Minmax Methods in the Calculus of Variations of Curves and Surfaces

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I Lecture 2

Deformations in Infinite Dimensional Manifolds.

The goal of this section is to present the main lines of *Palais deformation theory* exposed in [3].

We shall start by recalling some notions from **infinite dimensional Banach Manifolds** theory that will be useful to us. For a more systematic study we suggest the reader to consult [1].

In this lecture as in the previous one N^n denotes a closed C^∞ sub-manifold of the Euclidian Space $\mathbb{R}^m$.

We recall that a topological space is **Hausdorff** if every pair of points can be included in two disjoint open sets containing each exactly one of the two points. A topological space is called **normal** if for any two disjoint closed sets have disjoint open neighborhoods.

Definition I.1. *A C^p* **Banach Manifold** *$\mathcal{M}$ for $p \in \mathbb{N} \cup \{\infty\}$ is an Hausdorff topological space together with a covering by open sets $(U_i)_{i\in I}$, a family of Banach vector spaces $(E_i)_{i\in I}$ and a family of continuous mappings $(\varphi_i)_{i\in I}$ from U_i inton E_i such that*

i) for every $i \in I$

$$\varphi_i\; U_i \;\longrightarrow\; \varphi_i(U_i) \qquad \text{is an homeomorphism}$$

ii) for every pair of indices $i \neq j$ in I

$$\varphi_j \circ \varphi_i^{-1} \;:\; \varphi_i(U_i \cap U_j) \subset E_i \;\longrightarrow\; \varphi_j(U_i \cap U_j) \subset E_j$$

is a C^p diffeomorphism

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$\square$

Example 1. Let Σ^k be a closed oriented $k-$dimensional manifold and let $l \in \mathbb{N}$ and $p \geq 1$. Define

$$\mathcal{M} := W^{l,p}(\Sigma^k, N^n) := \left\{ \vec{u} \in W^{l,p}(\Sigma^k, \mathbb{R}^m) \quad ; \quad \vec{u}(x) \in N^n \quad \text{ for a.e. } x \in \Sigma^k \right\}$$

where on Σ^k we can choose any arbitrary reference smooth metric, they are all equivalent since Σ^k is compact. Assume $\mathbf{l\, p > k}$ then $W^{l,p}(\Sigma^k, N^n)$ defines a Banach Manifold. This comes mainly from the fact that, under our assumptions,

$$W^{l,p}(\Sigma^k, \mathbb{R}^m) \quad \hookrightarrow \quad C^0(\Sigma^k, \mathbb{R}^m) \quad . \tag{I.1}$$

The Banach manifold structure is then defined as follows. Choose $\delta > 0$ such that each geodesic ball $B_\delta^{N^n}(z)$ for any $z \in N^n$ is strictly convex and the exponential map

$$\exp_z \ : \ V_z \subset T_z N^n \ \longrightarrow \ B_\delta^{N^n}(z)$$

realizes a C^∞ diffeomorphism for some open neighborhood of the origin in $T_z N^n$ into the geodesic ball $B_\delta^{N^n}(z)$. Because of the embedding (I.1) there exists $\varepsilon_0 > 0$ such that

$$\forall\ \vec{u}\ ,\ \vec{v} \in W^{l,p}(\Sigma^k, N^n) \quad \|\vec{u} - \vec{v}\|_{W^{l,p}} < \varepsilon_0$$
$$\Longrightarrow \quad \|\mathrm{dist}_N(\vec{u}(x), \vec{v}(x))\|_{L^\infty(\Sigma^k)} < \delta \quad .$$

We equip the space $W^{l,p}(\Sigma^k, N^n)$ with the $W^{l,p}$ norm which makes it a metric space and for any $\vec{u} \in \mathcal{M} = W^{l,p}(\Sigma^k, N^n)$ we denote by $B_{\varepsilon_0}^{\mathcal{M}}(\vec{u})$ the open ball in $\mathcal{M}$ of center $\vec{u}$ and radius ε_0.

As a covering of $\mathcal{M}$ we take $(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}))_{\vec{u} \in \mathcal{M}}$. We denote by

$$E^{\vec{u}} := \Gamma_{W^{l,p}}\left(\vec{u}^{-1} TN\right) := \left\{ \vec{w} \in W^{l,p}(\Sigma^k, \mathbb{R}^m)\ ;\ \vec{w}(x) \in T_{\vec{u}(x)} N^n\ \forall\ x \in \Sigma^k \right\}$$

this is the Banach space of $W^{l,p}-$sections of the bundle $\vec{u}^{-1}TN$ and for any $\vec{u} \in \mathcal{M}$ and $\vec{v} \in B_{\varepsilon_0}^{\mathcal{M}}(\vec{u})$ we define $\vec{w}^{\,\vec{u}}(\vec{v})$ to be the following element of $E^{\vec{u}}$

$$\forall\ x \in \Sigma \qquad \vec{w}^{\,\vec{u}}(\vec{v})(x) := \exp^{-1}_{\vec{u}(x)}(\vec{v}(x))$$

It is not difficult to see that

$$\vec{w}^{\,\vec{v}} \circ (\vec{w}^{\,\vec{u}})^{-1} \ : \ \vec{w}^{\,u}\left(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}) \cap B_{\varepsilon_0}^{\mathcal{M}}(\vec{v})\right) \ \longrightarrow \ \vec{w}^{\,v}\left(B_{\varepsilon_0}^{\mathcal{M}}(\vec{u}) \cap B_{\varepsilon_0}^{\mathcal{M}}(\vec{v})\right)$$

defines a C^∞ diffeomorphism. $\square$

The goal of the present section is to construct, for C^1 lagrangians on some special Banach manifolds, a substitute to the gradient of such lagrangian in order to be able to deform each level set to a lower level set if there is no critical point in between. The strategy for constructing such **pseudo-gradient** will consist in pasting together "pieces" using **partitions of unity** of suitable regularity. To that aim we introduce the following notion.

