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Integrability by compensation in Besov-Morrey spaces and applications to Schrödinger systems with antisymmetric potentials

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Contents

Abstract

In the study of regularity of weak harmonic maps from the unit disc B^2 into the unit sphere $S^{m-1} \subset \mathbb{R}^m$, i. e. maps in $W_2^1(B^2, \mathbb{R}^m)$ such that the values are almost everywhere in S^{m-1} and which solve the following equation (in the sense of distributions)

$$-\Delta u = u|\nabla u|^2 \tag{1}$$

there are three major observations:

Conservation laws
It was observed by Shatah (see [46]) that u solves (1) if and only if the following conservation law holds

$$div(u^i \nabla u^j - u^j \nabla u^i) = 0 \text{ for all } i, j \in \{1, \ldots, m\}\,.$$

Antisymmetry
Due to the above conservation law and the fact that $\sum_j u^j \nabla u^j = 0$, Hélein (cf. [26]) rewrote the equation (1)as follows

$$\begin{aligned} -\Delta u^i &= \sum_{j=1}^m u^i \nabla u_j \cdot \nabla u^j \\ &= \sum_{j=1}^m (u^i \nabla u_j - u_j \nabla u^i) \cdot \nabla u^j \\ &= \sum_{j=1}^m \Omega_j^i \cdot \nabla u^j \end{aligned}$$

where obviously Ω_j^i is **antisymmetric**!

Higher integrability
Last but not least, from Wente's work ([66]) (see also Coifman et al. ([16]) and Tartar ([56])) it is known that the right hand side of $\Delta u^i = \sum_{j=1}^m \Omega_j^i \cdot \nabla u^j$ is not even in L^1 but belongs to the Hardy spaces $\mathfrak{H}^1$, a strict subspace of L^1.

Thus, the additional structure leads to **higher integrability**, which again gives rise to improved regularity results in dimension $n = 2$.

What has be observed in the context of harmonic maps into spheres was generalised by Rivière in ([38]) to the study of problems of the form

$$\Delta u = \Omega \cdot \nabla u. \tag{2}$$

where Ω is antisymmetric.
By construction of suitable **gauge transformations** he was able to rewrite such equations in **divergence form** and applied this fact in the study of regularity questions. Interestingly, his approach was also applicable - with the necessary modification - to the study of Willmore surfaces ([41]).
The above cite article [38] is the point of departure for the present thesis: We will prove a generalisation of Wente's result for arbitrary dimension (part II).
Once such an assertion is at hand, we will study the effect of this result in the framework of equations of the form of (2) (part III) and in part IV - motivated by a recent work of Rivière ([39]) - also to problems of the form

$$\Delta u = \Omega u.$$

In part V we discuss some related minor questions which arose in the previous parts.
The first part of the present work is devoted to some preliminary "warm up" results, and in the Appendix, we will provide all the necessary definitions, illustrated and enriched by the most important theorems and close with two alternative proofs of Wente's result, one of which contains the crucial ideas which lead to our generalisation mentioned above.

Zusammenfassung

Im Rahmen der Regularitätsfrage von schwach harmonischen Abbildungen von der Einheitskreisscheibe B^2 in die Sphäre $S^{m-1} \subset \mathbb{R}^m$, d. h. Abbildungen in $W_2^1(B^2, \mathbb{R}^m)$, so dass die Werte fast überall in S^{m-1} liegen und (im Sinne der Distributionen) die foldende Gleichung erfüllen

$$-\Delta u = u|\nabla u|^2 \tag{3}$$

haben sich drei essentielle Beobachtungen herauskristallisiert:

Erhaltungssätze
Shatah (vgl. [46]) hat bemerkt, dass u die Gleichung (3) genau dann erfüllt, wenn der folgende Erhaltungssatz gilt

$$div(u^i \nabla u^j - u^j \nabla u^i) = 0 \text{ for all } i, j \in \{1, \dots, m\}\,.$$

Antisymmetrie
Dank des obigen Erhaltungssatzes und unter Berücksichtigung der Tatsache $\sum_j u^j \nabla u^j = 0$, ist es Hélein (vgl. [26]) gelungen, die Gleichung(3) wie folgt umzuschreiben

$$\begin{aligned} -\Delta u^i &= \sum_{j=1}^m u^i \nabla u_j \cdot \nabla u^j \\ &= \sum_{j=1}^m (u^i \nabla u_j - u_j \nabla u^i) \cdot \nabla u^j \\ &= \sum_{j=1}^m \Omega^i_j \cdot \nabla u^j \end{aligned}$$

wobei offensichtlich Ω^i_j **antisymmetrisch** ist!

Höhere Integrabilität
Zu guter Letzt wissen wir aus der Arbeit von Wente (vgl.([66]) (siehe auch

Coifman et al. ([16]) und Tartar ([56])), dass die rechte Seite von $\Delta u^i = \sum_{j=1}^m \Omega_j^i \cdot \nabla u^j$ nicht nur in L^1 sondern sogar im Hardy Raum $\mathfrak{H}^1$ liegt, wobei letzterer ein strikter Unterraum von L^1 ist. Mit anderen Worten, die zusätzliche Struktur führt zu **höherer Integrabilität**, welche wiederum - in Dimension $n = 2$ - zu verbesserten Regularitätsresultaten führt.

Die oben beschriebenen Beobachtungen für harmonische Abbildungen mit Werten in einer Sphäre wurden von Rivière in ([38]) im Zusammenhang mit dem Studium von Problemen der Form

$$\Delta u = \Omega \cdot \nabla u. \tag{4}$$

wobei Ω antisymmtrisch ist, verallgemeinert.
Durch Konstruktion einer geeigneten **Eichtransformation** ist es ihm gelungen, Probleme dieser Form in **Divergenz-Form** umzuschreiben und diese Umformulierung im Kontext der Regularitätsfragen anzuwenden. Interessanterweise lässt sich dieser Ansatz - mit den notwendigen Anpassungen - sogar im Zusammenhang mit Willmore Flächen anwenden([41]).
Die oben zitierte Arbeit [38] bildet den Ausgangspunkt für die vorligende Dissertation: Wir werden eine Verallgemeinerung des Resultats von Wente für beliebige Dimension beweisen (Teil II).
Ist einmal ein solches Ergebnis zur Hand, werden wir es im Rahmen von Problemen, die sich in der Form von (4) formulieren lassen, anwenden (Teil III). In Teil IV werden wir uns schliesslich - motiviert durch einen aktuellen Artikel von Rivière ([39]) - Problemen der folgenden Form zuwenden

$$\Delta u = \Omega u.$$

In Teil V werden wir einige kleinere, verwandte Fragestellungen behandeln, die sich aus dem Kontext der erwähnten Probleme ergeben.
Der erste Teil der Arbeit ist einigen Vorbereitungen gewidmet, und im Appendix finden sich alle verwendeten Definitionen, zusammen mit den wichtigsten verwendeten Theoremen. Schliesslich präsentieren wir im Anhang zwei alternative Beweise des ursprünglichen Resultats von Wente, von denen einer bereits die grundlegenden Ideen enthält, die bei unserer Verallgemeinerung essentiell sind.

Introduction

In this section we will give an overview of the present work, describe the larger context of our results and point out the guidelines and major ideas whereas the technical details are carried out in the subsequent chapters.
For the sake of simplicity, in what follows we will use the abbreviation a_x for $\frac{\partial}{\partial x}a$.

Our work was motivated by Rivière's article about Schrödinger systems with antisymmetric potentials [38], i.e. systems of the form

$$-\Delta u = \Omega \cdot \nabla u \tag{5}$$

with $u \in W^{1,2}(\omega, \mathbb{R}^m)$ and $\Omega \in L^2(\omega, so(m) \otimes \Lambda^1\mathbb{R}^n)$, $\omega \subset \mathbb{R}^n$.
The differential equation (5) has to be understood in the following sense:
For all indexes $i \in \{1, \ldots, m\}$ we have $-\Delta u^i = \sum_{j=1}^m \Omega^i_j \cdot \nabla u^j$ and $L^2(\omega, so(m) \otimes \Lambda^1\mathbb{R}^n)$ means that $\forall\, i, j \in \{1, \ldots, m\}$, $\Omega^i_j \in L^2(\omega, \Lambda^1\mathbb{R}^n)$ and $\Omega^i_j = -\Omega^j_i$.
In particular, it was the result that in dimension $n = 2$ solutions to (5) are continuous which attracted our interest.

The interest for such systems originates in the fact that they "encode" all Euler-Lagrange equations for conformally invariant quadratic Lagrangians in dimension 2 (see [38] and also [25]).
In what follows we will take $\omega = B^n$, the n-dimensional unit ball, centred at the origin.

In the above cited work, there were three crucial ideas:

- **Antisymmetry of Ω**
  If we drop the assumption that Ω is symmetric, there may occur solutions which are not continuous as the following example shows:
  Let $n = 2$, $u^i = 2 \log \log \frac{1}{r}$ for $i = 1, 2$ and let

$$\Omega = \begin{pmatrix} \nabla u^1 & 0 \\ 0 & \nabla u^2 \end{pmatrix}$$

Obviously, u satisfies equation (5) with the given Ω but is not continuous.

- **Construction of conservation laws**
  In fact, once there exist $A \in L^\infty(B^n, M_m(\mathbb{R})) \cap W^{1,2}(B^n, M_m(\mathbb{R}))$ such that
  $$d^*(dA - A\Omega) = 0. \tag{6}$$
  for given $\Omega \in L^2(B^n, so(m) \otimes \Lambda^1\mathbb{R}^n)$, then any solution u of (5) satisfies the following conservation law
  $$d(*Adu + (-1)^{n-1}(*B) \wedge du) = 0 \tag{7}$$
  where B satisfies $-d^*B = dA - A\Omega$.
  The existence of such an A (and B) is proved by Rivière in [38] and relies on a **non linear Hodge decomposition** which can also be interpreted as a **change of gauge**. (see in our case theorem 24)

- **Understanding the linear problem**
  The proof of the above mentioned regularity result uses the result below for the linear problem:

  Theorem A *([66],[16], [56])*
  Let a, b satisfy $\nabla a, \nabla b \in L^2$ and let φ be the unique solution to
  $$\begin{cases} -\Delta\varphi = \nabla a \cdot \nabla^\perp b = *(da \wedge db) = a_x b_y - a_y b_x \text{ in } B_1^2(0) \\ \varphi = 0 \text{ on } \partial B_1^2(0). \end{cases} \tag{8}$$
  Then φ is continuous and it holds that
  $$||\varphi||_\infty + ||\nabla\varphi||_{2,1} + ||\nabla^2\varphi||_1 \le C||\nabla a||_2 \, ||\nabla b||_2. \tag{9}$$

  Note that the L^∞-part in the above estimate is the key point for the existence of A, B satisfying (6).

A more detailed explanation of these key points and their interplay can be found in Rivière's overview [40].

Our strategy to extend the cited regularity result to domains of arbitrary dimension is to find first of all a good generalisation of Wente's estimate. Here, the first question is to detect a suitable substitute for L^2 since obviously for $n \ge 3$ from the fact that $a, b \in W^{1,2}$ we can not conclude that φ is continuous. So we have to reduce our interest to a smaller space than L^2. A first idea is to look at the Morrey space $\mathcal{M}_2^n$, i.e. at the spaces of all functions $f \in L^2_{loc}(\mathbb{R}^n)$ such that

$$||f|\mathcal{M}_2^n|| = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{1-n/2} ||f|L^2(B(x_0,R))|| < \infty.$$

The choice of this space was motivated by the following observation (for details see [42]):
For stationary harmonic maps u we have the following monotonicity estimate

$$r^{2-n} \int_{B_r^n(x_0)} |\nabla u|^2 \leq R^{2-n} \int_{B_R^n(x_0)} |\nabla u|^2$$

for all $r \leq R$. From this, it is rather natural to look at the Morrey space $\mathcal{M}_2^n$.

Unfortunately, this first try is not successful as the following counterexample in dimension $n = 3$ shows:
Let $a = \frac{x_1}{|x|}$ and $b = \frac{x_2}{|x|}$. As required $\nabla a, \nabla b \in \mathcal{M}_2^3(B_1^3(0))$. The results in ([16]) imply that the unique solution φ of (8) satisfies $\nabla^2 \varphi \in \mathcal{M}_1^{\frac{3}{2}}$ but φ is not bounded!
Therefore, in [43] the attempt to construct conservation laws for (5) in the framework of Morrey spaces fails.

Another drawback is that C^∞ is not dense in $\mathcal{M}_2^n$. This point is particularly important if one has in mind the proof via paraproducts of Wente's L^∞ bound for the solution φ.
In this paper we shall study L^∞ estimates by replacing the Morrey spaces $\mathcal{M}_2^n$ by their "nearest" Littlewood Paley counterpart, the Besov-Morrey spaces $B^0_{\mathcal{M}_2^n,2}$, i.e. the spaces of $f \in \mathcal{S}'$ such that

$$\Big(\sum_{j=0}^{\infty} ||\mathcal{F}^{-1}\varphi_j \mathcal{F} f|\mathcal{M}_2^n(\mathbb{R}^n)||^2 \Big)^{\frac{1}{2}} < \infty$$

where $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ is a suitable partition of unity.
It turns out that we have a suitable density result at hand, see lemma 6. These spaces were introduced by Kozono and Yamazaki in [29] and applied to the study of the Cauchy problem for the Navier-Stokes equation and seminar heat equation (see also [32]).

Note, that we have the following **natural embeddings**: $B^0_{\mathcal{M}_2^n,2} \subset \mathcal{M}_2^n$ (see lemma 1) and on compact subsets $B^0_{\mathcal{M}_2^n,2}$ is a natural subset of L^2 (see lemma

5).

The success to which these Besov-Morrey spaces give rise relies crucially on the fact that **we first integrate and then sum**!
In the spirit of the scales of Triebel-Lizorkin and Besov spaces (definition are restated in the next section) where we have for $0 < q \leq \infty$ and $0 < p < \infty$

$$B^s_{p,\min\{p,q\}} \subset F^s_{p,q} \subset B^s_{\max\{p,q\}}$$

and due to the fact that for $1 < q \leq p < \infty$

$$||f||_{\mathcal{M}^p_q} \simeq \Big|\Big| \Big(\sum_{j=0}^{\infty} |\mathcal{F}^{-1}\varphi_j \mathcal{F} f|^2 \Big)^{\frac{1}{2}} \Big|\Big|_{\mathcal{M}^p_q}$$

it is obvious to exchange the order of summability and integrability in order to find a smaller space starting from a given one.

A more detailed exposition of the framework of Besov-Morrey spaces is given in the Appendix.

We have

Theorem B

i) Assume that $a, b \in B^0_{\mathcal{M}^n_2,2}$, and assume further that

$$a_x, a_y, b_x, b_y \in B^0_{\mathcal{M}^n_2,2} \textit{ where } x, y = z_i, z_j \textit{ with } i, j, \in \{1, \dots, n\}\,.$$

Then any solution of

$$-\Delta u = a_x b_y - a_y b_x$$

is continuous and bounded.

ii) Assume that a_x, a_y, b_x and b_y are distributions whose support is contained in B^n and belong to $B^0_{\mathcal{M}^n_2,2}$, $n \geq 3$.
Moreover, let u be a solution (in the sense of distributions) of

$$-\Delta u = a_x b_y - b_x a_y.$$

Then it holds

$$\nabla u \in B^0_{\mathcal{M}^n_2,1}.$$

iii) Assume that a_x, a_y, b_x *and* b_y *are distributions whose support in* B^n *and belong to* $B^0_{\mathcal{M}^n_2,2}$.
Moreover, let u *be a solution (in the sense of distributions) of*

$$-\Delta u = a_x b_y - b_x a_y.$$

Then it holds

$$\nabla^2 u \in B^{-1}_{\mathcal{M}^n_2,1} \subset B^{-2}_{\infty,1}.$$

Remark

- If we reduce our interest to dimension $n = 2$, our assumption in the theorem below coincide with the original ones in Wente's framework due to the fact that $\mathcal{M}^2_2 = L^2$ and $B^0_{2,2} = L^2 = F^0_{2,2}$.
- Obviously we have the a-priori bound

$$||u||_\infty \leq C\big(||a|B^0_{\mathcal{M}^n_2,2}|| + ||\nabla a|B^0_{\mathcal{M}^n_2,2}||\big)\big(||b|B^0_{\mathcal{M}^n_2,2}|| + ||\nabla b|B^0_{\mathcal{M}^n_2,2}||\big).$$

- Now, if we use a homogeneous partition of unity instead of an inhomogeneous as before, our result holds if we replace the spaces $B^0_{\mathcal{M}^n_2,2}$ by the spaces $\mathcal{N}^0_{n,2,2}$. For further information about these homogeneous function spaces we refer to Mazzucato's article [32].
- Note that the estimate $\nabla u \in B^0_{\mathcal{M}^n_2,1}$ implies that u is bounded and continuous.

As an application of what we did so far, we would like to present an adaptation of Rivière's construction of conservation laws via gauge transformation (see [38]) to our setting, more precisely we are able to prove the following assertion:

Theorem C *Let* $n \geq 3$. *There exist constants* $\varepsilon(m) > 0$ *and* $C(m) > 0$ *such that for every* $\Omega \in B^0_{\mathcal{M}^n_2,2}(B^n, so(m) \otimes \Lambda^1\mathbb{R}^n)$ *which satisfies*

$$||\Omega|B^0_{\mathcal{M}^n_2,2}|| \leq \varepsilon(m)$$

there exist $A \in L^\infty(B^n, Gl_m(\mathbb{R})) \cap B^1_{\mathcal{M}^n_2,2}$ *and* $B \in B^1_{\mathcal{M}^n_2,2}(B^n, M_m(\mathbb{R}) \otimes \Lambda^2\mathbb{R}^n)$ *such that*

i)

$$d_\Omega := dA - A\Omega = -d^*B = - * d * B$$

ii)

$$||\nabla A|B^0_{\mathcal{M}^n_2,2}||+||\nabla A^{-1}|B^0_{\mathcal{M}^n_2,2}||+\int_{B^n}||dist(A,SO(m))||^2_\infty \leq C(M)||\Omega|B^0_{\mathcal{M}^n_2,2}||$$

iii)

$$||\nabla B|B^0_{\mathcal{M}^n_2,2}|| \leq C(m)||\Omega|B^0_{\mathcal{M}^n_2,2}||.$$

This finally leads to the following regularity result:

Corollary D *Let the dimension n satisfy $n \geq 3$. Let $\varepsilon(m)$, Ω, A and B be as in theorem 25. Then any solution u of*

$$-\Delta u = \Omega \cdot \nabla u$$

satisfies the conservation law

$$d(*Adu + (-1)^{n-1}(*B) \wedge du) = 0.$$

Moreover, any distributional solution of $\Delta u = -\Omega \cdot \nabla u$ which satisfies in addition

$$\nabla u \in B^0_{\mathcal{M}^n_2,2}$$

is continuous.

Remark Note that the continuity assertion of the above corollary is already contained in [42], but our result differs from [42] (see also [44] for a modification of the proof of Rivière and Struwe) in so far, as on one hand we do not impose any smallness of the norm of the gradient of a solution and really construct A and B (see theorem 25) and not only construct Ω and ξ such that $P^{-1}dP + P^{-1}\Omega P = *d\xi$, but on the other hand work in a slightly smaller space.

A second major input comes from the natural question what happens if in equation (5) **the right hand side depends on u instead of ∇u**, more precisely if we replace the right hand side by the product of an antisymmetric matrix and the (vector valued) function itself, i. e. if we study

$$-\Delta u = \Omega u. \tag{10}$$

A first answer is again give by Rivière who proved in [39] that under the hypothesis that u belongs to $L^{\frac{n}{n-2}}(B^n, \mathbb{R}^m)$ and $\Omega \in L^{\frac{n}{2}}(B^n, so(m))$ where $n \geq 3$ the original equation (10) can be written in divergence form and that under the same assumptions $u \in L^\infty_{loc} \cap W^{2,\frac{n}{2}}_{loc}(B^n)$.

Apart from the structural link between the two problems (5) and (10) there is the deeper connection that in both cases the improvements rely on **gauge transformations** in the construction of which the fact that the potential Ω is antisymmetric is a crucial ingredient.
Our goal is to present a possible generalisation in so far that we start with the hypothesis that Ω is supported in the n-dimensional unit ball and belongs to the Morrey space $M_2^{\frac{n}{2}}$, i. e. satisfies

$$||f|M_2^{\frac{n}{2}}|| = \sup_{x_0\in\mathbb{R}^n}\sup_{1\geq R>0} R^{2-n/2}||f|L^2(B(x_0,R))|| < \infty.$$

More precisely, we will show that

Theorem E *Let $n \geq 4$ and let $m \in \mathbb{N}^*$.*
Assume that $u \in M_2^{\frac{n}{2}}(B^n, \mathbb{R}^m)$, $\Delta u \in L^1_{loc}$ and $\Omega \in M_2^{\frac{n}{2}}(B^n, so(m))$ such that

$$||\Omega||_{M_2^{\frac{n}{2}}} \leq \varepsilon$$

where ε is given by the theorem below.
Then

$$-\Delta u = \Omega u \tag{11}$$

is equivalent to

$$div(A\nabla u - \nabla A u) = 0 \tag{12}$$

where A is again given by the following theorem.

Theorem F *Let $n \geq 4$ and let $m \in \mathbb{N}^*$.*
Then there exists a constant $\varepsilon > 0$ but small enough such that in a neighbourhood of the origin, there exists a map

$$\begin{aligned}\mathcal{S} : M_2^{\frac{n}{2}(B^n,so(m))} &\rightarrow L^\infty(B^n)\cap W^2_{M_2^{\frac{n}{2}}}(B^n, Gl_m(\mathbb{R}))\\ \Omega &\rightarrow A\end{aligned}$$

with the following properties

i)

$$\Delta A + A\Omega = 0$$

ii)

$$||A||_{L^\infty}(B^n) = \sup_{x\in B^n, X\in S^{m^1}} |A(x)X| \leq 1$$

iii) A is invertible almost everywhere and $A^{-1} \in M_a^b$ where $\frac{4b}{n} \geq a$

iv) there exists a constant $C > 0$ *such that*

$$||A^{-1}\nabla A||_{M_4^n(B^n)} + ||\nabla A||_{W^1_{M_2^{\frac{n}{2}}}(B^n)} \leq C||\Omega||_{M_2^{\frac{n}{2}}}$$

provided that $||\Omega||_{M_2^{\frac{n}{2}}} \leq \varepsilon$.

Part I

Preliminaries

In this first part, we will collect and prove some additional, technical lemmas which will turn out to be not only interesting in itself, but also very helpful.

Chapter 1

Additional results for Besov-Morrey spaces

Once one is working in the framework of Besov-Morrey spaces, it is natural to ask for relations of these spaces to the standard Morrey spaces. A first answer to this question is given here:

Lemma 1. *Let* $1 < q \leq 2$, $1 < q \leq p < \infty$ *and* $r \leq q$. *Then*

$$B^0_{\mathcal{M}^p_q,r} \subset \mathcal{M}^p_q$$

and

$$N^0_{p_q,r} \subset M^p_q.$$

Proof of lemma 1:
We start with the following observation.
Let $x_0 \in \mathbb{R}^n$ and $r > 0$ and recall that $1 < q \leq 2$ and $r \leq q$. Then for $f \in B^0_{\mathcal{M}^p_q,r}$ we have

$$\begin{aligned}
\Big(\int_{B_r(x_0)} \big(\sum_{s=0}^{\infty} |f^s|^2\big)^{\frac{q}{2}}\Big)^{\frac{1}{q}} &\leq \Big(\int_{B_r(x_0)} \sum_{s=0}^{\infty} |f^s|^q\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=0}^{\infty} \int_{B_r(x_0)} |f^s|^q\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=0}^{\infty} ||f^s||^q_{L^q(B_r(x_0))}\Big)^{\frac{1}{q}} \\
&\leq \Big(\sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q} (r^{\frac{n}{q}-\frac{n}{p}})^q\Big)^{\frac{1}{q}} \\
&= \Big((r^{\frac{n}{q}-\frac{n}{p}})^q \sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q}\Big)^{\frac{1}{q}}.
\end{aligned}$$

And we continue

$$\begin{aligned}
\Big(\int_{B_r(x_0)} (\sum_{s=0}^{\infty} |f^s|^2)^{\frac{q}{2}}\Big)^{\frac{1}{q}} &\leq r^{\frac{n}{q}-\frac{n}{p}}\Big(\sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q}\Big)^{\frac{1}{q}} \\
&= r^{\frac{n}{q}-\frac{n}{p}}\Big(\sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q}\Big)^{\frac{1}{q}}. \\
&= r^{\frac{n}{q}-\frac{n}{p}}\Big(\sum_{s=0}^{\infty} ||f^s||^q_{\mathcal{M}^p_q}\Big)^{\frac{1}{q}} \\
&= r^{\frac{n}{q}-\frac{n}{p}}||f|B^0_{\mathcal{M}^p_q,q}|| \\
&\leq Cr^{\frac{n}{q}-\frac{n}{p}}||f|B^0_{\mathcal{M}^p_q,r}||
\end{aligned}$$

From the last inequality we have that for all $r > 0$ and for all $x_0 \in \mathbb{R}^n$

$$r^{\frac{n}{p}-\frac{n}{q}}\Big|\Big|(\sum_{s=0}^{\infty} |f^s|^2)^{\frac{q}{2}}\Big|\Big|_{L^q(B_r(x_0))} \leq C||f|B^0_{\mathcal{M}^p_q,r}||.$$

This last estimate together with the result quoted below from [31] implies that $f \in \mathcal{M}^p_q$.
The assertion in the case $f \in N^0_{p,q,r}$ is the same.

Proposition 2. *([31]) Let $f \in \mathcal{M}^p_q$ with $1 < q \leq p < \infty$. Then the two norms*

$$\Big|\Big|\Big(\sum_{s=0}^{\infty} |f^s|^2\Big)^{\frac{1}{2}}\Big|\Big|_{\mathcal{M}^p_q}$$

and

$$||f||_{\mathcal{M}^p_q}$$

are equivalent.
Similar for $f \in M^p_q$.

□

From this result we immediately deduce the following corollary.

Corollary 3. *Let $1 < q \leq 2$, $1 < q \leq p < \infty$ and $r \leq q$ and assume that $f \in B^0_{\mathcal{M}^p_q,r}$ has compact support. Then $f \in L^q$.*

This holds because of the preceding lemma and the fact that for a bounded domain Ω we have the embedding $M^p_q(\Omega) \subset L^q(\Omega)$.
Similar to the result that $W^{1,p} = F^1_{p,2}$, $1 < p < \infty$ we have the following lemma.

Lemma 4. *Assume that f is a compactly supported distribution. Then, if $1 < q \leq 2$, $1 < q \leq p < \infty$ and $r \leq q$, the following two norms are equivalent*

$$||f|B^0_{\mathcal{M}^p_q,r}|| + ||\nabla f|B^0_{\mathcal{M}^p_q,r}||$$

$$||f|B^1_{\mathcal{M}^p_q,r}||.$$

Proof of lemma 4:

i) In a first step we will show that if $f \in B^1_{\mathcal{M}^p_q,r}$ there exist a constant C - independent of f - such that

$$||f|B^0_{\mathcal{M}^p_q,r}|| + ||\nabla f|B^0_{\mathcal{M}^p_q,r}|| \leq C||f|B^1_{\mathcal{M}^p_q,r}||.$$

Obviously, we have that

$$||f|B^0_{\mathcal{M}^p_q,r}|| \leq ||f|B^1_{\mathcal{M}^p_q,r}||.$$

Moreover, we observe that

$$\begin{aligned}
||\nabla f|B^0_{\mathcal{M}^p_q,r}|| &= \Big(\sum_{j=0}^{\infty} ||(\nabla f)^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} \\
&\leq \Big(\sum_{j=1}^{\infty} ||(\nabla f)^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} + ||(\nabla f)^0||_{\mathcal{M}^p_q} \\
&\leq C\Big(\sum_{j=1}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} + C||f||_{\mathcal{M}^p_q} \\
&\quad \text{where for the first addend we used proposition 93} \\
&\quad \quad \text{with the necessary adaptations to our situation} \\
&\quad \text{and for the second addend we used lemma 98} \\
&\quad \quad \text{and the observation } \mathcal{F}^{-1}(\xi\varphi_0\hat{f}) = \mathcal{F}^{-1}(\xi\varphi_0) * f. \\
&\leq C||f|B^1_{\mathcal{M}^p_q,r}|| + C||f|B^0_{\mathcal{M}^p_q,r}|| \\
&\quad \text{because of lemma 1} \\
&\leq C||f|B^1_{\mathcal{M}^p_q,r}|| + C||f|B^1_{\mathcal{M}^p_q,r}|| \\
&\leq ||f|B^1_{\mathcal{M}^p_q,r}||
\end{aligned}$$

as desired.

ii) Now, we assume that f satisfies

$$||f|B^0_{\mathcal{M}^p_q,r}|| + ||\nabla f|B^0_{\mathcal{M}^p_q,r}|| < \infty.$$

We have to show that this last quantity controls

$$||f|B^1_{\mathcal{M}^p_q,r}||.$$

In fact, we calculate

$$\begin{aligned}
||f|B^1_{\mathcal{M}^p_q,r}|| &= \Big(\sum_{j=0}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} \\
&\leq C||f^0||_{\mathcal{M}^p_q} + C\Big(\sum_{j=1}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^q_p}\Big)^{\frac{1}{r}} \\
&\leq C||f^0|B^0_{\mathcal{M}^p_q,r} + C||\nabla f|B^0_{\mathcal{M}^p_q,r}|| \\
&\quad \text{again by proposition 93} \\
&\leq C(||f^0|B^0_{\mathcal{M}^p_q,r} + ||\nabla f|B^0_{\mathcal{M}^p_q,r}||).
\end{aligned}$$

□

Moreover, also the fact that for a compactly supported distribution the homogeneous and the inhomogeneous Sobolev norms are equivalent, we have the following result.

Lemma 5. *Let* $1 < q \leq 2$, $1 < q \leq p < \infty$, $2 \leq p$, $r \leq q$ *and* $n \geq 3$. *Assume that the distribution* f *has the following properties:* f *has compact support and* $\nabla f \in B^0_{\mathcal{M}^p_q,r}$. *Then*

$$f \in B^1_{\mathcal{M}^p_q,r}.$$

Proof of lemma 5:
According to lemma 4 it is enough to show that $f \in B^0_{\mathcal{M}^p_q,r}$. First of all, we observe that

$$\begin{aligned}
\Big|\Big|\sum_{j=1}^{\infty} f^j\Big|B^0_{\mathcal{M}^p_q,r}\Big|\Big| &\leq \Big|\Big|\sum_{j=1}^{\infty} f^j\Big|B^1_{\mathcal{M}^p_q,r}\Big|\Big| \\
&\leq C\Big(\sum_{j=0}^{\infty} 2^{jr}||f^j||^r_{\mathcal{M}^p_q}\Big)^{\frac{1}{r}} \\
&\leq C\Big(\sum_{j=0}^{\infty} ||(\nabla f)^j||^r_{\mathcal{M}^p_q}\Big)^{\frac{1}{r}} \\
&\leq ||\nabla f|B^0_{\mathcal{M}^p_q,r}||.
\end{aligned}$$

Now, it remains to estimate $||f^0||_{\mathcal{M}^p_q}$:
It holds

$$f^0 = \mathcal{F}^{-1}\Big(\sum_{i=1}^{n} \frac{\xi_i}{|\xi|^2}\xi_i \hat{f}\varphi_0\Big).$$

Next, due to lemma 1 and its corollary we know that $f \in L^q$ and in particular - since f has compact support $f \in L^1$ so $\xi_i \hat{f} \in L^\infty$ for all i. Moreover, thanks to our assumptions

$$\varphi_0 \frac{1}{|\xi|} \in L^{\frac{p}{p-1}} \quad \text{where} \quad \frac{p}{p-1} \in [1,2].$$

So, for all possible i

$$\varphi_0 \frac{\xi_i}{|\xi|^2} \xi_i \hat{f} \in L^{\frac{p}{p-1}}.$$

From this we conclude that

$$f^0 \in L^p \subset \mathcal{M}^p_q,$$

and finally

$$\begin{aligned} ||f^0|B^0_{\mathcal{M}^p_q,r}|| &\leq ||f^0||_{\mathcal{M}^p_q} + ||f^1||_{\mathcal{M}^p_q} \\ &\leq ||f^0||_{L^p} + C\Big|\Big| \sum_{j=1}^{\infty} f^j | B^0_{\mathcal{M}^p_q,r} \Big|\Big| \\ &\leq C||\nabla f|B^0_{\mathcal{M}^p_q,r}|| + C\Big|\Big| \sum_{j=1}^{\infty} f^j | B^0_{\mathcal{M}^p_q,r} \Big|\Big| \\ &\leq C||\nabla f|B^0_{\mathcal{M}^p_q,r}||. \end{aligned}$$

□

As a by-product of our studies we have the following density result.

