi(\vec{\sigma}))}{L^2(\Psi(\vec{\sigma}))}\right)$$

v) For every $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$dist(\vec{\sigma}, G) \geq \varepsilon \quad \Longrightarrow \quad L(\Psi(\vec{\sigma})) \leq L(\vec{\sigma}) - \alpha \quad .$$

where the distance is derived from the $W^{1,2}$ norm.

$\Box$

The proof of Birkhoff theorem is for instance given in [1].

We are now explaining how to "pull tight" the original minimal sequence in $\Omega_{\vec{\sigma}_0} \cap \Lambda$ by the mean of the **curve shortening map** Ψ in order to get property (I.6).

Let $\varepsilon > 0$ and let $\alpha > 0$ to be fixed later but chosen small enough in order, according to property v) of Ψ, to have at least

$$\mathrm{dist}(\vec{\gamma}, G) \geq \varepsilon/2 \quad \Longrightarrow \quad L(\Psi(\vec{\gamma})) \leq L(\vec{\gamma}) - \sqrt{\frac{2\pi}{W}}\,\alpha \quad . \tag{I.7}$$

Choose k large enough in such a way that

$$W \leq \max_{t\in[0,1]} E(\vec{\sigma}_k(t,\cdot)) < W + \frac{\alpha}{2}$$

We assume that both $\vec{\sigma}_k(t,\cdot)$ and $\Psi(\vec{\sigma}_k(t,\cdot))$ are in normal parametrization for all $t \in [0,1]$. Hence the previous assertion is implying that

$$\sqrt{2\pi\,W} \leq \max_{t\in[0,1]} L(\vec{\sigma}_k(t,\cdot)) < \sqrt{2\pi W} + \sqrt{\frac{2\pi}{W}}\,\frac{\alpha}{2}$$

Let t such that

$$E(\Psi(\vec{\sigma}_k(t,\cdot))) = (2\pi)^{-1} L^2(\Psi(\vec{\sigma}_k(t,\cdot))) \geq W - \frac{\alpha}{4} \tag{I.8}$$

This implies that

$$\sqrt{2\pi W} + \sqrt{\frac{2\pi}{W}}\,\frac{\alpha}{2} > L(\vec{\sigma}_k(t,\cdot)) \geq L(\Psi(\vec{\sigma}_k(t,\cdot))) \geq \sqrt{2\pi W} - \sqrt{\frac{2\pi}{W}}\,\frac{\alpha}{2}$$

We deduce from (I.7) that for such t

$$\mathrm{dist}(\vec{\sigma}_k(t,\cdot),G) \le \varepsilon/2 \quad . \tag{I.9}$$

From property v) we have also for the t satisfying (I.8)

$$\mathrm{dist}^2\left(\vec{\sigma}_k(t,\cdot),\Psi(\vec{\sigma}_k(t,\cdot))\right) \le \varphi\left(\frac{L^2(\vec{\sigma}_k(t,\cdot)) - L^2(\Psi(\vec{\sigma}_k(t,\cdot)))}{L^2(\Psi(\vec{\sigma}_k(t,\cdot)))}\right) \tag{I.10}$$

Because of the continuity of φ we can choose now α such that for all $x < 3\,\alpha/W$ we have $\varphi(x) \le \varepsilon^2/4$. Combining (I.9) and (I.10) gives property (I.6) for the "tighter" minimizing sequence $\vec{\gamma}_k := \Psi(\vec{\sigma}_k)$. The property (I.6) being satisfied we deduce the following theorem.

Theorem I.3. *Let $\vec{\sigma}_0$ be a sweepout of N^2 such that $W_{\vec{\sigma}_0} > 0$ then the number $W_{\vec{\sigma}_0}$ is the length of a closed geodesics in N^2 homotopic to $\vec{\sigma}_0$.* □

The proof of the existence of a **curve shortening map** Ψ is based on the crucial *local convexity property* of the Dirichlet energy for maps into N^n. We denote again by δ any positive number such that all the geodesic balls $B_\delta^{N^n}(z)$ are strictly convex in (N^n,h).