Definition I.2. *A topological Hausdorff space is called* **paracompact** *if every open covering admits a locally finite*[1] *open refinement.* □

We have the following result

Theorem I.1. [**Stones 1948**] *Every metric space is paracompact.* □

There is a more restrictive "separation axiom" for topological spaces than being **Hausdorff** which is called **normal**.

Definition I.3. *A topological space is called* **normal** *if any pair of disjoint closed sets have disjoint open neighborhoods.* □

we have the following proposition

Proposition I.1. *Every* **Hausdorff paracompact** *space is* **normal**. □

We have the following important lemma (which looks obvious but requires a proof in infinite dimension)

Lemma I.1. *Let $\mathcal{M}$ be a* **normal Banach Manifold** *and let (U, φ) be a chart on $\mathcal{M}$, i.e. $U \subset \mathcal{M}$ is an open subset of $\mathcal{M}$ and φ is an homeomorphism from U into an open set $\varphi(U) \subset E$ of a Banach Space $(E, \|\cdot\|_E)$. For any $x_0 \in U$ and for r small enough*

$$B_\varphi(x_0, r) := \{y \in U \ ; \ \|\varphi(x_0) - \varphi(y)\|_E \le r\} = \varphi^{-1}(\overline{B_r^E(x_0)})$$

is closed in $\mathcal{M}$ and it's interior is given by

$$B_\varphi^{int}(x_0, r) := \{y \in U \ ; \ \|\varphi(x_0) - \varphi(y)\|_E < r\} = \varphi^{-1}(B_r^E(x_0))$$

□

Proof of lemma I.1. Since the Banach manifold is assumed to be normal there exists two disjoint open sets V_1 and V_2 such that $\mathcal{M} \setminus U \subset V_1$ and $x_0 \in V_2$. Since φ is an homeomorphism, the preimage by φ^{-1} of V_2 is open

[1] *locally finite* means that any point posses a neighborhood which intersects only finitely many open sets of the subcovering

in the Banach space E. Choose now radii $r > 0$ small enough such that $\overline{B_r^E(x_0)} \subset \varphi(V_2)$, hence for such a r we have that

$$\mathcal{M} \setminus B_\varphi(x_0, r) = V_1 \cup \phi^{-1}\left(E \setminus \overline{B_r^E(x_0)}\right)$$

So $\mathcal{M} \setminus B_\varphi(x_0, r)$ is the union of two open sets. It is then open and $B_\varphi(x_0, r)$ is closed in $\mathcal{M}$. $\square$

Remark I.1. *As counter intuitive as it could be at a first glance, there are counter-examples of the closure of* $\varphi^{-1}(\overline{B_r^E(x)})$ *when* r *is not assumed to be small enough and even with* $\overline{B_r^E(x)} \subset \varphi(U)$ *! (see for instance [3]).*

We shall need the following lemma

Lemma I.2. *Let* $\mathcal{M}$ *be a* **normal Banach Manifold** *and let* (U, φ) *be a chart on* $\mathcal{M}$*, i.e.* $U \subset \mathcal{M}$ *is an open subset of* $\mathcal{M}$ *and* φ *is an homeomorphism from* U *into an open set* $\varphi(U) \subset E$ *of a Banach Space* $(E, \|\cdot\|_E)$*. For any* $x_0 \in U$ *and for* r *small enough such that* $\varphi^{-1}(\overline{B_r^E(x_0)})$ *is closed and included in* U *according to lemma I.1 then the function defined by*

$$\begin{cases} \forall\ x \in U \qquad g(x) := \inf\left\{\|\varphi(x) - \varphi(y)\|_E \quad ; \quad y \in U \setminus \varphi^{-1}(B_r^E(x_0))\right\} \\ \\ \forall\ x \in \mathcal{M} \setminus U \qquad g(x) = 0 \end{cases}$$

is locally Lipschitz on $\mathcal{M}$ *and strictly positive exactly on* $\varphi^{-1}(B_r^E(x_0))$. $\square$

Proof of lemma I.2. First of all we prove that g is globally lipschitz on U. Let $x, y \in U$ and let $\varepsilon > 0$. Choose $z \in U \setminus \varphi^{-1}(B_r^E(x_0))$ such that

$$\|\varphi(x) - \varphi(z)\|_E < g(x) + \varepsilon$$

We have by definition

$$\|\varphi(y) - \varphi(z)\|_E \geq g(y)$$

Combining the two previous inequalities give

$$\begin{aligned} g(y) - g(x) &\leq \|\varphi(y) - \varphi(z)\|_E - \|\varphi(y) - \varphi(z)\|_E + \varepsilon \\ &\leq \|\varphi(y) - \varphi(x)\|_E + \varepsilon \end{aligned}$$

exchanging the role of x and y gives the lipschitzianity of g on U. Take now $y \notin U$ since $\mathcal{M} \setminus U$ and $\varphi^{-1}(\overline{B_r^E(x_0)})$ are closed and disjoint, since $\varphi^{-1}(\overline{B_r^E(x_0)}) \subset U$, the normality of $\mathcal{M}$ gives the existence of two disjoint open neighborhoods containing respectively $\mathcal{M} \setminus U$ and $\varphi^{-1}(\overline{B_r^E(x_0)})$.

Hence there exists an open neighborhood of y which does not intersect $\varphi^{-1}(\overline{B_r^E(x_0)})$ and on which g is identically zero. This implies the local lipschitzianity of g. $\square$

One of the reasons why we care about paracompactness in our context comes from the following property.

