Minmax Methods in the Calculus of Variations of Curves and Surfaces

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I Lecture 3

A Viscosity Approach in the Calculus of Variations of Curves and Surfaces.

The heuristic of the method is simple. Consider the energy E to which one aims to apply a minmax procedure. If E does not satisfy the Palais-Smale condition on add to E a more coercive term multiplied by a small "viscosity" parameter σ in order for the obtained "smoothed" functional E^σ to satisfy Palais Smale. Apply Palais deformation theory of lecture 2 in order to obtain a minmax critical point of E^σ and make σ go to zero. As we will see the procedure, in contrast with the simplicity of it's main strategy, offers surprising difficulties we have to overcome.

I.1 An Attempt for constructing Geodesics using a Viscosity Approach

We consider again a closed sub-manifold N^n of $\mathbb{R}^m$. We consider the Banach manifold introduced in lecture 2 and given by

$$\mathcal{M} := W^{2,2}_{imm}(S^1, N^n)$$

This Banach manifold is equipped with a Finsler structure given for any $\vec{\gamma} \in \mathcal{M}$ and any $\vec{v} \in \Gamma_{W^{2,2}}(\vec{\gamma}^{-1}TN^n)$

$$\|\vec{v}\|_{\vec{\gamma}} := \left[\int_{S^1} \left[|\nabla^2 \vec{v}|^2_{g_{\vec{\gamma}}} + |\nabla \vec{v}|^2_{g_{\vec{\gamma}}} + |\vec{v}|^2\right] \, dvol_{g_{\vec{\gamma}}}\right]^{1/2} \tag{I.1}$$

We have seen that $(\mathcal{M}, \|\cdot\|)$ is complete for the induced Palais distance.

Introduce for any $\vec{\Phi} \in \mathcal{M}$

$$E^\sigma(\vec{\gamma}) := \int_{S^1} \left[1 + \sigma^2 \, |\vec{\kappa}_{\vec{\gamma}}|^2\right] \, dl_{\vec{\gamma}}$$

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where $\vec{\kappa}_{\vec{\gamma}}$ is the curvature of the immersion inside N^n given by

$$\vec{\kappa}_{\vec{\gamma}} = \nabla^h_{\dot{\vec{\gamma}}/|\dot{\vec{\gamma}}|}\left[\partial_{\dot{\vec{\gamma}}/|\dot{\vec{\gamma}}|}\vec{\gamma}\right]$$

We have the following proposition

Lemma I.1. *E^σ is C^1 on $\mathcal{M}$ and in constant speed parametrization we have*

$$\begin{array}{l}dE^\sigma\cdot\vec{v} = \displaystyle\int_{S^1}\left\langle\nabla^h\vec{w}, d\vec{\gamma}\right\rangle_{g_{\vec{\gamma}}}\ dl_{\vec{\gamma}} + \sigma^2\int_{S^1} 2 < \nabla^2\vec{v}, \nabla d\vec{\gamma} >_{g_{\vec{\gamma}}}\ dl_{\vec{\gamma}}\\[2mm] -\,3\,\sigma^2\displaystyle\int_{S^1} < \nabla\vec{v}, d\vec{\gamma} >_{g_{\vec{\gamma}}}\ |\vec{\kappa}_{\vec{\gamma}}|^2\ dl_{\vec{\gamma}} + \sigma^2\int_{S^1} 2 < R^h(\vec{v}, d\vec{\gamma})\,d\vec{\gamma}, \nabla^h d\vec{\gamma} >_{g_{\vec{\gamma}}}\ dl_{\vec{\gamma}}\end{array} \tag{I.2}$$

The proof can be proved in [1]. As a matter of illustration and for later purposes we compute the derivative of E^σ in the case $N^n = S^n$.

$$\begin{aligned}\vec{\kappa}_{\vec{\gamma}} &= P^T\left[\frac{1}{|\partial_\theta\vec{\gamma}|}\frac{\partial}{\partial\theta}\left[\frac{\partial_\theta\vec{\gamma}}{|\partial_\theta\vec{\gamma}|}\right]\right] = P^T\left[\frac{\partial^2_{\theta^2}\vec{\gamma}}{|\partial_\theta\vec{\gamma}|^2}\right] - P_T\left[\frac{\partial_\theta|\partial_\theta\vec{\gamma}|}{|\partial_\theta\vec{\gamma}|^3}\ \partial_\theta\vec{\gamma}\right]\\ &= \frac{\partial^2_{\theta^2}\vec{\gamma}}{|\partial_\theta\vec{\gamma}|^2} + \vec{\gamma}\end{aligned}$$