Lemma 6. *Let $1 \leq q \leq p < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. Then O_M is dense in $N^s_{p,q,r}$ respectively in $\mathcal{N}^s_{p,q,r}$ and $B^s_{\mathcal{M}^p_q,r}$ where O_M denotes the space of all C^∞-functions such that $\forall \beta \in \mathbb{N}^n$ there exist constants $C_\beta > 0$ and $m_\beta \in \mathbb{N}$ such that*

$$|\partial^\beta f(x)| \leq C_\beta (1 + |x|)^{m_\beta} \; \forall x \in \mathbb{R}^n.$$

Moreover, if $f \in N^s_{p,q,r}$ or $f \in B^s_{\mathcal{M}^q_p r,}$ with $s \geq 0$, $1 \leq q \leq 2$ and $1 \leqq\leq p \leq \infty$ has compact support, it can be approximated by elements in C^∞_0.

Proof of lemma 6:
Density of O_M in $N^s_{p,q,r}$ respectively in $B^s_{\mathcal{M}^p_q,r}$
The idea is to approximate $f \in N^s_{p,q,r}$ by $f_n := \sum_{k=0}^n f^k$.
From the definition of the spaces $N^s_{p,q,r}$ we immediately deduce that there exists $N \in \mathbb{N}$ such that

$$\Big(\sum_{j=N+1}^{\infty} 2^{sjr} ||f^j||^r_{M^p_q} \Big)^{\frac{1}{r}} < \varepsilon.$$

What concerns the first contributions, i.e. f^0-f^N, we know that

$$\sum_{j=0}^{N} f^j =: f_N \in O_M.$$

So,

$$||f - f_N|N^s_{p,q,r}|| \leq C\Big(\sum_{j=N+1}^{\infty} 2^{sjr}||f^j||^r_{M^p_q}\Big)^{\frac{1}{r}} < C\varepsilon$$

where C does not depend on f. This shows that f_N approximates f in the desired way.
The proof in the case $B^s_{\mathcal{M}^p_q,r}$ is the same - with the necessary modifications of course.

Density of O_M in $\mathcal{N}^s_{p,q,r}$
The idea is the same as above.
Observe that the definition implies that there exist integers n and m such that

$$\Big(\sum_{j \notin \{-n,\dots,0\dots m\}} 2^{sjr}||f_j||^r_{\mathcal{M}^s_{p,q}}\Big)^{1/r} \leq \frac{\varepsilon}{2}.$$

And as before, this gives us the result that O_M is dense in $\mathcal{N}^s_{p,q,r}$.

Another idea to prove the density of C^∞ in $N^s_{p,q,r}$ arises from the usual mollification:
We have to show that for any given ε and any given function $f \in N^s_{p,q,r}$ there exists a function $g \in C^\infty$ such that

$$||f - g|N^s_{p,q,r}|| \leq \varepsilon.$$

As indicated above, our candidate for g will be a function of the form

$$g = \varphi_\delta * f$$

where φ_δ is a mollifying sequence (and δ will be specified later on).
First of all, observe that due to Tonelli-Fubini we have $\varphi_\delta * f^j = (\varphi_\delta * f)^j$.
Now, as above we observe that the fact that f belongs to $N^s_{p,q,r}$ implies that there exists $N_0 \in \mathbb{N}$ such that

$$\Big(\sum_{N_0+1}^{\infty} 2^{jsr}||f^j|M^p_q||^r\Big)^{\frac{1}{r}} \leq \tilde{\varepsilon}$$

which together with lemma 98 immediately leads to the observation that

$$\Big(\sum_{N_0+1}^{\infty} 2^{jsr}||(f-f*\varphi_\delta)^j|M_q^p||^r\Big)^{\frac{1}{r}} \leq \frac{\varepsilon}{2}.$$

For the remaining contributions we first of all observe that

$$|f^j - f^j * \varphi_\delta| \leq ||\nabla f^j||_\infty \delta \leq C||f|N^s_{p,q,r}||2^j\delta.$$

In order to see this, note that $f^j \in N^s_{p,q,1}$ which together with propositions 93 and 94 and theorem 97 implies that

$$||\nabla f^j||_\infty \leq C||f|N^s_{p,q,r}||2^j.$$

In the case $j=0$ observe that

$$\begin{aligned}(\partial_{x_i} f)^0 &= \mathcal{F}^{-1}(i\xi_i \hat{f}\phi_0)\\ &= \mathcal{F}^{-1}(i\xi_i \hat{f}\phi_0(\phi_0+\phi_1))\\ &= f^0 * \mathcal{F}^{-1}(i\xi_i(\phi_0+\phi_1))\end{aligned}$$

which implies that

$$||\partial_{x_i} f^0|M_q^p|| \leq C||f^0|M_q^p||.$$

Apart from this observation, the argument is the same as the usual one known in the framework of Lebesgue spaces.
Now, we can calculate for any radius $R \in (0,1]$ and for any point $x_0 \in \mathbb{R}^n$

$$\begin{aligned}R^{\frac{n}{p}-\frac{n}{q}}||f^j - f^j*\varphi_\delta||_{L^q(B_R(x_0))} &= R^{\frac{n}{p}-\frac{n}{q}}\Big(\int_{B_R(x_0)} |f^j - f^j*\varphi_\delta|^q\Big)^{\frac{1}{q}}\\ &\leq CR^{\frac{n}{p}-\frac{n}{q}}\Big(||\nabla f^j||^q_\infty \delta^q R^n\Big)^{\frac{1}{q}}\\ &\leq CR^{\frac{n}{p}-\frac{n}{q}}\Big(||f|N^s_{p,q,r}||^q 2^{jq}\delta^q R^n\Big)^{\frac{1}{q}}\\ &= CR^{\frac{n}{p}}||f|N^s_{p,q,r}||\delta 2^j\\ &\leq C||f|N^s_{p,q,r}||\delta 2^j\end{aligned}$$

from which we conclude that

$$\begin{aligned}\Big(\sum_{j=0}^{N_0} 2^{jsr}||f^j - f^j*\varphi_\delta|M_q^p||^r\Big)^{\frac{1}{r}} &\leq \sum_{j=0}^{N_0} ||f|N^s_{p,q,r}||\delta 2^{N_0+N_0 sr}\\ &\leq (N_0+1)||f|N^s_{p,q,r}||\delta 2^{N_0+N_0 sr}\\ &\leq \frac{\varepsilon}{2}\end{aligned}$$

if we choose δ sufficiently small.
This shows that $f \in N^s_{p,q,r}$ can be approximated by compactly supported smooth function - the convolution $f * \varphi_\delta * f$ has compact support.
Now, we assume that $f \in B^s_{\mathcal{M}^p_q,r}$ where $s \geq 0$, $1 < q \leq 2$ and $1 \leq q \leq p \leq \infty$ has compact support. First of all, we observe that according to lemma 1 $f \in \mathcal{M}^p_q$ and since it has compact support, $f \in L^q$. From this we deduce that whenever $O \leq j \leq N_0$, $f^j \in B^s_{q,m}$ for all $s \in \mathbb{R}$ and arbitrary m and in particular, $f^j \in L^p$. So for each j there exists a δ_j such that

$$||f^j - f^j * \varphi_{\delta_j}||_q^m \leq \Big(\frac{\varepsilon}{2(N_0+1)}\Big)^m.$$

If we now choose δ small enough, then

$$\Big(\sum_{j=0}^{N_0} 2^{jsr}||f^j - f^j * \varphi_\delta|M^p_q||^r\Big)^{\frac{1}{r}} = \Big(\sum_{j=0}^{N_0} 2^{jsr}||(f - f*)^j\varphi_\delta|M^p_q||^r\Big)^{\frac{1}{r}} \leq \frac{\varepsilon}{2}.$$

The other frequencies are estimated as above.
Finally we observe that $f * \varphi_\delta$ is not only smooth but also compactly supported since it is a convolution of a compactly supported function with a compactly supported distribution.

□

Remark 7. A close look at the proof we just gave, shows that in fact

$$\cap_{m\geq 0} C^m$$

is dense in the above spaces.

Last, but not least we would like to mention a stability result which we will apply later on.

Lemma 8. *Let* $g \in B^0_{\mathcal{M}^n_2,2}$ *and* $f \in B^1_{\mathcal{M}^n_2,2,} \cap L^\infty$. *Then*

$$||gf|B^0_{\mathcal{M}^n_2,2}|| \leq C||g|B^0_{\mathcal{M}^n_2,2}||(||f|B^1_{\mathcal{M}^n_2,2}|| + ||f||_\infty),$$

i.e. $B^0_{\mathcal{M}^n_2,2}$ *is stable under multiplication with a function in* $B^1_{\mathcal{M}^n_2,2} \cap L^\infty$.

Proof of lemma 8:
We split the product fg into the three paraproducts $\pi_1(f,g)$, $\pi_2(f,g)$ and $\pi_3(f,g)$ and analyse each of them independently.

i) We start with $\pi_1(f,g) = \sum_{k=2}^{\infty}\sum_{l=0}^{k-2} f^l g^k$. It is easy to see that a simple adaptation of lemma 3.15 of [32] to our variant of Besov-Morrey, implies that it suffices to show that

$$\Big(\sum_{k=2}^{\infty} ||g^k \sum_{l=0}^{k-2} f^l||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \leq C||g|B^0_{\mathcal{M}_2^n,2}||(||f|B^1_{\mathcal{M}_2^n,2}|| + ||f||_\infty).$$

In fact, we calculate

$$\begin{aligned}
\Big(\sum_{k=2}^{\infty} ||g^k \sum_{l=0}^{k-2} f^l||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} &\leq \Big(\sum_{k=2}^{\infty} ||g^k (\sup_s |\sum_{l=0}^{s} f^l|)||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{k=2}^{\infty} ||g^k||^2_{\mathcal{M}_2^n} || \sup_s |\sum_{l=0}^{s} f^l|||^2_\infty\Big)^{\frac{1}{2}} \\
&\leq || \sup_s |\sum_{l=0}^{s} f^l|||_\infty (\sum_{k=2}^{\infty} ||g^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \\
&\leq || \sup_s |\sum_{l=0}^{s} f^l|||_\infty ||g|B^0_{\mathcal{M}_2^n,2}|| \\
&\leq ||f||_\infty ||g|B^0_{\mathcal{M}_2^n,2}|| \\
&\quad \text{because of lemma 4.4.2 of [43]} \\
&< \infty.
\end{aligned}$$

ii) Next, we study $\pi_2(f,g) = \sum_{k=0}^{\infty}\sum_{l=k-1}^{k+1} f^l g^k$. For our further calculations we fix $l = k$. We will see that what follows will not depend on this choice, so

$$||\pi_2(f,g)|B^0_{\mathcal{M}_2^n,2}|| \leq C \sup_{s\in\{-1,0,1\}} || \sum_{k=0}^{\infty} f^{k+s} g^k |B^0_{\mathcal{M}_2^n,2}||.$$

In fact, we will show a bit more, namely $\pi_2(f,g) \in B^1_{\mathcal{M}_1^{\frac{n}{2}},1}$. Again a simple adaptation of lemma 3.16 of [32] shows that we only have to estimate $\sum_{k=0}^{\infty} 2^k ||f^k g^k||_{\mathcal{M}_1^{\frac{n}{2}}}$. In fact, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} 2^k ||f^k g^k||_{\mathcal{M}_1^{\frac{n}{2}}} &\leq \sum_{k=0}^{\infty} 2^k ||f^k||_{\mathcal{M}_2^n} ||g^k||_{\mathcal{M}_2^n} \\
&\leq \Big(\sum_{k=0}^{\infty} 2^{2k} ||g^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \Big(\sum_{k=0}^{\infty} ||f^k||^2_{\mathcal{M}_2^n}\Big)^{\frac{1}{2}} \\
&\leq ||g|B^1_{\mathcal{M}_2^n,2}|| \; ||f|B^0_{\mathcal{M}_2^n,2}|| \\
&< \infty.
\end{aligned}$$

Once we have this, it implies together with theorem 97 - adapted to our variant of Besov-Morrey spaces - and the fact that $l^1 \subset l^2$ immediately that $\sum_{k=0}^{\infty} f^k g^k \in B^0_{\mathcal{M}^n_2,2}$. And finally we get that $\pi_2(f,g) \in B^0_{\mathcal{M}^n_2,2}$.

iii) The remaining addend is $\pi_3(f,g) = \pi_1(g,f)$. Again, as in i) it is enough to show that we can estimate $\Big(\sum_{l=2}^{\infty} ||f^l \sum_{k=0}^{l-2} g^k||^2_{\mathcal{M}^n_2}\Big)^{\frac{1}{2}}$ in the desired manner. In fact we observe that the following inequalities hold:

$$\begin{aligned}
\Big(\sum_{l=2}^{\infty} ||f^l \sum_{k=0}^{l-2} g^k||^2_{\mathcal{M}^n_2}\Big)^{\frac{1}{2}} &\leq \sum_{l=2}^{\infty} ||f^l \sum_{k=0}^{l-2} g^k||_{\mathcal{M}^n_2} \\
&\leq \sum_{l=2}^{\infty} ||f^l||_{\mathcal{M}^n_2} || \sum_{k=0}^{l-2} g^k||_{\infty} \\
&= \sum_{l=2}^{\infty} 2^l ||f^l||_{\mathcal{M}^n_2} 2^{-l} || \sum_{k=0}^{l-2} g^k||_{\infty} \\
&\leq \Big(\sum_{l=0}^{\infty} 2^{2l} ||f^l||^2_{\mathcal{M}^n_2}\Big)^{\frac{1}{2}} \Big(\sum_{l=0}^{\infty} 2^{-2l} || \sum_{k=0}^{l-2} g^k||^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C\Big(\sum_{l=0}^{\infty} 2^{2l} ||f^l||^2_{\mathcal{M}^n_2}\Big)^{\frac{1}{2}} \Big(\sum_{l=0}^{\infty} 2^{-2l} || \sum_{k=0}^{l} g^k||^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C||f|B^1_{\mathcal{M}^n_2,2}|| \Big(\sum_{l=0}^{\infty} 2^{-2l} || \sum_{k=0}^{l} g^k||^2_{\infty}\Big)^{\frac{1}{2}} \\
&\leq C||f|B^1_{\mathcal{M}^n_2,2}||\ ||g|B^{-1}_{\infty,2}|| \\
&\quad \text{according to lemma 4.4.2 of [43]} \\
&\leq C||f|B^1_{\mathcal{M}^n_2,2}||\ ||g|N^0_{n,2,2}|| \\
&\quad \text{due to theorem 97} \\
&\leq C||f|B^1_{\mathcal{M}^n_2,2}||\ ||g|B^0_{\mathcal{M}^n_2,2}|| \\
&< \infty.
\end{aligned}$$

If we put together all our results from i) to iii) we see that we have the estimate

$$||gf|B^0_{\mathcal{M}^n_2,2}|| \leq C||g|B^0_{\mathcal{M}^n_2,2}||(||f|B^1_{\mathcal{M}^n_2,2}|| + ||f||_{\infty})$$

as claimed.

□

Chapter 2

Two auxiliary lemmas for Morrey spaces

In this chapter we want to present some "stability results" which are similar to lemma 8. More precisely, we will prove the following assertion

Lemma 9. *Assume that* $h \in W^1_{M^p_q}$ *where* $1 \leq q \leq p < \infty$, $p < n$ *and* $\frac{4p}{n} \geq q$. *Moreover, let* $f \in M^n_4$. *Then*

$$hf \in M^p_q.$$

Proof of lemma 9:
First of all, we state that due to theorem 96 we have that

$$h \in M_s^{\frac{ps}{q}} \subset L^s \text{ where } s = \frac{nq}{n-p}.$$

Then by Hölder's inequality it is easy to see that

$$hf \in M^b_a \text{ where } \frac{1}{a} = \frac{1}{s} + \frac{1}{4} \text{ and } \frac{1}{b} = \frac{q}{ps} + \frac{1}{n}.$$

Finally, we have to show that $M^b_a \subset M^p_q$. This holds if

$$1 \leq q \leq \min\{a, p\} \leq \max\{a, p\} \leq b < \infty.$$

Observe

i) According to our hypothesis, we have $1 \leq q$ and $q \leq p$.

ii) In order to see that $q \leq a$,note that (use the information on the exponents we have!)

$$q \leq a \;\Leftrightarrow\; \frac{1}{q} - \frac{1}{a} \geq 0 \;\Leftrightarrow\; \frac{p}{nq} - \frac{1}{4} \geq 0 \;\Leftrightarrow\; \frac{4p}{n} \geq q$$

where again the last condition is satisfied thanks to our assumptions.

iii) Moreover, we calculate

$$\frac{1}{p} - \frac{1}{b} = \frac{1}{p} - \frac{1}{n} - \frac{q}{ps} = 0.$$

And similarly, we can verify that $b \geq a$.

These facts ensure that all the necessary requirements are fulfilled. This completes the proof of the lemma.

□

Lemma 10. *Assume that* $h \in W^2_{M^p_q}$ *where* $1 \leq q \leq p < \infty$, $p < \frac{n}{2}$ *and* $\frac{4p}{n} \geq q$. *Moreover, let* $f \in M_2^{\frac{n}{2}}$. *Then*

$$hf \in M^p_q.$$

The proof of this lemma is mutatis mutandis the same as the proof of lemma 9.

Part II

A generalisation of Wente's result

In this part we will study the following problem: Look at

$$-\Delta u = \frac{\partial}{\partial x_i} a \frac{\partial}{\partial x_j} b - \frac{\partial}{\partial x_i} b \frac{\partial}{\partial x_j} a =: J_{ij}(a,b). \tag{1}$$

and assume that the right hand side of the above equation belongs to some Besov-, Triebel-Lizorkin- or Besov-Morrey space. What can we say about the properties of u?
Our aim is to present a generalisation - stated above in the introduction as Theorem B - of the famous result of Wente, Tartar and Coifman-Lions-Meyer-Semmes (see [66], [56] and [16] for instance). For the sake of completeness, let us recall this result ($B_1^2(0)$ denotes the two-dimensional unit disc)

Theorem 11. *([66],[16], [56]) Let a and b be two function in $W^{1,2}(B_1^2(0),\mathbb{R})$. Moreover, let ϕ be the unique solution of*

$$\begin{cases} -\Delta\phi & = \nabla a \cdot \nabla^\perp b = \partial_x a \partial_y b - \partial_y a \partial_x b \quad \text{in } B_1^2(0) \\ \varphi & = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial B_1^2(0), \end{cases}$$

Then the following estimates hold

$$||\phi||_\infty + ||\nabla\phi||_{2,1} + ||\nabla^2\phi||_1 \le C||\nabla a||_2 ||\nabla b||_2$$

Note that the above equation is **scalar**!

So if we start with slightly modified spaces in which a and b shall lie, we still have continuity of solutions to (1). More precisely we have

Theorem 12. *Assume that*

$$\Big(\sum_{s=0}^{\infty} ||a_x^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} < \infty \; x = z_i, z_j$$

and

$$\Big(\sum_{s=0}^{\infty} ||b_y^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} < \infty \; y = z_i, z_j$$

as well as

$$\Big(\sum_{s=0}^{\infty} ||a^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} < \infty$$

and

$$\Big(\sum_{s=0}^{\infty} ||b^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} < \infty.$$

Then any solution of

$$-\Delta u = J_{ij}(a,b)$$

where as above $i, j, \in \{1,\dots,n\}$ is continuous.

Remark 13. Note that the hypothesis of the preceding result can be reformulated as follows:
We assume that $\left(\sum_{s=0}^{\infty}||a_x^s|M_2^n||^2\right)^{\frac{1}{2}} < \infty$, i.e. that

$$\begin{aligned}
&\Big(\sum_{s=0}^{\infty}||a^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{s=0}^{\infty}(\sup_{x_0\in\mathbb{R}^n}\sup_{0<r} r^{\frac{n}{n}-\frac{n}{2}}||a^s|L^2(B(x_0,r))||)^2\Big)^{\frac{1}{2}} \\
&< \infty.
\end{aligned}$$

But this last requirement is equivalent to the following

$$\Big(\sum_{s=0}^{\infty}(\sup_{x_0\in\mathbb{R}^n}\sup_{0<r} r^{2-n}\int_{B(x_0,r)}|a^s|^2)^2\Big)^{\frac{1}{2}} < \infty.$$

As in the above cited two-dimensional case, we are also able to give estimates for the first and second derivatives: Similar to the situation we had in theorem 11, we will assume in addition that the supports of a_x, a_y, b_x and b_y are contained in the n-dimensional unit ball B^n. Moreover, assume that $n \geq 3$. We start with the following estimates for the gradient of u.

Proposition 14. *Assume that a_x, a_y, b_x and b_y are distributions whose support is contained in B^n and belong to $B^0_{\mathcal{M}_2^n,2}$, $n \geq 3$. Moreover, let u be a solution (in the sense of distributions) of*

$$-\Delta u = a_x b_y - b_x a_y.$$

Then we it holds

$$\nabla u \in B^0_{\mathcal{M}_2^n,1}$$

and

$$\nabla u \in B^0_{2,\frac{4}{3}}.$$

In particular, $\nabla u \in L^{2-\varepsilon}$ for all $\varepsilon > 0$.

Lemma 15. *Assume that a_x, a_y, b_x and b_y are distributions whose support in B^n and belong to $B^0_{\mathcal{M}_2^n,2}$. Moreover, let u be a solution (in the sense of distributions) of*

$$-\Delta u = a_x b_y - b_x a_y.$$

Then it holds

$$\nabla^2 u \in B^{-1}_{\mathcal{M}_2^n,1} \subset B^{-2}_{\infty,1}.$$

Before we will come to the proofs of these results, let us discuss a duality result on which the here presented statements heavily rely.

Chapter 1

Duality result

Here, we want to present a description of predual spaces of particular Besov-Morrey spaces which we shall encounter later again.

Proposition 16. *The dual space of* $b^0_{L^1(H^{n-2}_\infty),\infty}$ *is the space* $B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$.

Remark 17. The above result has the same flavour as (see for instance [43])

$$(b^0_{\infty,\infty})^* = B^0_{1,1}$$

Proof of proposition 16:
We have to show the two inclusion relations.
We start with $(b^0_{L^1(H^{n-2}_\infty),\infty})^* \supset B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$:
Assume that $f \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1} \subset \mathcal{S}'$ and assume that $\psi \in b^0_{L^1(H^{n-2}_\infty),\infty}$. By density we may assume that $\psi \in \mathcal{S}$. We have to show that $f \in (b^0_{L^1(H^{n-2}_\infty),\infty})^*$. To this end let $\sum_{k=0}^\infty \check{\varphi}_k * \psi_k$ be a representation of ψ with

$$\sup_k ||\psi_k||_{L^1(H^{n-2}_\infty)} \leq 2||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}||.$$

Note that in our case - as a tempered distribution - f acts on ψ and we estimate

$$\begin{aligned} |f(\psi)| &= |f(\sum_{k\geq 0} \check{\varphi}_k * \psi_k)| \\ &= \Big|f\Big(\sum_{k=o}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big)\Big| \\ &= \Big|\sum_{k=0}^{\infty} f\Big(\mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big)\Big| = \Big|\sum_{k=0}^{\infty} \int f\mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k)\Big| \end{aligned}$$

and further

$$\begin{aligned}
|f(\psi)| &= \Big|\sum_{k=0}^{\infty} \psi_k \mathcal{F}(\varphi_k \mathcal{F}^{-1} f)\Big| = \Big|\sum_{k=0}^{\infty} \int \psi_k \, df\Big| \\
&\quad \text{where } df = \mathcal{F}(\varphi_k \mathcal{F}^{-1} f)\, d\lambda \text{ with } \lambda \text{ the Lebesgue measure} \\
&\leq \sum_{k=0}^{\infty} |\psi_k \mathcal{F}(\varphi_k \mathcal{F}^{-1} f)| \\
&\leq \sup_{k\geq 0} ||\psi_k||_{L^1(H_\infty^{n-2})} \sum_{k=o}^{\infty} ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}^{\frac{n}{2}}} \\
&\quad \text{recall proposition 103} \\
&= \sup_{k\geq 0} ||\psi_k||_{L^1(H_\infty^{n-2})} \sum_{k=o}^{\infty} ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\
&\quad \text{cf. also remark 105} \\
&\leq C \sup_{k\geq 0} ||\psi_k||_{L^1(H_\infty^{n-2})} \sum_{k=o}^{\infty} ||\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\
&\leq C||\psi|b^0_{L^1(H_\infty^{n-2}),\infty}|| \;\; ||f|B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| \\
&< \infty \\
&\quad \text{thanks to our assumptions.}
\end{aligned}$$

Now we show the other inclusion, $(b^0_{L^1(H_\infty^{n-2}),\infty})^* \subset B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$:
We start with $f \in (b^0_{L^1(H_\infty^{n-2}),\infty})^*$ and we have to show that f belongs also to $B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$: First of all, note that f gives also rise to elements of $(L^1(H_\infty^{n-2}))^*$ as follows: Each $\psi \in b^0_{L^1(H_\infty^{n-2}),\infty}$ can be seen as a sequence $\{\psi_k\}_{k=0}^{\infty} \subset L^1(H_\infty^{n-2})$, and of course $\check{\varphi}_k * \psi_k \in b^0_{L^1(H_\infty^{n-2}),\infty}$ $\forall k \in \mathbb{N}$. Moreover, for each $k \in \mathbb{N}$ we have - again by density of $\mathcal{S}$ -

$$\begin{aligned}
f(\delta_{kj}(\check{\varphi}_j * \psi_j)) &= < f, \delta_{kj}\psi >_{(b^0_{L^1(H_\infty^{n-2}),\infty})^*, b^0_{L^1(H_\infty^{n-2}),\infty}} \\
&= < f, \check{\varphi}_k * \psi_k >_{(b^0_{L^1(H_\infty^{n-2}),\infty})^*, b^0_{L^1(H_\infty^{n-2}),\infty}} \\
&= < f, \check{\varphi}_k * \psi_k >_{\mathcal{S}',\mathcal{S}} \\
&= < f, \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) >_{\mathcal{S}',\mathcal{S}} \\
&= < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{S}',\mathcal{S}} \\
&= < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{M}^{\frac{n}{2}}, L^1(H_\infty^{n-2})} .
\end{aligned}$$

Next we will construct a special element of $b^0_{L^1(H_\infty^{n-2}),\infty}$:
Let $0 < \varepsilon$ small.
We choose ψ_k such that

- $\psi_k \in \mathcal{S}$: Remember that we have density!
- $||\psi_k||_{L^1(H_\infty^{n-2})} \leq 1$
- $0 < \; < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{M}^{\frac{n}{2}}, L^1(H_\infty^{n-2})}$
-

$$\begin{aligned} < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), \psi_k >_{\mathcal{M}^{\frac{n}{2}}, L^1(H_\infty^{n-2})} &\geq ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}^{\frac{n}{2}}} - \varepsilon 2^{-k} \\ &= ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{(L^1(H_\infty^{n-2}))^*} - \varepsilon 2^{-k} \\ &= \sup_{\substack{u \in L^1(H_\infty^{n-2}) \\ ||u||_{L^1(H_\infty^{n-2})} \leq 1}} | < \mathcal{F}(\varphi_k \mathcal{F}^{-1} f), u > | - \varepsilon 2^{-k}. \end{aligned}$$

Note that like that $\psi = \sum_{k=0}^{\infty} \check{\varphi}_k * \psi_k \in b^0_{L^1(H_\infty^{n-2}),\infty}$ with

$$||\psi | b^0_{L^1(H_\infty^{n-2}),\infty}|| \leq 1.$$

If we put now all this together we find - recall that f acts linearly! -

$$\begin{aligned} \sum_{k=0}^{\infty} ||f^k||_{\mathcal{M}_1^{\frac{n}{2}}} &= \sum_{k=0}^{\infty} ||\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\ &= C \sum_{k=0}^{\infty} ||\mathcal{F}(\varphi_k \mathcal{F}^{-1} f)||_{\mathcal{M}_1^{\frac{n}{2}}} \\ &\leq 2\varepsilon + f(\psi) \\ &\quad \psi \text{ as constructed above} \\ &\leq 2\varepsilon + ||f|(b^0_{L^1(H_\infty^{n-2}),\infty})^*|| \;\; ||\psi|b^0_{L^1(H_\infty^{n-2}),\infty}|| \\ &\leq 2\varepsilon + ||f|(b^0_{L^1(H_\infty^{n-2}),\infty})^*||. \end{aligned}$$

Since this holds for all $0 < \varepsilon$ we let ε tend to zero and get the desired inclusion.
All together we established the duality result we claimed above.

□

Chapter 2

Proof of the generalisation of Wente's result

2.1 Proof of theorem 12:

The proof we present here, is in close analogy to an alternative proof of the original Wente result in dimension two. This alternative approach is presented in Appendix B.1 and to emphasise the parallels, we sometimes refer to corresponding steps there.

The proof of this assertion is split into several parts: In a first step we show that $\pi_1(a_x, b_y),\ \pi_3(a_x, b_y),\ \pi_3(a_y, b_x)$ and $\pi_1(a_y, b_x) \in B^{-2}_{\infty,1}$ and
$\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^{-2}_{\infty,1}$. (The reader who is not familiar with paraproducts can find the necessary definitions and details in Appendix B1.)
Once we have this we show in a second step that under this hypothesis the solution u of

$$-\Delta u = f \text{ where} f \in B^{-2}_{\infty,1}$$

is continuous.

Analysis of $\pi_1(a_x, b_y) = \sum_{k=2}^{\infty} \sum_{l=0}^{k-2} a_x^l b_y^k$:

We will show that $\pi_1(a_x, b_y) \in B^{-2}_{\infty,1}$.
Our hypotheses together with theorem 97 ensures us that $a_x,\ b_y \in B^{-1}_{\infty,2}$.
Next, due to proposition 123 it is enough to prove that

$$||2^{-2j} c_j | l^1(L^\infty)|| < \infty$$

where $c_j := \sum_{t=0}^{k-2} a_x^t b_y^j$.