Proposition I.2. *Let $(\mathcal{O}_\alpha)_{\alpha\in A}$ be an arbitrary covering of a C^1 paracompact Banach manifold $\mathcal{M}$. Then there exists a locally lipschitz partition of unity subordinated to $(\mathcal{O}_\alpha)_{\alpha\in A}$, i.e. there exists $(\phi_\alpha)_{\alpha\in A}$ where ϕ_α is locally lipschitz in $\mathcal{M}$ and such that*

i)

$$Supp(\phi_\alpha) \subset \mathcal{O}_\alpha$$

ii)

$$\phi_\alpha \geq 0$$

iii)

$$\sum_{\alpha\in A} \phi_\alpha \equiv 1$$

where the sum is locally finite.

$\square$

Proof of proposition I.2. To each point x in $\mathcal{O}_\alpha$ we assign an open neighborhood of the form $\varphi_i^{-1}(B_r^{E_i}(\varphi(x)))$ included in $\mathcal{O}_\alpha$ for r small enough given by lemma I.1. From the total union of all the families

$$(\varphi_i^{-1}(B_r^{E_i}(\varphi(x))))_{x\in\mathcal{O}_\alpha}$$

where $\alpha \in A$ we extract a locally finite sub covering that we denote $(\varphi_i^{-1}(B_r^{E_i}(\varphi(x_i))))_{i\in I_\alpha}$ and $\alpha \in A$ (we can have possibly $I_\alpha = \emptyset$). To each open set $\varphi_i^{-1}(B_r^{E_i}(\varphi(x_i)))$ we assign the function g_α^i given by lemma I.2 which happens to be strictly positive on $\varphi_i^{-1}(B_r^{E_i}(\varphi(x_i)))$ and zero outside. The functions g_α^i are locally lipschitz and since the family $(\varphi_i^{-1}(B_r^{E_i}(\varphi(x_i))))_{i\in I_\alpha}$ is locally finite, $\sum_\alpha \sum_{i\in I_\alpha} g_\alpha^i$ is locally lipschitz too. we take

$$\phi_\alpha := \frac{\sum_{i\in I_\alpha} g_\alpha^i}{\sum_{\alpha\in A}\sum_{i\in I_\alpha} g_\alpha^i}$$

with the convention that $\phi_\alpha \equiv 0$ on $\mathcal{M}$ if $I_\alpha = \emptyset$. The family $(\phi_\alpha)_{\alpha\in A}$ solves i), ii) and iii) and proposition I.2. $\square$

We introduce more structures in order to be able to perform deformations in Banach Manifolds.

Definition I.4. *A Banach manifold $\mathcal{V}$ is called C^p-* **Banach Space Bundle** *over another Banach manifold $\mathcal{M}$ if there exists a Banach Space E, a submersion π from $\mathcal{V}$ into $\mathcal{M}$, a covering $(U_i)_{i\in I}$ of $\mathcal{M}$ and a family of homeomorphism from $\pi^{-1}U_i$ into $U_i \times E$ such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\tau_i} & U_i \times E \\ & \pi\searrow & \downarrow \rho \\ & & U_i \end{array}$$

where ρ is the canonical projection from $U_i \times E$ onto U_i. The restriction of τ_i on each fiber $\mathcal{V}_x := \pi^{-1}(\{x\})$ for $x \in U_i$ realizes a continuous isomorphism onto E_i. Moreover the map

$$x \in U_i \cap U_j \ \longrightarrow\ \tau_i \circ \tau_j^{-1}\Big|_{\pi^{-1}(x)} \in \mathcal{L}(E, E)$$

is C^p. □

Definition I.5. *Let $\mathcal{M}$ be a normal Banach manifold and let $\mathcal{V}$ be a Banach Space Bundle over $\mathcal{M}$. A* **Finsler structure** *on $\mathcal{V}$ is a continuous function*

$$\|\cdot\| \ :\ \mathcal{V} \longrightarrow \mathbb{R}$$

such that for any $x \in \mathcal{M}$

$$\|\cdot\|_x := \|\cdot\||_{\pi^{-1}(\{x\})} \qquad \text{is a norm on } \mathcal{V}_x \quad .$$

Moreover for any local trivialization τ_i over U_i and for any $x_0 \in U_i$ we define on $\mathcal{V}_x$ the following norm

$$\forall\ \vec{w} \in \pi^{-1}(\{x\}) \qquad \|\vec{w}\|_{x_0} := \|\tau_i^{-1}\left(x_0, \rho(\tau_i(\vec{w}))\right)\|_{x_0}$$

and there exists $C_{x_0} > 1$ such that

$$\forall\ x \in U_i \qquad C_{x_0}^{-1}\ \|\cdot\|_x \leq \|\cdot\|_{x_0} \leq C_{x_0}\ \|\cdot\|_x \quad .$$

□

Definition I.6. *Let $\mathcal{M}$ be a normal C^p Banach manifold. $T\mathcal{M}$ equipped with a Finsler structure is called a* **Finsler Manifold**. □

Remark I.2. *A Finsler structure on $T\mathcal{M}$ defines in a canonical way a dual Finsler structure on $T^*\mathcal{M}$.* □

Example. Let Σ^2 be a closed oriented $2-$dimensional manifold and N^n be a closed sub-manifold of $\mathbb{R}^m$. For $q > 2$ we define

$$\mathcal{M} := W^{2,q}_{imm}(\Sigma^2, N^n) := \left\{\vec{\Phi} \in W^{2,q}(\Sigma^2, N^n) \ ; \ \text{rank}\,(d\Phi_x) = 2 \quad \forall x \in \Sigma^2\right\}$$

The set $W^{2,q}_{imm}(\Sigma^2, N^n)$ as an open subset of the normal Banach Manifold $W^{2,q}(\Sigma^2, N^n)$ inherits a Banach Manifold structure. The tangent space to $\mathcal{M}$ at a point $\vec{\Phi}$ is the space $\Gamma_{W^{2,q}}(\vec{\Phi}^{\,-1}TN^n)$ of $W^{2,q}-$sections of the bundle $\vec{\Phi}^{\,-1}TN^n$, i.e.