Take $\vec{\gamma}_s$ and denote $\partial_s\vec{\gamma}|_{s=0} = \vec{v}$. We have

$$\begin{aligned}2\,\vec{\kappa}_{\vec{\gamma}}\cdot\partial_s\vec{\kappa}_{\vec{\gamma}} &= 2\,\vec{\kappa}_{\vec{\gamma}}\cdot\left[\frac{\partial^2_{\theta^2}\vec{v}}{|\partial_\theta\vec{\gamma}|^2} + \vec{v}\right] - 4\,\vec{\kappa}_{\vec{\gamma}}\cdot\partial^2_{\theta^2}\vec{\gamma}\ \partial_\theta\vec{v}\cdot\partial_\theta\vec{\gamma}\\ &= 2\,\vec{\kappa}_{\vec{\gamma}}\cdot\left[\frac{\partial^2_{\theta^2}\vec{v}}{|\partial_\theta\vec{\gamma}|^2} + \vec{v}\right] - 4\,|\vec{\kappa}_{\vec{\gamma}}|^2 < d\vec{v}, d\vec{\gamma} >_{g_{\vec{\gamma}}}\end{aligned}$$

We have seen

$$\partial_s dl_{\vec{\gamma}_s}|_{s=0} = < d\vec{v}, d\vec{\gamma} >_{g_{\vec{\gamma}}}\ dl_{\vec{\gamma}} \quad .$$

Combining all the previous gives in constant speed parametrization $|\partial_\theta\vec{\gamma}| \equiv L/2\pi$

$$\frac{L}{2\pi}\ dE^\sigma(\vec{\gamma})\cdot\vec{v} = \int_{S^1}\left[-\ddot{\vec{\gamma}} + \sigma^2\left[2\,\ddot{\vec{\kappa}} + \frac{L^2}{2\pi}\,\vec{\kappa} + 3\,\partial_\theta\left(|\vec{\kappa}|^2\ \dot{\vec{\gamma}}\right)\right]\right]\cdot\vec{v}\ d\theta \tag{I.3}$$

We have the following **Palais Smale Property "modulo gauge changes"**.

Proposition I.1. *Let $\sigma > 0$ and $\vec{\gamma}_j$ be a sequence in $\mathcal{M} := W^{2,2}_{imm}(S^1, N^n)$, where the space of $W^{2,2}$ immersions into N^n is equipped with the Finsler structure given by (I.1) such that*

$$E^\sigma(\vec{\gamma}_j) \longrightarrow \beta(\sigma) \quad \textit{and} \quad DE^\sigma_{u_j} \longrightarrow 0 \quad ,$$

then there exists a subsequence $u_{j'}$ and a sequence $\psi_{j'}$ of $W^{2,2}-$diffeomorphisms of S^1 such that

$$\vec{\gamma}_{j'}\circ\psi_{j'}\longrightarrow\vec{\sigma}_\infty \quad \textit{for the Palais distance.}$$

If one assume furthermore that $u_{j'}$ stays within a ball of finite radius in $\mathcal{M}$ for the Palais distance, then one can take $\psi_{j'}$ to be the identity. □

Proof of proposition I.1. The proof of this result can be found in [1]. As a matter of illustration we present it in the sphere case when $N^n=S^n$. We take ψ_j such that the parametrization is of constant speed : $|\partial_\theta(\vec{\gamma}_j\circ\psi_j)|\equiv L_j/2\pi$. We omit to write explicitly the composition with ψ_j and we assume that $\vec{\gamma}_j$ itself is in constant speed parametrization. As we saw, the geodesic curvature in S^n is given by

$$\vec{\kappa}_{\vec{\gamma}_j}=\frac{\partial^2_{\theta^2}\vec{\gamma}_j}{|\partial_\theta\vec{\gamma}_j|^2}+\vec{\gamma}_j=\vec{k}_j+\vec{\gamma}_j$$

where $\vec{k}_j$ is the vector curvature of the same immersion but viewed as an immersion into the ambient space $\mathbb{R}^m$. The Fenchel theorem gives

$$2\pi\le\int_{S^1}|\vec{k}_j|^2\ dl_{\vec{\gamma}_j}\le L_j^{1/2}\ \left[\int_{S^1}|\vec{k}_j|^2\ dl_{\vec{\gamma}_j}\right]^{1/2}\le L_j^{1/2}\ \left[\int_{S^1}\left[|\vec{\kappa}_{\vec{\gamma}_j}|^2+1\right]\ dl_{\vec{\gamma}_j}\right]^{1/2}$$

Hence the length L_j is bounded from above and from below by a positive number. Hence, in constant speed parametrization the assumption that $E^\sigma(\vec{\gamma}_j)$ is uniformly bounded reads

$$\limsup_{j\to+\infty}\int_{S^1}\left|\partial^2_{\theta^2}\vec{\gamma}_j+\frac{L_j^2}{4\pi^2}\,\vec{\gamma}_j\right|^2\ d\theta<+\infty$$

And this implies that there exists a subsequence $\vec{\gamma}_{j'}$ such that

$$\vec{\gamma}_{j'}\ \rightharpoonup\ \vec{\gamma}_\infty\quad\text{weakly in } W^{2,2}(S^1)\quad .$$