We actually have

$$\begin{aligned}
||2^{-2j}c_j|l^1(L^\infty)|| &= \sum_{j=0}^{\infty} 2^{-2j}||\sum_{t=0}^{j-2} a_x^t b_y^j||_\infty \\
&\leq \sum_{j=0}^{\infty} 2^{-2j}||\sum_{t=0}^{j-2} a_x^t||_\infty ||b_y^j||_\infty \\
&= \sum_{j=0}^{\infty} 2^{-j}||\sum_{t=0}^{j-2} a_x^t||_\infty 2^{-j}||b_y^j||_\infty \\
&\leq \Big(\sum_{j=0}^{\infty} 2^{-2j}||\sum_{t=0}^{j-2} a_x^t||_\infty^2\Big)^{\frac{1}{2}} \Big(\sum_{j=0}^{\infty} 2^{-2j}||b_y^j||_\infty^2\Big)^{\frac{1}{2}} \\
&\quad \text{due to Hölder's inequality} \\
&= ||2^{-j}|\sum_{t=0}^{j-2} a_x^t|\ |l^2(L^\infty)|| ||b_y|B_\infty^{-1}|| \\
&\leq C||2^{-j}|\sum_{t=0}^{j} a_x^t|\ |l^2(L^\infty)|| ||b_y|B_\infty^{-1}|| \\
&\leq C||a_x|B_{\infty,2}^{-1}|| ||b_y|B_\infty^{-1}|| \\
&\quad \text{because of lemma 126} \\
&< \infty \\
&\quad \text{thanks to our hypothesis.}
\end{aligned}$$

This shows that in fact $\pi_1(a_x, b_y) \in B_{\infty,1}^{-2}$. Similarly one proves that also $\pi_1(a_y, b_x)$, $\pi_3(a_x, b_y)$ and $\pi_1(a_y, b_x)$ belong to the same space.

In remains to analyse the contribution where the frequencies are comparable. This is our next goal.

Analysis of $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$:

In stead of first applying the embedding result of Kozono/Yamazaki which embeds Morrey-Besov spaces into Besov spaces and then analysing a certain quantity, we invert the order of these steps in order to estimate $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$.
In comparison with the proof of proposition (117) we will not test the above expression with a element from its dual space, but we will use the result concerning predual spaces of Morrey spaces, proposition 16.

Next, there will follow some technical lemmas:
Now, lemma 129 has the following counterpart in this modified context. Recall that $\mathcal{S}$ is dense in $b^0_{L^1(H^{n-2}_\infty),\infty}$:

Lemma 18. *Let $\phi \in \Phi(\mathbb{R}^n)$ and assume that $\psi \in \mathcal{S} \cap L^1(H^{n-2}_\infty)$ with representation $\{\psi_k\}_{k=0}^\infty$, i.e. $\sum_{k=0}^\infty \check{\varphi}_k * \psi_k = \psi$, such that*

$$\sup_k ||\psi_k||_{L^1(H^{n-2}_\infty)} \leq 2||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}||.$$

Then

$$\begin{aligned}
\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * \psi_k\Big|\Big|_{L^1(H^{n-2}_\infty)} &= \Big|\Big|\frac{\partial}{\partial x}(\check{\varphi}_k * \psi_k)\Big|\Big|_{L^1(H^{n-2}_\infty)} \\
&\leq C2^s||\psi_k||_{L^1(H^{n-2}_\infty)} \leq C2^s||\psi|b^0_{L^1(H^{n-2}_\infty),\infty}||.
\end{aligned}$$

Proof of lemma 18:
For the proof of this lemma, we need the a corollary of Adams, see below (cf. [1]).
It holds

$$\begin{aligned}
\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * \psi_k\Big|\Big|_{L^1(H^{n-2}_\infty)} &\leq ||\,|\frac{\partial}{\partial x}\check{\varphi}_k| *|\,\psi_k|\,||_{L^1(H^{n-2}_\infty)} \\
&\leq C \sup_{\substack{\mu\in\mathcal{M}^{\frac{n}{2}}_+ \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{\int |\frac{\partial}{\partial x}\check{\varphi}_k| *|\,\psi_k|\;d\mu\right\} \\
&= C \sup_{\substack{\mu\in\mathcal{M}^{\frac{n}{2}}_+ \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{\int\int |\frac{\partial}{\partial x}\check{\varphi}_k|(x-y)|\psi_k|(y)\;d\lambda(y)d\mu(x)\right\} \\
&= C \sup_{\substack{\mu\in\mathcal{M}^{\frac{n}{2}}_+ \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{\int |\psi_k|(y)\int |\frac{\partial}{\partial x}\check{\varphi}_k|(x-y)\;d\mu(x)d\lambda(y)\right\} \\
&\quad \text{by Tonelli's theorem} \\
&= C \sup_{\substack{\mu\in\mathcal{M}^{\frac{n}{2}}_+ \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{\int |\psi_k|(y)\int |\frac{\partial}{\partial x}\check{\varphi}_k|(y-x)\;d\mu(x)d\lambda(y)\right\} \\
&\quad \text{note that } \varphi_k \text{ can be chose radial which together} \\
&\quad \text{with theorem 52 asserts that } \check{\varphi}_k \text{ is radial} \\
&\quad \text{and } \frac{\partial}{\partial x}\check{\varphi}_k \text{ too}
\end{aligned}$$

and we continue

$$\begin{aligned}
\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * \psi_k\Big|\Big|_{L^1(H_\infty^{n-2})} &\leq C \sup_{\substack{\mu\in\mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{\int |\psi_k|(y)\frac{\partial}{\partial x}\check{\varphi}_k|(y-x) * \mu(y)\, d\lambda(y)\right\} \\
&= C \sup_{\substack{\mu\in\mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{\int |\psi_k|(y)\, d\nu(y)\right\} \\
&\quad \text{where } \nu := \frac{\partial}{\partial x}\check{\varphi}_k\lambda * \mu \\
&\leq C \sup_{\substack{\mu\in\mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{||\psi_k||_{L^1(H_\infty^{n-2})}\, ||\frac{\partial}{\partial x}\check{\varphi}_k\lambda * \mu||_{\mathcal{M}^{\frac{n}{2}}}\right\} \\
&\leq C \sup_{\substack{\mu\in\mathcal{M}_+^{\frac{n}{2}} \\ ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\leq 1}} \left\{||\psi_k||_{L^1(H_\infty^{n-2})}\, ||\frac{\partial}{\partial x}\check{\varphi}_k||_{L^1}\, ||\mu||_{\mathcal{M}^{\frac{n}{2}}}\right\} \\
&\quad \text{by lemma 98} \\
&\leq C||\psi_k||_{L^1(H_\infty^{n-2})}\, ||\frac{\partial}{\partial x}\check{\varphi}_k||_{L^1} \\
&\leq C2^k||\psi_k||_{L^1(H_\infty^{n-2})} \\
&\quad \text{as in the case of proposition 117} \\
&\leq C2^k||\psi|b^0_{L^1(H_\infty^{n-2}),\infty}||
\end{aligned}$$

what we had to prove.

□

Corollary 19. *([1]) If $f(x)\geq 0$ is lower semi-continuous on $\mathbb{R}^n$, then*

$$||f||_{L^1(H_\infty^d)} = \int f\, dH_\infty^d \sim \sup\left\{\int f\, d\mu|\mu\in\mathcal{M}_+^{\frac{n}{n-d}} \text{ and } ||\mu||_{\mathcal{M}^{\frac{n}{n-d}}}\leq 1\right\}.$$

Next, recall that the idea which lead to lemma 132 was independent on any norm! So it is easy to see that the corresponding result holds also here.

Now, we can start with the estimate of $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$. Our goal is to show that $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$ belongs to $B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$. Making use of the above duality result, see proposition 16, we will first show that

$$\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1}\ \forall s\in\mathbb{N}$$

then we establish

$$\sum_{s=0}^{\infty} \Big|\Big| \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \Big| B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \Big|\Big| < \infty.$$

This ensures that

$$\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}.$$

First of all, let us fix $t = s + j$ where $j \in \{-1, 0, 1\}$.
In order to show that $a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1}$ it suffices to show that for all $\psi \in b^0_{L^1(H_\infty^{n-2}),\infty}$ with $||\psi|b^0_{L^1(H_\infty^{n-2}),\infty}|| \leq 1$ the following inequality holds

$$\int_{\mathbb{R}^n} \psi \ d(a_x^t b_y^s - a_y^t b_x^s) = \int_{\mathbb{R}^n} \psi(a_x^t b_y^s - a_y^t b_x^s) \ d\lambda < \infty$$

where as before λ denotes the Lebesgue measure.
Moreover, in the subsequent calculations we assume that for ψ we have a representation $\{\psi_k\}_{k=0}^{\infty}$, i.e. $\sum_{k=0}^{\infty} \check{\varphi}_k * \psi_k = \psi$, such that

$$\sup_k ||\psi_k||_{L^1(H_\infty^{n-2})} \leq 2||\psi|b^0_{L^1(H_\infty^{n-2}),\infty}|| \leq 2$$

and again, recall that we have density of $\mathcal{S}$ in $b^0_{L^1(H_\infty^{n-2}),\infty}$.
In this case we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(a_x^t b_y^s - a_y^t b_x^s) &= \int_{\mathbb{R}^n} \psi \frac{\partial}{\partial x}(a^t b_y^s) - \psi \frac{\partial}{\partial y}(a^t b_x^s) \\
&= \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial x}(a^t b_y^s) \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) \Big) \\
&\quad - \frac{\partial}{\partial y}(a^t b_x^s) \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) \Big) \Big] \\
&\quad \text{because of the same reason as in lemma 132} \\
&= \int_{\mathbb{R}^n} \Big[- a^t b_y^s \frac{\partial}{\partial x} \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) \Big) \\
&\quad + a^t b_x^s \frac{\partial}{\partial y} \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F}\psi_k) \Big) \Big] \\
&\quad \text{by a simple integration by parts.}
\end{aligned}$$

And furthermore

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(a_x^t b_y^s - a_y^t b_x^s) &= \int_{\mathbb{R}^n} \Big[-a^t b_y^s \Big(\sum_{k=0}^{s+3} \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k \Big) \\
&\quad + a^t b_x^s \Big(\sum_{k=0}^{s+3} \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k \Big) \Big] \\
&= \sum_{k=0}^{s+3} \int_{\mathbb{R}^n} \Big[-a^t b_y^s \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k \\
&\quad + a^t b_x^s \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k \Big] \\
&\leq \sum_{k=0}^{s+3} \Big(||a^t b_y^s | \mathcal{M}^{\frac{n}{2}}|| \, || \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \\
&\quad + ||a^t b_x^s | \mathcal{M}^{\frac{n}{2}}|| \, || \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \Big) \\
&\quad \text{by proposition 103} \\
&= \sum_{k=0}^{s+3} \Big(||a^t b_y^s | \mathcal{M}_1^{\frac{n}{2}}|| \, || \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \\
&\quad + ||a^t b_x^s | \mathcal{M}_1^{\frac{n}{2}}|| \, || \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \Big) \\
&\leq \sum_{k=0}^{s+3} \Big(||a^t | \mathcal{M}_2^{n}|| \, ||b_y^s | \mathcal{M}_2^{n}|| \, || \frac{\partial}{\partial x} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \\
&\quad + ||a^t | \mathcal{M}_2^{n}|| \, ||b_x^s | \mathcal{M}_2^{n}|| \, || \frac{\partial}{\partial y} \check{\varphi}_k * \psi_k | L^1(H_\infty^{n-2})|| \Big) \\
&\quad \text{because of Hölder's inequality with Morrey norms} \\
&\quad \text{see also remark below} \\
&\leq \sum_{k=0}^{s+3} \Big(||a^t | \mathcal{M}_2^{n}|| \, ||b_y^s | \mathcal{M}_2^{n}|| \; 2^k ||\psi | b^0_{L^1(H_\infty^{n-2}),\infty}|| \\
&\quad + ||a^t | \mathcal{M}_2^{n}|| \, ||b_x^s | \mathcal{M}_2^{n}|| \; 2^k ||\psi | b^0_{L^1(H_\infty^{n-2}),\infty}|| \Big) \\
&\quad \text{according to lemma 18} \\
&\leq C 2^s ||a^t | \mathcal{M}_2^{n}|| \, ||b_y^s | \mathcal{M}_2^{n}|| + C 2^s ||a^t | \mathcal{M}_2^{n}|| \, ||b_x^s | \mathcal{M}_2^{n}|| \\
&< \infty \\
&\quad \text{due to our assumptions.}
\end{aligned}$$

Thus we have seen that for all $s \in \mathbb{N}$

$$a_x^t b_y^s - a_y^t b_x^s \in (b^0_{L^1(H_\infty^{n-2}),\infty})^* = B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}.$$

Next, we study

$$\sum_{s=0}^{\infty} \Big|\Big| \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \Big| B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \Big|\Big|.$$

What concerns this latter quantity, we will assume for the sake of simplicity that $t = s$. Then we can estimate

$$\begin{aligned}
\sum_{s=0}^{\infty} ||a_x^s b_y^s - a_y^s b_x^s | B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| &= ||a_x^0 b_y^0 - a_y^0 b_x^0 | B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| + \sum_{s=1}^{\infty} ||a_x^s b_y^s - a_y^s b_x^s | B^0_{\mathcal{M}_1^{\frac{n}{2}},1}|| \\
&\leq C||a^0|\mathcal{M}_2^n|| \; ||b_y^0|\mathcal{M}_2^n|| + C||a^0|\mathcal{M}_2^n|| \; ||b_x^0|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} 2^s ||a^s|\mathcal{M}_2^n|| \; ||b_y^s|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} 2^s ||a^s|\mathcal{M}_2^n|| \; ||b_x^s|\mathcal{M}_2^n|| \\
&\leq C||a^0|\mathcal{M}_2^n|| \; ||b_y^0|\mathcal{M}_2^n|| + C||a^0|\mathcal{M}_2^n|| \; ||b_x^0|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} ||a_x^s|\mathcal{M}_2^n|| \; ||b_y^s|\mathcal{M}_2^n|| \\
&\quad + C\sum_{s=1}^{\infty} ||a_y^s|\mathcal{M}_2^n|| \; ||b_x^s|\mathcal{M}_2^n|| \\
&\quad \text{similar to lemma 39} \\
&\quad \text{cf. also theorem 2.9 in [29]} \\
&\leq C||a^0|\mathcal{M}_2^n|| \; ||b_y^0|\mathcal{M}_2^n|| + C||a^0|\mathcal{M}_2^n|| \; ||b_x^0|\mathcal{M}_2^n|| \\
&\quad + C\Big(\sum_{s=1}^{\infty} ||a_x^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \Big(\sum_{s=1}^{\infty} ||b_y^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \\
&\quad + C\Big(\sum_{s=1}^{\infty} ||a_y^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \Big(\sum_{s=1}^{\infty} ||b_x^s|\mathcal{M}_2^n||^2\Big)^{\frac{1}{2}} \\
&\quad \text{by Hölder's inequality} \\
&< \infty \\
&\quad \text{thanks to our hypothesis.}
\end{aligned}$$

All together we have seen that

$$\sum_{s=0}^{\infty} a_x^s b_y^s - a_y^s b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}.$$

Now, since the above estimate is independent of the choice of j we immediately conclude that

$$\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in N^0_{\frac{n}{2},1,1}$$

Now, as we know that $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^0_{\mathcal{M}_1^{\frac{n}{2}},1} \subset N^0_{\frac{n}{2},1,1}$ we apply the embedding result of Kozono/Yamazaki, theorem 97, and find that

$$\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^{-2}_{\infty,1}.$$

Remark 20. Assume that $f, g \in \mathcal{M}_2^n$. Then we have for all $0 < r$ and for all $x \in \mathbb{R}^n$

$$\begin{aligned} ||fg||_{L^1(B_r(x))} &\leq ||f||_{L^2(B_r(x))}||g||_{L^2(B_r(x))} \\ &\leq C_1 r^{\frac{n}{2}-1} C_2 r^{\frac{n}{2}-1} \\ &= C r^{n-2}. \end{aligned}$$

According to the definition, this shows that $fg \in \mathcal{M}_1^{\frac{n}{2}}$.

Conclusion

Finally, by the same arguments as in the proof of proposition 38 we conclude that any solution of

$$-\Delta u = f$$

where $f \in B^{-2}_{\infty,1}$ is bounded continuous since due to Sickel/Triebel [47] we know

$$B^0_{\infty,1} \subset C.$$

□

Corollary 21. *A careful look at the proof of theorem 12 and at the proof of theorem 116 reveals that the assertion of theorem 12 holds also under the hypothesis*

$$a_x \in \mathcal{N}^0_{n,2,2}\ ,\ x = z_i, z_j$$

and

$$b_x \in \mathcal{N}^0_{n,2,2}\ ,\ x = z_i, z_j.$$

Remark 22. Note that in the result above, we just give a sufficient condition. If one wants to study necessary conditions in order to obtain continuity of solutions to (1), one might start by studying limit cases for embeddings of Besov (and respectively Triebel-Lizorkin) spaces into C, the space of continuous and bounded functions, probably by working on bounded domains instead of the whole space $\mathbb{R}^n$.
The choice of Besov-Morrey spaces was motivated by the work of Rivière and Struwe (see [42]).

Now, if one compares the above results with theorem 11 one realises that so far, we didn't say anything about the derivatives of a solution u of

$$-\Delta u = a_x b_y - b_x a_y.$$

2.2 Proof of proposition 14:

We will prove the two estimates separately:

i) In a first step we show that $a_x b_y - a_y b_x \in B^{-1}_{\mathcal{M}_2^n,1}$:
From the proof of theorem 12 we know that

$$\sum_{k=0}^{\infty} \sum_{s=k-1}^{k+1} a_x^k b_y^s - a_y^k b_x^s \in B^0_{\mathcal{M}_2^n,1} \subset B^{-1}_{\mathcal{M}_2^n,1}.$$

Next, we observe that

$$\begin{aligned} ||\pi_3(a_x, b_x)|B^{-1}_{\mathcal{M}_2^n,1}|| &\leq C \sum_{s=0}^{\infty} 2^{-s} \Big|\Big| \sum_{k=0}^{s-2} a_x^s b_y^k \Big|\Big|_{\mathcal{M}_2^n} \\ &\quad \text{by a simple modification of lemma 3.16 in [32]} \\ &\leq C \sum_{s=0}^{\infty} 2^{-s} ||a_x^s||_{\mathcal{M}_2^n} \Big|\Big| \sum_{k=0}^{s-2} b_y^k \Big|\Big|_\infty \\ &\leq C \Big(\sum_{s=0}^{\infty} ||a_x^s||^2_{\mathcal{M}_2^n} \Big)^{\frac{1}{2}} \Big(\sum_{s=0}^{\infty} 2^{-2s} \Big|\Big| \sum_{k=0}^{s-2} b_y^k \Big|\Big|^2_\infty \Big)^{\frac{1}{2}} \\ &\leq C ||a_x|B^0_{\mathcal{M}_2^n,2}|| \Big(\sum_{s=0}^{\infty} 2^{-2s} \Big|\Big| \sum_{k=0}^{s} b_y^k \Big|\Big|^2_\infty \Big)^{\frac{1}{2}} \\ &\leq C ||a_x|B^0_{\mathcal{M}_2^n,2}||\ ||b_y|B^{-1}_{\mathcal{M}_2^n,2}|| \\ &\quad \text{according to lemma 4.4.2 of [43]} \\ &\leq ||a_x|B^0_{\mathcal{M}_2^n,2}||\ ||b_y|B^0_{\mathcal{M}_2^n,2}||. \end{aligned}$$

Now, since

$$\partial_{x_i} u = \mathcal{F}^{-1}\Big(i\frac{\xi_i}{|\xi|^2}\mathcal{F}(\Delta u)\Big)$$

we note first, that due to the facts that $\Delta u \in F^0_{1,2} \subset L^1$ and $r^{-1} \in L^{\frac{n}{n-1}}$ for $n \geq 3$,

$$(\nabla u)^0 \in L^n \subset \mathcal{M}_2^n$$

which implies that $(\nabla u)^0 \in B^0_{\mathcal{M}_2^n,2}$.
Second, for $s \geq 1$ we have

$$||(\nabla u)^s||_{\mathcal{M}_2^n} \leq C2^{-s}||(\Delta u)^s||_{\mathcal{M}_2^n}$$

which leads to the conclusion - remember the first step! - that $\sum_{s\geq 1}(\nabla u)^s \in B^0_{\mathcal{M}_2^n,1}$.
Alternatively one could observe that

$$\partial^{|\alpha|}\Big(\frac{\xi_i}{|\xi|^2}\Big) \leq C|\xi|^{-1-|\alpha|}$$

information, which together with theorem 2.9 in [29] leads to the same conclusion as above, namely that

$$\nabla u \in B^0_{\mathcal{M}_2^n,1}.$$

ii) On the other hand theorem 116 and the proof of theorem 12 imply that

$$a_x b_y - a_y b_x \in B^{-2}_{\infty,1} \cap F^0_{1,2}$$

from which we deduce - similarly to the proof of proposition 38 - that

$$\nabla u \in B^{-1}_{\infty,1} \cap F^1_{1,2}.$$

Now, interpolation by the complex method (see e.g. proposition 2.5.2 in [43]) between the two spaces $B^{-1}_{\infty,1}$ and $F^1_{1,2}$ leads to the conclusion that

$$\nabla u \in B^0_{2,\frac{4}{3}} \subset L^2.$$

These estimates complete the proof.

□

2.3 Proof of lemma 15:

This proof is very similar to the one of proposition 14.
In stead of the observation $\partial^{|\alpha|}\left(\frac{\xi_i}{|\xi|^2}\right) \leq C|\xi|^{-1-|\alpha|}$ here we use theorem 2.9 of [29] together with the fact that

$$\partial^{|\alpha|}\left(\frac{\xi_i \xi_j}{|\xi|^2}\right) \leq C|\xi|^{-|\alpha|}.$$

□

Part III

Gauge and regularity results for problems of the form $-\Delta u = \Omega \cdot \nabla u$ with Ω antisymmetric

As an application of what we did so far, we would like to present a generalisation (cf. theorem C and corollary D in the introduction) of the following regularity result of Rivière (see [38])

Theorem 23. *([38]) Let $m \in \mathbb{N}$. For every $\Omega = (\Omega^i_j)_{1\leq i,j\leq m}$ in $L^2(B^2_1(0), so(m)\otimes \Lambda^1\mathbb{R}^2)$ every weak solution u of*

$$-\Delta u = \Omega \cdot \nabla u$$

is continuous.

Note that $L^2(B^2_1(0), so(m) \otimes \Lambda^1\mathbb{R}^2)$ means that $\forall\ i, j \in \{1, \ldots, m\}\,,\ \Omega^i_j \in L^2(B^2_1(0), \Lambda^1\mathbb{R}^2)$ and $\Omega^i_j = -\Omega^j_i$. Moreover, the above equation has to be understood in the following sense: For all indexes $i \in \{1, \ldots, m\}$ we have $-\Delta u^i = \sum_{j=1}^m \Omega^i_j \cdot \nabla^j$.

As a possible generalisation we have

Theorem 24. *Let the dimension n satisfy $n \geq 3$. For every $m \in \mathbb{N}$ there exists a constant $\varepsilon(m) > 0$ such that for every $\Omega \in B^0_{\mathcal{M}^n_2,2}(B^n, so(m)\otimes\Lambda^1\mathbb{R}^n)$ with*

$$||\Omega|B^0_{\mathcal{M}^n_2,2}|| \leq \varepsilon(m)$$

any distributional solution of

$$-\Delta u = \Omega \cdot \nabla u$$

which satisfies in addition

$$\nabla u \in B^0_{\mathcal{M}^n_2,2}$$

is continuous.

This result crucially relies on the following gauge result.

Theorem 25. *Let $n \geq 3$. There exist constants $\varepsilon(m) > 0$ and $C(m) > 0$ such that for every $\Omega \in B^0_{\mathcal{M}^n_2,2}(B^n, so(m) \otimes \Lambda^1\mathbb{R}^n)$ which satisfies*

$$||\Omega|B^0_{\mathcal{M}^n_2,2}|| \leq \varepsilon(m)$$

there exist $A \in L^\infty(B^n, Gl_m(\mathbb{R}))\cap B^1_{\mathcal{M}^n_2,2}$ and $B \in B^1_{\mathcal{M}^n_2,2}(B^n, M_m(\mathbb{R})\otimes\Lambda^2\mathbb{R}^n)$ such that

i)

$$d_\Omega := dA - A\Omega = -d^*B = - * d * B$$

ii)

$$C(M)||\Omega|B^0_{\mathcal{M}^n_2,2}|| \geq ||\nabla A|B^0_{\mathcal{M}^n_2,2}|| + ||\nabla A^{-1}|B^0_{\mathcal{M}^n_2,2}|| + \int_{B^n} ||dist(A, SO(m))||^2_\infty$$

iii)

$$||\nabla B|B^0_{\mathcal{M}^n_2,2}|| \leq C(m)||\Omega|B^0_{\mathcal{M}^n_2,2}||.$$

Remark 26. Note that our result differs from the generalisation in [42] (see also [44] for a modification of the proof of Rivière and Struwe) in so far, as on one hand we do not impose any smallness of the norm of the gradient of a solution and really construct A and B (see theorem 25 and not only construct Ω and ξ such that $P^{-1}dP + P^{-1}\Omega P = *d\xi$, but on the other hand work in a slightly smaller space.

Chapter 1

Proof of theorem 24

Assume theorem 25 to be true and let A and B be as there.
Then we have
$$\begin{cases} *d*(Adu) = -d^*B \cdot \nabla u \\ d(Adu) = dA \wedge du. \end{cases}$$
These equations together with a classical Hodge decomposition for Adu

$$Adu = d^*E + dD \text{ with} E, D \in W^{1,2}$$

lead to the following equations

$$\begin{cases} \Delta D = -d^*B \cdot \nabla u \\ \Delta E = dA \wedge du. \end{cases}$$

Since the right hand sides are made of Jacobeans we conclude that $D, E \in B^0_{\infty,1}$. Next, we observe that

$$du = A^{-1}(d^*E + dD) \in B^0_{\mathcal{M}^n_2,1} \subset B^{-1}_{\infty,1}.$$

This holds because $A^{-1} \in B^1_{\mathcal{M}^n_2,2} \cap L^\infty$ (see also theorem 25) and $dD, d^*E \in B^0_{\mathcal{M}^n_2,1}$ (see also proposition 14).The proof of the above fact is the same as the proof of the assertion of lemma 8. In a last step we note that (recall the reasons why proposition 38, theorem 12 and proposition 14 hold) thanks to the information we have so far

$$u \in B^0_{\infty,1} \subset C$$

which completes the proof.

□

Chapter 2

Proof of theorem 25

Lemma 27. *There exist constants* $\varepsilon(m) > 0$ *and* $C(m) > 0$ *such that for every* $\Omega \in B^0_{\mathcal{M}^n_2,2}(B^n, so(m) \otimes \Lambda^1\mathbb{R}^n)$ *which satisfies*

$$||\Omega|B^0_{\mathcal{M}^n_2,2}|| \leq \varepsilon(m)$$

there exist $\xi \in B^1_{\mathcal{M}^n_2,2}(B^n, so(m) \otimes \Lambda^{n-2}\mathbb{R}^n)$ *and* $P \in B^1_{\mathcal{M}^n_2,2}(B^n, SO(m))$ *such that*

i)
$$*d\xi = P^{-1}dP + P^{-1}\Omega P \text{ in } B^n$$

ii)
$$\xi = 0 \text{ on } \partial B^n$$

iii)
$$||\xi|B^1_{\mathcal{M}^n_2,2}|| + ||P|B^1_{\mathcal{M}^n_2,2}|| \leq C(m)||\Omega|B^0_{\mathcal{M}^n_2,2}||.$$

The proof of this lemma is a straightforward adaptation of the corresponding assertion in [42].

Now, let $\varepsilon(m), P$ and ξ be as in lemma 27. Note that since $P \in SO(m)$ we have also $P^{-1} \in B^1_{\mathcal{M}^n_2,2}$. Our goal is to find A and B such that

$$dA - A\Omega = -d^*B. \tag{2.1}$$

If we set $\tilde{A} := AP$ then, according to equation (2.1) it has to satisfy

$$d\tilde{A} + (d^*B)P = \tilde{A} + d\xi.$$

As a intermediate step we will first study the following problem

$$\begin{cases} \Delta\hat{A} & = d\hat{A}\cdot *d\xi - d^*B\cdot\nabla P \text{ in} B^n \\ d(d^*B) & = d\hat{A}\wedge dP^{-1} - d*(\hat{A}d\xi P^{-1}) - d*(d\xi P^{-1}) \\ \frac{\partial\hat{A}}{\partial\nu}, & = 0 \text{ and } B = 0 \text{ on } \partial B^n \\ \int_{B^n}\hat{A} & = id. \end{cases}$$

For this system we have the a-priori-estimates (recall theorem 12, proposition 14 with its proof, lemma 8 and the fact that we are working on a bounded domain)

$$\begin{aligned} ||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| + ||\hat{A}||_\infty &\leq C||\xi|B^1_{\mathcal{M}^n_2,2}||\;||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| \\ &\quad + C||P|B^1_{\mathcal{M}^n_2,2}||\;||B|B^1_{\mathcal{M}^n_2,2}|| \end{aligned}$$

and

$$\begin{aligned} ||B|B^1_{\mathcal{M}^n_2,2}|| &\leq C||P^{-1}|B^1_{\mathcal{M}^n_2,2}||\;||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| + C||\xi|B^1_{\mathcal{M}^n_2,2}||\;||\hat{A}||_\infty \\ &\quad + C||\xi|B^1\mathcal{M}^n_2,2||. \end{aligned}$$

Since the used norms of ξ and P - as well as of P^{-1} - can be bounded in terms of $C||\Omega|B^0_{\mathcal{M}^n_2,2}||$ the above estimates together with standard fix point theory guarantee the existence of $\hat{A}$ and B such that they solve the above system and in addition satisfy

$$||\hat{A}|B^1_{\mathcal{M}^n_2,2}|| + ||\hat{A}||_\infty + ||B|B^1_{\mathcal{M}^m_2,2}|| \leq C||\Omega|B^0_{\mathcal{M}^m_2,2}||. \tag{2.2}$$

Next, similar to the proof of theorem 24 we decompose for some D

$$d\hat{A} - \hat{A}*d\xi + d^*BP = d^*D.$$

Then we set $\tilde{A} := \hat{A} + id$, which satisfies for some $n-2$-form F

$$d\tilde{A} - \tilde{A}*d\xi + d^*BP = d^*D - *d\xi =: *dF.$$

It is not difficult to show that $*d(*dFP^{-1}) = 0$ together with $F = 0$ on ∂B^n imply that $F \equiv 0$ (see also a similar assertion in [38] and remember that on bounded domains $B^0_{\mathcal{M}^n_2,2} \subset L^2$).
From this we conclude that in fact $\tilde{A}$ satisfies the desired equation. If wet finally set $A := \tilde{A}P^{-1}$ and let B as given in the above system we get that in fact these A and B solve the required relation (2.1).
So far, we have proved parts ii) and iii) of theorem 25 (recall also estimate (2.2)).

Moreover, the invertibility of A follows immediately from its construction, likewise the estimates for ∇A and ∇A^{-1}.
Last but not least, the relation $A = \hat{A}P^{-1} + idP^{-1}$ implies that

$$||A - SO(m)||_\infty \le C||\hat{A}||_\infty \le C||\Omega|B^0_{\mathcal{M}_2^n,2}||.$$

This completes the proof of theorem 25.

$\square$

Part IV

Gauge results for $-\Delta u = \Omega \cdot u$ with Ω antisymmetric

The goal of this part is to apply an idea which is somehow similar to the one arising in the preceding part where we used an appropriate gauge transformation in order to rewrite the equation we want to study in divergence form - as a conservation law - which enabled us to make a more refined regularity analysis.

Our problem here is the following one:

$$-\Delta u = \Omega u \tag{3}$$

where Ω is antisymmetric.