$$T_{\vec{\Phi}}\mathcal{M} = \left\{\vec{w} \in W^{2,q}(\Sigma^2, \mathbb{R}^m) \ ; \ \vec{w}(x) \in T_{\vec{\Phi}(x)}N^n \quad \forall\, x \in \Sigma^2\right\} \quad .$$

We equip $T_{\vec{\Phi}}\mathcal{M}$ with the following norm

$$\|\vec{v}\|_{\vec{\Phi}} := \left[\int_\Sigma \left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}} + |\nabla\vec{v}|^2_{g_{\vec{\Phi}}} + |\vec{v}|^2\right]^{q/2} \ dvol_{g_{\vec{\Phi}}}\right]^{1/q} + \|\, |\nabla\vec{v}|_{g_{\vec{\Phi}}}\, \|_{L^\infty(\Sigma)}$$

where we keep denoting, for any $j \in \mathbb{N}$, ∇ to be the connection on $(T^*\Sigma)^{\otimes^j} \otimes \vec{\Phi}^{-1}TN$ over Σ defined by $\nabla := \nabla^{g_{\vec{\Phi}}} \otimes \vec{\Phi}^*\nabla^h$ and $\nabla^{g_{\vec{\Phi}}}$ is the Levi Civita connection on $(\Sigma, g_{\vec{\Phi}})$ and ∇^h is the Levi-Civita connection on N^n. We check for instance that $\nabla^2\vec{v}$ defines a C^0 section of $(T^*\Sigma)^2 \otimes \vec{\Phi}^{-1}TN$.

Observe that, using Sobolev embedding and in particular due to the fact $W^{2,q}(\Sigma, \mathbb{R}^m) \hookrightarrow C^1(\Sigma, \mathbb{R}^m)$ for $q > 2$, the norm $\|\cdot\|_{\vec{\Phi}}$ as a function on the Banach tangent bundle $T\mathcal{M}$ is obviously continuous.

Proposition I.3. *The norms $\|\cdot\|_{\vec{\Phi}}$ defines a C^2-Finsler structure on the space $\mathcal{M}$.* □

Proof of proposition I.3. We introduce the following trivialization of the Banach bundle. For any $\vec{\Phi} \in \mathcal{M}$ we denote $P_{\vec{\Phi}(x)}$ the orthonormal projection in $\mathbb{R}^m$ onto the $n-$dimensional vector subspace of $\mathbb{R}^m$ given by $T_{\vec{\Phi}(x)}N^n$ and for any $\vec{\xi}$ in the ball $B^{\mathcal{M}}_{\varepsilon_1}(\vec{\Phi})$ for some $\varepsilon_1 > 0$ and any $\vec{v} \in T_{\vec{\xi}}\mathcal{M} = \Gamma_{W^{2,q}}(\vec{\xi}^{-1}TN)$ we assign the map $\vec{w}(x) := P_{\vec{\Phi}(x)}\vec{v}(x)$. It is straightforward to check that for $\varepsilon_1 > 0$ chosen small enough the map which to $\vec{v}$ assigns $\vec{w}$ is an isomorphism from $T_{\vec{\xi}}\mathcal{M}$ into $T_{\vec{\Phi}}\mathcal{M}$ and that there exists $k_{\vec{\Phi}} > 1$ such that $\forall \vec{v} \in TB^{\mathcal{M}}_{\varepsilon_1}(\vec{\Phi})$

$$k_{\vec{\Phi}}^{-1}\,\|\vec{v}\|_{\vec{\xi}} \le \|\vec{w}\|_{\vec{\Phi}} \le k_{\vec{\Phi}}\,\|\vec{v}\|_{\vec{\xi}}$$

This concludes the proof of proposition I.3. □

Theorem I.2. [Palais 1970] *Let $(\mathcal{M}, \|\cdot\|)$ be a Finsler Manifold. Define on $\mathcal{M}\times\mathcal{M}$*

$$d(p,q) := \inf_{\omega\in\Omega_{p,q}} \int_0^1 \left\|\frac{d\omega}{dt}\right\|_{\omega(t)}\ dt$$

where

$$\Omega_{p,q} := \left\{\omega \in C^1([0,1],\mathcal{M})\ ;\ \omega(0) = p \quad \omega(1) = q\right\} \quad .$$

Then d defines a distance on $\mathcal{M}$ and $(\mathcal{M}, d)$ defines the same topology as the one of the Banach Manifold. d is called **Palais distance** *of the Finsler manifold $(\mathcal{M}, \|\cdot\|)$.* □

Contrary to the first appearance the non degeneracy of d is not straightforward and requires a proof (see [3]). This last result combined with theorem I.1 gives the following corollary.

Corollary I.1. *Let $(\mathcal{M}, \|\cdot\|)$ be a Finsler Manifold then $\mathcal{M}$ is paracompact.* □

The following result is going to play a central role in this course

Proposition I.4. *Let $\mathcal{M}$ be the space*

$$W^{2,q}_{imm}(\Sigma^2, N^n) := \left\{\vec{\Phi} \in W^{2,q}(\Sigma^2, N^n)\ ;\ rank(d\Phi_x) = 2 \quad \forall x \in \Sigma^2\right\}$$

where Σ^2 is a closed oriented surface and N^n a closed sub-manifold of $\mathbb{R}^m$. The Finsler Manifold given by the structure

$$\|\vec{v}\|_{\vec{\Phi}} := \left[\int_\Sigma \left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}} + |\nabla\vec{v}|^2_{g_{\vec{\Phi}}} + |\vec{v}|^2\right]^{q/2}\ dvol_{g_{\vec{\Phi}}}\right]^{1/q} + \|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)}$$

is complete for the Palais distance. □

We have also.