Observe that in this constant speed parametrization the assumption we have for any $\vec{v}\in T_{\vec{\gamma}_j}\mathcal{M}$

$$\|\vec{v}\|_{\vec{\gamma}_j}\le 1\iff$$

$$\int_{S^1}|\nabla^h_{\partial\vec{\gamma}_j}(\nabla^h_{\partial\vec{\gamma}_j}\vec{v})|^2|\partial_\theta\vec{\gamma}_j|^{-3}\ d\theta+|\nabla^h_{\partial_\theta\vec{\gamma}_j}\vec{v}|^2|\partial_\theta\vec{\gamma}_j|^{-1}\ d\theta+|\vec{v}|^2\ d\theta\le 1$$

We have

$$\nabla^h_{\partial_\theta\vec{\gamma}_j}\vec{v}=P^T(\vec{\gamma}_j)(\partial_\theta\vec{v})=\partial_\theta\vec{v}-\vec{\gamma}_j\cdot\partial_\theta\vec{v}\ \vec{\gamma}_j=\partial_\theta\vec{v}+\partial_\theta\vec{\gamma}_j\cdot\vec{v}\ \vec{\gamma}_j$$

and

$$\nabla^h_{\partial_\theta\vec{\gamma}_j}\left(\nabla^h_{\partial_\theta\vec{\gamma}_j}\vec{v}\right)=P^T(\vec{\gamma}_j)\left(\partial_\theta\left(\nabla^h_{\partial_\theta\vec{\gamma}_j}\vec{v}\right)\right)$$

$$=\partial^2_{\theta^2}\vec{v}+2\ \partial_\theta\vec{\gamma}_j\cdot\partial_\theta\vec{v}\ \vec{\gamma}_j+\partial^2_{\theta^2}\vec{\gamma}\cdot\vec{v}\ \vec{\gamma}_j+\partial_\theta\vec{\gamma}_j\cdot\vec{v}\ \partial_\theta\vec{\gamma}_j$$

Hence, using in particular the embedding $W^{1,2}\hookrightarrow C^0$ it is not difficult to see that there exists a constant $C>0$ independent of j such that

$$\|\vec{v}\|_{W^{2,2}(S^1,\mathbb{R}^m)}\le C\quad\Longrightarrow\quad\|\vec{v}\|_{\vec{\gamma}_j}\le 1$$

Combining this fact with the assumptions together with (I.3) gives

$$\sup_{\left\{\|\vec{v}\|_{W^{2,2}}\le 1\ ;\ \vec{v}\cdot\vec{\gamma}_j\equiv 0\right\}}\int_{S^1}\left[-\ddot{\vec{\gamma}}_j+\sigma^2\left[2\,\ddot{\vec{\kappa}}_j+\frac{L_j^2}{2\pi}\,\vec{\kappa}_j+3\,\partial_\theta\left(|\vec{\kappa}_j|^2\ \dot{\vec{\gamma}}_j\right)\right]\right]\cdot\vec{v}\ d\theta$$

$$\downarrow$$

$$0$$

Let $\vec{w}\in W^{2,2}(S^1,\mathbb{R}^m)$ there exists $C_1>0$ such that $\|\vec{w}\|_{W^{2,2}}\le C_1\Rightarrow\|\vec{v}\|_{W^{2,2}}\le 1$ where $\vec{v}=\vec{w}-\vec{\gamma}_j\cdot\vec{w}\ \vec{\gamma}_j$. Observe that we have successively

$$-\ddot{\vec{\gamma}}_j\cdot\vec{\gamma}_j=|\dot{\vec{\gamma}}_j|^2\quad,\quad\vec{\kappa}_j\cdot\vec{\gamma}_j=0\quad,\quad\partial_\theta\left(|\vec{\kappa}_j|^2\ \dot{\vec{\gamma}}_j\right)\cdot\vec{\gamma}_j=-\,|\vec{\kappa}_j|^2\,|\dot{\vec{\gamma}}_j|^2$$

and

$$\ddot{\vec{\kappa}}_j\cdot\vec{\gamma}_j=|\vec{\kappa}_j|^2\ |\dot{\vec{\gamma}}_j|^2$$