Announced as theorem F in the introduction, we will prove

Theorem 28. *Let $n \geq 4$ and let $m \in \mathbb{N}^*$.*
Then there exists a constant $\varepsilon > 0$ but small enough such that in a neighbourhood of the origin, there exists a map

$$\begin{array}{rcl} \mathcal{S} : M_2^{\frac{n}{2}(B^n, so(m))} & \rightarrow & L^\infty(B^n) \cap W^2_{M_2^{\frac{n}{2}}}(B^n, Gl_m(\mathbb{R})) \\ \Omega & \rightarrow A & \end{array}$$

with the following properties

i)
$$\Delta A + A\Omega = 0$$

ii)
$$||A||_{L^\infty(B^n)} = \sup_{x \in B^n, X \in S^{m^1}} |A(x)X| \leq 1$$

iii) A is invertible almost everywhere and $A^{-1} \in M_a^b(B^n$ where $\frac{4b}{n} \geq a$

iv) there exists a constant $C > 0$ such that
$$||A^{-1}\nabla A||_{M_4^n(B^n)} + ||\nabla A||_{W^1_{M_2^{\frac{n}{2}}}(B^n)} \leq C||\Omega||_{M_2^{\frac{n}{2}}(B^n}$$

provided that $||\Omega||_{M_2^{\frac{n}{2}}(B^n} \leq \varepsilon$.

In what follows, all norm are taken on B^n unless other domains are indicated.

Notation: Similarly to the classical Sobolev spaces $W^{k,p}$ of distributions such that their derivatives up to order k belong to L^p, we denote by $\mathbf{W}^{\mathbf{k}}_{\mathbf{M}^{\mathbf{p}}_{\mathbf{q}}}$ those distributions which satisfy the requirement that their derivatives up to order k are in M_q^p. In addition, a distribution u belongs to $\mathbf{W}^{\mathbf{k}}_{\mathbf{0},\mathbf{M}^{\mathbf{p}}_{\mathbf{q}}}(\Omega)$ if $u \in W^k_{M_q^p}(\Omega)$ and u is zero on the boundary $\partial\Omega$ (in the sense of distributions).

Chapter 1

An intermediate transformation

Similar to the intermediate construction of ξ and P (see lemma 27) in the context of our gauge result for problems of the form

$$-\Delta u = \Omega \cdot \nabla u$$

also in our new situation where we study prob;ems of the form

$$-\Delta u = \Omega u$$

the construction of the gauge is splitted into two steps.
In a first step, presented in this chapter, we will show the existence of $P \in W^2_{M_2^{\frac{n}{2}}}(B^n, SO(m))$ with the property that in the sense of tempered distributions it holds

$$\frac{1}{2}[\Delta P P^{-1} - P \Delta P^{-1}] + P \Omega P^{-1} = 0$$

with $P = id$ on the boundary ∂B^n, again in the sense of tempered distributions. More precisely, we will prove the following lemma:

Lemma 29. *Let $n \geq 4$ and let $m \in \mathbb{N}^*$. Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for every $\Omega \in M_2^{\frac{n}{2}}(B^n, so(m))$ satisfying*

$$||\Omega||_{M_2^{\frac{n}{2}}(B^n)} \leq \varepsilon_0$$

there exists $P \in W^2_{M_2^{\frac{n}{2}}}(B^n, SO(m))$ such that

$$\begin{cases} \frac{1}{2}[\Delta P P^{-1} - P \Delta P^{-1}] + P \Omega P^{-1} = 0 \text{ in } \mathcal{D}'(B^n) \\ P = id \text{ in } \mathcal{D}'(\partial B^n) \end{cases} \tag{1.1}$$

Moreover, it holds

$$||P - id|| W^2_{0,M_2^{\frac{n}{2}}(B^n)} \leq C ||\Omega||_{M_2^{\frac{n}{2}}(B^n)}. \tag{1.2}$$

1.1 Proof of lemma 29

We set

$\mathcal{U}_{\varepsilon}^{q,p} := \left\{\Omega \in M_q^p(B^n, so(m))\ ,\ ||\Omega||_{M_q^p} < \varepsilon\right\}$ where $n > p > \dfrac{n}{2}$, $\dfrac{4p}{n} \geq q$ and $q > 2$.

Claim:
There exist ε_0 and $C > 0$ such that

$$\mathcal{V}_{\varepsilon_0,C}^{q,p} := \left\{ \begin{array}{l} \Omega \in \mathcal{U}_{\varepsilon_0}^{q,p} \text{ such that } \exists P = \exp(U) \text{ satisfying (1.1) and (1.2)} \\ \text{and } ||P - id||_{W^2_{0,M_q^p}} \leq C||\Omega||_{M_q^p} \end{array} \right\}$$

is open as well as closed in $\mathcal{U}_{\varepsilon_0}^{q,p}$ with respect to the M_q^p-norm.
Thus - due to the fact that obviously $\mathcal{U}_{\varepsilon_0}^{q,p}$ is connected - we find that we have $\mathcal{U}_{\varepsilon_0}^{q,p} = \mathcal{V}_{\varepsilon,C}^{q,p}$.

This claim immediately implies the assertion of lemma 29:
Let ε_0 be given by the claim above and let Ω satisfy

$$||\Omega||_{M_2^{\frac{n}{2}}} < \varepsilon_0.$$

Once we are given such an Ω a simple mollification procedure yields a sequence $\{\Omega_k\} \subset \mathcal{U}_{\varepsilon}^{q,p}$ which converges strongly to Ω in $M_2^{\frac{n}{2}}$.
Now, let P_k be the matrix associated to Ω_k via the preceding claim. In particular, we have for all radii R

$$\begin{aligned} ||P_k - id||_{W^{2,2}(B_R^n)} &\leq ||P_k - id||_{W^2_{M_2^{\frac{n}{2}}}} R^{\frac{n}{2}-2} \\ &\leq C\ ||\Omega_k||_{M_2^{\frac{n}{2}}} R^{\frac{n}{2}-2} \\ &\leq C||\Omega||_{M_2^{\frac{n}{2}}} R^{\frac{n}{2}-2} \\ &< C\varepsilon_0 R^{\frac{n}{2}-2} \\ &\phantom{<} \text{ uniformly in } k! \end{aligned} \tag{1.3}$$

In particular, the sequence $P_k - id$ is uniformly bounded in $W^{2,2}$.
Thus, there exists a subsequence which converges weakly in $W^{2,2}$ to a limit $P \in W^{2,2}$. This implies also that

$$\Delta P_k \rightharpoonup \Delta P \text{ in } L^2.$$

This together with the fact that weak limits are unique and the uniform bounds for given radii, allows us to deduce that the limit P belongs in fact to $W^2_{M_2^{\frac{n}{2}}}$ and the following estimate holds

$$||P - id||_{W^2_{M_2^{\frac{n}{2}}}} \leq C||\Omega||_{M_2^{\frac{n}{2}}}.$$

Moreover, from the embedding result, theorem 95, we know that

$$W^2_{M_2^{\frac{n}{2}}} \hookrightarrow M_p^q \text{ for all } p < \infty,\ p \geq q.$$

A direct adaptation of the classical result about compact embeddings (see e.g. [12], p.169) shows that also the above embeddings are compact. All the information we thus have at hand, allow us to pass to the limit in equation (1.1) and we have shown that P satisfies the required equation (in the distributional sense).

□

Proof of the claim:
In the proof of the claim, we will need the following two lemmas which will be proved in the next two sections.

Lemma 30. *Let $n \geq 4$ and let $m \in \mathbb{N}^*$. Then there exist ε_1 and C_1 such that for any $P \in W^2_{M_2^{\frac{n}{2}}}(B^n, SO(m))$ which satisfy*

$$P = id \text{ on } \partial B^n \text{ and } ||P - id||_{W^2_{0,M_2^{\frac{n}{2}}}(B^n)} \leq \varepsilon_1$$

it holds

$$||P - id||_{W^2_{0,M_2^{\frac{n}{2}}}(B^n)} \leq C_1||P^{-1}\Delta P - \Delta P^{-1}P||_{M_2^{\frac{n}{2}}(B^n)}.$$

Moreover, if $P \in W^2_{M_q^p}(B^n, SO(m))$ with $P = id$ on ∂B^n with

$$||P - id||_{W^2_{0,M_2^{\frac{n}{2}}}(B^n)} \leq \varepsilon_1$$

it holds

$$||P - id||_{W^2_{0,M_q^p}(B^n)} \leq C_1||P^{-1}\Delta P - \Delta P^{-1}P||_{M_q^p(B^n)}.$$

And for the map

$$\begin{aligned} F^{P_0} : W^2_{0,M_q^p}(B^n, so(m)) &\rightarrow M_q^p(B^n, so(m)) \\ V &\mapsto (P_0 \exp(V))^{-1}\Delta(P_0 \exp(V)) \\ &\quad -\Delta(P_0 \exp(V))^{-1}P_0 \exp(V) \end{aligned}$$

where $P_0 \in W^2_{M_q^p}(B^n, so(m))$ we have

Lemma 31. *There exists $\varepsilon_2 > 0$ such that for any $U_0 \in W^2_{0,M^p_q}(B^n, so(m))$ which satisfies*

$$|| \exp(U_0) - id||_{W^2_{M^p_q}} \leq \varepsilon_2$$

we have that $dF_0^{P_0=\exp(U_0)}$ is invertible from $W^2_{0,M^p_q}(B^n, so(m))$ to $M^p_q(B^n, so(m))$.

Now, back to the proof of the claim. First we will show

Closedness of $\mathcal{V}^{q,p}_{\varepsilon,C}$ in $\mathcal{U}^{q,p}_{\varepsilon_0}$:
This assertion is shown by exactly the same arguments as we used in order to prove that the claim implies lemma 29. Note that in fact closedness holds for all $\varepsilon_0 > 0$ and for all $C > 0$.

Openness of $\mathcal{V}^{q,p}_{\varepsilon,C}$ in $\mathcal{U}^{q,p}_{\varepsilon_0}$:
We will show that there exist ε_0 and $C > 0$ such that openness holds.
Let $P_0 \in W^2_{M^p_q}(B^n, so(m))$ be given and look at the following map

$$\begin{aligned} F^{P_0} : W^2_{0,M^p_q}(B^n, so(m)) &\to M^p_q(B^n, so(m)) \\ V &\mapsto (P_0 \exp(V))^{-1}\Delta(P_0 \exp(V)) \\ &\quad -\Delta(P_0 \exp(V))^{-1} P_0 \exp(V). \end{aligned}$$

Now, observe that

i) Due to the fact that $W^2_{M^p_q} \hookrightarrow C$, cf. theorem 95, the map

$$V \mapsto \exp(V)$$

is a smooth map from $W^2_{0,M^p_q}(B^n, so(m))$ to $W^2_{M^p_q}(B^n(SO(m))$.

ii) Obviously, the Laplace operator is a smooth, linear map from $W^2_{M^p_q}$ to M^p_q.

iii) Recall that the map

$$\begin{aligned} W^2_{M^p_q} \times M^p_q &\to M^p_q \\ (A, B) &\mapsto AB \end{aligned}$$

is also smooth (remember that $W^2_{M^p_q} \hookrightarrow C$).

These observation allow us to conclude that the map F^{P_0} is C^1. And thus it makes sense to look at its differential, and at the origin we calculate for $\xi \in W^2_{0,M^p_q}(B^n, so(m))$

$$\frac{1}{2} dF_0^{P_0} \cdot \xi = L^{P_0} \cdot \xi := \Delta\xi + [P_0^{-1}\nabla P_0, \nabla\xi] + [\Omega_0, \xi]$$

where $2\Omega_0 := P_0^{-1}\Delta P_0 - \Delta P_0^{-1} P_0$ and $[\cdot,\cdot]$ denotes the standard commutator. Once we have the invertibility of dF^{P_0} at the origin, which actually holds true thanks to lemma 31, we can complete the proof of the openness of $\mathcal{V}^{q,p}_{\varepsilon,C}$ in $\mathcal{U}^{q,p}_{\varepsilon_0}$ as follows:
Assume that ε_0 is smaller than ε_1 from lemma 30 and smaller than ε_2 from lemma 31 and let C be equal to C_1 given in lemma 30. Moreover, assume that $C\varepsilon_0 < \varepsilon_2$.
We have to show that in a neighbourhood of a given $\Omega_0 \in \mathcal{V}^{q,p}_{\varepsilon_0,C}$ for each Ω there exists a $P = \exp(U)$ such that

$$\frac{1}{2}[\Delta P P^{-1} - P\Delta P^{-1}] + P\Omega P^{-1} = 0$$

and the desired estimates

$$\begin{aligned} ||P - id||_{W^2_{M_2^{\frac{n}{2}}}} &\leq C||\Omega||_{M_2^{\frac{n}{2}}} \\ ||P - id||_{W^2_{0,M_q^p}} &\leq C||\Omega||_{M_q^p} \end{aligned}$$

hold.
First of all, note that $\frac{1}{2}[\Delta P P^{-1} - P\Delta P^{-1}] + P\Omega P^{-1} = 0$ can be rewritten as

$$2\Omega = \Delta P^{-1} P - P^{-1}\Delta P.$$

In addition, observe that

$$F^{P_0}(0) = P_0^{-1}\Delta P_0 - \Delta P_0^{-1} P_0 = -2\Omega_0.$$

Thanks to lemma 31 we are allowed to apply the local inversion theorem and thus can prove that in fact, for every Ω which lies in a small enough neighbourhood of Ω_0 there exists a $P = \exp(U)$ which solves the desired equation.
It remains to show the claimed estimates:
They are immediate consequences of the construction of P and lemma 30.
Thus we have proved the desired openness and the proof of lemma 29 is complete.

□

1.2 Proof of lemma 30

To start with we rewrite

$$P^{-1}\Delta P = \frac{1}{2}[P^{-1}\Delta P - \Delta P^{-1} P] + \frac{1}{2}[P^{-1}\Delta P + \Delta P^{-1} P].$$

Note moreover that

$$\begin{aligned} P^{-1}\Delta P = \Delta P^{-1}P &= div(P^{-1}\nabla P + \nabla P^{-1}P) - 2\nabla P^{-1}\cdot\nabla P \\ &= -2\nabla P^{-1}\cdot\nabla P. \end{aligned}$$

Now, we estimate

$$\begin{aligned} ||P^{-1}\Delta P + \Delta P^{-1}P||_{M_2^{\frac{n}{2}}} &\leq 2||\nabla P||_{M_4^n}||\nabla P||_{M_4^n} \\ &\quad \text{because of the above calculation} \\ &\quad \text{and the fact that } P\in SO(m) \\ &\leq 2\varepsilon_1||\nabla P||M_4^n \\ &\quad \text{thanks to our hypothesis.} \end{aligned}$$

In addition, observe that

$$||\nabla P||_{M_4^n} \leq C||\nabla^2 P||_{M_2^{\frac{n}{2}}} \leq C||\Delta P||_{M_2^{\frac{n}{2}}}$$

(remember theorem 96; to see the second inequality, recall the corresponding assertion in the classical framework of Sobolev spaces, see e.g. [51], and work non balls of radius R).
If we put together what we know so far, we find

$$\begin{aligned} ||\Delta P||_{M_2^{\frac{n}{2}}} &= ||PP^{-1}\Delta P||_{M_2^{\frac{n}{2}}} \\ &\leq ||P||_\infty ||P^{-1}\Delta P||_{M_2^{\frac{n}{2}}} \\ &\leq ||P^{-1}\Delta P||_{M_2^{\frac{n}{2}}} \\ &\quad \text{since } P\in SO(m) \\ &\leq \frac{1}{2}||P^{-1}\Delta P - \Delta P^{-1}P||_{M_2^{\frac{n}{2}}} + \frac{1}{2}||P^{-1}\Delta P + \Delta P^{-1}P||_{M_2^{\frac{n}{2}}} \\ &\leq \frac{1}{2}||P^{-1}\Delta P - \Delta P^{-1}P||_{M_2^{\frac{n}{2}}} + \frac{1}{2}2\varepsilon_1||\nabla P||_{M_4^n} \\ &\leq \frac{1}{2}||P^{-1}\Delta P - \Delta P^{-1}P||_{M_2^{\frac{n}{2}}} + \frac{1}{4}||\Delta P||_{M_2^{\frac{n}{2}}} \\ &\quad \text{provided that } \varepsilon_1 \text{ is small enough.} \end{aligned}$$

Thus we have

$$||\Delta P||_{M_2^{\frac{n}{2}}} \leq \frac{2}{3}||P^{-1}\Delta P - \Delta P^{-1}P||_{M_2^{\frac{n}{2}}}$$

which finally leads to the first assertion of our lemma.
Next, we have - again due to our above rewriting -

$$\begin{aligned} ||P^{-1}\Delta P + \Delta P^{-1}P||_{M_q^p} &\leq 2C||\nabla P||_{M_4^n}||\nabla P||_{W^1_{M_q^p}} \\ &\quad \text{thanks to lemma 9} \\ &\leq 2C\varepsilon_1||\nabla P||_{W^1_{M_q^p}} \\ &\quad \text{according to our assumption} \\ &\leq C\varepsilon_1||\Delta P||_{M_q^p} \end{aligned}$$

and we can complete the proof of the second assertion exactly in the same way as for the first assertion.

□

1.3 Proof of lemma 31

We have to show that there exists ε_2 such that if

$$||\exp(U_0) - id||_{W^2_{M_2^{\frac{n}{2}}}} \leq \varepsilon_2$$

then there exists a constant C_{U_0} such that for each $\omega \in M_q^p(B^n, so(m))$ there exists a unique $\xi \in W^2_{0,M_q^p}(B^n, so(m))$ such that it holds

$$\begin{cases} L^{P_0}\xi & = \omega \\ ||\xi||_{W^2_{0,M_q^p}(B^n,so(m))} & \leq C_{U_0}||\omega||_{M_q^p(B^n,so(m))}. \end{cases}$$

Recall that for $\xi \in W^2_{0,M_q^p}(B^n, so(m))$

$$\frac{1}{2}dF_0^{P_0} \cdot \xi = L^{P_0} \cdot \xi := \Delta\xi + [P_0^{-1}\nabla P_0, \nabla\xi] + [\Omega_0, \xi].$$

Due to the following embeddings

$$W^2_{M_q^p} \hookrightarrow C(B^n) \hookrightarrow L^\infty(B^n)$$

it is immediately clear that

$$[\Omega_0, \xi] \in M_q^p.$$

Next, we will look at the commutator $[P_0^{-1}\nabla P_0, \nabla\xi] = P_0^{-1}\nabla P_0\nabla\xi - \nabla\xi P_0^{-1}\nabla P_0$. We estimate

$$\begin{aligned} ||[P_0^{-1}\nabla P_0, \nabla\xi]||_{M_q^p} &\leq 2||\nabla P_0 \nabla\xi||_{M_q^p} \\ &\leq 2C||\nabla P_0||_{M_4^n}||\nabla\xi||_{W^1_{M_q^p}} \\ &\quad \text{thanks to lemma 9} \\ &\leq C||P_0 - id||_{W^2_{0,M_2^{\frac{n}{2}}}}||\xi||_{W^2_{0,M_q^p}} \end{aligned}$$

So if we put together what we know so far, we find that L^{P_0} is a continuous map from $W^2_{0,M_q^p}(B^n, so(m))$ to $M_q^p(B^n, so(m))$.
Next, assume for the moment that $\xi \in W^2_{M_t^s}$ where

$$\frac{4s}{n} \geq t \text{ and } \frac{1}{p} + \frac{1}{s} < \frac{4}{n}.$$

The second requirement can be fulfilled because of the following reason:
Our hypotheses, $p > \frac{n}{2}$ can be rephrased as $\frac{1}{p} = \frac{2}{n} - \delta$ for some positive δ. Thus, we may set

$$\frac{1}{s} = \frac{2}{n} + \frac{\delta}{2}.$$

Then it is easily checked that for this choice of s, we have $\frac{1}{p} + \frac{1}{s} < \frac{4}{n}$.
And we estimate, due to lemma 10

$$||[\Omega_0, \xi]||_{M_t^s} \leq C||\Omega_0||_{M_2^{\frac{n}{2}}}||\xi||_{W^2_{0,M_t^s}}$$

and due to lemma 9 and theorem 96

$$||[P_0^{-1}\nabla P_0, \nabla\xi]||_{M_t^s} \leq C||P_0 - id||_{W^2_{0,M_2^{\frac{n}{2}}}}||\xi||_{W^2_{0,M_t^s}}.$$

Thus, if $||P_0 - id||_{W^2_{0,M_2^{\frac{n}{2}}}}$ is small enough, we can conclude that for every $\omega \in M_t^s(B^n)$ there exists a unique solution $\xi \in W^2_{0,M_t^s}(B^n)$ of $L^{P_0}\xi = \omega$. Once we assume in addition that ω is $so(m)$-valued, it holds

$$P^{P_0}(\xi + \xi^t) = 0.$$

because

$$(P_0^{-1}\nabla P_0)^t = -P_0^{-1}\nabla P_0 \text{ and } \Omega_0^t = -\Omega_0.$$

But since the solution is unique - what we have shown above - this implies that $\xi^t = -\xi$.
All in all, we have seen that

$$\begin{aligned} L^{P_0} : W^2_{0,M_t^s}(B^n, so(m)) &\rightarrow M_t^s(B^n, so(m)) \\ \xi &\mapsto \Delta\xi + [P_0^{-1}\nabla P_0, \nabla\xi] + [\Omega_0, \xi] \end{aligned}$$

is an isomorphism.
Note that due to the particular restrictions on s and t, we have in particular $t \leq \frac{4s}{n} < 2 < q$ and thus

$$W^2_{0,M^p_q} \subset W^2_{0,M^s_t}.$$

Next, observe that for $\xi \in W^2_{M^s_t}$ and $\Omega_0 \in M^p_q$ one has

$$\Omega_0 \xi \in M^b_a \text{ with } \frac{1}{a} = \frac{1}{q} + \frac{n-2s}{nt} \text{ and } \frac{1}{b} = \frac{1}{p} + \frac{n-2s}{ns}.$$

For such a and b it holds

$$\frac{n}{b} = \frac{n}{p} + \frac{n-2s}{s} = n\Big(\frac{1}{p} + \frac{1}{s}\Big) - 2 < n\frac{4}{n} - 2 = 2$$

and thus - by theorem 95

$$W^2_{M^b_a} \hookrightarrow L^\infty.$$

So, we have

$$\begin{aligned} ||\Delta^{-1}([\Omega_0, \xi])||_\infty &\leq C||[\Omega_0, \xi]||_{M^b_a} \\ &\leq C||\Omega_0||_{M^p_q}||\xi||_{W^2_{0,M^s_t}} \end{aligned}$$

where Δ^{-1} denotes the inverse Laplacian on B^n with zero Dirichlet boundary condition.
Similarly, one finds

$$||\Delta^{-1}([P_0^{-1}\nabla P_0, \nabla\xi])||_\infty \leq C||P_0 - id||_{W^2_{0,M^p_q}}||\xi||_{0,W^2_{0,M^s_t}}$$

These last inequalities implies that in fact the unique solution $\xi \in W^2_{0,M^s_t}(B^n, so(m))$ of $L^{P_0}\omega = \xi$ for given $\omega \in M^p_q$ is bounded with the following bound

$$||\xi||_\infty \leq C\Big[1 + ||P_0 - id||_{W^2_{0,M^p_q}}\Big]||\omega||_{M^p_q}.$$

Then we have also

$$\begin{aligned} ||[\Omega_0, \xi]||_{M^p_q} &\leq C||\Omega_0||_{M^p_q}||\xi||_\infty \\ &\leq C||\Delta P_0||_{M^p_q}\Big[1 + ||P_0 - id||_{W^2_{0,M^p_q}}\Big]||\omega||_{M^p_q}. \end{aligned}$$

Now, observe that for any $\zeta \in W^2_{0,M^p_q}$ we have (recall lemma 9)

$$||[P_0^{-1}\nabla P_0, \nabla\zeta]||_{M^p_q} \leq C||P_0 - id||_{W^2_{0,M^{\frac{n}{2}}_2}}||\zeta||_{W^2_{0,M^p_q}}.$$

So, if $||P_0 - id||_{W^2_{0,M_2^{\frac{n}{2}}}}$ is small enough the map

$$\begin{aligned} H^{P_0} : W^2_{0,M^p_q}(B^n, so(m)) &\rightarrow M^p_q(B^n, so(m)) \\ \xi &\mapsto \Delta\xi + [P_0^{-1}\nabla P_0, \nabla\xi] \end{aligned}$$

is an isomorphism. Furthermore, the same arguments as in the case of L^{P_0} reveal also that H^{P_0} is an isomorphism from $W^2_{0,M^s_t}(B^n, so(m))$ to $M^s_t(B^n, so(m))$. Finally, let

$$\zeta := (H^{P_0})^{-1}\big(\omega - [\Omega_0, \xi]\big)$$

which leads to the conclusion $\zeta = \xi$ because $H^{P_0}(\zeta - \xi) = 0$ and hence $\xi \in W^2_{0,M^p_q}(B^n, so(m))$ with the estimate

$$||\xi||_{W^2_{0,M^p_q}} \leq C\Big[1 + ||\Delta P_0||_{M^p_q}\big[1 + ||P_0 - id||_{W^2_{0,M^p_q}}\big]\Big]||\omega||_{M^p_q}.$$

Chapter 2

Some useful estimates

In this chapter we would like to present some supplementary estimates which will turn out to be useful in the proof of theorem 28, in particular in lemma 34.

More precisely, we have

Lemma 32. *Let a and b satisfy $\frac{4b}{n} \geq a \geq 1$, $b < \infty$ where the dimension n satisfies $n \geq 4$.*
Then there exist $\delta > 0$ and $C > 0$ such that for any $\Omega \in M_q^p(B^n, M_m(\mathbb{R}))$ and for any $A \in W^2_{M_q^p}(B^n, Gl(\mathbb{R}))$ where $n > p > \frac{n}{2}$, $p \geq q > 2$ and $\frac{4p}{n} \geq q$ with $A^{-1} \in L^\infty(B^n)$ and

$$||\Omega||_{M_2^{\frac{n}{2}}} + ||A^{-1}\nabla A||_{M_4^n} \leq \delta$$

and

$$\begin{cases} -\Delta A + A\Omega = 0 \ \text{ in } B^n \\ A = id \text{ on } \partial B^n. \end{cases}$$

the following estimates hold

i)

$$||A^{-1}\nabla A||_{M_4^n} \leq C||\Omega||_{M_2^{\frac{n}{2}}}$$

ii)

$$||A^{-1} - id||_{M_a^b} \leq C||\Omega||_{M_2^{\frac{n}{2}}}$$

iii)

$$||A^{-1}\nabla A||_{M_q^p} \leq C||\Omega||_{M_q^p}$$

iv)

$$||A^{-1} - id||_\infty \leq C||\Omega||_{M^p_q}.$$

Proof of lemma 32:
First of all, we observe that

$$\begin{cases} -\Delta A + A\Omega = 0 \text{ in } B^n \\ A = id \text{ on } \partial B^n. \end{cases}$$

is equivalent to

$$\begin{cases} d^*(A^{-1}dA) = -\Omega - A^{-1}dA \cdot A^{-1}dA \text{ in } B^n \\ d(A^{-1}dA) = A^{-1}dA \wedge A^{-1}dA \text{ in } B^n \\ \iota_{\partial B^n} A^{-1}dA = 0. \end{cases} \tag{2.1}$$

From that we may infer that

$$\begin{aligned} ||A^{-1}\nabla A||_{M^n_4} &\leq C||\Omega||_{M_2^{\frac{n}{2}}} + C||(A^{-1}\nabla A)^2||_{M_2^{\frac{n}{2}}} \\ &\leq C||\Omega||_{M_2^{\frac{n}{2}}} + C||A^{-1}\nabla A||^2_{M^n_4} \\ &\leq C||\Omega||_{M_2^{\frac{n}{2}}} + C\delta||A^{-1}\nabla A||_{M^n_4} \end{aligned}$$

and thus, for δ small enough such that $C\delta < \frac{1}{2}$ we have

$$||A^{-1}\nabla A||_{M^n_4} \leq C||\Omega||_{M_2^{\frac{n}{2}}}.$$

Next, a short calculation - taking into account the equation which is satisfied by A - reveals that

$$\begin{cases} -\Delta(A^{-1} - id) = [\Omega + 2(A^{-1}\nabla A)^2](A^{-1} - id) + [\Omega + 2(A^{-1}\nabla A)^2] \text{ in } B^n \\ A^{-1} - id = 0 \text{ on } \partial B^n. \end{cases}$$

Now, observe that our assumption guarantee that

$$M^b_a \times M_2^{\frac{n}{2}} \to M^s_t$$

such that

$$W^2_{M^s_t} \hookrightarrow M_r^{\frac{sr}{t}} \hookrightarrow M^b_a \text{ where } r = \frac{nt}{n-2s}.$$

Then we estimate

$$\begin{aligned}
||A^{-1}-id||_{M^b_a} &\leq C||A^{-1}-id||_{M_r^{\frac{sr}{t}}} \\
&\leq C||\Delta(A^{-1}-id)||_{M^s_t} \\
&\leq C||[\Omega+2(A^{-1}\nabla A)^2](A^{-1}-id)||_{M^s_t} \\
&\quad +||\Omega+2(A^{-1}\nabla A)^2||_{M^s_t} \\
&\leq C||\Omega+2(A^{-1}\nabla A)^2||_{M_2^{\frac{n}{2}}}||A^{-1}-id||_{M^b_a} \\
&\quad +C||\Omega+2(A^{-1}\nabla A)^2||_{M_2^{\frac{n}{2}}} \\
&\quad \text{note that in particular we have } M^s_t\subset M_2^{\frac{n}{2}} \\
&\leq \frac{1}{2}||A^{-1}-id||_{M^b_a}+C||\Omega+2(A^{-1}\nabla A)^2||_{M_2^{\frac{n}{2}}} \\
&\quad \text{once } \delta \text{ is small enough, recall also our hypothesis}
\end{aligned}$$

and thus we continue

$$\begin{aligned}
||A^{-1}-id||_{M^b_a} &\leq 2C||\Omega+2(A^{-1}\nabla A)^2||_{M_2^{\frac{n}{2}}} \\
&\leq 2C||\Omega||_{M_2^{\frac{n}{2}}}+4C||(A^{-1}\nabla A)^2||_{M_2^{\frac{n}{2}}} \\
&\leq 2C||\Omega||_{M_2^{\frac{n}{2}}}+4C||A^{-1}\nabla A||^2_{M^n_4} \\
&\leq 2C||\Omega||_{M_2^{\frac{n}{2}}}+4C||\Omega||^2_{M_2^{\frac{n}{2}}} \\
&\quad \text{due to the first assertion} \\
&\quad \text{already proved above} \\
&\leq 2C||\Omega||_{M_2^{\frac{n}{2}}}+4C||\Omega||_{M_2^{\frac{n}{2}}}\delta \\
&\quad \text{because of our hypothesis} \\
&\leq C||\Omega||_{M_2^{\frac{n}{2}}}.
\end{aligned}$$

This shows that the second claimed inequality holds as well.
Furthermore, from the system (2.1) we deduce for $r=\frac{nq}{n-p}$

$$\begin{aligned}
||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}} &\leq C||\Omega||_{M^p_q}+C||(A^{-1}\nabla A)^2||_{M^p_q} \\
&\leq C||\Omega||_{M^p_q}+C||A^{-1}\nabla A||_{M^n_4}||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}} \\
&\leq C||\Omega||_{M^p_q}+C||\Omega||_{M_2^{\frac{n}{2}}}||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}} \\
&\quad \text{thanks to the first assertion of our lemma} \\
&\leq C||\Omega||_{M^p_q}+C\delta||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}} \\
&\quad \text{according to our hypothesis}
\end{aligned}$$

and finally find

$$||A^{-1}\nabla A||_{M_q^p} \le C||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}} \le C||\Omega||_{M_q^p}.$$

Last, but not least we observe

$$\nabla(A^{-1} - id) = \nabla A^{-1}AA^{-1} = -A^{-1}\nabla A(A^{-1} - id) - A^{-1}\nabla A.$$

And we restrict our possible choice of a and b by the following additional requirement (compare also to the proof of lemma 31)

$$\frac{1}{p} + \frac{1}{b} < \frac{2}{n}.$$

Under this supplementary hypothesis it holds

$$M_r^{\frac{pr}{q}} \times M_a^b \hookrightarrow M_v^u \text{ such that } W^1_{M_v^u} \hookrightarrow L^\infty \text{ where } r = \frac{nq}{n-p}.$$

So we obtain

$$\begin{aligned}
||A^{-1} - id||L^\infty &\le C||A^{-1} - id||_{W^1_{M_v^u}} \\
&\le ||\nabla(A^{-1} - id)||_{M_v^u} \\
&\le C||A^{-1}\nabla A(A^{-1} - id)||_{M_v^u} + C||A^{-1}\nabla A||_{M_v^u} \\
&\le C||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}}||A^{-1} - id||_{M_a^b} + ||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}} \\
&\le C||\Omega||_{M_q^p}||A^{-1} - id||_{M_a^b} + C||\Omega||_{M_q^p} \\
&\quad \text{thanks to assertion iii)} \\
&\le C||\Omega||_{M_q^p}||\Omega||_{M_2^{\frac{n}{2}}} + ||\Omega||_{M_q^p} \\
&\quad \text{due to assertion ii)} \\
&\le C\delta||\Omega||_{M_q^p} + ||\Omega||_{M_q^p} \\
&\le C||\Omega||_{M_q^p},
\end{aligned}$$

and estimate iv) holds, too.