Proposition I.5. *For N^n a closed sub-manifold of $\mathbb{R}^m$ and $p > 1$ we define on*

$$\mathcal{M} := W^{2,p}_{imm}(S^1, N^n) := \left\{\vec{\gamma} \in W^{2,p}(S^1, N^n)\ ;\ rank(d\gamma_x) = 1 \quad \forall x \in S^1\right\}$$

the following Finsler structure

$$\|\vec{v}\|_{\vec{\gamma}} := \left[\int_{S^1} \left[|\nabla^2\vec{v}|^2_{g_{\vec{\gamma}}} + |\nabla\vec{v}|^2_{g_{\vec{\gamma}}} + |\vec{v}|^2\right]^{p/2}\ dvol_{g_{\vec{\gamma}}}\right]^{1/p}$$

Then $(\mathcal{M}, \|\cdot\|)$ is complete for the Palais distance . □

We shall present only the proof of proposition I.4. The proof of proposition th-complete-S1 is very similar and can be found in [2].

Proof of proposition I.4. For any $\vec{\Phi} \in \mathcal{M}$ and $\vec{v} \in T_{\vec{\Phi}}\mathcal{M}$ we introduce the tensor in $(T^*\Sigma)^{\otimes^2}$ given in coordinates by

$$\begin{aligned}\nabla\vec{v}\,\dot{\otimes}\,d\vec{\Phi} + d\vec{\Phi}\,\dot{\otimes}\,\nabla\vec{v} &= \sum_{i,j=1}^{2}\left[\nabla_{\partial_{x_i}}\vec{v}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi}\cdot\nabla_{\partial_{x_j}}\vec{v}\right]\ dx_i\otimes dx_j\\ &= \sum_{i,j=1}^{2}\left[\nabla^h_{\partial_{x_i}\vec{\Phi}}\vec{v}\cdot\partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi}\cdot\nabla^h_{\partial_{x_j}\vec{\Phi}}\vec{v}\right]\ dx_i\otimes dx_j\end{aligned}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^m$. Observe that we have

$$\left|\nabla\vec{v}\,\dot{\otimes}\,d\vec{\Phi} + d\vec{\Phi}\,\dot{\otimes}\,\nabla\vec{v}\right|_{g_{\vec{\Phi}}} \le 2\ |\nabla\vec{v}|_{g_{\vec{\Phi}}}$$

Hence, taking a C^1 path $\vec{\Phi}_s$ in $\mathcal{M}$ one has for $\vec{v} := \partial_s\vec{\Phi}$

$$\begin{aligned}\||d\vec{v}\dot{\otimes}d\vec{\Phi} + d\vec{\Phi}\dot{\otimes}d\vec{v}|^2_{g_{\vec{\Phi}}}\|_{L^\infty(\Sigma)} &= \left\|\sum_{i,j,k,l=1}^{2} g^{ij}_{\vec{\Phi}}\ g^{kl}_{\vec{\Phi}}\ \partial_s(g_{\vec{\Phi}})_{ik}\ \partial_s(g_{\vec{\Phi}})_{jl}\right\|_{L^\infty(\Sigma)}\\ = \left\|\ |\partial_s(g_{ij}dx_i\otimes dx_j)|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)} &= \left\|\ |\partial_s g_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)}\end{aligned} \tag{I.2}$$

Hence

$$\int_0^1 \left\|\ |\partial_s g_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right\|_{L^\infty(\Sigma)}\ ds \le 2\ \int_0^1 \|\partial_s\vec{\Phi}\|_{\vec{\Phi}_s}\, ds \tag{I.3}$$

We now use the following lemma

Lemma I.3. *Let M_s be a C^1 path into the space of positive n by n symmetric matrix then the following inequality holds*

$$Tr\,(M^{-2}(\partial_s M)^2) \ge \|\partial_s \log M\|^2 = Tr\,((\partial_s\log M)^2)$$

Proof of lemma I.3. We write $M = \exp A$ and we observe that

$$\mathrm{Tr}\,(\exp(-2A)(\partial_s\exp A)^2) = \mathrm{Tr}\,(\partial_s A)^2$$

Then the lemma follows. □

Combining the previous lemma with (I.2) and (I.3) we obtain in a given chart

$$\int_0^1 \|\partial_s\log(g_{ij})\|\ ds \le \int_0^1 \sqrt{\mathrm{Tr}\,((\partial_s\log g_{ij})^2)}\,ds \le 2\ \int_0^1 \|\partial_s\vec{\Phi}\|_{\vec{\Phi}_s}\,ds \tag{I.4}$$

This implies that in the given chart the log of the matrix $(g_{ij}(s))$ is uniformly bounded for $s\in[0,1]$ and hence $\vec{\Phi}_1$ is an immersion. It remains to show that it has a controlled $W^{2,q}$ norm. We introduce $p = q/2$ and denote

$$\text{Hess}_p(\vec{\Phi}) := \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \; dvol_{g_{\vec{\Phi}}}$$

and we compute

$$\begin{aligned}\frac{d}{ds}(\text{Hess}_p(\vec{\Phi})) = p \int_\Sigma \partial_s |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} \, [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1} \; dvol_{g_{\vec{\Phi}}} \\ + \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \; \partial_s(dvol_{g_{\vec{\Phi}}})\end{aligned} \tag{I.5}$$

Classical computations give

$$\partial_s(dvol_{g_{\vec{\Phi}}}) = \left\langle \nabla \partial_s \vec{\Phi}, d\vec{\Phi} \right\rangle_{g_{\vec{\Phi}}} \; dvol_{g_{\vec{\Phi}}}$$