Lemma I.2. *Let I be an interval in S^1 such that $|I| \le \delta/(2\pi L)$ for some positive number L and let $\vec{\sigma}_1$ be a Lipschitz map on I such that $|\partial_\theta\vec{\sigma}_1| \le L$ and $\vec{\sigma}_2$ be the minimizing geodesic with the same end points, then we have*

$$dist^2(\vec{\sigma}_1,\vec{\sigma}_2) \le C\ \left[E(\vec{\sigma}_1) - E(\vec{\sigma}_2)\right]$$

where $C > 0$ only depends on N^n. □

Birkhoff Approach to the Construction of Minmax Minimal S^2.

We shall now consider maps u from S^2 into a closed (at least C^2) submanifold N^n of $\mathbb{R}^m$ and look for critical points to the area functional given by

$$A(\vec{u}) := \int_{S^2} dvol_{g_{\vec{u}}}$$

where $dvol_{g_{\vec{u}}}$ denotes the 2-form on S^2 given in local coordinates by

$$dvol_{g_{\vec{u}}} := \sqrt{|\partial_{x_1}\vec{u}|^2\,|\partial_{x_2}\vec{u}|^2 - (\partial_{x_1}\vec{u}\cdot\partial_{x_2}\vec{u})^2}\ dx_1\wedge dx_2 \quad .$$

Hence, when $\vec{u}$ is a C^1 immersion, the first variation of A is a one form on $\Gamma(\vec{u}^{-1}TN^n)$ (i.e. the space of $\mathbb{R}^m$ valued maps $\vec{w}$ such that $\vec{w}(x) \in T_{\vec{u}(x)}N^n$

for every $x \in S^2$), given by

$$
\begin{aligned}
dA(\vec{u}) \cdot \vec{w} := & \int_{S^2} \frac{\partial_{x_1}\vec{w} \cdot \partial_{x_1}\vec{u}\ |\partial_{x_2}\vec{u}|^2 + \partial_{x_2}\vec{w} \cdot \partial_{x_2}\vec{u}\ |\partial_{x_1}\vec{u}|^2}{|\partial_{x_1}\vec{u}|^2\, |\partial_{x_2}\vec{u}|^2 - (\partial_{x_1}\vec{u} \cdot \partial_{x_2}\vec{u})^2}\ dvol_{g_{\vec{u}}} \\
& - \int_{S^2} \frac{[\partial_{x_1}\vec{w} \cdot \partial_{x_2}\vec{u}\ + \partial_{x_2}\vec{w} \cdot \partial_{x_1}\vec{u}]\ \partial_{x_1}\vec{u} \cdot \partial_{x_2}\vec{u}}{|\partial_{x_1}\vec{u}|^2\, |\partial_{x_2}\vec{u}|^2 - (\partial_{x_1}\vec{u} \cdot \partial_{x_2}\vec{u})^2}\ dvol_{g_{\vec{u}}}
\end{aligned}
\tag{I.11}
$$

where we assume that $\vec{w}$ is supported in a chart. Using more intrinsic notations this gives

$$
\begin{aligned}
dA(\vec{u}) \cdot \vec{w} &= \int_{S^2} \langle d\vec{w}, d\vec{u} \rangle_{g_{\vec{u}}}\ dvol_{g_{\vec{u}}} = \int_{S^2} \left\langle \vec{w}, (\nabla^h)^{*_{g_{\vec{u}}}} d\vec{u} \right\rangle_{g_{\vec{u}}}\ dvol_{g_{\vec{u}}} \\
&= \int_{S^2} \left\langle \vec{w}, P^T(d^{*_{g_{\vec{u}}}} d\vec{u}) \right\rangle_{g_{\vec{u}}}\ dvol_{g_{\vec{u}}} = -\int_{S^2} \left\langle \vec{w}, P^T(\Delta_{g_{\vec{u}}} \vec{u}) \right\rangle_{g_{\vec{u}}}\ dvol_{g_{\vec{u}}}
\end{aligned}
\tag{I.12}
$$

where $g_{\vec{u}}$ simply denotes the pull-back by $\vec{u}$ of the induced metric h on N^n and is given by

$$
\forall\, X\, , Y \qquad g_u(X, Y) = du(X) \cdot du(Y)
$$

and it defines a C^∞ metric on S^2 and where $\Delta_{g_{\vec{u}}}$ is the negative **Laplace Beltrami operator** for the induced metric $g_{\vec{u}}$ given in local coordinates by

$$
\Delta_{g_{\vec{u}}} \phi = \sqrt{det(g^{kl})}\ \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(\frac{g^{ij}}{\sqrt{det(g^{kl})}} \frac{\partial \phi}{\partial x_j} \right)
$$

where we omit the subscript $\vec{u}$ and $(g^{kl})_{k,l=1,2}$ is the inverse matrix to $g_{ij} := \partial_{x_i}\vec{u} \cdot \partial_{x_j}\vec{u}$.