Combining all the previous implies that

$$-\ddot{\vec{\gamma}}_j-|\dot{\vec{\gamma}}_j|^2\,\vec{\gamma}_j+\sigma^2\left[2\,\ddot{\vec{\kappa}}_j+\frac{L_j^2}{2\pi}\,\vec{\kappa}_j+3\,\partial_\theta\left(|\vec{\kappa}_j|^2\ \dot{\vec{\gamma}}_j\right)+\ |\vec{\kappa}_j|^2\,|\dot{\vec{\gamma}}_j|^2\,\vec{\gamma}_j\right]$$

$$\downarrow\qquad\text{strongly in } W^{-2,2}(S^1)$$

$$0$$

Wedging by $\vec{\gamma}_j$ gives that

$$\partial_\theta\left((1-2\sigma^2)\,\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j+2\,\sigma^2\ \vec{\gamma}_j\wedge\dot{\vec{\kappa}}_j-2\,\sigma^2\ \dot{\vec{\gamma}}_j\wedge\vec{\kappa}_j+3\,\sigma^2\,|\vec{\kappa}_j|^2\,\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j\right)\tag{I.4}$$

converges to zero in $W^{-2,2}$. Hence there exists a converging sequence of constant 2-vectors $\vec{C}_j$ such that

$$(1-2\sigma^2)\,\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j+2\,\sigma^2\ \vec{\gamma}_j\wedge\dot{\vec{\kappa}}_j-2\,\sigma^2\ \dot{\vec{\gamma}}_j\wedge\vec{\kappa}_j+3\,|\vec{\kappa}_j|^2\,\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j-\vec{C}_j\to 0$$

in $W^{-1,2}$. Since $L^1(S^1)$ embeds in a compact way into $W^{-1,2}(S^1)$ we deduce that $\vec{\gamma}_j\wedge\vec{\kappa}_j$ is strongly pre-compact in $L^2(S^1)$ and since $\vec{\kappa}_j=(\vec{\gamma}_j\wedge\vec{\kappa}_j)\llcorner\vec{\gamma}_j$

we have that $\vec{\kappa}_j$ and thus $\partial^2_{\theta^2}\vec{\gamma}_j$ is pre-compact in L^2. This gives the strong convergence of $\vec{\gamma}_j$ to $\vec{\gamma}_\infty$ in $W^{2,2}$ and this easily imply that

$$DE^\sigma(\vec{\gamma}_\infty) = 0 \quad .$$

This concludes the proof of the theorem. □

Let $\mathcal{A}$ be an admissible family of $W^{2,2}_{imm}(S^1, N^n)$ and denote

$$\beta(0) := \inf_{A\in\mathcal{A}} \sup_{\vec{\gamma}\in A} \int_{S^1} dl_{\vec{\gamma}}$$

Assume $\beta(0) > 0$.

Example. $\mathcal{A}$ is the sub-family of

$$C^0((0,1), W^{2,2}_{imm}(S^1, N^2) \cap C^0([0,1], W^{1,1}(S^1, N^2)$$

of maps $\vec{\gamma}(t,\cdot)$ which are constant at $t = 0$ and $t = 1$ and which realize a non trivial sweep-out of N^2 (i.e. a non zero class of $\pi_2(N^2)$). □

For any $A \in \mathcal{A}$ we define

$$A_0 := \left\{\vec{\gamma} \in A \ \text{ s.t. } \ \int_{S^1} dl_{\vec{\gamma}} \geq \beta(0)/2\right\} \quad .$$

and $\mathcal{A}_0 = \{A \in \mathcal{A} \ ; \ A_0 \neq \emptyset\}$. Let

$$\beta(\sigma) := \inf_{A\in\mathcal{A}_0} \sup_{\vec{\gamma}\in A_0} E^\sigma(\vec{\gamma}) \quad .$$

It is straightforward to prove that

$$\lim_{\sigma\to 0} \beta(\sigma) = \beta(0)$$

Because of all the previous we have the existence of $\vec{\gamma}^\sigma$ such that

$$E^\sigma(\vec{\gamma}^\sigma) = \beta(\sigma) \quad \text{and} \quad DE^\sigma(\vec{\gamma}^\sigma) = 0$$

In constant speed parametrization we have a sequence σ_j such that

$$\vec{\gamma}^{\sigma_j} \rightharpoonup \vec{\gamma}^{\sigma_\infty} \quad \text{weakly in} \quad (W^{1,\infty})^*$$

We are now facing the following difficulty.

Do we have $\beta_0 = L(\vec{\gamma}^{\sigma_\infty})$ and $\vec{\gamma}^{\sigma_\infty}$ is a geodesic ?

The answer to this question is a-priori negative. We have

Proposition I.2. *Let $N^n = S^2$, the unit sphere of the 3-dimensional euclidian Space. Under the previous notations, there exists $\sigma_j \to 0$ and $\vec{\gamma}_j \in W^{2,2}_{imm}(S^1, S^2)$ such that*

$$\limsup_{j\to+\infty} E^{\sigma_j}(\vec{\gamma}_j) < +\infty \quad \textit{and} \quad DE^{\sigma_j}_{\vec{\gamma}_j} = 0 \quad ,$$

moreover $\vec{\gamma}_j$ is in normal parametrization and

$$\vec{\gamma}_j \rightharpoonup \vec{\gamma}_\infty \quad \textit{weakly in} \quad (W^{1,\infty})^*$$

but

$$\dot{\vec{\gamma}}_j \underset{j\to\infty}{\not\longrightarrow} \dot{\vec{\gamma}}_\infty \quad a.e.$$

Moreover, for every measurable $I \subset S^1$ such that $\mathcal{L}^1(I) \neq 0$

$$L(\vec{\gamma}_\infty \llcorner I) < \liminf_{j\to+\infty} L(\vec{\gamma}_j \llcorner I)$$

and $\vec{\gamma}_\infty$ is not a geodesic. □

I.2 Struwe's Monotonicity Trick.