So we have

$$\begin{aligned}\left| \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \; \partial_s(dvol_{g_{\vec{\Phi}}}) \right| \le \| \, |\nabla \partial_s \vec{\Phi}|_{g_{\vec{\Phi}}} \, \|_{L^\infty(\Sigma)} \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \; dvol_{g_{\vec{\Phi}}} \\ \le \|\partial_s \vec{\Phi}\|_{\vec{\Phi}} \; \int_\Sigma [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p \; dvol_{g_{\vec{\Phi}}}\end{aligned} \tag{I.6}$$

In local charts we have

$$|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} = \sum_{i,j,k,l=1}^2 g^{ij}_{\vec{\Phi}} \, g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k} \vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l} \vec{\Phi} \right\rangle_h$$

Thus in bounding $\int_\Sigma \partial_s |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}} \, [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1} \; dvol_{g_{\vec{\Phi}}}$ we first have to control terms of the form

$$\left| \int_\Sigma \sum_{i,j,k,l=1}^2 \partial_s g^{ij}_{\vec{\Phi}} \, g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k} \vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l} \vec{\Phi} \right\rangle_h \; [1 + |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1} \; dvol_{g_{\vec{\Phi}}} \right| \tag{I.7}$$

We write

$$\begin{aligned}&\sum_{i,j,k,l=1}^2 \partial_s g^{ij}_{\vec{\Phi}} \, g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k} \vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l} \vec{\Phi} \right\rangle_h \\ &= \sum_{i,j,k,l,t,r=1}^2 \partial_s g^{ij}_{\vec{\Phi}} \, g_{jt} \, g^{tr} g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k} \vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l} \vec{\Phi} \right\rangle_h \\ &= - \sum_{i,j,k,l,=1}^2 \left(\sum_{t,r=1}^2 \partial_s g_{jt} \, g^{tr} \right) \; g^{ij}_{\vec{\Phi}} \, g^{kl}_{\vec{\Phi}} \, \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k} \vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l} \vec{\Phi} \right\rangle_h\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\int_\Sigma \sum_{i,j,k,l=1}^2 \partial_s g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}}\ \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h\ [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\ dvol_{g_{\vec{\Phi}}}\right| \\
&\quad\leq\ \|\,|\partial_s g_{\vec{\Phi}}|_{g_{\vec{\Phi}}}\|_{L^\infty(\Sigma)}\ \int_\Sigma [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p\ dvol_{g_{\vec{\Phi}}} \\
&\quad\leq\ \|\partial_s\vec{\Phi}\|_{\vec{\Phi}_s}\ \int_\Sigma [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^p\ dvol_{g_{\vec{\Phi}}}
\end{aligned} \tag{I.8}$$

We have also

$$\begin{aligned}
&\partial_s\left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h \\
&\quad=\left\langle \nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right), \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h + \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right)\right\rangle_h
\end{aligned}$$

By definition we have

$$\nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right) = \nabla^h_{\partial_{x_i}\vec{\Phi}}\left(\nabla^h_{\partial_s\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right) + R^h(\partial_{x_i}\vec{\Phi},\partial_s\vec{\Phi})\partial_{x_k}\vec{\Phi}$$

where we have used the fact that $[\partial_s\vec{\Phi},\partial_{x_i}\vec{\Phi}] = \vec{\Phi}_*[\partial_s,\partial_{x_i}] = 0$. Using also that $[\partial_s\vec{\Phi},\partial_{x_k}\vec{\Phi}] = 0$, since ∇^h is torsion free, we have finally

$$\nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right) = \nabla^h_{\partial_{x_i}\vec{\Phi}}\left(\nabla^h_{\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi}\right) + R^h(\partial_{x_i}\vec{\Phi},\partial_s\vec{\Phi})\partial_{x_k}\vec{\Phi} \tag{I.9}$$

where R^h is the Riemann tensor associated to the Levi-Civita connection ∇^h. We have

$$\nabla^h_{\partial_{x_i}\vec{\Phi}}\left(\nabla^h_{\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi}\right) = (\nabla^h)^2_{\partial_{x_i}\vec{\Phi}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi} + \nabla^h_{\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi} \tag{I.10}$$

Hence

$$\begin{aligned}
\left\langle \nabla^h_{\partial_s\vec{\Phi}}\left(\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}\right), \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h &= \left\langle (\nabla^h)^2_{\partial_{x_i}\vec{\Phi}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h \\
+\left\langle \nabla^h_{\nabla^h_{\partial_{x_i}\vec{\Phi}}\partial_{x_k}\vec{\Phi}}\partial_s\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h &+ \left\langle R^h(\partial_{x_i}\vec{\Phi},\partial_s\vec{\Phi})\partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}}\partial_{x_l}\vec{\Phi}\right\rangle_h
\end{aligned} \tag{I.11}$$

Combining all the previous gives then

$$\begin{aligned}
&\left|\int_\Sigma \sum_{i,j,k,l=1}^{2} g^{ij}_{\vec{\Phi}}\, g^{kl}_{\vec{\Phi}}\, \partial_s \left\langle \nabla^h_{\partial_{x_i}\vec{\Phi}} \partial_{x_k}\vec{\Phi}, \nabla^h_{\partial_{x_j}\vec{\Phi}} \partial_{x_l}\vec{\Phi} \right\rangle_h \ dvol_{g_{\vec{\Phi}}}\right| \\
&\quad \le C \int_\Sigma \left| \left\langle \nabla^2 \partial_s\vec{\Phi}, \nabla d\vec{\Phi} \right\rangle_{g_{\vec{\Phi}}} \right| \ [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\ dvol_{g_{\vec{\Phi}}} \\
&\qquad + C \int_\Sigma |\nabla \partial_s\vec{\Phi}|_{g_{\vec{\Phi}}}\ |\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}\ [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\ dvol_{g_{\vec{\Phi}}} \\
&\qquad + C\,\|R^h\|_{L^\infty(N^n)} \int_\Sigma |\partial_s\vec{\Phi}|_h\ |\nabla d\vec{\Phi}|_{g_{\vec{\Phi}}}\ [1+|\nabla d\vec{\Phi}|^2_{g_{\vec{\Phi}}}]^{p-1}\ dvol_{g_{\vec{\Phi}}}
\end{aligned} \tag{I.12}$$