□

Chapter 3

The final gauge transform

3.1 Two technical lemmas towards the proof of our gauge result

The first technical lemma is the following

Lemma 33. *Let the dimension n satisfy $n \geq 4$ and let $m \in \mathbb{N}^*$.*
Then there exists $\varepsilon_0 > 0$ such that for every $P \in W^1_{M_4^n}(B^n, SO(m))$ with

$$||\nabla P||_{M_4^n} < \varepsilon_0$$

and for every $Q \in W^2_{M_1^{\frac{n}{2}}}$ such that

$$\begin{cases} -\Delta Q - 2\nabla Q \cdot \nabla P P^{-1} - Q(\nabla P P^{-1})^2 = 0 \text{ in } B^n \\ Q = id \text{ on } \partial B^n. \end{cases}$$

the following holds: $Q \in L^\infty \cap W^2_{M_2^{\frac{n}{2}}}(B^n, M_m(\mathbb{R}))$ and

$$\sup_{X \in \mathbb{R}^m} ||QX||_{L^\infty(B^n)} \leq 1.$$

Proof of lemma 33:
First of all, we will show that for all fixed $X \in \mathbb{R}^m$ it holds

$$\Delta(X^t Q Q^t X) \geq 0.$$

Note that

$$X^t Q Q^t X =< X^t Q, Q^t X >=< (Q^t X)^t, Q^t X >= |Q^t X|^2 \in \mathbb{R}$$

where $< \cdot, \cdot >$ denotes the standard Euclidean inner product.
In what follows, recall that P has values in $SO(m)$, so we have in particular $(\nabla PP^{-1})^t = -\nabla PP^{-1}$!
Now, for fixed X we calculate in the sense of distributions - this is possible due to the fact that we assume $Q \in W^2_{M_1^{\frac{n}{2}}}$ -

$$\begin{aligned}
\Delta(X^tQQ^tX) &= X^t\Delta QQ^tX + X^tQ\Delta Q^tX + 2X^t\nabla Q\cdot\nabla Q^tX \\
&= -2X^t\nabla Q\cdot\nabla PP^{-1}Q^tX - X^tQ(\nabla PP^{-1})^2Q^tX \\
&\quad +2X^tQ(\nabla PP^{-1})\cdot\nabla Q^tX - X^tQ(\nabla PP^{-1})^2Q^tX \\
&\quad +2X^t\nabla Q\cdot\nabla Q^tX
\end{aligned}$$

where we used the fact that Q solves

$$-\Delta Q - 2\nabla Q\cdot\nabla PP^{-1} - Q(\nabla PP^{-1})^2 = 0.$$

Furthermore, observe that

$$\begin{aligned}
-2X^t\nabla Q\cdot(\nabla PP^{-1})Q^tX &= -2(\nabla PP^{-1}Q^tX)^t\cdot(X^t\nabla Q)^t \\
&= 2X^tQ(\nabla PP^{-1})\cdot\nabla Q^tX
\end{aligned}$$

because for Y and Z in $\mathbb{R}^m$ we have

$$Y^tZ = Z^tY.$$

And so we find

$$\begin{aligned}
\Delta(X^tQQ^tX) &= 4X^tQ(\nabla PP^{-1})\cdot\nabla Q^tX \\
&\quad -2X^tQ(\nabla PP^{-1})^2\cdot Q^tX \\
&\quad +2X^t\nabla Q\cdot\nabla Q^tX \\
&\geq -2X^tQ(\nabla PP^{-1})\cdot(\nabla PP^{-1})^tQ^tX - 2X^t\nabla Q\cdot\nabla Q^tX \\
&\quad -2X^tQ(\nabla PP^{-1})^2\cdot Q^tX \\
&\quad +2X^t\nabla Q\cdot\nabla Q^tX \\
&\quad \text{because of Cauchy-Schwarz} \\
&= 2X^tQ(\nabla PP^{-1})^2Q^tX - 2X^t\nabla Q\cdot\nabla Q^tX \\
&\quad -2X^tQ(\nabla PP^{-1})^2\cdot Q^tX \\
&\quad +2X^t\nabla Q\cdot\nabla Q^tX \\
&\quad \text{since } (\nabla PP^{-1})^t = -\nabla PP^{-1} \\
&= 0.
\end{aligned}$$

Thus, the classical maximum principle implies that

$$\sup_{X\in\mathbb{R}^m} ||Q^t X||^2_{L^\infty(B^n)} \leq 1,$$

i. e. $Q \in L^\infty(B^n)$.
Hence

$$Q(\nabla P P^{-1})^2 \in M_2^{\frac{n}{2}}.$$

Next, by lemma 9 we find the a-priori bounds for $1 \leq a \leq b < n$ with $\frac{4b}{n} \geq a$

$$\begin{aligned} ||\nabla Q \cdot \nabla P||_{M_a^b} &\leq C||\nabla P||_{M_4^n} ||Q - id||_{W^2_{0,M_a^b}} \\ &\leq C\varepsilon_0 ||Q - id||_{W^2_{0,M_a^b}} \\ &\quad \text{due to our hypothesis.} \end{aligned}$$

But from that we conclude that

$$\begin{aligned} K_P : W^2_{0,M_a^b}(B^n, M_m(\mathbb{R})) &\rightarrow M_a^b(B^n, M_m(\mathbb{R})) \\ \eta &\mapsto -\Delta\eta - 2\nabla\eta \cdot \nabla P P^{-1} \end{aligned} \tag{3.1}$$

where

$$1 \leq a \leq b < n \text{ and } \frac{4b}{n} \geq a.$$

is an isomorphism.
So in particular for $a = 1, b = \frac{n}{2}$ and $a = 2, b = \frac{n}{2}$.
Finally, we look at the following special choice of η, namely at $\eta = Q - id$. The fact that the above map is an isomorphism implies that actually $Q \in W^2_{M_2^{\frac{n}{2}}}$ because

$$\begin{aligned} -\Delta\eta - 2\nabla\eta \cdot \nabla P P^{-1} &= -\Delta Q - 2\nabla Q \cdot \nabla P P^{-1} \\ &= Q(\nabla P P^{-1})^2 \in M_2^{\frac{n}{2}} \\ &\quad \text{since } -\Delta Q - 2\nabla Q \cdot \nabla P P^{-1} - Q(\nabla P P^{-1})^2 = 0. \end{aligned}$$

And we have the obvious estimate

$$||Q - id||_{W^2_{0,M_2^{\frac{n}{2}}}} \leq ||\nabla P||^2_{M_4^n}.$$

□

And as a last preparation of the proof of theorem 28, we will establish the next lemma.

Lemma 34. *Assume that*

$$2 < q \leq p,\ \frac{n}{2} < p < n,\ \text{and}\ \frac{4p}{n} \geq q$$

and let $1 \leq a \leq b < \infty$ *satisfy* $\frac{4b}{n} \geq a$.
Then there exist $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all* $\Omega \in M^p_q(B^n, so(m))$ *with*

$$||\Omega||_{M_2^{\frac{n}{2}}} < \varepsilon_0$$

there exists $A \in W^2_{M^p_q}(B^n, Gl_m(\mathbb{R}))$ *such that* $A^{-1} \in L^\infty(B^n)$ *and*

$$\begin{cases} -\Delta A + A\Omega = 0 \text{ in } B^n \\ A = id \text{ on } \partial B^n. \end{cases} \tag{3.2}$$

and the following estimates hold

i)
$$||A^{-1}\nabla A||_{M_4^n} \leq C||\Omega||_{M_2^{\frac{n}{2}}}$$

ii)
$$||A^{-1} - id||_{M_a^b} \leq C||\Omega||_{M_2^{\frac{n}{2}}}$$

iii)
$$||A^{-1}\nabla A||_{M_r^{\frac{pr}{q}}} \leq C||\Omega||_{M_q^p} \text{ where } r = \frac{nq}{n-p}$$

iv)
$$||A^{-1} - id||_\infty \leq C||\Omega||_{M_q^p}.$$

Proof of lemma 34:
From the proofs of lemma 31 and lemma 29 we know that the map

$$\begin{aligned} \mathcal{J} : \mathcal{U}^{q,p}_{\varepsilon_0} &\to W^2_{M^p_q}(B^n, SO(m)) \\ \Omega &\mapsto P \end{aligned}$$

is continuous for $q > 2$, $\frac{n}{2} < p < n$ with $\frac{4p}{n} \geq q$.
Now, we will show that the following map

$$\begin{aligned} \mathcal{I} : M^{\frac{np}{n-p}}_{\frac{nq}{n-p}}(B^n, \mathbb{R} \otimes so(m)) &\to W^2_{M^p_q}(B^n, M_m(\mathbb{R})) \\ \eta &\mapsto Q \end{aligned}$$

such that
$$\begin{cases} -\Delta Q - 2\nabla Q \cdot \eta - Q\eta^2 = 0 \text{ in } B^n \\ Q = id \text{ on } \partial B^n \end{cases}$$

is continuous for $2 < q \leq p$, $\frac{n}{2} < p < n$ and $\frac{4p}{n} \geq q$, provided that

$$||\eta||_{M_4^n} \leq \varepsilon_0 \ , \ \varepsilon_0 \text{ small enough.}$$

In a next step we shall prove that

$$\begin{array}{rcl} L_\eta : W^2_{0,M_q^p}(B^n, M_m(\mathbb{R})) & \rightarrow & M_q^p(B^n, M_m(\mathbb{R})) \\ u & \mapsto & -\Delta u - 2\nabla u \cdot \eta - u\eta^2 \end{array}$$

is continuous and invertible once ε_0 is small enough.
Observe that for $1 \leq a \leq b < \frac{n}{2}$ with $\frac{4b}{n} \geq a$, according to lemma 9 we have

$$||\nabla u \cdot \nabla \eta||_{M_a^b} \leq C||u||_{W^1_{0,M_a^b}} ||\nabla \eta||_{M_4^n}$$

and similarly, due to lemma 10

$$||u(\eta)^2||_{M_a^b} \leq C||u||_{W^2_{0,M_a^b}} ||\eta||^2_{M_4^n}.$$

Thus, for such a and b

$$L_\eta : W^2_{0,M_a^b}(B^n, M_m(\mathbb{R})) \rightarrow M_a^b(B^n, M_m(\mathbb{R}))$$

is an isomorphism, provided that ε_0 is small enough.
Next, let $f \in M_q^p(B^n, M_m(\mathbb{R})) \subset M_a^b(B^n, M_m(\mathbb{R}))$ (where p and q are as above) and let u be the unique solution of $L_\eta u = f$ in $W^2_{0,M_a^b}(B^n, M_m(\mathbb{R}))$ where $\frac{1}{p} + \frac{1}{b} < \frac{4}{n}$ (cf. also the proof of lemma 31).
From the fact that u solves $L_\eta u = f$ we find

$$-tr(\Delta u u^t) - 2tr(\nabla u \cdot \eta u^t) - tr(u\eta^2 u^t) = tr(f u^t)$$

and thus

$$\Delta \frac{|u|^2}{2} + |\nabla u|^2 + 2 < \nabla u, u\eta > + |u\eta|^2 = < f, u >$$

where $< \cdot, \cdot >$ denotes the scalar product on $M_m(\mathbb{R})$ given by $< A, B >= tr(AB^t)$.
And finally we have

$$\Delta \frac{|u|^2}{2} + < f, u > \geq 0.$$

Now, let φ be a solution of the following problem

$$\begin{cases} -\Delta \varphi = < f, u > \text{ in } B^n \\ \varphi = 0 \text{ on } \partial B^n. \end{cases}$$

Thanks to the assumptions on the exponents we can estimate

$$\begin{aligned} ||\varphi||_\infty &\leq C||f||_{M^p_q}||u||_{M^b_a} \\ &\leq C||f||_{M^p_q}||u||_{M^p_q} \\ &= C||f||^2_{M^p_q}. \end{aligned}$$

All the information we have so far yields - together with the maximum principle - that

$$-C||f||^2_{M^p_q} \leq -||\varphi||_\infty \leq \frac{|u|^2}{2} + \varphi \leq 0$$

and finally

$$||u||_\infty \leq C||f||_{M^p_q}.$$

Again from the fact that $L_\eta u = f$, and the fact that u is bounded, we find

$$\begin{aligned} ||\Delta u - 2\nabla u \cdot \nabla\eta||_{M^p_q} &= ||f + u\eta^2||_{M^p_q} \\ &\leq ||f||_{M^p_q} + ||u||_\infty||\eta^2||_{M^p_q} \\ &\leq ||f||_{M^p_q} + C||f||_{M^p_q}||\eta^2||_{M^p_q} \\ &\leq C||f||_{M^p_q}(1 + ||\eta^2||_{M^p_q}) \\ &\leq C||f||_{M^p_q}(1 + ||\eta||_{M^n_4}||\eta||_{M^{\frac{np}{n-p}}_{\frac{4q}{4-q}}}) \\ &\leq C||f||_{M^p_q}(1 + ||\eta||_{M^n_4}||\eta||_{M^{\frac{np}{n-p}}_{\frac{nq}{n-q}}}) \\ &\quad \text{according to our assumption on the exponents.} \end{aligned}$$

Moreover, we have the following a-priori estimate

$$\begin{aligned} ||\nabla u\eta||_{M^p_q} &\leq C||\eta||_{M^n_4}||\nabla u||_{M^{\frac{np}{n-p}}_{\frac{4q}{4-q}}} \\ &\leq C\varepsilon_0||\nabla u||_{M^{\frac{np}{n-p}}_{\frac{4q}{4-p}}} \\ &\quad \text{due to our assumption} \\ &\leq C\varepsilon_0||\nabla u||_{M^{\frac{np}{n-p}}_{\frac{nq}{n-p}}} \\ &\leq C\varepsilon_0||u||_{W^2_{M^p_q}}. \end{aligned}$$

Thus, our solution u is even in $W^2_{M^p_q}$, L_η is invertible also from M^p_q to W^2_{0,M^p_q} (and continuous), provided that ε_0 is small enough.

Next, we will prove that the map $\mathcal{I}$ is continuous.
Let $\delta \in M^{\frac{np}{n-p}}_{\frac{nq}{n-p}}(B^n, \mathbb{R} \otimes so(m))$ such that

$$||\eta + \delta||_{M_4^n} \leq \varepsilon_0.$$

Furthermore, let $Q+q$ solve $L_{\eta+\delta}(Q+q) = 0$ with $Q+q = id$ on the boundary ∂B^n. Then it holds

$$L_{\eta+\delta} q = -L_{\eta+\delta} Q + L_\eta Q = 2\nabla Q \cdot \delta + Q((\eta+\delta)^2 - \eta^2)$$

with zero boundary condition. Due to the invertibility we have shown above, we infer that $q \in W^2_{0,M^p_q}$ with the estimate

$$\begin{aligned} ||q||_{W^2_{0,M^p_q}} &\leq C||\nabla Q||_{M^{\frac{np}{n-p}}_{\frac{4q}{4-q}}} ||\delta||_{M_4^n} \\ &\quad + ||Q||_\infty ||\delta||_{M^{\frac{np}{n-p}}_{\frac{4q}{4-q}}} (||\eta||_{M_4^n} + ||\delta||_{M_4^n}) \\ &\leq C||\nabla Q||_{M^{\frac{np}{n-p}}_{\frac{nq}{n-p}}} ||\delta||_{M^{\frac{np}{n-p}}_{\frac{nq}{n-p}}} \\ &\quad + C||Q||_{W^2_{M^p_q}} ||\delta||_{M^{\frac{np}{n-p}}_{\frac{nq}{n-p}}} (||\eta||_{M_4^n} + ||\delta||_{M_4^n}). \end{aligned}$$

and thus $\mathcal{I}$ is continuous.
And hence - from the construction - also the map

$$\begin{aligned} \mathcal{K} : \mathcal{U}^{q,p}_{\varepsilon_0} &\rightarrow W^2_{M^p_q}(B^n, M_m(\mathbb{R})) \\ \Omega &\mapsto A := QP \end{aligned}$$

is continuous for $2 < q \leq p$, $\frac{n}{2} < p < n$ with $\frac{4p}{n} \geq q$.

Next, let p and q be as before, let further be $\varepsilon_0 > 0$, $C > 0$ and $1 \leq a \leq b < \infty$ with $\frac{4b}{n} \geq a$. Then set

$$\mathcal{W}^{q,p,a,b}_{\varepsilon_0,C} := \left\{ \begin{array}{l} \Omega \in \mathcal{U}^{q,p}_{\varepsilon_0} \text{ such that } A := \mathcal{K}(\Omega) \text{ satisfies 3.2} \\ \text{and the estimates i) up to iv)} \end{array} \right\}$$

We claim that for all choices of a, b, p and q which respect the supplementary requirements there exist $\varepsilon_0 > 0$ and $C > 0$ such that $\mathcal{U}^{q,p}_{\varepsilon_0} = \mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$. For this purpose it suffices to show that there exist $\varepsilon_0 > 0$ and $C > 0$ such that $\mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ is closed and open in $\mathcal{U}^{q,p}_{\varepsilon_0}$ and not empty.

$\mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ is not empty for $C > 0$ and $\varepsilon_0 := \frac{\delta}{2}$ where δ is given by lemma 32:
Due to the fact that $\mathcal{K}$ is continuous, in a M^p_q-neighbourhood of zero, we have

that $||\nabla A||_{M_4^n}$, $||A-id||_\infty$ and $||A^{-1}-id||_\infty$ are small. Hence, $||A^{-1}\nabla A||_{M_4^n} < \frac{\delta}{2}$ once the neighbourhood is small enough. Thanks to lemma 32, this shows that for our choice of ε_0, $\mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ is not empty.

$\mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ is closed in $\mathcal{U}^{q,p}_{\varepsilon_0}$:
Let $\Omega_k \in \mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ such that

$$\Omega_k \to \Omega_\infty \text{ in } M^p_q.$$

The continuity of the map $\mathcal{K}$ implies that $A_k := \mathcal{K}(\Omega_k)$ converge strongly to a limit $A_\infty := \mathcal{K}(\Omega_\infty)$ in $W^2_{M^p_q}$. Our assumption that $\Omega_k \in \mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ and the convergence of the sequence give that

$$||A_k^{-1}\nabla A_K||_{M_4^n} \leq C||\Omega_k||_{M_2^{\frac{n}{2}}} \leq C\varepsilon_0$$

and

$$||A_k^{-1} - id||_\infty \leq C||\Omega_k||_{M^p_q} \leq C$$

which implies that $||A_k^{-1}||_\infty$ and $||\nabla A_k^{-1}A_k||_{M_4^m} = ||A_k^{-1}\nabla A_k||_{M_4^n}$ are uniformly bounded. Thus, $||\nabla A_k^{-1}||_{M_4^m}$ is uniformly bounded as well. Hence, there exists a subsequence which converges strongly in $L^s(B^n)$, $s < 4^*$. Finally, from the identity $A_kA_k^{-1} = A_k^{-1}A_k$ we find that A_k^{-1} has to converge to A_∞^{-1}, and thus, the required properties hold, i. e. $\Omega_\infty \in \mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$.

$\mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ is open in $\mathcal{U}^{q,p}_{\varepsilon_0}$:
Let ε_0 be small enough and let C be given by lemma 32. Moreover, let $\Omega \in \mathcal{W}^{q,p,a,b}_{\varepsilon_0,C}$ and $A := \mathcal{K}(\Omega)$.
In particular, we have

$$||A^{-1}||_\infty < \infty.$$

Now, let $a \in W^2_{0,M^p_q}(B^n, M_m(\mathbb{R}))$ and we have $A + a = A(id + A^{-1}a)$. From that we obtain - if $||a||_{W^2_{M^p_q}}$ is small enough

$$||(A+a)^{-1} - A^{-1}||_\infty \leq C||A^{-1}||_\infty||a||_{W^2_{M^p_q}}.$$

In addition, we estimate

$$\begin{aligned}||(A+a)^{-1}\nabla(A+a) - A^{-1}\nabla A||_{M_4^n} &\leq ||[(A+a)^{-1} - A^{-1}]\nabla A||_{M_4^n}\\ &\quad +||(A+a)^{-1}\nabla a||_{M_4^n}\\ &\leq C||\nabla A||_{M_4^n}||A^{-1}||_\infty||a||_{W^2_{M^p_q}}\\ &\quad +||\nabla a||_{M_4^n}||(A+a)^{-1}||_\infty\\ &\leq C||a||_{W^2_{M^p_q}}.\end{aligned}$$

Since $\mathcal{K}$ is continuous and

$$||A^{-1}\nabla A||_{M_4^n} \leq C||\Omega||_{M_2^{\frac{n}{2}}} < C\varepsilon_0$$

there exists a radius ρ_Ω such that for every $\omega \in M_q^p$ with $||\omega||_{M_q^p} < \rho_\Omega$ we have

$$||\mathcal{K}(\Omega+\omega)^{-1}\nabla(\mathcal{K}(\Omega+\omega))||_{M_4^n} \leq 2C\varepsilon_0.$$

If ε_0 and ρ_Ω are small enough, such that $\rho_\Omega + 2C\varepsilon_0 + \varepsilon_0 < \delta$ we can apply lemma 32 in order to see that $\mathcal{K}(\Omega+\omega) = A+a$, for some a as above, satisfies the required properties.

□

3.2 Proof of theorem 28

Before we come to the proof of theorem 28, let us establish the fact, that what we have seen in lemma 34 holds also for the limit case $p = \frac{n}{2}$ and $q = 2$. More precisely, we will show the following lemma

Lemma 35. *Let* $1 \leq a, b < \infty$ *be such that* $\frac{4b}{n} \geq a$. *There exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all* $\Omega \in M_2^{\frac{n}{2}}(B^n, so(m))$ *which satisfy*

$$||\Omega||_{M_2^{\frac{n}{2}}(B^n)} \leq \varepsilon_0$$

there exists $A \in L^\infty \cap W^2_{M_2^{\frac{n}{2}}}(B^n, Gl_m(\mathbb{R}))$ *with the following properties*

- $$A^{-1} \in M_a^b(B^n)$$

- $$\begin{cases} -\Delta A + A\Omega = 0 \text{ in } B^n \\ A = id \text{ on } \partial B^n. \end{cases}$$

Moreover, the following estimates hold

$$||A^{-1}\nabla A||_{M_4^n} \leq C||\Omega||_{M_2^{\frac{n}{2}}}$$

$$||A^{-1} - id||_{M_2^b} \leq C||\Omega||_{M_2^{\frac{n}{2}}}.$$

Proof of lemma 35:
Let $\frac{n}{2} < p < n$, $q > 2$ and $\frac{4p}{n} \geq q$. Moreover, assume that $\Omega \in M_2^{\frac{n}{2}}$ with

$$||\Omega||_{M_2^{\frac{n}{2}}} < \varepsilon_0$$

where ε_0 is given by lemma 34. In addition, let Ω_k be a sequence in M_q^p, which converges to the given Ω (in $M_2^{\frac{n}{2}}$) and satisfies

$$||\Omega_k||_{M_2^{\frac{n}{2}}} < \varepsilon_0.$$

Note that this can be achieved by a standard mollification argument.
Now, recall that the for the A_k, associated to Ω_k, we have the following estimates (see lemma 33 and lemma 34):

- $$||A_k||_\infty \leq 1 \text{ cf. lemma 33, } P \in SO(m)$$
- $$||\nabla A_k||_{M_4^n} \leq C||\Omega_k||_{M_2^{\frac{n}{2}}} \leq C\varepsilon_0$$
- $$||A_k^{-1}\nabla A_k||_{M_4^n} \leq C||\Omega_k||_{M_2^{\frac{n}{2}}} \leq C\varepsilon_0$$
- $$||A_k^{-1} - id||_{M_a^b} \leq C||\Omega_k||_{M_2^{\frac{n}{2}}} \leq C\varepsilon_0.$$

From the last inequalities we deduce that the sequence $\left\{A_k^{-1}\right\}$ is uniformly bounded, namely it holds

$$||A_k^{-1}||_{M_a^b} \leq C$$

From the first and the second inequality we infer that there exists a subsequence $\left\{A_k'\right\} \subset W^{1,4}$ which converges weakly to some limit A (in $W^{1,4}$). Obviously, A and Ω fulfil

$$\begin{cases} -\Delta A + A\Omega = 0 \text{ in } B^n \\ A = id \text{ on } \partial B^n. \end{cases}$$

Now, note that

$$||\nabla A_k^{-1} A_k||_{M_4^n} = ||A_k^{-1}\nabla A_k||_{M_4^n} \leq C\varepsilon_0.$$

From this follows that

$$\begin{aligned}
||\nabla A_k^{-1}||_{M_d^c} &= ||\nabla A_k^{-1} A_k A_k^{-1}||_{M_d^c} \\
&\leq ||\nabla A_k^{-1} A_k||_{M_4^n} ||A_k^{-1}||_{M_a^b} \\
&= ||\nabla A_k^{-1} A_k||_{M_4^n} ||A_k^{-1}||_{M_a^b} \\
&\leq C\varepsilon_0
\end{aligned}$$

where

$$\frac{1}{c} = \frac{1}{n} + \frac{1}{b} \text{ and } \frac{1}{d} = \frac{1}{4} + \frac{1}{a}.$$

Due to this last estimate, we may assume - if necessary by passing to a subsequence, which we still denote $\left\{A_k^{-1}\right\}$ - that our subsequence satisfies $A_k^{-1} \in W^{1,d}$ and that the A_k^{-1} converge weakly to some limit $\tilde{A}$.
Hence the sequence $\left\{A_k^{-1}\right\}$ converge strongly in L^s where $s < d^* = \frac{nd}{n-d}$ (d^* is positive because according to our hypothesis $n \geq 4 > d$).
On the other hand, we know that - since $\{A_k\}$ converges weakly in $W^{1,4}$ - this same sequence converges strongly to A in L^r where $r < 4^* = \frac{4n}{n-4} \leq d^*$.
These facts imply that in the following equality we can pass to the limit -at least in the sense of distributions -

$$A_k^{-1} A_k = A - k A_k^{-1} = id.$$

This finally leads to the conclusion that

$$\tilde{A} = A^{-1}.$$

This shows that A is invertible and thus has to be bounded.
And we are able to conclude that in fact A belongs to $W^2_{M_2^{\frac{n}{2}}}(B^n, Gl_m(\mathbb{R}))$, namely we have that

$$\begin{aligned}
||A||_{W^2_{M_2^{\frac{n}{2}}}} &\leq C||\Delta A||_{M_2^{\frac{n}{2}}} \\
&\leq C||A\Omega||_{M_2^{\frac{n}{2}}} \\
&\quad \text{because of } \Delta A + A\Omega = 0 \\
&\leq C||A||_\infty ||\Omega||_{M_2^{\frac{n}{2}}} \\
&< \infty \\
&\quad \text{according to our hypothesis that } ||\Omega||_{M_2^{\frac{n}{2}}} \leq \varepsilon_0 \\
&\quad \text{and the fact that } A \text{ is bounded.}
\end{aligned}$$

The estimates i) and ii) follow from lemma 34.

□

Proof of theorem 28:

First of all, we would like to point out that all the following equalities take place in the sense of distributions.
The underlying calculations make - at first instance - make formally seance, and since all the involved quantities are at least in L^1, they can be made rigorous in the seance of distributions.
Let $\Omega \in M_2^{\frac{n}{2}}(B^n, so(m))$ and let $u \in M_2^{\frac{n}{2}}$ be a solution of

$$-\Delta u = \Omega u.$$

Moreover, let $P \in W^2_{M_2^{\frac{n}{2}}}(B^n, SO(m))$ be given by lemma 29. Then it holds

$$-\Delta(Pv) = \Delta P v - P\Delta v - 2\,div(\nabla P v).$$

Now, set $w := Pv$. A short calculation yields that $-\Delta v = \Omega v$ is equivalent to

$$-\Delta w = \left[\Delta P P^{-1} + P\Omega P^{-1}\right] w - 2\,div(\nabla P P^{-1} w).$$

Due to the fact that P satisfies $\frac{1}{2}\left[\Delta P P^{-1} - P\Delta P^{-1}\right] + P\Omega P^{-1} = 0$, the above equality is equivalent to

$$-\Delta w - \frac{1}{2}\left[\Delta P P^{-1} + P\Delta P^{-1}\right] w + 2\,div(\nabla P P^{-1} w) = 0.$$

Now, observe that

$$\begin{aligned} -\left[\Delta P P^{-1} + P\Delta P^{-1}\right] &= -div(\nabla P P^{-1} + P\nabla P^{-1}) + 2\nabla P \cdot \nabla P^{-1} \\ &= 2\nabla P \cdot \nabla P^{-1} \\ &\quad \text{since } \nabla P P^{-1} = -P\nabla P^{-1} \\ &= 2\nabla P(-P^{-1}\nabla P P^{-1}) \\ &= -2(\nabla P P^{-1})^2 \end{aligned}$$

where

$$-2(\nabla P P^{-1})^2 := -2\sum_{j=1}^{n}(\partial_{x_j} P P^{-1})^2.$$

Thus our original equation is equivalent to

$$-\Delta w - (\nabla P P^{-1})^2 w + 2\,div(\nabla P P^{-1} w) = 0$$

where

$$(\nabla PP^{-1})^2 \in M_2^{\frac{n}{2}}(B^n, Sym_m^+),$$

i. e. $(\nabla PP^{-1})^2$ has values in the space of symmetric, non-negative $m \times m$-matrices. The fact that this quantity belongs to $M_2^{\frac{n}{2}}$ is a consequence of the facts that on one hand, $P \in SO(m)$ and on the other hand that, according to theorem 96, $W^1_{M_2^{\frac{n}{2}}}$ embeds into M_4^n and we have the estimates (see also lemma 29)

$$||\nabla P||_{M_4^n} \leq C||P - id||_{W^2_{M_2^{\frac{n}{2}}}} \leq C||\Omega||_{M_2^{\frac{n}{2}}} \leq C\varepsilon < \varepsilon_0.$$

Note that the last estimate gives us an indication how small ε has to be.
Next, we consider the matrix Q which is associated to P via lemma 29 Multiplying the last equation by Q from the left leads to

$$\begin{aligned} 0 &= -Q\Delta w - Q(\nabla PP^{-1})^2 w + 2Qdiv(\nabla PP^{-1}w) \\ &= -Q\Delta w - \left[Q(\nabla PP^{-1})^2 + 2\nabla Q \cdot \nabla PP^{-1}\right] w + 2div(Q\nabla PP^{-1}w) \\ &= -Q\Delta w + \Delta Qw + 2div(Q\nabla PP^{-1}w) \\ &\quad \text{because } -\Delta Q - 2\nabla Q \cdot \nabla PP^{-1} - Q(\nabla PP^{-1})^2 = 0 \\ &= div(-Q\nabla w + \nabla Qw + 2Q\nabla PP^{-1}) \end{aligned}$$

Remember that we sat $w = Pu$, so we can re-substitute $u = P^{-1}w$ which leads to

$$div((QP)\nabla u - \nabla(QP)u) = 0.$$

Finally we set $A := QP$ and obtain

$$A\Delta u - \Delta Au = 0$$

which can be rephrased - since $-\Delta u = \Omega u$ - as

$$\Delta A\Omega u - \Delta Au = 0$$

and which finally implies the claimed property

$$\Delta A + A\Omega = 0$$

since we assume that $u \neq 0$. Note that in the last step we do not claim an equivalence!
The further properties of A stated in our theorem follow from lemma 33 , lemma 35 and lemma 29 and obviously from the construction of A.