Theorem I.1. *Let $(\mathcal{M}, \|\cdot\|)$ be a complete Finsler manifold. Let E^σ be a family of C^1 functions for $\sigma \in [0,1]$ on $\mathcal{M}$ such that for every $\vec{\gamma} \in \mathcal{M}$*

$$\sigma \longrightarrow E^\sigma(\vec{\gamma}) \quad \textit{and} \quad \sigma \longrightarrow \partial_\sigma E^\sigma(\vec{\gamma}) \tag{I.5}$$

are increasing and continuous functions with respect to σ. Assume moreover that

$$\|DE^\sigma_{\vec{\gamma}} - DE^\tau_{\vec{\gamma}}\|_{\vec{\gamma}} \leq C(\sigma)\ \delta(|\sigma - \tau|)\ f(E^\sigma(\vec{\gamma})) \tag{I.6}$$

where $C(\sigma) \in L^\infty_{loc}((0,1))$, $\delta \in L^\infty_{loc}(\mathbb{R}_+)$ and goes to zero at 0 and $f \in L^\infty_{loc}(\mathbb{R})$. Assume that for every σ the functional E^σ satisfies the Palais Smale condition. Let $\mathcal{A}$ be an admissible family of $\mathcal{M}$ and denote

$$\beta(\sigma) := \inf_{A\in\mathcal{A}} \sup_{\vec{\gamma}\in A} E^\sigma(\vec{\gamma})$$

Then there exists a sequence $\sigma_j \to 0$ and $\vec{\gamma}_j \in \mathcal{M}$ such that

$$E^{\sigma_j}(\vec{\gamma}_j) = \beta(\sigma_j) \quad , \quad DE^{\sigma_j}(\vec{\gamma}_j) = 0$$

Moreover $\vec{\gamma}_j$ satisfies the so called "entropy condition"

$$\partial_{\sigma_j} E^{\sigma_j}(\vec{\gamma}_j) = o\left(\frac{1}{\sigma_j\ \log\left(\frac{1}{\sigma_j}\right)}\right) \quad .$$

□

Before proving theorem I.1 we are going to apply this result to the case of $W^{2,2}-$immersions of curve into a closed sub-manifold N^n

Theorem I.2. *In the Finsler manifold $W^{2,2}_{imm}(S^1, N^n)$ equipped with the Finsler structure (I.1) we consider the family of C^1 functions on $\mathcal{M}$ given by*

$$E^\sigma(\vec{\gamma}) := \int_{S^1} \left[1 + \sigma^2\, |\vec{\kappa}_{\vec{\gamma}}|^2\right]\, dl_{\vec{\gamma}}$$

then for any admissible family of $W^{2,2}_{imm}(S^1, N^n)$ we denote

$$\beta(\sigma) := \inf_{A \in \mathcal{A}}\ \sup_{\vec{\gamma} \in A}\ E^\sigma(\vec{\gamma})$$

and assume $\beta(0) > 0$. Then there exists a sequence $\sigma_j \to 0$ and $\vec{\gamma}_j \in \mathcal{M}$ such that

$$E^{\sigma_j}(\vec{\gamma}_j) = \beta(\sigma_j) \quad , \quad DE^{\sigma_j}(\vec{\gamma}_j) = 0$$

moreover

$$\sigma_j^2 \int_{S^1} |\vec{\kappa}_{\vec{\gamma}_j}|^2\, dl_{\vec{\gamma}_j} = o\left(\frac{1}{\log\left(\frac{1}{\sigma_j}\right)}\right) \tag{I.7}$$

and

$$\dot{\gamma}_j \longrightarrow \dot{\gamma}_\infty \quad .$$

moreover $\vec{\gamma}_\infty$ <u>is a geodesic</u> satisfying

$$L(\vec{\gamma}_\infty) = \beta(0) \quad .$$

□

Proof of theorem I.2. We aim to apply first theorem I.1. Conditions (I.5) are clearly fulfilled. Regarding condition (I.6) a short computation, starting from the explicit expression of the derivative of E^σ given by lemma I.1, implies that for any $\vec{\gamma} \in W^{2,2}_{imm}(S^1, N^n)$ and any $\vec{v} \in T_{\vec{\gamma}}\mathcal{M}$

$$|DE^\sigma_{\vec{\gamma}} \cdot \vec{v} - DE^\tau_{\vec{\gamma}} \cdot \vec{v}| \le C_{N^n}\, |\tau^2 - \sigma^2|\, \int_{S^1} |\nabla(d\vec{\gamma})|^2_{g_{\vec{\gamma}}}\, dl_{\vec{\gamma}}\ \|\vec{v}\|_{\vec{\gamma}}$$

Hence, all the conditions for applying theorem I.1 are fulfilled and we obtain a sequence $\sigma_j \to 0$ together with a sequence of critical points $\vec{\gamma}_j$ of E^{σ_j} such that $\beta(\sigma_j) = E^{\sigma_j}(\vec{\gamma}_j)$ and the entropy condition (I.7) is fulfilled.