Combining all the above we finally obtain that

$$\left|\partial_s \mathrm{Hess}_p(\vec{\Phi})\right| \le C\, \|\partial_s\vec{\Phi}\|_{\vec{\Phi}}\ \left[\mathrm{Hess}_p(\vec{\Phi}) + \mathrm{Hess}_p(\vec{\Phi})^{1-1/2p}\right] \tag{I.13}$$

Combining (I.4) and (I.13) we deduce using Gromwall lemma that if we take a C^1 path from $[0,1)$ into $\mathcal{M}$ with finite length for the Palais distance d, the limiting map $\vec{\Phi}_1$ is still a $W^{2,q}-$immersion of Σ into N^n, which proves the completeness of $(\mathcal{M}, d)$. □

Definition I.7. *Let $\mathcal{M}$ be a C^2 Finsler Manifold and E be a C^1 function on $\mathcal{M}$. Denote*

$$\mathcal{M}^* := \{u \in \mathcal{M} \quad ; \quad DE_u \neq 0\} \quad .$$

A **pseudo-gradient** *is a Lipschitz continuous section $X\ :\ \mathcal{M}^* \to T\mathcal{M}^*$ such that*

i)

$$\forall\, u \in \mathcal{M}^* \qquad \|X(u)\|_u < 2\ \|DE_u\|_u$$

ii)

$$\forall\, u \in \mathcal{M}^* \quad \|DE_u\|^2_u < \langle X(u), DE_u\rangle_{T_u\mathcal{M}^*, T^*_u\mathcal{M}^*}$$

□

The following result is mostly using the existence of a Lipschitz partition of unity for any covering of a Finsler Manifold (combine proposition I.2 and corollary I.1).

Proposition I.6. *Every C^1 function on a Finsler Manifold admits a pseudo-gradient.* □

The following definition is central in Palais deformation theory.

Definition I.8. *Let E be a C^1 function on a Finsler manifold $(\mathcal{M}, \|\cdot\|)$ and $\beta \in E(\mathcal{M})$. On says that E fulfills the* **Palais-Smale condition** *at the level β if for any sequence u_n staisfying*

$$E(u_n) \longrightarrow \beta \quad \text{ and } \quad \|DE_{u_n}\|_{u_n} \longrightarrow 0 \quad ,$$

then there exists a subsequence $u_{n'}$ and $u_\infty \in \mathcal{M}$ such that

$$d(u_{n'}, u_\infty) \longrightarrow 0 \quad .$$

and hence $E(u_\infty) = \beta$ and $DE_{u_\infty} = 0$. □

Example. Let $\mathcal{M}$ be $W^{1,2}(S^1, N^n)$ for the Finsler structure given by

$$\forall\ \vec{w} \in \Gamma_{W^{1,2}}(\vec{u}^{\,-1}TN^n) \qquad \|\vec{w}\|_{\vec{u}} := \|\vec{w}\|_{W^{1,2}(S^1)}$$

Then the Dirichlet Energy satisfies the Palais Smale condition for every level set. □

Definition I.9. *A family of subsets $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ of a Banach manifold $\mathcal{M}$ is called* **admissible family** *if for every homeomorphism Ξ of $\mathcal{M}$ isotopic to the identity we have*

$$\forall\, A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}$$

□

Example 1. A closed 2 dimensional sub-manifold N^2 of $\mathbb{R}^m$ being given and $\alpha \in \pi_2(N^2) \neq 0$, considering the Banach Manifold $\mathcal{M} := W^{1,2}(S^1, N^2)$ we can take

$$\mathcal{A} := \left\{ \begin{array}{c} u \in C^0([0,1], W^{1,2}(S^1, N^2))\ ;\ u(0,\cdot) \text{ and } u(1,\cdot) \text{ are constant} \\ \text{and} \quad u(t,\theta)\ [0,1]\times S^1 \longrightarrow N^2 \quad \text{realizes } \alpha \end{array} \right\} \tag{I.14}$$

i.e. for $N^2 \simeq S^2$ $\mathcal{A}$ corresponds to a class of sweep-outs of the form Ω_{σ_0}. □

Example 2. Consider $\mathcal{M} := W^{2,q}_{imm}(S^2, \mathbb{R}^3)$ and take $c \in \pi_1(\mathrm{Imm}(S^2, \mathbb{R}^3)) = \mathbb{Z}_2 \times \mathbb{Z}$ then the following family is admissible

$$\mathcal{A} := \left\{ \vec{\Phi} \in C^0([0,1], W^{2,q}_{imm}(S^2, \mathbb{R}^3))\ ;\ \vec{\Phi}(0,\cdot) = \vec{\Phi}(1,\cdot) \quad \text{ and } [\vec{\Phi}] = c \right\}$$

□

We can now state the main theorem in this section.