□

Chapter 4

Applications

In this section we want to apply the previous gauge result in order to write problems of the form

$$-\Delta u = \Omega u$$

where Ω is antisymmetric in divergence form, see, theorem E in the introduction.

We will prove that

Theorem 36. *Let $n \geq 4$ and let $m \in \mathbb{N}^*$.*
Assume that $u \in M_2^{\frac{n}{2}}(B^n, \mathbb{R}^m)$, $\Delta u \in L^1_{loc}$ and $\Omega \in M_2^{\frac{n}{2}}(B^n, so(m))$ such that

$$||\Omega||_{M_2^{\frac{n}{2}}} \leq \varepsilon$$

where ε is given by theorem 28.
Then

$$-\Delta u = \Omega u \tag{4.1}$$

is equivalent to

$$div(A\nabla u - \nabla A u) = 0 \tag{4.2}$$

where A is again given by theorem 28.

Proof of theorem 36:
First, we will show that the assumption that u solves $-\Delta u = \Omega u$ together with the properties of A we have at hand, implies that $div(A\nabla u - \nabla A u) = 0$: In particular we have that $A \in L^\infty \cap W^{1,2}$, thus we may calculate (use density

of C^∞ in the usual Lebesgue and Sobolev spaces)

$$\begin{aligned} div(A\nabla u - \nabla A u) &= A\Delta u - \Delta A u \\ &= A\Delta u + A\Omega u \text{ because } \Delta A + A\Omega = 0 \\ &= -A\Omega u + A\Omega u \text{ because } -\Delta u = \Omega u \\ &= 0. \end{aligned}$$

Next we will show that also the reverse implication holds, once we have the additional assumption that $u \in L^1_{loc}$:
As above, we have that almost everywhere

$$0 = div(A\nabla u - \nabla A u) = A\Delta u - \Delta A u.$$

Since A is almost everywhere invertible, this implies that

$$0 = \Delta u - A^{-1}\Delta A u = \Delta u + \Omega u \; a.e..$$

This later statement is obviously equivalent to

$$-\Delta u = \Omega u \;\; a.e.$$

Remark 37.

i) In dimension $n = 1$ and $n = 2$ it is obvious that the hypothesis $v, \Omega \in L^2$ immediately imply that $v \in L^\infty \cap W^{2,1}$.

ii) In arbitrary dimension, the fact that under the hypothesis that $u, \Omega \in M_2^{\frac{n}{2}}$ imply in particular that $\Delta u \in L^1$. Such an assumption was studied in [39] (in particular proof of theorem I.4) and leads to the conclusion that
$$u \in L^\infty \cap W^2_{M_2^{\frac{n}{2}}} \text{ locally.}$$

Part V

Some minor miscellaneous results

Chapter 1

Regularity in low dimensions

We start with the low-dimensional case.
Thanks to proposition 117 we know that - for arbitrary dimension! - under appropriate assumptions on a and b the Jacobean belongs to $F^0_{1,2}$. In dimension $n = 1$ or $n = 2$ this is enough to conclude that u is continuous. More precisely we have

Proposition 38. *Let $n = 1, 2$ and assume that $f \in F^0_{1,2}(\mathbb{R}^n)$. Then any solution u of (1) is continuous and bounded.*

Proof of proposition 38:
Recall that for $f \in F^0_{1,2} = \mathfrak{h}^1$

$$f = \sum_{k=0}^{\infty} f^k \text{ in } \mathcal{S}'.$$

This enables us to rewrite our equation (1) as

$$\Delta u = f^0 + \sum_{k \geq 1} f^k.$$

The advantage of the latter decomposition consists in the separation of the contribution to f whose Fourier support is contained in the unit ball around the origin from the other contributions! And the solution u can be written as

$$\begin{aligned} u &= \Delta^{-1} f^0 + \Delta^{-1}(\sum_{k \geq 1} f^k) \\ &=: u_1 + u_2. \end{aligned}$$

Our strategy is to show that u_1 as well as u_2 is continuous, so also their sum is continuous.

What concerns u_1, observe that due to the Paley-Wiener theorem f^0 is analytic, so in particular continuous. This implies immediately - by classical results (see e.g. [24]) - that u_1 is continuous.
But since $f^0 \in B^s_{1,2}$ for all $s \in \mathbb{R}$ this can be improved via the following observations:

- Since $f^0 \in B^0_{1,2}$ we know that $f^0 \in B^{-1}_{2,2} \subset F^{-1}_{2,2} = W^{-1,2}$ (see theorem 74). So $u_1 \in W^{1,2}$ (see e.g. [8]) and in particular $u_1 \in L^2$.

- Moreover, we have that $||u_1|\dot{B}^{s+2}_{2,1}|| \leq ||f^0|\dot{B}^s_{2,1}||$ due to proposition 86 (proposition 85 assures us that f^0 belongs to $\dot{B}^s_{2,1}$ for $s > 0$). Once we have this, we estimate for $s > 0$

$$\begin{aligned} ||u_1|B^s_{2,1}|| &\leq C(||u_1||_2 + ||u_1|\dot{B}^{s+2}_{2,1}||) \\ &\quad \text{due to proposition 85} \\ &\leq C(||u_1||_2 + ||f^0|\dot{B}^s_{2,1}||) \\ &\leq C(||u_1||_2 + C||f^0|B^s_{2,1}||) \\ &< \infty \\ &\quad \text{again by the fact that } f^0 \in B^s_{1,2}\ \forall s \text{ and theorem 74} \end{aligned}$$

  Now, this information together with theorem 75 below of Sickel and Triebel leads to the conclusion that u_1 is not only continuous but also bounded!

Next, we will show that u_2 is bounded and continuous. In order to reach this goal, we show that $u_2 \in B^0_{\infty,1}$: We find the following estimates

$$\begin{aligned} ||u_2|B^0_{\infty,1}|| &= \sum_{s=0}^{\infty} ||u_2^s||_\infty \\ &= \sum_{s=0}^{\infty} 2^{-2s}2^{2s}||u_2^s||_\infty \\ &= C\sum_{s=0}^{\infty} 2^{-2s}||(\Delta u_2)^s||_\infty \end{aligned}$$

This last passage holds thanks to the lemma below (see [55] for instance).
For $s = 0$ we observe

$$\mathcal{F}(-\Delta u_2) = \mathcal{F}(\sum_{k\geq 1} f^k)$$

which implies

$$supp(\mathcal{F}(u_2)) \subset (B_1(0))^c$$

because of the fact that

$$supp(\mathcal{F}(\sum_{k\geq 1} f^k)) \subset (B_1(0))^c.$$

So in this case too, we can apply lemma 39 in order to conclude that also for $s = 0$ we have

$$||u_2^0||_\infty \leq C||(\Delta u_2)^0||_\infty.$$

Lemma 39. *([55]) Let g be a function such that its Fourier image is supported in an annulus A with radii $r_1, r_2 \simeq 2^s$. Then*

$$2^{ms}||g||_p \simeq ||\nabla^m g||_p \;\;,\; 1 \leq p \leq \infty.$$

Back to our estimate, we continue

$$\begin{aligned}
||u_2|B^0_{\infty,1}|| &\leq C\sum_{s=0}^{\infty} 2^{-2s}||(\Delta u_2)^s||_\infty \\
&= C\sum_{s=0}^{\infty} 2^{-2s}||(\sum_{k\geq 1} f^k)^s||_\infty \\
&= C\sum_{s=0}^{\infty} 2^{-2s}||\mathcal{F}^{-1}(\sum_{k=s-1}^{s+1} \varphi_s\varphi_k\hat{f})||_\infty \\
&\leq \sum_{s=0}^{\infty} 2^{-2s}||f^s||_\infty \\
&\quad \text{thanks to a Fourier multiplier result} \\
&\quad \text{similar to the one we state in the second chapter} \\
&\quad \text{for further details we refer to [61]} \\
&= C||f|B^{-2}_{\infty,1}|| \\
&\leq C||f|F^0_{1,2}|| \\
&\quad \text{because of theorem 74} \\
&\quad \text{if } n=1, \text{ use in addition } F^0_{1,2} \subset F^{-1}_{1,2} \\
&< \infty \\
&\quad \text{according to our assumption.}
\end{aligned}$$

This shows that u_2 belongs to $B^0_{\infty,1}(\mathbb{R}^2)$.
Alternatively one could make use of the lifting property, proposition 86, to show that $u_2 \in F^2_{1,2} \subset C$.
The last ingredient is the embedding result 75 due to Sickel/Triebel (see [47]).
Recall that C denotes the space of all uniformly continuous functions on $\mathbb{R}^n$.

In our case we us the embedding of $B^0_{\infty,1}(\mathbb{R}^2)$ into C. Thus also u_2 is continuous and bounded.
This leads immediately to the assertion we claimed because u as a sum of two bounded continuous functions is again continuous and bounded.

$\square$

In higher dimension, the embedding result of Sickel and Triebel (see theorem 75 below) does not help us any longer in order to conclude that a solution of (1) where $f \in F^0_{1,2}$ is continuous.
But as we saw in Part II, this problem can be avoided once we start with adapted hypothesis involving Morrey-Besov spaces.

Chapter 2

Regularity results for $-\Delta u = ab$

Last, but not least, if we drop the improved algebraic structure of the Jacobean and in stead of this look at

$$-\Delta u = ab \tag{2.1}$$

we have the following assertion.

Proposition 40. *Assume that* $a \in N^0_{p_1,q_1,r_1}$ *and* $b \in N^0_{p_2,q_2,r_2}$ *with*

$$\begin{aligned} n\Big(\frac{1}{p_1} + \frac{1}{p_2}\Big) &< 2 \\ \frac{1}{q_1} + \frac{1}{q_2} &= 1 \\ \frac{1}{r_1} + \frac{1}{r_2} &= 1. \end{aligned}$$

Then any solution of (2.1) is continuous.

Proof of proposition 40:
To start with, note that our problem

$$-\Delta u = ab \tag{2.2}$$

can be rewritten as

$$\begin{aligned} -\Delta u_1 &= \pi_1(a,b) \\ -\Delta u_2 &= \pi_2(a,b) \\ -\Delta u_3 &= \pi_3(a,b). \end{aligned}$$

Analysis of $-\Delta u_1 = \pi_1(a,b)$ ***respectively*** $-\Delta u_3 = \pi_3(a,b)$

First, observe that thanks to the our assumptions there exists

$$0 < \varepsilon < \ 2 - n\Big(\frac{1}{p_1} + \frac{1}{p_2}\Big).$$

Our next goal is to show that $\pi(a,b) \in N^{-\varepsilon}_{p,1,1}$ where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$: In order to reach this goal, let us recall the following analogue of proposition 123 in the framework of Besov-Morrey spaces (see for instance [32]).

Lemma 41. *([32]) Let $\{f_k\}$, $k \geq 0$ be a sequence of tempered distributions such that for some $A > 0$ supp $\mathcal{F}u_0 \subset B_{2A}(0)$ and supp $\mathcal{F}u_k \subset \left\{\xi \in \mathbb{R}^n | A2^{k-1} < |\xi| < A^{k+}\right\}$ for $k > 0$. Then*

$$||\sum_{k=0}^{\infty} f_k | N^s_{p,q,r}|| \leq C(A)(||f_0|M^p_q|| + ||\left\{2^{sk}||f_k|M^q_p||\right\}^{\infty}_{k=1}|l^1||).$$

According to this, it is enough to show that

$$\sum_{s=0}^{\infty} 2^{-\varepsilon s}||\sum_{k=0}^{s-2} a^k b^s||_{M^p_1} < \infty.$$

In fact, we have

$$\begin{aligned}
\sum_{s=0}^{\infty} 2^{-\varepsilon s}||\sum_{k=0}^{s-2} a^k b^s||_{M^p_1} &\leq \sum_{s=0}^{\infty} 2^{-\varepsilon s}||\sum_{k=0}^{s-2} a^k||_{M^{p_1}_{q_1}}||b^s||_{M^{p_2}_{p_2}} \\
&\leq \Big(\sum_{s=0}^{\infty} 2^{-r_1\varepsilon s}||\sum_{k=0}^{s-2} a^k||^{r_1}_{M^{p_1}_{q_1}}\Big)^{\frac{1}{r_1}} \Big(\sum_{s=0}^{\infty} ||b^s||^{r_2}_{M^{p_2}_{p_2}}\Big)^{\frac{1}{r_2}} \\
&\leq C||2^{-\varepsilon s}|\sum_{k=0}^{s} f^k| \ \ |l^{r_1}(M^{p_1}_{q_1})||||b_y|N^0_{q_2,p_2,r_2}|| \\
&\leq C||f|N^{-\varepsilon}_{p_1,q_1,r_1}||||b_y|N^0_{q_2,p_2,r_2}||) \\
&\quad \text{here we use lemma 126 with the modification} \\
&\quad \text{that instead of Lebesgue-norms we have Morrey-norms} \\
&\quad \text{the proof in this different setting is the same} \\
&\leq C||f|N^0_{p_1,q_1,r_1}||||b_y|N^0_{q_2,p_2,r_2}|| \\
&< \infty \\
&\quad \text{according to our hypothesis.}
\end{aligned}$$

Now, we apply the same procedure as in the previous proof and conclude that the solution $u_{1,1}$ of

$$-\Delta u_{1,1} = \sum_{k \geq 1} f^k \tag{2.3}$$

where $f \in N^{-\varepsilon}_{p,1,1}$ belongs to $B^{2-\varepsilon-n/p}_{\infty,1}$ which embeds to C since we assumed that

$$2 - \frac{n}{p} - \varepsilon > 0.$$

Moreover, as before we have that the solution $u_{1,0}$ of

$$-\Delta u_{1,0} = f^0$$

with $f \in N^{-\varepsilon}_{p,1,1}$ is continuous.
So we find that $u_1 = u_{1,0} + u_{1,1}$ is continuous.
The problem

$$-\Delta u_3 = \pi_3(a,b)$$

is treated in exactly the same manner, so also u_3 is continuous.
It remains to study the second equation:

Analysis of $-\Delta u_2 = \pi_2(a,b)$

In the case of this remaining part where the frequencies of a and b are comparable, we split our problem further:
Since

$$\pi_2(a,b) = \sum_{s=0}^{\infty}\sum_{k=s-1}^{s+1} a^k b^s = \sum_{s=0}^{\infty}\sum_{k=s-1}^{s+1} (\check{\varphi}_0 * a^k b^s + (1-\varphi_0)^{\vee} * a^k b^s) =: \tilde{\pi}_{2,0}(a,b) + \tilde{\pi}_{2,1}(a,b)$$

in what follows, we look at the two following equations

$$\begin{aligned} -\Delta u_{2,0} &= \tilde{\pi}_{2,0}(a,b) \\ -\Delta u_{2,1} &= \tilde{\pi}_{2,1}(a,b). \end{aligned} \tag{2.4}$$

What concerns the first equation, we can immediately conclude that the solution $u_{2,0}$ is continuous: This is obtained from the fact that *supp* $\mathcal{F}\tilde{\pi}_{2,0}(a,b) \subset B_2(0)$ which implies that $\tilde{\pi}_{2,0}(a,b) \in C^\infty$ together with classical regularity theory (see e.g. [24]).
In the case of the second equation we shall show that $u_{2,1} \in B^{\varepsilon}_{\infty,\infty}$ where as before $\varepsilon > 0$ is such that

$$0 < \varepsilon < \ 2 - n\Big(\frac{1}{p_1} + \frac{1}{p_2}\Big).$$

So we find

$$
\begin{aligned}
||u_{2,1}|B^{\varepsilon}_{\infty,\infty}|| &= \sup_{s\geq 0} 2^{\varepsilon}||u^s_{2,1}||_{\infty} \\
&= \sup_{s\geq 0} 2^{\varepsilon-s}2^s||u^s_{2,1}||_{\infty} \\
&\leq \sup_{s\geq 0} 2^{\varepsilon-s}||(-\Delta u_{2,1})^s||_{\infty} \\
&= ||\tilde{\pi}_{2,1}|B^{2-\varepsilon}_{\infty,\infty}|| \\
&\leq ||\tilde{\pi}_{2,1}|B^{-n(\frac{1}{p}}_{\infty,\infty}|| \\
&\leq ||\tilde{\pi}_{2,1}|N^0_{p,1,\infty}|| \\
&\quad \text{due to the embedding result of Kozono/Yamazaki (see theorem (97))} \\
&= \sup_{s\geq 0} ||\check{\varphi}_s * (\sum_{l=0}^{\infty}\sum_{k=l-1}^{l+1}(1-\varphi_0)^{\vee} * a^k b^l)|M^p_1|| \\
&\leq \sup_{s\geq 0}\sum_{l=0}^{\infty} C \max_{k\in\{l-1,\dots,l+1\}} ||\check{\varphi}_s * (1-\varphi_0)^{\vee} * a^k b^l|M^p_1|| \\
&\leq C\sup_{s\geq 0}\sum_{l=0}^{\infty} C \max_{k\in\{l-1,\dots,l+1\}} ||(1-\varphi_0)^{\vee} * a^k b^l|M^p_1|| \\
&\quad \text{due to multiplier result of Kozono/Yamazaki (see lemma 98} \\
&= C\sum_{l=0}^{\infty} \max_{k\in\{l-1,\dots,l+1\}} ||*(\psi(2^{(}-l-3\cdot)-\varphi_0)^{\vee} * a^k b^l|M^p_1|| \\
&\leq \sum_{l=0}^{\infty} \max_{k\in\{l-1,\dots,l+1\}} ||a^k b^l|M^p_1|| \\
&\quad \text{by the same result as in the second last step} \\
&\leq \sum_{l=0}^{\infty} \max_{k\in\{l-1,\dots,l+1\}} ||a^k|M^{p_1}_{q_1}||||b^l|M^{p_2}_{q_2}|| \\
&\leq C\sum_{l=0}^{\infty} ||a^l|M^{p_1}_{q_1}||||b^l|M^{p_2}_{q_2}|| \\
&\leq C\Big(\sum_{l=0}^{\infty}||a^k|M^{p_1}_{q_1}||^{r_1}\Big)^{\frac{1}{r_1}}\Big(\sum_{l=0}^{\infty}||b^l|M^{p_2}_{q_2}||^{r_2}\Big)^{\frac{1}{r_2}} \\
&\leq C||a|N^0_{p_1,q_1,r_1}||||b|N^0_{p_2,q_2,r_2} \\
&< \infty \ \text{according to our hypothesis.}
\end{aligned}
$$

□

Chapter 3

Products in $\mathfrak{h}^1$

A natural question that arises when we look at the above proposition is the following: Is it possible to give up the additional algebraic structure, i.e. the determinant structure, and replace it by other assumptions in order to obtain the assertion that a certain product of first derivatives has an improved integrability property, i.e. we wish to find conditions under which $a_x b_y \in \mathfrak{h}^1$, or more generally, under which conditions a product belongs to $\mathfrak{h}^1$.
One possible answer in this direction is given in the lemma below.

Lemma 42. *Assume that there exist $\varepsilon > 0$ and $p > 1$ such that $a \in F^{\varepsilon}_{p,2}(\mathbb{R}^n)$ and $b \in L^{p'}(\mathbb{R}^n)$ where*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then

$$ab \in F^{\varepsilon}_{1,2} \subset F^{0}_{1,2} = \mathfrak{h}^1.$$

Note that - in case that ε is an integer - what we said at the beginning of this section holds also in the context of this lemma. In this case we may apply in addition Sobolev's embedding theorem.
Since the proof of this result is on one hand an immediate consequence of classical results and on the other hand implied by the technique applied in order to prove the previous results, we postpone its proof to the appendix.

Proof of lemma 42:
Let us first of all explain what we can obtain as an immediate consequence of the technique we used in the proof of proposition 117.
We will start with discussing the case $a \in W^{2,p}(\mathbb{R}^n)$ and $b \in W^{1,p'}(\mathbb{R}^n)$ where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Our intermediate goal is to show that $a_x b_y \in F^0_{1,2} = \mathfrak{h}^1$.

We start with the estimates for $\pi_1(a_x, b_y)$ and $\pi_3(a_x, b_y)$.

As before, thanks to proposition 123 it remains to show that $||c_k \mid L^1(l^2)|| < \infty$ where $c_k = \sum_{t=0}^{k-2} a_x^t b_y^k$:
First note that $\sup_{k\geq 0} |\sum_{t=0}^{k} a_x^t| \in L^p$. This holds since $a_x \in L^p = F_{p,2}^0$ which together with lemma 126 gives that $|| \sup_{k\geq 0} |\sum_{t=0}^{k} a_x^t| \,||_p \leq C||a_x||_p < \infty$. Apart from that we have that $||(\sum_{k=0}^{\infty}(b_y^k)^2)^{\frac{1}{2}}||_{p'} = ||b_y|F_{p',2}^0|| \leq C||b_y||_{p'}$. So we can estimate

$$\begin{aligned} ||c_k \mid L^1(l^2)|| &= ||(\sum_{k=0}^{\infty}(c_k)^2)^{\frac{1}{2}}||_1 \\ &= ||(\sum_{k=0}^{\infty}(\sum_{t=0}^{k-2} a_x^t b_y^k)^2)^{\frac{1}{2}}||_1 \\ &\leq ||\sup_{s\geq 0}|\sum_{t=0}^{s} a_x^t|(\sum_{k=0}^{\infty}(b_y^k)^2)^{\frac{1}{2}}||_1 \\ &\leq C||a_x||_p \, ||b_y||_{p'} \end{aligned}$$

where in the last step we used Hölder's inequality.

So summarised we have the following estimate for $\sum_{s=2}^{\infty}\sum_{t=0}^{s-2} a_x^t b_y^s$

$$||\sum_{s=2}^{\infty}\sum_{t=0}^{s-2} a_x^t b_y^s||_{\mathfrak{h}^1} \leq C||c_k \mid L^1(l^2)|| \leq C||a_x||_p \, ||b_y||_{p'}.$$

$\pi_3(a_x, b_y)$ $\pi_1(a_y, b_x)$ and $\pi_3(a_y, b_x)$ can be estimated in exactly the same way.

In contrast to the case of the determinant, here we have to estimate $\pi_2(a_x, b_y)$. This can be done as follows:
We estimate

$$\Big|\Big|\sum_{k=0}^{s+3} \check{\varphi}_k * f_k\Big|\Big|$$

similarly to what we have done in the proof of lemma 131. More precisely we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\sum_{s=0}^{\infty} a_x^t b_y^s\Big) h &= \sum_{s=0}^{\infty}\int_{\mathbb{R}^n} a_x^t b_y^s \big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}\big(\varphi_k \mathcal{F} f_k\big) \\ &\leq \sum_{s=0}^{\infty}\int_{\mathbb{R}^n} |a_x^t||b_y^s|(s+4)||\check{\varphi}_1||_1||h||_{bmo} \\ &\quad \text{as in the proof of lemma 131} \end{aligned}$$

and continue

$$\begin{aligned}
\int_{\mathbb{R}^n}\Big(\sum_{s=0}^{\infty} a_x^t b_y^s\Big)h &\le C\sum_{s=0}^{\infty}\int_{\mathbb{R}^n}|a_x^t||b_y^s|2^s||\check{\varphi}_1||_1||h||_{bmo} \\
&\le C||h||_{bmo}\int_{\mathbb{R}^n}\sum_{s=0}^{\infty}2^s|a_x^t||b_y^s| \\
&\le C||h||_{bmo}(||\nabla^2 a||_p + ||a_x||_p)||\nabla b||_{p'}. \\
&\quad \text{again by lemma 128}
\end{aligned}$$

This proof shows that the assumption $a \in W^{2,2}(\mathbb{R}^n)$ can be weakened as follows: Since for all $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that

$$s + 4 \le C(\varepsilon)2^{\varepsilon s} \tag{3.1}$$

we just need that a_x belongs to $F^{\varepsilon}_{p,2}(\mathbb{R}^n) = H^{\varepsilon}_p(\mathbb{R}^n)$ for some $\varepsilon > 0$ where $H^{\varepsilon}_p(\mathbb{R}^n)$ denotes the Bessel-potential space of all tempered distributions f for which

$$||\mathcal{F}^{-1}(1+|x|^2)^{\frac{\varepsilon}{2}}\mathcal{F}f||_p < \infty.$$

And finally, since we did not integrate by parts we can even prove the stronger result that under the hypothesis that $a \in F^{\varepsilon}_{p,2}(\mathbb{R}^n)$ and $b \in L^{p'}(\mathbb{R}^n)$ where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$ the product $ab \in F^0_{1,2}$: The contributions π_1 and π_3 are estimated as before and for π_2 we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\Big(\sum_{s=0}^{\infty} a^t b^s\Big)h &= \sum_{s=0}^{\infty}\int_{\mathbb{R}^n} a^t b^s\big(\sum_{k=0}^{s+3}\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}f_k\big) \\
&\le \sum_{s=0}^{\infty}\int_{\mathbb{R}^n}|a^t||b^s|(s+4)||\check{\varphi}_1||_1||h||_{bmo} \\
&\quad \text{similarly to the proof of lemma 131} \\
&\le C\sum_{s=0}^{\infty}\int_{\mathbb{R}^n}|a^t||b^s|2^{\varepsilon s}||\check{\varphi}_1||_1||h||_{bmo} \\
&\le C||h||_{bmo}\int_{\mathbb{R}^n}\sum_{s=0}^{\infty}2^{\varepsilon s}|a^t||b^s| \\
&\le C||h||_{bmo}||a||_{F^{\varepsilon}_{p,2}}||b||_{p'}
\end{aligned}$$

As claimed in section 1, it is even possible to obtain a better result.
For the estimates of $\pi_1(a,b)$ and $\pi_3(a,b)$ we apply the result below (see [43]) where we set $s = \varepsilon,\ r_1 = p,\ r_2 = p'$ and $q = 2$:

Proposition 43. *([43]) Let $s \in \mathbb{R}$ and $0 < q \leq \infty$.*
Assume that

$$\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Let $0 < p < \infty$, $0 < r_1 < \infty$ and $0 < r_2 < \infty$. Then

$$||\sum_{l=2}^{\infty} \sum_{k=0}^{l+2} f^l g^k | F^s_{p,q}|| \leq C ||f| F^s_{r_1,q}|| \cdot ||g| F^0_{r_2,2}||$$

where C is independent of f and g.
If $r_2 = \infty$ then we have

$$||\sum_{l=2}^{\infty} \sum_{k=0}^{l+2} f^l g^k | F^s_{p,q}|| \leq C ||f| F^s_{r_1,q}|| \cdot ||g||_{L^\infty}.$$

Similar conclusions hold if we replace the F-spaces by B-spaces.

What concerns $\pi_2(a, b)$ the claim follows immediately from the next proposition (again, see [43] for instance):

Proposition 44. *([43]) Assume that*

$$\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}.$$

Let $s_1, s_2 \in \mathbb{R}$, $0 < p, r_1, r_2 < \infty$ and $0 < q, q_1, q_2 \leq \infty$.
Suppose

$$s_1 + s_2 > n \cdot \max(0, \frac{1}{p} - 1).$$

Further, let $q \geq p$. Then

$$\max_{-1 \leq j \leq 1} ||\sum_{k=0}^{\infty} f^{k+j} g^k | F^{s_1+s_2}_{p,q}|| \leq C ||f| F^{s_1}_{r_1,q_1}|| \cdot ||g| F^{s_2}_{r_2,q_2}||$$

where C is independent of f and g.
The same conclusion holds also if we replace all the F-spaces by B-spaces, even for $0 < p, r_1, r_2 \leq \infty$.

This completes our proof.

□

Remark 45. If we look at a bounded domain $I \subset \mathbb{R}$ instead of $\mathbb{R}^n$ and at $a \in W^{2,2}(\mathbb{R})$ and $b \in L^2(\mathbb{R})$, lemma 42 can be proved alternatively as follows: Note that the Sobolev embeddings tell us that $a_x \in L^\infty(I)$ which implies that $a_x b \in L^2(I) = F^0_{2,2}(I)$ and the embedding $F^0_{2,2}(I) = L^2(I) \hookrightarrow F^0_{1,2}(I) = \mathfrak{h}^1(I)$ - a special case of the theorem 78 - immediately leads to the conclusion that $a_x b_x \in \mathfrak{h}^1(I)$.

Chapter 4

Integrability properties of products of derivatives

The subsequent discussion is motivated by lemma 42 and the following result of Evans-Müller (see [20]; alternatively obtained also by Semmes (see [45])).

Theorem 46. *([20],[45]) Let $u \in W^{1,2}_{loc}(\mathbb{R}^2)$ be a weak solution of*

$$-\Delta u = f \text{ in } \mathbb{R}^2$$

where $f \in L^1_{loc}(\mathbb{R}^2)$ and

$$f \geq 0.$$

Then

$$u_x u_y, u_x^2 - u_y^2 \in \mathfrak{H}^1_{loc}(\mathbb{R}^2).$$

What about a global version of this result? Let us say a few words concerning this question:
Assume that $u \in W^{1,2}(\mathbb{R}^2)$ is a weak solution of

$$-\Delta u = f \text{ in } \mathbb{R}^2$$

with $f \in L^1(\mathbb{R}^2)$and

$$f \geq 0.$$

In this case the Fourier transform gives

$$|\xi|^{-2}\hat{f} = \hat{u} \in L^2(\mathbb{R}^2).$$

This together with the fact that $\hat{f}$ is continuous (since $f \in L^1(\mathbb{R}^2)$) implies immediately that

$$\hat{f}(0) = \int_{\mathbb{R}^2} f = 0.$$

But from this we conclude that $f = 0$ a.e. since we assumed that f is positive.
Now, what about harmonic tempered distributions? Since the only harmonic tempered distributions u are polynomials, they are not in $W^{1,2}$ unless they are identically zero. But for each bounded open C^∞-domain Ω we have that $u_x u_y \in L^p(\Omega)$ for all $p \geq 1$. So due to theorem 78 we immediately conclude that $u_x u_y \in F^0_{1,2;loc}$!
This embedding is in fact the key point in what follows.

Since our approach involves estimates which depend on the dimension we are working in, what follows is grouped according to the dimension of the underlying Euclidean space.