In order to simplify the presentation we give the rest of the argument in the particular case $N^n = S^n$ (the general case is presented in [1]). In that case we can use the expression (I.4) and infer the existence of a sequence of constant 2-vectors $\vec{C}_j$ such that in constant speed parametrization one has.

$$(1-2\sigma_j^2)\,\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j+2\,\sigma^2\,\vec{\gamma}_j\wedge\dot{\vec{\kappa}}_j-2\,\sigma_j^2\,\dot{\vec{\gamma}}_j\wedge\vec{\kappa}_j+3\,\sigma_j^2\,|\vec{\kappa}_j|^2\,\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j=\vec{C}_j$$

Because of the entropy condition (I.7) we have that

$$L_j\;\to\beta(0)>0$$

and hence $|\vec{\gamma}_j|\equiv L_j/2\pi\to\beta(0)/2\pi$. Using this fact we deduce that in normal parametrization

$$\sigma_j^2\,|\vec{\kappa}_j|^2\;\longrightarrow\;0\quad\text{ strongly in } L^1(S^1)$$

Hence there exists a sequence of 2-vector valued function $\vec{F}_j\;\longrightarrow\;0$ strongly in $L^1(S^1)$ such that

$$\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j=-\,2\,\sigma_j^2\,\vec{\gamma}_j\wedge\dot{\vec{\kappa}}_j+\vec{F}_j+\vec{C}_j\quad.$$

Integrating this identity over S^1 gives

$$\int_{S^1}\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j=2\pi\;\vec{C}_j+o(1)$$

hence

$$\limsup_{j\to+\infty}|\vec{C}_j|\le\beta(0)/2\pi$$

Taking now the scalar product with $\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j$ and integrating over S^1 gives

$$\frac{L_j^2}{2\pi}=-\,2\,\sigma_j^2\,\int_{S^1}(\vec{\gamma}_j\wedge\dot{\vec{\kappa}}_j)\cdot(\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j)+o(1)+\int_{S^1}\vec{C}_j\cdot\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j\;d\theta$$

This implies that

$$|\vec{C}_j|\;\longrightarrow\;\beta(0)/2\pi$$

and consequently

$$\lim_{j\to+\infty}\left|\,\left|\int_{S^1}\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j\;d\theta\right|-\int_{S^1}|\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j|\;d\theta\right|=0$$

we deduce the strong convergence of $\vec{\gamma}_j\wedge\dot{\vec{\gamma}}_j$ in L^1 from which the theorem follows. □

Remark I.1. *Observe that the argument of the proof is similar to a "compensated compactness type argument" as it has been originally introduced by Luc Tartar in [2].* □

Proof of theorem I.1. Since $\beta(\sigma)$ is a non decreasing function of σ Lebesgue theorem implies that it is differentiable almost everywhere. Denoting by $D\beta$ the distributional derivative of β we have the existence of an L^1 non negative function $\beta'(\sigma)$ which coincides with the derivative of β almost everywhere and a non negative Radon measure μ on $[0,1]$ such that

$$D\beta(\sigma) = \beta'(\sigma)\ d\mathcal{L}^1 \llcorner [0,1] + \mu$$

we have moreover the existence of a Lebesgue zero measure subset B of $[0,1]$ such that $\mu(B) = \mu([0,1])$. We then deduce that

$$\int_0^\tau \beta'(\sigma)\ d\sigma \le \beta(\tau) - \beta(0) \quad .$$

Then there exists a sequence of point of differentiability for β in $(0,1)$ that we denote σ_j such that

$$\sigma_j \to 0 \quad \text{ and } \quad \beta'(\sigma_j) \le \frac{o(1)}{\sigma_j\ \log\frac{1}{\sigma_j}}$$

Let now σ be a point of differentiability of β and fix $\varepsilon > 0$. Since β is differentiable at σ we have for τ close enough to σ an larger than σ

$$\beta(\tau) \le \beta(\sigma) + (\beta'(\sigma) + \varepsilon)\ (\tau - \sigma) \quad . \tag{I.8}$$

Take now $\tau > \sigma$ close enough to σ in such a way that (I.8) holds and let $A \in \mathcal{A}$ and $\vec{\gamma} \in A$ such that

$$\left\{ \begin{array}{l} \beta(\sigma) \le E^\sigma(\vec{\gamma}) + \varepsilon\ (\tau - \sigma) \\ \\ E^\tau(\vec{\gamma}) \le \beta(\tau) + \varepsilon(\tau - \sigma) \quad . \end{array} \right. \tag{I.9}$$