Theorem I.3. [Palais 1970] *Let $(\mathcal{M}, \|\cdot\|)$ be a Banach manifold together with a $C^{1,1}-$Finsler structure. Assume $\mathcal{M}$ is complete for the induced Palais distance d and let $E \in C^1(\mathcal{M})$ satisfying the Palais-Smale condition $(PS)_\beta$ for the level set β. Let $\mathcal{A}$ be an admissible family in $\mathcal{P}(\mathcal{M})$ such that*

$$\inf_{A\in\mathcal{A}} \sup_{u\in A} E(u) = \beta$$

then there exists $u \in \mathcal{M}$ satisfying

$$\left\{\begin{array}{l} DE_u = 0 \\ \\ E(u) = \beta \end{array}\right. \tag{I.15}$$

□

Proof of theorem I.3. We argue by contradiction. Assuming there is no u satisfying (I.1) then Palais Smale condition $(PS)_\beta$ implies

$$\exists\, \delta_0 > 0\,, \exists\, \epsilon_0 > 0 \quad \beta-\varepsilon < E(u) < \beta+\varepsilon \quad \Longrightarrow \quad \|DE_u\|_u \geq \delta \quad . \tag{I.16}$$

Let $u \in \mathcal{M}^*$. Because of the Local lipschitz nature of a fixed pseudo-gradient given by proposition I.6 there exists a maximal time $t^u_{max} \in (0, +\infty]$ such that

$$\left\{\begin{array}{l} \displaystyle\frac{d\phi_t(u)}{dt} = -\ X(\phi_t(u))\ \eta(E(\phi_t(u))) \quad \text{in } [0, t^u_{max}) \\ \\ \phi_0(u) = u \end{array}\right.$$

where $1 \geq \eta(t) \geq 0$ is supported in $[\beta - \varepsilon_0, \beta + \varepsilon]$ and is equal to one on $[\beta - \varepsilon_0/2, \beta + \varepsilon_0/2]$.

We have for any $0 \leq t_1 < t_2 < t^u_{max}$ we have

$$\begin{aligned} d(\phi_{t_1}(u), \phi_{t_2}(u)) &\leq \int_{t_1}^{t_2} \left\|\frac{d\phi_t(u)}{dt}\right\|_{\phi_t(u)}\ dt \\ &\leq 2\int_{t_1}^{t_2} \eta(E(\phi_t(u)))\ \|DE_{\phi_t(u)}\|_{\phi_t(u)}\ dt \\ &\leq |t_2 - t_1|^{1/2}\ \left[\int_{t_1}^{t_2} \eta(E(\phi_t(u)))\ \|DE_{\phi_t(u)}\|^2_{\phi_t(u)}\ dt\right]^{1/2} \end{aligned}$$

and

$$\begin{aligned} &\int_{t_1}^{t_2} \eta(E(\phi_t(u)))\ \|DE_{\phi_t(u)}\|^2_{\phi_t(u)}\ dt \\ &\qquad \leq -\int_{t_1}^{t_2} \eta(E(\phi_t(u)))\ \left\langle X(\phi_t(u)), DE_{\phi_t(u)}\right\rangle\ dt \\ &\qquad \leq E(\phi_{t_1}(u)) - E(\phi_{t_2}(u)) \end{aligned}$$

Hence

$$d(\phi_{t_1}(u),\phi_{t_2}(u)) \leq\ 2\,|t_2-t_1|^{1/2}\ [E(\phi_{t_1}(u))-E(\phi_{t_2}(u))]^{1/2}$$

Hence, assuming $t^u_{max}<+\infty$, $\phi_t(u)$ realizes a Cauchy sequence as $t\rightarrow t^u_{max}$. Since $\mathcal{M}$ is complete, the only possibility for the extinction of the flow is that $\lim_{t\rightarrow t^u_{max}}\phi_t(u)$ belongs to $\mathcal{M}^\ast$. But the flow is constant in time outside $E^{-1}([\beta-\varepsilon_0,\beta+\varepsilon])$ hence $t^u_{max}=+\infty$.

Hence for any $t\in\mathbb{R}_+$ ϕ_t is an homeomorphism of $\mathcal{M}$ isotopic to the identity and, since $\mathcal{A}$ is admissible

$$\forall\ A\in\mathcal{A}\quad\forall\ t\in[0,+\infty)\qquad\qquad \phi_t(A)\in\mathcal{A}\quad .$$

Let u now such that $\beta\le E(u)\le\beta+\varepsilon_0/2$. For any $\tau>0$ such that $E(\phi_t(u))\ge\beta-\varepsilon_0/2$ we have (taking $\delta_0<1$)

$$-\,\tau\,\delta_0\le E(\phi_t(u))-E(u)=\int_0^\tau\frac{d\phi_t(u)}{dt}\ dt\le-\,2\,\tau\,\delta_0^2$$

Hence for any $\tau\,\delta_0\le\varepsilon_0/2$ we have[2]

$$E(\phi_\tau(u))\le E(u)-\ 2\,\tau\,\delta_0^2\quad .$$

In particular

$$E(\phi_{\varepsilon_0/2\delta_0}(u))\le E(u)-\ \delta_0\,\varepsilon_0$$

Choose $A\in\mathcal{A}$ such that

$$\sup_{u\in A}E(u)<\beta+\delta_0\,\varepsilon_0$$

Hence we have for $t_0=\varepsilon_0/2\delta_0$

$$\sup_{\phi_{t_0}(u)\in\phi_{t_0}(A)}\quad E(\phi_{t_0}(u))<\beta$$

which is a contradiction. $\Box$

Application. We take $\mathcal{M}:=W^{1,2}(S^1,N^2)$ where $N^2\simeq S^2$. Let any sweep-out $\vec{\sigma}_0$ of N^2 corresponding to a non zero element of $\pi_2(N^2)$. Then

$$W_{\vec{\sigma}_0}=\inf_{\vec{\sigma}\in\Omega_{\vec{\sigma}_0}\cap\Lambda}\quad\max_{t\in[0,1]}E(\vec{\sigma}(t,\cdot))$$

is achieved by a closed geodesic. This gives a new proof of Birkhoff existence result. $\Box$

[2]Observe that this kind of inequality is reminiscent to the condition v) of the definition of Birkhoff curve shortening process.

Now, what about surfaces ? The Dirichlet energy of maps into a submanifold of is <u>not satisfying the Palais Smale</u> anymore in 2 dimension. So Palais Deformation theory does not apply directly to the construction of minimal surfaces by working with the Dirichlet energy. We would also like to go beyond the Colding-Minicozzi framework which is restricted to spheres.

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