In dimension 1 we have:

Proposition 47. *Let $u \in W^{1,2}(\mathbb{R})$ be a weak solution of*

$$-\Delta u = f \text{ in } \mathbb{R}$$

where $f \in L^1(\mathbb{R})$. Then

$$u_x u_x = u_x^2 \in F^0_{1,2}(\mathbb{R}).$$

Proof of proposition 47:
The proof of this proposition is quite similar to the proof of proposition 117. The estimates of $\pi_1(u_x, u_x)$ and $\pi_3(u_x, u_x)$ can be obtained in exactly the same way as in the case of proposition 117.
What concerns the remaining term, according to what we have seen so far, it is sufficient to have a closer look at

$$\sum_{s=0}^{\infty} \int_{\mathbb{R}} u_x^t u_x^s \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \tag{4.1}$$

where $t = s + j$ with $j \in \{-1, 0, 1\}$. In this case an integration by part gives

$$\begin{aligned} \sum_{s=0}^{\infty} \int_{\mathbb{R}} u_x^t u_x^s \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) = & -\sum_{s=0}^{\infty} \int_{\mathbb{R}} u^t u_x^s \frac{\partial}{\partial x} \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \\ & -\sum_{s=0}^{\infty} \int_{\mathbb{R}} u^t u_{xx}^s \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \quad (4.2) \\ =: & \quad I + II. \quad (4.3) \end{aligned}$$

Now, the desired estimate for I is obtained as in the proof of proposition 117.
In case of II, the corresponding estimate is based on the two following observations.

Lemma 48. *Let u and f have the same meaning as above.*
Then for $s = 0, 1, 2, \ldots$ the following estimate holds

$$||u^s||_2 \leq C||f||_1 2^{-\frac{3s}{2}}$$

Proof of lemma 48:
We have

$$u^s = \mathcal{F}^{-1}(\varphi_s \hat{u}) = \mathcal{F}^{-1}(\varphi_s |\xi|^{-2} \hat{f})$$

where

$$\varphi_s \hat{f} \in L^\infty$$

with

$$||\varphi_s \hat{f}||_\infty \leq C||f||_1$$

and

$$||\,|\xi|^{-2} \chi_{supp\ \varphi_s}||_2 \leq C 2^{-\frac{3s}{2}}.$$

This last bound is obtained as follows

$$\begin{aligned} ||\,|\xi|^{-2} \chi_{supp\ \varphi_s}||_2 &= C\Big(\int_{supp\ \varphi_s} r^{-4} dr\Big)^{\frac{1}{2}} \\ &= C\Big(r^{-3}|_{2^{s-1}}^{2^{s+1}}\Big)^{\frac{1}{2}} \\ &= C 2^{-\frac{3s}{2}}. \end{aligned}$$

□

Moreover we will need one additional information:

Lemma 49. *Let u and f have the same meaning as above.*
Then for $s = 0, 1, 2, \ldots$ the following estimate holds

$$||u^s_{xx}||_2 \leq C||f||_1 2^{\frac{s}{2}}$$

Proof of lemma 49:
We have

$$u^s_{xx} = \mathcal{F}^{-1}(\varphi_s \xi^2 \hat{u}) = \mathcal{F}^{-1}(\varphi_s \hat{f})$$

where

$$\hat{f} \in L^\infty$$

with

$$||\hat{f}||_\infty \leq C||f||_1$$

and

$$||\varphi_s||_2 \leq 2^{\frac{s}{2}}||\varphi_1||_2.$$

So

$$\begin{aligned} ||u^s_{xx}||_2 &= C||\varphi_s \hat{f}||_2 \\ &\leq C||f||_1 \; ||\varphi_s||_2 \\ &\leq C||f||_1 2^{\frac{s}{2}}. \end{aligned}$$

□

So we estimate II in the following manner

$$\begin{aligned} \sum_{s=0}^{\infty}\int_{\mathbb{R}} u^t u^s_{xx}\Big(\sum_{k=0}^{s+3}\mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k)\Big) &\leq \sum_{s=0}^{\infty}\int_{\mathbb{R}} |u^t| \; |u^s_{xx}| \sum_{k=0}^{s+3} |\mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k)| \\ & \text{remember that } t = s + j \text{ where } j \in \{-1, 0, 1\} \\ &\leq \sum_{s=0}^{\infty} ||u^s||_2 \; ||u^s_{xx}||_2 \; \sum_{k=0}^{s+3} \Big|\Big|\check{\varphi}_k * f_k\Big|\Big|_\infty \\ &\leq \sum_{s=0}^{\infty} C||f||_1 2^{-\frac{3s}{2}} ||u^s_{xx}||_2 \; \sum_{k=0}^{s+3} \Big|\Big|\check{\varphi}_k * f_k\Big|\Big|_\infty \\ & \text{by lemma 48} \\ &\leq \sum_{s=0}^{\infty} C||f||_1 2^{-\frac{3s}{2}} ||f||_1 2^{\frac{s}{2}} \sum_{k=0}^{s+3} \Big|\Big|\check{\varphi}_k * f_k\Big|\Big|_\infty \\ & \text{by lemma 49} \\ &\leq \sum_{s=0}^{\infty} C||f||_1 2^{-\frac{3s}{2}} ||f||_1 2^{\frac{s}{2}} (s+4)||h||_{bmo} \\ & \text{as we have seen earlier} \\ &\leq C\sum_{s=0}^{\infty} 2^{-\frac{3s}{2}} 2^{\frac{s}{2}} 2^{\varepsilon s} \\ & \text{for any } \varepsilon \text{ (see also 3.1)} \\ &\leq C\sum_{s=0}^{\infty} 2^{\varepsilon s - s} \\ &< \infty. \end{aligned}$$

This finally completes the proof of the global version of our result in one dimension.

□

Remark 50. Again let us say why these arguments fail in higher dimension. First of all, note that in general we have for dimension n

$$||u^s||_2 \leq C||f||_1 2^{\frac{s(n-4)}{2}}.$$

Moreover, also the estimate from lemma 49 are different in dimension n. In particular, we have

$$||u^s_{xx}||_2 \leq C||f||_1 2^{\frac{sn}{2}}.$$

This shows that already in dimension 2 the additional powers of 2 which arise from the *bmo*-contribution, namely $2^{\varepsilon s}$ can no longer be absorbed.

Now, let us come pass to higher dimensions.

First of all, as an immediate consequence of lemma 42 we have

Lemma 51. *Assume that* $u \in W^{1,p}(\mathbb{R}^n)$ *with* $p \in (1, \infty)$ *is a solution in* $\mathbb{R}^n$ *of*

$$-\Delta u = f$$

where

$$f \in L^{p'}(\mathbb{R}^n).$$

Let $x = z_i$ *where* $i \in \{1, \dots, n\}$ *and* $y = z_j$ *where* $j \in \{1, \dots, n\}$ *then*

$$u_x u_y \in F^1_{1,2}(\mathbb{R}^n) \subset F^0_{1,2}(\mathbb{R}^n) = \mathfrak{h}^1(\mathbb{R}^n).$$

Proof of lemma 51:
As mentioned above, lemma 51 is an immediate consequence of lemma 42 and the well known fact that

$$\Delta u \in L^p \text{ with } p \in (1, \infty) \Rightarrow \nabla^2 u \in L^p.$$

□

From the article of Evans/Müller (see [20]) we learn that in the case of a radially symmetric right hand side, i.e. in the case

$$-\Delta u = f$$

where $f \in L^1$ is radially symmetric, the sign condition $f \geq 0$ can be dropped. This phenomena leads to the question: What is particular about radially symmetric integrable functions? In fact the subsequent theorem (see e.g. [53]) provides us with an answer:

Theorem 52. *([53]) Suppose $f \in L^1(\mathbb{R}^n)$, where $n \geq 2$, is a radial function, i.e. $f(x) = f_0(|x|)$ for a.e. $x \in \mathbb{R}^n$. Then the Fourier transform $\hat{f}$ is also radial and has the form $\hat{f}(x) = F_0(|x|)$ for all $x \in \mathbb{R}^n$, where*

$$F_0(|x|) = F_0(r) = 2\pi r^{-[(n-2)/2]} \int_0^\infty f_0(s) J_{(n-2)/2}(2\pi rs) s^{n/2}\, ds.$$

Here J_k with $k \in \mathbb{R}$ greater than $-\frac{1}{2}$ denotes the Bessel function

$$J_k(t) = \frac{(t/2)^k}{\Gamma[(2k+1)/2]\Gamma(\frac{1}{2})} = \int_{-1}^1 e^{its}(1-s^2)^{(2k-1)/2}\, ds$$

for $t > 0$.
This fact leads us to the the following result.

Proposition 53. *Assume that $u \in W^{1,2}(\mathbb{R}^n)$, with $n \geq 2$, is a solution of*

$$-\Delta u = f$$

where $f \in \mathcal{S}'$ with $\hat{f}$ bounded and

$$\hat{f} = O(|\xi|^{-(n-2)/2-\varepsilon})$$

for some $\varepsilon > 0$. Let $x = z_i$ where $i \in \{1, \dots, n\}$ and $y = z_j$ where $j \in \{1, \dots, n\}$ then

$$u_x u_y \in B^{\varepsilon_1}_{1,1} \subset F^0_{1,2}$$

where $\varepsilon_1 < \varepsilon$.

Proof of proposition 53:
In order to establish the assertion we need the following lemma.

Lemma 54. *Let l be chosen big enough such that for $\xi \geq 2^{l-1}$*

$$|\hat{f}(\xi)| \leq C|\xi|^{-(n-2)/2-\varepsilon}.$$

Then for $s \geq l$ we have

$$||u^s_v||_2 \leq C2^{-s\varepsilon}$$

where $v = z_i$ where $i \in \{1, \dots, n\}$.

Proof of lemma 54:
In fact, we can estimate

$$\begin{aligned}
||u_v^s||_2 &= ||\mathcal{F}^{-1}(\varphi_s \mathcal{F}(u_v))||_2 \\
&= ||\mathcal{F}^{-1}(\varphi_s \xi_i \hat{u})||_2 \\
&\leq C||\varphi_s \xi_i \hat{u}||_2 \\
&= C||\varphi_s \xi_i \hat{f} |\xi|^{-2}||_2 \\
&\leq C||\varphi_s \xi_i |\xi|^{-(n-2)/2-\varepsilon} |\xi|^{-2}||_2 \\
&\leq ||\varphi_s||_\infty \;\; || \, |\xi|^{-(n-2)/2-\varepsilon-1} \chi_{supp\ \varphi_s}||_2 \\
&\leq C \;\; || \, |\xi|^{-(n-2)/2-\varepsilon-1} \chi_{supp\ \varphi_s}||_2 \\
&\leq C\Big(\int_{2^{s-1}\leq r\leq 2^{s+1}} r^{-(n-2)-2\varepsilon-2} r^{n-1}\, dr\Big)^{\frac{1}{2}} \\
&\leq C\Big(\int_{2^{s-1}}^{2^{s+1}} r^{-1-2\varepsilon}\, dr\Big)^{\frac{1}{2}} \\
&\leq C 2^{-s\varepsilon}
\end{aligned}$$

□

We start with the estimate of $\pi_1(u_x, u_y)$: Due to proposition 123 it is enough to estimate $||2^{s\varepsilon_1} \sum_{t=0}^{s-2} u_x^t u_y^s \mid l^1(L^1(\mathbb{R}^n))|| < \infty$. We have

$$\begin{aligned}
||2^{s\varepsilon_1} \sum_{t=0}^{s-2} u_x^t u_y^s | l^1(L^1(\mathbb{R}))|| &= \sum_{s=0}^{\infty} 2^{s\varepsilon_1} || \sum_{k=0}^{s-2} u_x^k u_x^s ||_1 \\
&\leq \sum_{s=0}^{\infty} 2^{s\varepsilon_1} \Big(\sum_{k=0}^{s-2} ||u_x^k u_x^s||_1 \Big) \\
&\leq \sum_{s=0}^{\infty} 2^{s\varepsilon_1} \Big(\sum_{k=0}^{\infty} ||u_x^k||_2 ||u_x^s||_2 \Big) \\
&\leq \sum_{s=0}^{\infty} 2^{s\varepsilon_1} \Big(C2^{-s\varepsilon} \sum_{k=0}^{\infty} C2^{-s\varepsilon} \Big) \\
&\leq C \sum_{s=0}^{\infty} 2^{s(\varepsilon_1-\varepsilon)} \\
&< \infty.
\end{aligned}$$

(remember that we assumed $\varepsilon_1 < \varepsilon$)

The corresponding estimate for $\pi_3(u_x, u_y)$ is derived in the same way.

It remains to establish the announced estimate for $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} u_x^t u_y^s \in B_{1,1}^{\varepsilon_1}(\mathbb{R}^n) \subset F_{1,2}^0(\mathbb{R}^n)$.
In order to analyse the remaining contribution, we can apply the following theorem which guarantees that this term too can be handled as well. In order to state this result, let us start with a technical definition.

Definition 55 ($\mathcal{B}_*^p(\mathbb{R}^n)$)**.** *Let $0 < p \leq \infty$. $\mathcal{B}_*^{\mathbf{p}}(\mathbb{R}^n)$ is the set of all sequences b with the following properties. $b = \{b_k\}_{k=0}^{\infty}$ is a sequence of elements $b_k \in \mathcal{S}'(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that*

$$supp\ \mathcal{F}b_k \subset \left\{\xi \mid |\xi| \leq 2^k\right\} \ for\ k \geq 0.$$

Proposition 56. *([43]) Suppose $b \in \mathcal{B}_*^p(\mathbb{R}^n)$.*

*i) Let $s > n \cdot \max(0, \frac{1}{p} - 1, \frac{1}{q} - 1)$, $0 < p < \infty$ and $0 < q \leq \infty$.
If $||2^{js}b_j|L^p(\mathbb{R}^n, l^q)|| = A < \infty$, then the series $\sum_{j=0}^{\infty} b_j$ converge in $\mathcal{S}'(\mathbb{R}^n)$ to a limit $f \in F_{p,q}^s(\mathbb{R}^n)$, and the estimate $||f|F_{p,q}^s(\mathbb{R}^n)|| \leq CA$ takes place with some constant C independent of b.*

*ii) Let $s > n \cdot \max(0, \frac{1}{p} - 1)$, $0 < p \leq \infty$ and $0 < q \leq \infty$.
If $||2^{js}b_j|l^q(L^p(\mathbb{R}^n))|| = A < \infty$, then the series $\sum_{j=0}^{\infty} b_j$ converge in $\mathcal{S}'(\mathbb{R}^n)$ to a limit $f \in B_{p,q}^s(\mathbb{R}^n)$, and the estimate $||f|B_{p,q}^s(\mathbb{R}^n)|| \leq CA$ takes place with some constant C independent of b.*

For a proof of this proposition see [43].

Again we set $t = s$ for the moment and assume that $\varepsilon_1 > 0$. Thanks to proposition 56 it is enough to show that

$$\sum_{s=0}^{\infty} 2^{s\varepsilon_1}||u_x^s u_y^s||_1 < \infty.$$

In fact,

$$\begin{aligned}
\sum_{s=0}^{\infty} 2^{s\varepsilon_1}||u_x^s u_y^s||_1 &= \sum_{s=0}^{l-1} 2^{s\varepsilon_1}||u_x^s u_y^s||_1 + \sum_{s=l}^{\infty} 2^{s\varepsilon_1}||u_x^s u_y^s||_1 \\
&\quad \text{where } l \text{ is as in the lemma above} \\
&\leq C + \sum_{s=l}^{\infty} 2^{s\varepsilon_1}||u_x^s u_y^s||_1 \\
&\quad \text{since the first addend is a finite sum of finite addends} \\
&\leq C + \sum_{s=l}^{\infty} 2^{s\varepsilon_1} 2^{-2s\varepsilon} \\
&\quad \text{due to the preceding lemma and Hölder's inequality} \\
&\leq C + C\sum_{s=l}^{\infty} 2^{-\varepsilon_2} \\
&\quad \text{for some } \varepsilon_2 > 0 \\
&\quad \text{remember that we assumed } \varepsilon_1 < 2\varepsilon \\
&< \infty.
\end{aligned}$$

The assertion about $\pi_2(u_x, u_y)$ in the case $\varepsilon = 0$ follows immediately from the calculation above because $B^{\varepsilon_1}_{1,1} \subset B^0_{1,1}$.
This completes the proof of proposition 53 since similar estimates hold also if we look at $t = s + j$ with $j \in \{-1, 1\}$.

□

Note that in comparison to the result of Evans/Müller and Semmes here we have a global result for arbitrary dimension $n \geq 2$ with better estimates. Note that in the radially symmetric case with $n = 2$ the above cited theorem together with the following lemma (see again [53]) give that

$$\hat{f} = O(|\xi|^{-\frac{1}{2}}).$$

Lemma 57. *([53])* $J_m(r) = \sqrt{2/\pi r}\cos(r - \pi m/2 - \pi/4) + O(r^{-3/2})$ *as* $r \to \infty$. *In particular,*

$$J_m(r) = O(r^{-1/2}) \text{ as } r \to \infty.$$

Apart from this, we know that for any $f \in L^1$

$$|\hat{f}(\xi)| \to 0 \text{ uniformly as } |\xi| \to \infty ,$$

and in particular for $\varphi \in C_0^\infty$ we have

$$|\hat{\varphi}(\xi)| \leq C||\nabla\varphi||_\infty \, R^{-1}$$

if $|\xi| = R$.

Appendix A

Definitions and standard results

In this first appendix we summarise all the relevant definitions of function spaces and state those assertions related to them which are important for our work.
The aim is twofold, one hand there is the sake of completeness, on the other hand, we would like to provide a self-contained presentation of our research, which of course necessitates an introduction to the framework of function spaces in which our estimates take place.

A.1 Hardy spaces and BMO/bmo

Definition 58 (Hardy spaces)**.** *Let f be a tempered distribution and let $0 < p \leq \infty$. Then f belongs to the (homogeneous) Hardy space $\mathfrak{H}^{\mathbf{p}}(\mathbb{R}^{\mathbf{n}})$ if there is an element $\varphi \in \mathcal{S}$ with $\int \varphi \, dx \neq 0$ such that $M_\varphi f \in L^p(\mathbb{R}^n)$ where the maximal function $M_\varphi f$ is defined as follows*

$$M_\varphi f(x) = \sup_{t>0} |(f * \varphi_t)(x)|$$

where

$$\varphi_t(x) = t^{-n}\varphi(t^{-1}x), \ t > 0.$$

Remark 59. If we set $||M_\varphi f||_{L^p} = ||f||_{\mathfrak{H}^p}$ this defines a norm on $\mathfrak{H}^p$ if $p \geq 1$.

Remark 60. The following alternative but equivalent characterisation of $\mathfrak{H}^1(\mathbb{R}^n)$ will be useful when we compare homogeneous function spaces with their non-homogeneous analogues.
We say that a function f belongs to $\mathfrak{H}^1$ if it is in L^1 and in addition all its Riesz transforms $R_j f$ belong to L^1 too where the Riesz transforms are

defined as follows

$$(R_j f)^\wedge(\xi) = \hat{f}(\xi)\frac{\xi_j}{|\xi|}\ j = 1, \dots, n$$

and where $\wedge$ denotes the Fourier transform.
Note that this alternative definition of $\mathfrak{H}^1(\mathbb{R}^n)$ provides us with a rather easy necessary condition for a function to belong to this space: If $R_j f$ belongs to L^1 then $(R_j f)^\wedge$ is continuous which implies that we must have $\hat{f}(0) = 0$. This finally implies that $\int f\, dx = 0$.
So it is obvious that $\mathfrak{H}^1$ is strictly contained in L^1 since for example the characteristic function of the unit ball belongs to L^1 but it can not belong to $\mathfrak{H}^1$ because its mean value does not vanish.

Definition 61 (Non-homogeneous Hardy spaces $\mathfrak{h}^{\mathbf{p}}(\mathbb{R}^{\mathbf{n}})$)**.** *Let f be a tempered distribution and let $0 < p \leq \infty$. Then f belongs to $\mathfrak{h}^{\mathbf{p}}(\mathbb{R}^{\mathbf{n}})$ if there is an element $\varphi \in \mathcal{S}$ with $\int \varphi\, dx \neq 0$ such that $M_\varphi^{(1)} f \in L^p(\mathbb{R}^n)$ where the maximal function $M_\varphi^{(1)} f$ is defined as follows*

$$M_\varphi^{(1)} f(x) = \sup_{0<t\leq 1} |(f * \varphi_t)(x)|$$

where

$$\varphi_t(x) = t^{-n}\varphi(t^{-1}x),\ t > 0$$

Remark 62. There is a counterpart for remark 60 in the case of the non-homogeneous Hardy spaces. We say that $f \in \mathfrak{h}^1$ if $f \in L^1$ and $r_j f \in L^1$ for $j = 1, \dots, n$ where $r_j f$ denotes the non-homogeneous Riesz transform defined as follows

$$r_j f = \mathcal{F}^{-1}(\psi\frac{x_j}{|x|}\mathcal{F} f)\ j = 1, \dots n$$

where ψ is an infinitely differentiable function such that $\psi(x) = \psi(-x)$ and

$$\psi(x) = 0 \text{ if } |x| \leq 1 \text{ and } \psi(x) = 1 \text{ if } |x| \geq 2$$

So we see immediately that $\mathfrak{h}^1$ too is a subspace of L^1.
Moreover we have equivalence of the two norms $||f|||_{\mathfrak{h}^1} = ||M_\varphi^{(1)} f||_{L^1}$ and $||f||_{L^1} + \sum_{i=1}^n ||r_i f||_{L^1}$.
For instance see [61].

Last but not least, we have the local Hardy spaces $\mathfrak{H}^p_{loc}$:

Definition 63 (Local Hardy spaces)**.** *Let U be an open subset of $\mathbb{R}^n$ and let $0 < p \leq \infty$. Then a distribution $f \in \mathcal{D}'(U)$ belongs to the local (homogeneous) Hardy space $\mathfrak{H}^{\mathbf{p}}_{\mathbf{loc}}(\mathbf{U})$ if for each compact subset K of U there is an $\varepsilon > 0$ such that*

$$\int_K \Big(\sup_{0<t<\varepsilon} \sup_{\varphi \in \mathcal{T}} |\varphi_t * f(x)| \Big)^p < \infty$$

where φ_t has the same meaning as in the definition of the homogeneous Hardy spaces and

$$\mathcal{T} = \{\varphi \in C^\infty(\mathbb{R}^n) \mid supp\, \varphi \subset B(0,1) \text{ and } ||\nabla\varphi||_\infty \leq 1\}\,.$$

In other words, a distribution in $\mathfrak{H}^p_{loc}$ coincides locally with a function in $\mathfrak{H}^p$.

For more details about other equivalent definitions of these spaces see e.g. [21], [45], [51] and [61].

So far we have seen substitutes for L^1, next let us state the definitions of substitutes for L^∞.

Definition 64 (BMO)**.** *Let f be a locally integrable function. Then f belongs to* **BMO** *if the inequality*

$$\frac{1}{|B|}\int_B |f(x) - f_B|\, dx \leq A$$

holds for all balls B where $f_B = |B|^{-1}\int_B f\, dx$ denotes the mean value of f over the ball B.
The smallest bound A for which the above inequality is satisfied is taken to be the norm of f in BMO, $||f||_{BMO}$.

As $\mathfrak{H}^p$ has a non-homogeneous counterpart, namely $\mathfrak{h}^p$, also BMO has a non-homogeneous counterpart:

Definition 65 (bmo)**.** *Let f be a locally integrable function on $\mathbb{R}^n$ and let Q be a cube in $\mathbb{R}^n$ and denote*

$$f_Q = \frac{1}{|Q|}\int_Q f(x)\, dx$$

the mean value of f with respect to Q. Then **bmo** *consists of all $f \in L^1_{loc}(\mathbb{R}^n)$ which satisfy the following inequality*

$$||f||_{bmo} = \sup_{|Q|\leq 1} \frac{1}{|Q|}\int_Q |f(x) - f_Q|\, dx + \sup_{|Q|>1} \frac{1}{|Q|}\int_Q |f(x)|\, dx < \infty$$

The duality coupling of the Hardy space $\mathfrak{H}^1$ with BMO is stated in the following theorem due to Fefferman-Stein:

Theorem 66 (Fefferman-Stein,[21]). *a) Suppose $f \in BMO$. Then the linear functional l given as*

$$l(g) = \int_{\mathbb{R}^n} f(x)g(x)\,dx\ ,\ g \in \mathfrak{H}^1$$

initially defined on the dense subspace of $\mathfrak{H}^1$ atoms, has a unique bounded extension to $\mathfrak{H}^1$ and satisfies

$$||l|| \leq C||f||_{BMO}$$

b) Conversely, every continuous linear functional l on $\mathfrak{H}^1$ can be realised as above, with $f \in BMO$, and with

$$||f||_{BMO} \leq C'||l||$$

So roughly speaking $(\mathfrak{H}^1)^* = BMO$.

For a proof of this result see [21] or [52].

And its counterpart in the case of non-homogeneous spaces:

Theorem 67. *([61])* $(\mathfrak{h}^1)^* = bmo$

See for instance [61].

A.2 Besov and Triebel-Lizorkin spaces

Apart from these rather classical function spaces from above we shall work with the so called Besov and Triebel-Lizorkin spaces.

A.2.1 Non-homogeneous Besov and Triebel-Lizorkin spaces

In order to define them we have to introduce some additional notions:

Definition 68 ($\mathbf{\Phi}(\mathbb{R}^\mathbf{n})$). *Let $\mathbf{\Phi}(\mathbb{R}^\mathbf{n})$ be the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ such that*

$$\begin{cases} \text{supp}\ \varphi_0 \subset \{x|\ |x| \leq 2\} \\ \text{supp}\ \varphi_j \subset \{x|\ 2^{j-1} \leq |x| \leq 2^{j+1}\} \quad \text{if } j = 1, 2, 3, \dots, \end{cases}$$

for every multi-index α there exists a positive number C_α such that

$$2^{j|\alpha|}|D^\alpha \varphi_j(x)| \leq C_\alpha \text{ for all } j = 1, 2, 3, \ldots \text{ and all } x \in \mathbb{R}^n$$

and

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \; \forall x \in \mathbb{R}^n$$

Remark 69.

- Note that in the above expression $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ the sum is locally finite!
- Example of a system φ which belongs to $\Phi(\mathbb{R}^n)$:
  We start with an arbitrary $C_0^\infty(\mathbb{R}^n)$ function ψ which has the following properties: $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq \frac{3}{2}$. We set $\varphi_0(x) = \psi(x)$, $\varphi_1(x) = \psi(\frac{x}{2}) - \psi(x)$, and $\varphi_j(x) = \varphi_1(2^{-j+1}x)$, $j \geq 2$. Then it is easy to check that this family φ satisfies the requirements of our definition.
  Moreover, we have $\sum_{j=0}^{n} \varphi_j(x) = \psi(2^{-n}x)$, $n \geq 0$.
  By the way, other examples of $\varphi \in \Phi$, apart from this one, can be found in [43], [61] or [15].)

Now, we can state the definitions of the above mentioned Besov and Triebel-Lizorkin spaces.

Definition 70 (Besov spaces and Triebel-Lizorkin spaces). *Let $-\infty < s < \infty$, let $0 < q \leq \infty$ and let $\varphi \in \Phi(\mathbb{R}^n)$.*

i) If $0 < p \leq \infty$ then the (non-homogeneous) Besov spaces $\mathbf{B^s_{p,q}}(\mathbb{R}^\mathbf{n})$ consist of all $f \in \mathcal{S}'$ such that the following inequality holds

$$||f|B^s_{p,q}(\mathbb{R}^n)||^\varphi = ||2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f|l^q(L^p(\mathbb{R}^n))|| < \infty$$

ii) If $0 < p < \infty$ then the (non-homogeneous) Triebel-Lizorkin spaces $\mathbf{F^s_{p,q}}(\mathbb{R}^\mathbf{n})$ consist of all $f \in \mathcal{S}'$ such that the following inequality holds

$$||f|F^s_{p,q}(\mathbb{R}^n)||^\varphi = ||2^{js}\mathcal{F}^{-1}\varphi_j\mathcal{F}f|L^p(\mathbb{R}^n, l^q)|| < \infty$$

iii) If $p = \infty$ then the spaces $\mathbf{F^s_{\infty,q}}(\mathbb{R}^\mathbf{n})$ consist of all $f \in \mathcal{S}'$ such that $\exists \{f_k(x)\}_{k=0}^{\infty} \subset L^\infty(\mathbb{R}^n)$ such that the following holds

$$f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\varphi_k\mathcal{F}f_k \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$||2^{sk} f_k | L^\infty(\mathbb{R}^n, l^q)|| < \infty.$$

Moreover we set

$$||f | F^s_{\infty,q}(\mathbb{R}^n)||^\varphi = \inf ||2^{sk} f_k | L^\infty(\mathbb{R}^n, l^q)||$$

where the infimum is taken over all admissible representations of f.

Here $\mathcal{F}$ denotes the Fourier transform and

$$||f_k | l^q(L^p(\mathbb{R}^n))|| = \left(\sum_{k=0}^{\infty} \big(\int |f_k(x)|^p dx \big)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

and

$$||f_k | L^p(\mathbb{R}^n, l^q)|| = \left(\int \big(\sum_{k=0}^{\infty} |f_k(x)|^q \big)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Recall that the spaces $B^s_{p,q}$ and $F^s_{p,q}$ are independent of the choice of φ (see [61]).

As a short orientation in view of other function spaces we will recall some results concerning $F^s_{p,q}$ spaces (proofs can be found e.g. in [61]).

Proposition 71. *([61])*

$$\begin{aligned} W^{k,p} &= F^k_{p,2} \ \ \textit{for } k < \infty \textit{ and } 1 < p < \infty \\ bmo &= F^0_{\infty,2} \\ \mathfrak{h}^p &= F^0_{p,2} \ \ \textit{for } 0 < p < \infty \end{aligned}$$

Where the norm of the space $F^0_{p,2}$ is equivalent to $|| \cdot ||_{\mathfrak{h}^p} = ||M^{(1)}_\varphi(\cdot)||_{L^p}$ if $p \geq 1$ (equivalent metric in the case $p < 1$), in the first two cases too, we have equivalent norms.

Apart from this relation to well-known function spaces we recall the following results within the framework of Besov respectively Triebel-Lizorkin spaces. We start with a Fourier multiplier result (see for instance [61]).

Theorem 72. *([61]) Let $-\infty < s < \infty$ and $0 < q \leq \infty$. Let A be either $B^s_{p,q}(\mathbb{R}^n)$ with $0 < p \leq \infty$ or $F^s_{p,q}(\mathbb{R}^n)$ with $0 < p < \infty$. If the natural number N is sufficiently large, then there exists a positive number C such that*

$$||\mathcal{F}^{-1} m \mathcal{F} f \mid A|| \leq C ||m||_N ||f \mid A||$$

holds for all infinitely differentiable functions $m(x)$ *and all* $f \in A$ *where*

$$||m||_N = \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{\frac{|\alpha|}{2}} |D^\alpha m(x)|.$$

If we want to study sequences of functions in stead of single functions, the following assertion is quite useful (again see for instance [61]):

Theorem 73. *([61]) Let* $0 < p < \infty$, $0 < q \leq \infty$. *Let* $\Omega = \{\Omega_k\}_{k=0}^\infty$ *be a sequence of compact subsets of* $\mathbb{R}^n$. *Let moreover* $d_k > 0$ *be the diameter of* Ω_k. *if* $r > \frac{n}{2} + \frac{n}{\min(p,q)}$, *then there exists a constant* C *such that*

$$||\mathcal{F}^{-1} M_k \mathcal{F} f_k | L^p(l^q)|| \leq C \sup_l ||M_l(d_l \cdot)|F^r_{2,2}|| ||f_k|L^p(l^q)||$$

holds for all systems $\{f_k\}_{k=0}^\infty \subset L^p(l^q)$ *such that supp* $\mathcal{F} f_k \subset \Omega_k$ *and for all sequences* $\{M_k\}_{k=0}^\infty \subset F^r_{2,2}$.