The **Dirichlet energy** is given by

$$
E(\vec{u}) := \frac{1}{2} \int_{S^2} |d\vec{u}|^2_{g_{S^2}}\ dvol_{g_{S^2}}
$$

It is conformally invariant in the sense that for any metric $g_0 = e^{2\lambda} g_{S^2}$ proportional to g_{S^2} one has

$$
E(\vec{u}) := \frac{1}{2} \int_{S^2} |d\vec{u}|^2_{g_0}\ dvol_{g_0}
$$

Hence in local **conformal coordinates** x for g_{S^2} (which always exist around every point) one has

$$
\frac{1}{2} \int_{x^{-1}(D^2)} |d\vec{u}|^2_{g_{S^2}}\ dvol_{g_{S^2}} = \int_{D^2} |\partial_{x_1}\vec{u}|^2 + |\partial_{x_2}\vec{u}|^2\ dx_1 \wedge dx_2 \quad .
$$

In any local coordinates we have

$$
A(\vec{u}) = \int_{D^2} |\partial_{x_1}\vec{u} \wedge \partial_{x_2}\vec{u}|\ dx_1 \wedge dx_2
$$

Hence the link between the area and the Dirichlet energy is given clearly by

$$A(\vec{u}) \leq E(\vec{u}) \quad . \tag{I.13}$$

where g_{S^2} is the standard metric on S^2 and $dvol_{g_{S^2}}$ it's volume form. For a map in $W^{1,2}(S^2, \mathbb{R}^m)$ we have equality in (I.13) if and only if u is **weakly conformal** with respect to g_{S^2} that is to say in conformal coordinates

$$\left\{ \begin{array}{l} |\partial_{x_1}\vec{u}| = |\partial_{x_2}\vec{u}| \quad \text{ a.e.} \\[3mm] \partial_{x_1}\vec{u} \cdot \partial_{x_2}\vec{u} = 0 \text{ a.e.} \end{array} \right. \tag{I.14}$$

The first variation of the Dirichlet energy is given by

$$\begin{array}{l} \forall \vec{w} \in \Gamma(\vec{u}^{-1}TN^n) \quad dE(\vec{u}) \cdot \vec{w} := \displaystyle\int_{S^2} \langle d\vec{w}, d\vec{u} \rangle_{g_{S^2}} \; dvol_{g_{S^2}} \\[5mm] \qquad = -\displaystyle\int_{S^2} \vec{w} \cdot \Delta_{g_{S^2}} \vec{u} \; dvol_{g_{S^2}} \end{array}$$

and critical points to E satisfy the **harmonic map equation**

$$P^T(\Delta_{g_{S^2}}\vec{u}) = 0 \quad \Longleftrightarrow \quad \Delta_{g_{S^2}}\vec{u} - \left\langle d(P^T(\vec{u})), d\vec{u} \right\rangle_{g_{S^2}} = 0 \quad . \tag{I.15}$$

We deduce the following proposition

Proposition I.1. *Conformal harmonic immersions are critical points of the area functional.* □

Assuming for a moment again that $\vec{u}$ is a C^∞ immersion of S^2, the uniformization theorem (see for instance [**?**]) gives the existence of a diffeomorphism $\Psi \in \mathrm{Diff}(S^2)$ such that

$$\Psi^* g_u = e^{2\lambda} g_{S^2} \quad \Longleftrightarrow \quad u \circ \Psi \quad \text{ is conformal}$$

where λ is a smooth function on S^2. Hence we have

$$E(u \circ \Psi) = A(u \circ \Psi)$$

For a general $u \in W^{1,2}(S^2, \mathbb{R}^n)$ a result by Morrey [2] gives, for any $\varepsilon > 0$, the existence of an "almost conformal parametrization" Ψ_ε such that