We claim that under the two assumptions (I.8 and (I.9) we have

$$\partial_\sigma E^\sigma(\vec{\gamma}) \le \beta'(\sigma) + 3\,\varepsilon \quad . \tag{I.10}$$

Indeed, combining (I.8) and (I.9) together with the fact that $E^\tau(\vec{\gamma})$ is non decreasing in τ we have

$$\beta(\sigma) - \varepsilon\ (\tau - \sigma) \le E^\sigma(\vec{\gamma}) \le E^\tau(\vec{\gamma}) \le \beta(\tau) + \varepsilon(\tau - \sigma) \le \beta(\sigma) + (\beta'(\sigma) + 2\,\varepsilon)\ (\tau - \sigma)$$

which gives

$$\frac{E^\tau(\vec{\gamma}) - E^\sigma(\vec{\gamma})}{\tau - \sigma} \le \beta'(\sigma) + 3\,\varepsilon$$

using the fact that $\partial_\sigma E^\sigma(\vec{\gamma})$ is non decreasing we deduce the claim (I.10).

We are now going to construct for any $\sigma_k \to \sigma^+$ and $\vec{\gamma}_k$ such that

$$\lim_{k\to+\infty} \|DE^{\sigma_k}(\vec{\gamma}_k)\|_{\vec{\gamma}_k} = 0 \tag{I.11}$$

and

$$\left\{\begin{array}{l} \beta(\sigma) \le E^{\sigma}(\vec{\gamma}_k) + \varepsilon\ (\sigma_k - \sigma) \\ \\ E^{\sigma_k}(\vec{\gamma}_k) \le \beta(\sigma_k) + \varepsilon(\sigma_k - \sigma) \quad . \end{array}\right. \tag{I.12}$$

Given an arbitrary sequence $\sigma_j \to \sigma^+$ we assume that there exists δ such that for k large enough and for all $\vec{\gamma}$ satisfying

$$\left\{\begin{array}{l} \beta(\sigma) \le E^{\sigma}(\vec{\gamma}) + \varepsilon\ (\sigma_k - \sigma) \\ \\ E^{\sigma_k}(\vec{\gamma}) \le \beta(\sigma_k) + \varepsilon(\sigma_k - \sigma) \quad . \end{array}\right. \tag{I.13}$$

one has

$$\|DE^{\sigma_k}(\vec{\gamma})\|_{\vec{\gamma}} > \delta \quad . \tag{I.14}$$

We take a pseudo-gradient $X^{\sigma_k}(\vec{\gamma})$ on $\mathcal{M}^*$ given by proposition **??** and we consider a cut-off function $\chi \in C^\infty(\mathbb{R})$ supported on $\mathbb{R}_+$ and such that $\chi \equiv 1$ on $[1, +\infty)$. We consider then the following cut off of the pseudo-gradient

$$\tilde{X}^{\sigma_k}(\vec{\gamma}) := \chi\left(\frac{E^{\sigma}(\vec{\gamma}) - \beta(\sigma) + \varepsilon\,(\sigma_k - \sigma)}{\varepsilon\,(\sigma_k - \sigma)}\right)\ X^{\sigma_k}(\vec{\gamma})$$

and consider the flow given by

$$\left\{\begin{array}{l} \dfrac{d\phi_t^k(\vec{\gamma})}{dt} = -\ \tilde{X}^{\sigma_k}(\phi_t^k(\vec{\gamma})) \quad \text{in } [0, t_{max}^{\vec{\gamma}}) \\ \phi_0^k(\vec{\gamma}) = \vec{\gamma} \end{array}\right.$$

We have for any $\vec{\gamma}$ and for $t < t_{max}^{\vec{\gamma}}$

$$\begin{aligned} \frac{dE^{\sigma}(\phi_t^k(\vec{\gamma}))}{dt} &= -\,DE^{\sigma}(\phi_t^k(\vec{\gamma}))\cdot\tilde{X}^{\sigma_k}(\phi_t^k(\vec{\gamma})) \\ &= -\,\chi\left(\frac{E^{\sigma}(\phi_t^k(\vec{\gamma})) - \beta(\sigma) + \varepsilon\,(\sigma_k - \sigma)}{\varepsilon\,(\sigma_k - \sigma)}\right)\ DE^{\sigma}(\phi_t^k(\vec{\gamma}))\cdot X^{\sigma_k}(\phi_t^k(\vec{\gamma})) \\ &= -\,\chi\left(\frac{E^{\sigma}(\phi_t^k(\vec{\gamma})) - \beta(\sigma) + \varepsilon\,(\sigma_k - \sigma)}{\varepsilon\,(\sigma_k - \sigma)}\right)\ DE^{\sigma_k}(\phi_t^k(\vec{\gamma}))\cdot X^{\sigma_k}(\phi_t^k(\vec{\gamma})) \\ &\quad +\ \chi_k(\phi_t^k(\vec{\gamma}))\left[DE^{\sigma_k}(\phi_t^k(\vec{\gamma})) - DE^{\sigma}(\phi_t^k(\vec{\gamma}))\right]\cdot X^{\sigma_k}(\phi_t^k(\vec{\gamma})) \end{aligned}$$