$$E(u \circ \Psi_\varepsilon) \leq A(u \circ \Psi_\varepsilon) + \varepsilon$$

This reinforces the general idea that, similarly as the 1-dimensional case, minimizing the area A, at least for the disc or the 2 sphere which both posses a unique conformal structure, should be equivalent to minimize the energy E which is more coercive and much more compatible with calculus of variations arguments. This is what Douglas and Radò did for solving the Plateau problem in the 30th (see for instance [1]). The strategy consisting of minimizing the energy instead of the area has been implemented successfully by Sacks and Uhlenbeck in [3] for giving the corresponding result to the Cartan result theorem I.1 in 2 dimensions. Precisely we have.

Theorem I.4. **[Sacks Uhlenbeck 1980]** *Any non trivial free homotopy class of* $pi_2(N^n)$ *(the quotient of* $\pi_2(N, x_0)$ *by the* π_1 *action) can be represented by a sum of spheres, images of* S^2 *by conformal harmonic maps into* N^n *realizing immersed possibly branched minimal* S^2 *in* N^n*.* □

This paper is the pioneered work which has triggered a whole theory, called **concentration compactness theory**, which plays a central role in the analysis of conformally invariant PDE. Contrary to the 1 dimensional case the control of the energy doeas not give C^0 norm control : the embedding

$$W^{1,2} \hookrightarrow C^0$$

which was true on S^1 is missed (from very "little") in 2-dimension. As a consequence the minimizing sequence could "split" into separated spheres, "forgetting" the based point of the 2-homotopy class we are working in, and realizes at the limit a so called **bubble tree**, notion which has played a central role in the analysis of **pseudo-holomorphic curve**, **Yang-Mills Fields**, **Yamabe**...etc

In a series of works presented in a synthetic way in chapter 5 of [1], Colding and Minicozzi follow the general scheme of Birkhoff approach for performing minmax method for constructing minimal spheres when $\pi_2(N^n) = 0$. One of the up-shot of their method is the construction of a 2-dimensional generalization of Birkhoff curve shortening process where minimizing sequences are "pulled tight" by successive local replacements by energy minimizing harmonic maps. The process is successful due to the following local convexity result which is the 2-dimensional counterpart of lemma I.2

Theorem I.5. **[Colding Minicozzi 2008]** *There exists* $\varepsilon_1 > 0$ *depending only on* N^n *such that for any* $\vec{u}$ $W^{1,2}(D^2, \mathbb{R}^m)$ *such that* $u(x) \in N^n$ *for almost every* $x \in D^2$ *and for any harmonic smooth map into* N^n *such that*

$$\int_{D^2} |\nabla \vec{v}|^2 \, dx^2 < \varepsilon_1$$

we have that

$$\frac{1}{2}\int_{D^2} |\nabla(\vec{u}-\vec{v})|^2\ dx^2 \le \int_{D^2} |\nabla\vec{u}|^2 - |\nabla\vec{v}|^2\ dx^2 \quad .$$

□

There is at least one main reason which makes the 2-D situation **considerably more difficult** than Birkhoff one dimensional framework. This is coming from the lack of embedding

$$W^{1,2}(S^2) \not\hookrightarrow C^0(S^2)$$

we were mentioning above. Moreover, if working with the energy instead of the area has obvious advantages, one has to pay a price at some point. This price is related to the possibility for the energy to be dissipated between the "bubbles" in the so called "neck region". The proof of the absence of **neck energy** can be very technical and we will come back to this problematic later in the course while working with elastic energies of surfaces. Despite the abundant difficulties, Colding and Minicozzi have been able to implement the Birkhoff approach for 2-spheres and one of their main results in this direction is the following.

Theorem I.6. [Colding Minicozzi 2008] *Let β be a non trivial class in $\pi_3(N^n)$ then*

$$W := \inf_{\vec{u}(t,\cdot)\in\Omega_\beta} \max_{t\in[0,1]} Area(\vec{u}(t,\cdot)) > 0$$

and there exists finitely many conformal harmonic maps $(\vec{u}_j)_{j=1\cdots Q}$ from S^2 into N^n such that

$$W = \sum_{j=1}^{Q} E(\vec{u}_j) \quad .$$

□

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