where

$$\chi_k(\phi_t^k(\vec{\gamma})) := \chi\left(\frac{E^\sigma(\phi_t^k(\vec{\gamma})) - \beta(\sigma) + \varepsilon\,(\sigma_k - \sigma)}{\varepsilon\,(\sigma_k - \sigma)}\right) \quad .$$

Starting with $\vec{\gamma}$ satisfying

$$E^{\sigma_k}(\vec{\gamma}) \leq \beta(\sigma_k) + \varepsilon(\sigma_k - \sigma)$$

since the flow is decreasing the E^{σ_k} energy i.e.

$$E^\sigma(\phi_t^k(\vec{\gamma})) \leq E^{\sigma_k}(\phi_t^k(\vec{\gamma})) \leq E^{\sigma_k}(\vec{\gamma}) \leq \beta(\sigma_k) + \varepsilon(\sigma_k - \sigma)$$

and since

$$\tilde{X}^{\sigma_k}(\phi_t^k(\vec{\gamma})) \neq 0 \implies \beta(\sigma) \leq E^\sigma(\phi_t^k(\vec{\gamma})) + \varepsilon\ (\sigma_k - \sigma)$$

The energy $E^\sigma(\phi_t^k(\vec{\gamma}))$ is uniformly bounded from above and from below all along the flow and then, from our assumptions, we can choose k large enough in such a way that

$$\left[DE^{\sigma_k}(\phi_t^k(\vec{\gamma})) - DE^\sigma(\phi_t^k(\vec{\gamma}))\right] \cdot X^{\sigma_k}(\phi_t^k(\vec{\gamma})) \leq \delta^2/2$$

Since

$$DE^{\sigma_k}(\phi_t^k(\vec{\gamma})) \cdot X^{\sigma_k}(\phi_t^k(\vec{\gamma})) \geq \delta^2$$

for $\beta(\sigma) \leq E^\sigma(\phi_t^k(\vec{\gamma})) + \varepsilon\ (\sigma_k - \sigma)$, the energy E^σ also decreases along the flow and, unless

$$E^\sigma(\vec{\gamma}) < \beta(\sigma) - \varepsilon\ (\sigma_k - \sigma)$$

in which case the flow is constant, we must have

$$\beta(\sigma) \leq E^\sigma(\phi_t^k(\vec{\gamma})) + \varepsilon\ (\sigma_k - \sigma)$$

for all time. Arguing as in the proof of Palais theorem **??**, because of our assumption (I.14) under the condition (I.13), if we start with $\vec{\gamma}$ satisfying

$$E^{\sigma_k}(\vec{\gamma}) \leq \beta(\sigma_k) + \varepsilon(\sigma_k - \sigma)$$

the flow cannot extinct on a critical point of $\mathcal{M} \setminus \mathcal{M}^*$ and then exists for all time and $t^{\vec{\gamma}}_{max} = +\infty$. Taking now a point $A \in \mathcal{A}$ such that

$$\sup_{\vec{\gamma} \in A} E^{\sigma_k}(\vec{\gamma}) \leq \beta(\sigma_k) + \varepsilon(\sigma_k - \sigma)$$

we consider $\phi_t^k(A)$. Because of the above there will be a finite time T such that

$$\sup_{\vec{\gamma} \in A} \frac{E^\sigma(\phi_T^k(\vec{\gamma})) - \beta(\sigma) + \varepsilon\,(\sigma_k - \sigma))}{\varepsilon\ (\sigma_k - \sigma)} < 1$$

which implies

$$\sup_{\vec{\gamma}\in A} E^\sigma(\phi_T^k(\vec{\gamma}) < \beta(\sigma) \quad .$$

Since $\phi_T^k(A) \in \mathcal{A}$ we have reached a contradiction and we have proved the existence of $\vec{\gamma}_k$ satisfying both (I.11) and (I.12). We have then

$$\partial_\sigma E^\sigma(\vec{\gamma}_k) \le \beta'(\sigma) + 3\,\varepsilon \quad \text{ and } \quad \lim_{k\to+\infty} \|DE^{\sigma_k}(\vec{\gamma}_k)\|_{\vec{\gamma}_k} = 0$$

Because of the assumption (I.6) we deduce that $\vec{\gamma}_k$ is a Palais Smale sequence and since $\varepsilon = o(1/\sigma\ \log(\sigma^{-1}))$ as $\sigma \to 0$, the theorem I.1 follows. □

References

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