Apart from these facts, we shall also recall the elementary embedding results:

Theorem 74. *([61],[43]) Let* $s \in \mathbb{R}$.

i) Suppose in addition $0 < q_0 \leq q_1 \leq \infty$ *and* $\varepsilon > 0$. *Then*

$$B^{s+\varepsilon}_{p,q_0} \subset B^s_{p,q_1} \text{ if } 0 < p \leq \infty$$

and

$$F^{s+\varepsilon}_{p,q_0} \subset F^s_{p,q_1} \text{ if } 0 < p < \infty$$

ii) Let $0 < q \leq \infty$ *and* $0 < p < \infty$. *Then*

$$B^s_{p,\min\{p,q\}} \subset F^s_{p,q} \subset B^s_{p,\max\{p,q\}}.$$

iii) The assertion i) and ii) remain valid in the case of spaces on domains.

iv) Let $0 < p_0 \leq p_1 \leq \infty$, $0 < q \leq \infty$ *and* $-\infty < s_1 < s_0 < \infty$. *Then*

$$B^{s_0}_{p_0,q} \subset B^{s_1}_{p_1,q} \text{ if } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

v) Let $0 < p_0 < p_1 < \infty$, $0 < q \leq \infty$ *and* $-\infty < s_1 < s_0 < \infty$. *Then*

$$F^{s_0}_{p_0,q} \subset F^{s_1}_{p_1,q} \text{ if } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

and even

$$F^{s_0}_{p_0,\infty} \subset F^{s_1}_{p_1,q} \text{ if } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Lat, but not least, we may mix B- and F-spaces:

vi) Let $0 < q \leq \infty$ *and suppose in addition that* $0 < p < \infty$ *and* $0 < q, u, v \leq \infty$. *Then*

$$B^s_{p,u} \subset F^s_{p,q} \subset B^s_{p,v}$$

if and only if $0 < u \leq \min(p, q)$ *and* $\max(p, q) \leq v \leq \infty$.

vii) Let $0 < p_0 < p < p_1 \leq \infty$ *and suppose*

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}.$$

Then

$$B^{s_0}_{p_0,u} \subset F^s_{p,q} \subset B^{s_1}_{p_1,v}$$

if and only if $0 < u \leq p \leq v \leq \infty$.

For instance see [61] and [43].
On a more advanced level we have the following embedding assertion

Theorem 75. *([47])*

i) Let $s \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *Then the following assertions are equivalent*

a) $F^s_{p,q} \subset L^\infty$

b) $F^s_{p,q} \subset C$

c)

$$\text{either } s > \frac{n}{p} \text{ or } s = \frac{n}{p} \text{ and } 0 < p \leq 1.$$

ii) Let $s \in \mathbb{R}$, $0 < p \leq \infty$ *and* $0 < q \leq \infty$. *Then the following assertions are equivalent*

a) $B^s_{p,q} \subset L^\infty$

b) $B^s_{p,q} \subset C$

c)

$$\text{either } s > \frac{n}{p} \text{ or } s = \frac{n}{p} \text{ and } 0 < q \leq 1.$$

A.2.2 Besov and Triebel-Lizorkin spaces on domains

Instead of the whole space $\mathbb{R}^n$ one can also consider bounded open domains $\Omega \subset \mathbb{R}^n$. In this latter case we have the following definitions.

Definition 76 (Besov and Triebel-Lizorkin spaces on domains)**.** *Let Ω be a bounded open C^∞-domain $\subset \mathbb{R}^n$. Moreover let $-\infty < s < \infty$ and let $0 < q \leq \infty$.*

i) If $0 < p \leq \infty$ then the (non-homogeneous) Besov spaces $\mathbf{B^s_{p,q}(\Omega)}$ consist of all $f \in \mathcal{D}'(\Omega)$ such that $\exists g \in B^s_{p,q}(\mathbb{R}^n)$ with $g|_\Omega = f$ and we set

$$||f|B^s_{p,q}(\Omega)|| = \inf ||g|B^s_{p,q}(\mathbb{R}^n)||$$

where the infimum is taken over all admissible representations of f.

ii) If $0 < p < \infty$ then the (non-homogeneous) Triebel-Lizorkin spaces $\mathbf{F^s_{p,q}(\Omega)}$ consist of all $f \in \mathcal{D}'(\Omega)$ such that $\exists g \in F^s_{p,q}(\mathbb{R}^n)$ with $g|_\Omega = f$ and we set

$$||f|F^s_{p,q}(\Omega)|| = \inf ||g|F^s_{p,q}(\mathbb{R}^n)||$$

where the infimum is taken over all admissible representations of f.

Remark 77. There is no difficulty to extend the definition of $F^s_{p,q}(\Omega)$ to the cases $p = \infty$ and $0 < q < \infty$. We just have to start with the definition of the spaces $F^s_{\infty,q}(\mathbb{R}^n)$.

Concerning the relation between spaces on domains and spaces on the whole space, some of the most important facts are summarised in the following theorem.

Theorem 78. *([61]) Let Ω be a bounded C^∞-domain $\subset \mathbb{R}^n$.*

i) Let $0 < p_0 \leq \infty$, $0 < p_1 \leq \infty$, $0 < q_0 \leq \infty$, $0 < q_1 \leq \infty$ and $-\infty < s_1 < s_0 < \infty$. Then

$$B^{s_0}_{p_0,q_0}(\Omega) \subset B^{s_1}_{p_1,q_0}(\Omega) \text{ if } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$$

and

$$B^{s_0}_{p_0,q_0}(\Omega) \subset B^{s_1}_{p_1,q_1}(\Omega) \text{ if } s_0 - \frac{n}{p_0} > s_1 - \frac{n}{p_1}.$$

ii) Let $0 < p_0 < \infty$, $0 < p_1 < \infty$, $0 < q_0 \leq \infty$, $0 < q_1 \leq \infty$ and $-\infty < s_1 < s_0 < \infty$. Then

$$F^{s_0}_{p_0,q_0}(\Omega) \subset F^{s_1}_{p_1,q_1}(\Omega) \text{ if } s_0 - \frac{n}{p_0} \geq s_1 - \frac{n}{p_1}.$$

iii) *Let* $-\infty < s < \infty$. *Then*

$$B^s_{p_0,q}(\Omega) \subset B^s_{p_1,q}(\Omega) \text{ if } 0 < p_1 \le p_0 \le \infty \text{ and } 0 < q \le \infty$$

and

$$F^s_{p_0,q}(\Omega) \subset F^s_{p_1,q}(\Omega) \text{ if } 0 < p_1 \le p_0 < \infty \text{ and } 0 < q < \infty$$

For a proof of this theorem, see [61], p.197.

A.2.3 Local Besov and Treibel-Lizorkin spaces

Next, we look at the following local spaces:

Definition 79 (Local Besov and Triebel-Lizorkin spaces). *Let* $-\infty < s < \infty$ *and let* $0 < q \le \infty$.

i) *If* $0 < p \le \infty$ *then the local (non-homogeneous) Besov spaces* $\mathbf{B^s_{p,q;loc}}$ *consist of all* $f \in \mathcal{D}'(\mathbb{R}^n)$ *such that for all bounded open* C^∞*-domains* Ω *the restriction of* f *to* Ω, $f|_\Omega$, *belongs to* $B^s_{p,q}(\Omega)$.

ii) *If* $0 < p \le \infty$ *then the local (non-homogeneous) Triebel-Lizorkin spaces* $\mathbf{F^s_{p,q;loc}}$ *consist of all* $f \in \mathcal{D}'(\mathbb{R}^n)$ *such that for all bounded open* C^∞*-domains* Ω *the restriction of* f *to* Ω, $f|_\Omega$, *belongs to* $F^s_{p,q}(\Omega)$.

In other words, elements in a local Besov or Triebel-Lizorkin space coincide locally with an element of the corresponding space on the whole space $\mathbb{R}^n$.

A.2.4 Homogeneous Besov and Triebel-Lizorkin spaces

Since it turned out that the homogeneous counterparts of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ are as well involved in our studies, let us recall their definitions in order to facilitate understanding:
First of all, we need an appropriate extension of $\mathcal{S}'$, given by the following definition.

Definition 80 ($\mathcal{Z}(\mathbb{R}^\mathbf{n})$ and $\mathcal{Z}'(\mathbb{R}^\mathbf{n})$). $\mathcal{Z}(\mathbb{R}^\mathbf{n})$ *is defined to be the set of all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *such that*

$$(D^\alpha \mathcal{F}\varphi)(0) = 0 \text{ for every multi-index } \alpha.$$

And $\mathcal{Z}'(\mathbb{R}^\mathbf{n})$ *is the topological dual of* $Z(\mathbb{R}^n)$.

Once we have this, we can define the homogeneous counterpart of $\Phi(\mathbb{R}^n)$:

Definition 81 ($\dot{\mathbf{\Phi}}(\mathbb{R}^{\mathbf{n}})$)**.** *Let $\dot{\mathbf{\Phi}}(\mathbb{R}^{\mathbf{n}})$ be the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=-\infty}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ such that*

$$\textit{supp}\ \varphi_j \subset \left\{x \mid 2^{j-1} \leq |x| \leq 2^{j+1}\right\} \textit{ if } j \textit{ is an integer},$$

for every multi-index α there exists a positive number C_α such that

$$2^{j|\alpha|}|D^\alpha \varphi_j(x)| \leq C_\alpha \textit{ for all integers } j \textit{ and all } x \in \mathbb{R}^n$$

and

$$\sum_{j=-\infty}^{\infty} \varphi_j(x) = 1\ \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Remark 82.

i) Note that every $\varphi \in \dot{\Phi}(\mathbb{R}^n)$ generates a $\phi \in \Phi(\mathbb{R}^n)$ by setting

$$\begin{aligned} \phi_0 &= \sum_{j=-\infty}^{0} \varphi_j \\ \phi_k &= \varphi_k \text{ for } k \geq 1. \end{aligned}$$

ii) Again, let us give an example of such a partition of unity: We start with a function $\psi \in \mathcal{S}$ such that *supp* $\psi \subset \{x \mid 1 \leq |x| \leq 4\}$ and $\psi(x) = 1$ on $\left\{x \mid \frac{1}{2} \leq |x| \leq 2\right\}$. If we define

$$\varphi_k(x) = \frac{\psi(2^{-k+1}x)}{\sum_{j\in\mathbb{Z}} \psi(2^j x)} = \varphi_1(2^{-k+1}x)\ k \in \mathbb{Z}$$

then $\{\varphi_j\}_{j=-\infty}^{\infty} \in \dot{\Phi}(\mathbb{R}^n)$.

iii) Another possibility to construct a system $\{\varphi_j\}_{j=-\infty}^{\infty} \in \dot{\Phi}(\mathbb{R}^n)$ arises from the non-homogeneous example we gave in remark 69: We start with the same ψ as before and set

$$\varphi_j(x) := \psi(2^{-j}x) - \psi(2^{-j+1}x)\ \ j \in \mathbb{Z}.$$

It is not difficult to check that this really defines a system $\{\varphi_j\}_{j=-\infty}^{\infty} in \dot{\Phi}(\mathbb{R}^n)$ and that once we define $\phi(x) := \varphi_0(x)$ we have that $\varphi_j = \phi(2^{-j}x)$ Moreover, observe that

$$\sum_{k\leq j} \varphi_k = \psi(2^{-j}x)\ \ j \in \mathbb{Z}.$$

See also [55].

So finally we are able to define the homogeneous counterparts of the spaces $B^s_{p,q}$ and $F^s_{p,q}$:

Definition 83 (Homogeneous Besov and Triebel-Lizorkin spaces)**.** *Let* $-\infty < s < \infty$*, let* $0 < q \leq \infty$ *and let* $\varphi \in \dot{\Phi}(\mathbb{R}^n)$*.*

i) If $0 < p \leq \infty$ *then the homogeneous Besov spaces* $\dot{\mathbf{B}}^{\mathbf{s}}_{\mathbf{p,q}}(\mathbb{R}^{\mathbf{n}})$ *consist of all* $f \in \mathcal{Z}'(\mathbb{R}^n)$ *such that the following inequality holds*

$$||f|\dot{B}^s_{p,q}(\mathbb{R}^n)||^\varphi = \Big(\sum_{j=-\infty}^{\infty} 2^{jsq}||\mathcal{F}^{-1}\varphi_j\mathcal{F}f|L^p(\mathbb{R}^n)||^q\Big)^{\frac{1}{q}} < \infty$$

with the necessary modification in the case $q = \infty$*.*

ii) If $0 < p < \infty$ *then the homogeneous Triebel-Lizorkin spaces* $\dot{\mathbf{F}}^{\mathbf{s}}_{\mathbf{p,q}}(\mathbb{R}^{\mathbf{n}})$ *consist of all* $f \in \mathcal{Z}'(\mathbb{R}^n)$ *such that the following inequality holds*

$$||f|\dot{F}^s_{p,q}(\mathbb{R}^n)||^\varphi = \Big|\Big|\Big(\sum_{j=-\infty}^{\infty} 2^{jsq}|(\mathcal{F}^{-1}\varphi_j\mathcal{F}f)(\cdot)|^q\Big)^{\frac{1}{q}}|L^p(\mathbb{R}^n)\Big|\Big| < \infty$$

with the necessary modification in the case $q = \infty$*.*

iii) If $p = \infty$ *and* $1 < q < \infty$ *then the spaces* $\dot{\mathbf{F}}^{\mathbf{s}}_{\infty,\mathbf{q}}(\mathbb{R}^{\mathbf{n}})$ *consist of all* $f \in \mathcal{Z}'(\mathbb{R}^n)$ *such that* $\exists\, \{f_k(x)\}_{k=-\infty}^{\infty} \subset L^\infty(\mathbb{R}^n)$ *such that the following holds*

$$f = \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}\varphi_k\mathcal{F}f_k \text{ in } \mathcal{Z}'(\mathbb{R}^n)$$

and

$$\Big|\Big|\Big(\sum_{k=-\infty}^{\infty} 2^{skq}|f_k(x)|^q\Big)^{\frac{1}{q}}|L^\infty(\mathbb{R}^n)\Big|\Big| < \infty.$$

Again, remember that the spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ do not depend of the choice of φ.
At this stage, let us recall the most important facts about these homogeneous function spaces.

Theorem 84. *([61])*

i)

$$\dot{W}^{k,p} = \dot{F}^k_{p,2} \text{ for } k < \infty \text{ and } 1 < p < \infty$$

i.e.

$$\sum_{|\alpha|=m} ||D^\alpha f|L^p(\mathbb{R}^n)|| = ||f|\dot{W}^{m,p}(\mathbb{R}^n)||$$

is an equivalent norm on $\dot{F}^m_{p,2}(\mathbb{R}^n)$ *if* $k < \infty$ *and* $1 < p < \infty$*.*

ii)
$$\mathfrak{H}^p = \dot{F}^0_{p,2} = \mathfrak{h}^p = F^0_{p,2} \quad \text{for } 1 < p < \infty.$$

For a proof and further details see [61], chapter 5, in particular p. 242. See also [61], p. 88.
Last, but not least, we have the following relation between homogeneous and non-homogeneous Besov respectively Triebel-Lizorkin spaces (see for instance [61] vol. II):

Proposition 85. *([61]) Let $s > n \max(0, \frac{1}{p} - 1)$. Then*

$$F^s_{p,q} = L^p \cap \dot{F}^s_{p,q}$$

and

$$||f||_p + ||f|\dot{F}^s_{p,q}||$$

is an equivalent norm to

$$||f|F^s_{p,q}||.$$

The same conclusions hold also for Besov spaces.

Related to that, we know the following isomorphism between different homogeneous Besov respectively Triebel-Lizorkin spaces.

Proposition 86. *([61]) The mapping $f \to \dot{I}_\sigma f$, defined es*

$$\dot{I}_\sigma f(\cdot) = \mathcal{F}^{-1}(|\xi|^\sigma \mathcal{F} f),\ \sigma \in \mathbb{R},\ f \in \mathcal{Z}',$$

is isomorphic from $\dot{F}^s_{p,q}$ onto $\dot{F}^{s-\sigma}_{p,q}$ and from $\dot{B}^s_{p,q}$ onto $\dot{B}^{s-\sigma}_{p,q}$.

A.3 Besov-Morrey spaces

In stead of combining L^p-norms ans l^q-norm one can also combine Morrey- (respectively Morrey-Campanato-) norms with l^q-norms. This idea was first introduced and applied by Kozono and Yamazaki in [29].
In order to make the whole notation clear and to avoid misunderstanding we will recall some definitions.
We start with the definition of Morrey spaces

Definition 87 (Morrey spaces)**.** *Let $1 \leq q \leq p < \infty$.*

i) The Morrey spaces $\mathcal{M}^{\mathbf{p}}_{\mathbf{q}}(\mathbb{R}^{\mathbf{n}})$ consist of all $f \in L^q_{loc}(\mathbb{R}^n)$ such that

$$||f|\mathcal{M}^p_q|| = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{n/p-n/q} ||f|L^q(B(x_0, R))|| < \infty$$

ii) The local Morrey spaces $\mathbf{M^p_q}(\mathbb{R}^\mathbf{n})$ *consist of all* $f \in L^q_{loc}(\mathbb{R}^n)$ *such that*

$$||f|M^p_q|| = \sup_{x_0\in\mathbb{R}^n} \sup_{0<R\leq 1} R^{n/p-n/q}||f|L^q(B(x_0,R))|| < \infty$$

where $B(x_0, R)$ *denotes the closed ball in* $\mathbb{R}^n$ *with centre* x_0 *and radius* R*.*

Note that it is easy to see that the spaces $\mathcal{M}^p_q$ and M^p_q coincide on compactly supported functions.

Apart from these spaces of regular distributions, i.e. function belonging to L^1_{loc}, in the case $q = 1$ we are even allowed to look at measures in stead of functions. More precisely we have the following measure spaces of Morrey type. They will become useful later on in a rather technical context.

Definition 88 (Measure spaces of Morrey type)**.** *Let* $1 \leq p < \infty$*.*

i) The measure spaces of Morrey type $\mathcal{M}^\mathbf{p}(\mathbb{R}^\mathbf{n}) = \mathcal{M}^p$ *consist of all Radon measures* μ *such that*

$$||\mu|\mathcal{M}^p|| = \sup_{x_0\in\mathbb{R}^n} \sup_{R>0} R^{n/p-n}|\mu|(B(x_0,R)) < \infty.$$

ii) The local measure spaces of Morrey type $\mathbf{M_q}(\mathbb{R}^\mathbf{n}) = M^p$ *consist of all Radon measures* μ *such that*

$$|\mu|\mathcal{M}^p|| = \sup_{x_0\in\mathbb{R}^n} \sup_{0<R\leq 1} R^{n/p-n}|\mu|(B(x_0,R)) < \infty$$

where as above $B(x_0, R)$ *denotes the closed ball in* $\mathbb{R}^n$ *with centre* x_0 *and radius* R*.*

Remember that all the spaces we have seen so far, i. e. $\mathcal{M}^p_q$, M^p_q, $\mathcal{M}^p$ and M^p are Banach spaces with the norms indicated before. Moreover, $\mathcal{M}^p_1$ and M^p_1 can be considered as closed subspaces of $\mathcal{M}^p$ and M^p respectively, consisting of all those measures which are absolutely continuous with respect to the Lebesgue measure.

For details, see e.g. [29].

Once we have the above definition of Morrey spaces (of regular distributions), we now define the Besov-Morrey spaces in the same way as we constructed the Besov spaces, of course with the necessary changes.

Definition 89 (Besov-Morrey spaces)**.** *Let* $1 \leq q \leq p < \infty$*,* $1 \leq r \leq \infty$ *and* $s \in \mathbb{R}$*.*

i) *Let* $\varphi \in \dot{\Phi}(\mathbb{R}^n)$. *The homogeneous Besov-Morrey spaces* $\mathcal{N}^{\mathbf{s}}_{\mathbf{p},\mathbf{q},\mathbf{r}}$ *consist of all* $f \in \mathcal{Z}'$ *such that*

$$||f|\mathcal{N}^s_{p,q,r}(\mathbb{R}^n)||^{\varphi} = \Big(\sum_{j=-\infty}^{\infty} 2^{jsr}||\mathcal{F}^{-1}\varphi_j\mathcal{F}f|\mathcal{M}^p_q(\mathbb{R}^n)||^r\Big)^{\frac{1}{r}} < \infty.$$

ii) *Let* $\varphi \in \Phi(\mathbb{R}^n)$. *The inhomogeneous Besov-Morrey spaces* $\mathbf{N}^{\mathbf{s}}_{\mathbf{p},\mathbf{q},\mathbf{r}}$ *consist of all* $f \in \mathcal{S}'$ *such that*

$$||f|N^s_{p,q,r}(\mathbb{R}^n)||^{\varphi} = \Big(\sum_{j=0}^{\infty} 2^{jsr}||\mathcal{F}^{-1}\varphi_j\mathcal{F}f|M^p_q(\mathbb{R}^n)||^r\Big)^{\frac{1}{r}} < \infty.$$

Note that since $L^p(\mathbb{R}^n) = \mathcal{M}^p_p(\mathbb{R}^n)$ the framework of the $\mathcal{N}^s_{p,q,r}(\mathbb{R}^n)$ can be seen as a generalisation of the framework of the homogeneous Besov spaces. In our further work we will crucially use still another variant of spaces which are defined via Paley-Littlewood decomposition. We will use the decomposition into frequencies of positive power but measure the single contributions in a homogeneous Morrey norm:

Definition 90 (The spaces $B^s_{\mathcal{M}^p_q,r}$). *i) Let* $1 \leq q \leq p < \infty$, $1 \leq r \leq \infty$ *and* $s \in \mathbb{R}$. *Let* $\varphi \in \Phi(\mathbb{R}^n)$. *The spaces* $\mathbf{B}^{\mathbf{s}}_{\mathcal{M}^{\mathbf{p}}_{\mathbf{q}},\mathbf{r}}$ *consist of all* $f \in \mathcal{S}'$ *such that*

$$||f|B^s_{\mathcal{M}^p_q,r}(\mathbb{R}^n)||^{\varphi} = \Big(\sum_{j=0}^{\infty} 2^{jsr}||\mathcal{F}^{-1}\varphi_j\mathcal{F}f|\mathcal{M}^p_q(\mathbb{R}^n)||^r\Big)^{\frac{1}{r}} < \infty.$$

ii) *The spaces* $\mathbf{B}^{\mathbf{s}}_{\mathcal{M}^{\mathbf{p}}_{\mathbf{q}},\mathbf{r}}(\Omega)$ *where* Ω *is a bounded domain in* $\mathbb{R}^n$ *consist of all* $f \in B^s_{\mathcal{M}^p_q,r}$ *which in addition have compact support contained in* Ω.

Remark 91.

i) Again, as in the case of Besov and Triebel-Likorkin spaces, all the spaces defined above do not depend on the choice of φ.

ii) Previously we mentioned that our interest in these latter spaces was motivated by the work of Rivière and Struwe (see [43]) let us say a few words about this. In [43] the authors used the homogeneous Morrey space $L^{2,n-2}_1$ with norm

$$||u||^2_{L^{2,n-2}_1} = \sup_{x_0\in\mathbb{R}^n}\sup_{r>0}\Big(\frac{1}{r^{n-2}}\int_{B-r(x_0)}|\nabla u|^2\Big).$$

Note that $u \in L_1^{2,n-2}$ is equivalent to the fact that for all radii $r > 0$ and all $x_0 \in \mathbb{R}^n$ we have the inequality

$$||\nabla u||_{L^2(B_r(x_0))} \leq Cr^{(n-2)/p} = Cr^{\frac{n}{2}-\frac{2}{2}}$$

but this latter estimate is again equivalent to the fact that $\nabla u \in \mathcal{M}_2^n$. Finally we remember that $\mathcal{M}_2^n = \mathcal{N}_{n,2,2}^0$ (see for instance [32]) and note that $\nabla u \in \mathcal{N}_{n,2,2}^0$ is equivalent to $u \in \mathcal{N}_{n,2,2}^1$ since for all s - even for the negative ones - we have the equivalence $2^s||u^s||_{\mathcal{M}_2^n} \simeq ||(\nabla u)^s||_{\mathcal{M}_2^n}$ because we always avoid the origin in the Fourier space and also near the origin work with annuli with radii $r \simeq 2^s$.

Before we continue, let us state a few facts concerning the spaces $B^s_{\mathcal{M}_q^p,r}$ which are interesting and important.

Lemma 92. *i) The spaces $B^s_{\mathcal{M}_q^p,r}$ are complete for all possible choices of indexes.*

ii) a) Let $s > 0$, $1 \leq q \leq p < \infty$, $1 \leq r \leq \infty$ and $\lambda > 0$. Then

$$||f(\lambda\cdot)|B^s_{\mathcal{M}_q^p,r}|| \leq C\lambda^{-\frac{n}{p}} \sup\{1,\lambda\}^s \, ||f|B^s_{\mathcal{M}_q^p,r}||.$$

b) Let $s = 0$, $1 \leq q \leq p < \infty$, $1 \leq r \leq \infty$ and $\lambda > 0$. Then

$$||f(\lambda\cdot)|B^s_{\mathcal{M}_q^p,r}|| \leq C\lambda^{-\frac{n}{p}}(1 + |\log\lambda|)^{\alpha}||f|B^s_{\mathcal{M}_q^p,r}||$$

where

$$\alpha = \frac{1}{r} \text{ if } \lambda > 1 \text{ and } \alpha = 1 - \frac{1}{r} = \frac{1}{r'} \text{ if } 0 < \lambda < 1.$$

The first assertion is obtained by the same proof as the corresponding claim for the spaces $N^s_{p,q,r}$ in [29].
The second fact is a variation of a well known proof given in [10].

At this stage, let us recall the the following results of [29]:

Proposition 93. *([29]) There exists a positive constant C such that the following holds. Let m be a real number, and let j be an integer. Suppose that $P(\xi)$ is a C^∞-function on $D_{j-1} \cup D_j \cup D - j + 1$ - where D_j is a annulus with radii proportional to 2^j - such that the estimate $|(\partial^{|\alpha|}P/\partial\xi^\alpha)(\xi)| \leq A2^{(m-|\alpha|)j}$ holds for $\xi \in D - j - 1 \cup D_j \cup D_{j+1}$ with some constant A for every $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \leq [n/2] + 1$. Suppose further that p and q satisfy $1 \leq q \leq p\infty$. Then, for every $u \in M_q^p$ [resp.$u \in \mathcal{M}_q^p$] such that supp $\mathcal{F}u \subset D_j$, we have $\mathcal{F}^{-1}(P(\xi)\mathcal{F}u) \in M_q^p$ and $||\mathcal{F}^{-1}(P(\xi)\mathcal{F}u)|M_q^p|| \leq CA2^{mj}||u|M_q^p||$. [resp. $\mathcal{F}^{-1}(P(\xi)\mathcal{F}u) \in \mathcal{M}_q^p$ and $||\mathcal{F}^{-1}(P(\xi)\mathcal{F}u)|\mathcal{M}_q^p|| \leq CA2^{mj}||u|\mathcal{M}_q^p||$.]*

Proposition 94. *([29]) There exists a positive constant C such that the following holds. Suppose that $P(\xi)$ is a C^∞-function on $B_4(0)$ such that the estimate $|(\partial^{|\alpha|}P/\partial\xi^\alpha)(\xi)| \leq A$ holds for $\xi \in B_4(0)$ with some constant A for every $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \leq [n/2]+1$. Suppose further that p and q satisfy $1 \leq q \leq p < \infty$. Then, for every $u \in M_q^p$ [resp.$u \in \mathcal{M}_q^p$] such that supp $\mathcal{F}u \subset B_2(0)$, we have $\mathcal{F}^{-1}(P(\xi)\mathcal{F}u) \in M_q^p$ and $||\mathcal{F}^{-1}(P(\xi)\mathcal{F}u)|M_q^p|| \leq CA||u|M_q^p||$. [resp. $\mathcal{F}^{-1}(P(\xi)\mathcal{F}u) \in \mathcal{M}_q^p$ and $||\mathcal{F}^{-1}(P(\xi)\mathcal{F}u)|\mathcal{M}_q^p|| \leq CA||u|\mathcal{M}_q^p||$.]*

For further information about the Besov-Morrey spaces, see [29], [31] and [32].
A natural question that arises when we work in function spaces different from L^p- spaces or Sobolev-spaces is to ask for an analogy to the well-known Sobolev embedding results.
In the framework of Morrey-spaces there is the following result of Campanato (see [14])

Theorem 95. *([14]) Let Ω be a ball in $\mathbb{R}^n$ and assume that $u \in W^k_{M_q^p}$ with $1 \leq q \leq p < \infty$. Then it holds*

i) If $\frac{n}{p} \geq k$, then $u \in M^{\frac{n\beta}{n-\mu}}$ where

$$\beta < \beta^* = \frac{n}{n-k},\ if\ \frac{n}{p} > k$$
$$\beta < \beta^* = \infty,\ if\ \frac{n}{p} = k$$

and

$$\mu < n = k\beta - \frac{m\beta}{p}.$$

ii) If there exists h, $1 \leq h \leq k$ such that

$$h > \frac{n}{p}$$

then we have

$$\sum_{|r|=k-h} \sup_{\bar{\Omega}} |D^r u| \leq C||u||_{W^k_{M_q^p}}.$$

In addition, we have the following refinement due to Adams (see [3]) at our disposal

Theorem 96. *([3]) Let $k \geq 1$ be a natural number and assume that $\nabla^k u \in M_q^p(\Omega)$ where $\Omega \subset \mathbb{R}^n$ and $1 < q < \frac{nq}{kp}$. Then it holds*

$$u \in M^{\frac{np}{n-kp}}_{\frac{nq}{n-kp}}.$$

In the larger context of the Besov-Morrey-spaces in Kozono/Yamazaki the following generalisation is presented:

Theorem 97. *([29])*

i) Let p, q and s be real numbers such that $1 \leq q \leq p < \infty$, and let $r \in [1, \infty]$. Then we have the following continuous embeddings: $N^s_{p,q,r} \subset B^{s-n/p}_{\infty,r}$ and $\mathcal{N}^s_{p,q,r} \subset \dot{B}^{s-n/p}_{\infty,r}$.

ii) For every $\theta \in (0,1)$ the following embeddings hold: $N^s_{p,q,r} \subset N^{s-n(1-\theta)/p}_{p/\theta,q/\theta,r}$ and $\mathcal{N}^s_{p,q,r} \subset \mathcal{N}^{s-n(1-\theta/p}_{p/\theta,q/\theta,r}$.

The last assertion of Kozono/Yamazaki we would like to quote is the following:

Lemma 98. *([29]) Let ν be a Radon measure on $\mathbb{R}^n$ such that its total variation on $\mathbb{R}^n$ is $A < \infty$. Ten we have the following : Suppose that $1 \leq q \leq p < \infty$. Then, for every $u \in M^p_q$, [resp. $u \in \mathcal{M}^p_q$] we have $\nu * u \in M^p_q$ and $||\nu * u|M^p_q|| \leq A||u|M^p_q||$. [resp. $\nu * u \in \mathcal{M}^p_q$ and $||\nu * u|\mathcal{M}^p_q|| \leq A||u|\mathcal{M}^p_q||$]. The same conclusions hold also for M^p [resp. $\mathcal{M}^p$].*

Furthermore we have the following embedding result which relates the spaces $B^0_{\mathcal{M}^p_q,r}$ to the Morrey spaces with the same indexes respectively, similar for the spaces $N^0_{p,q,r}$.

A.4 Spaces involving Choquet integrals

In the preceding section devoted to Hardy spaces and also in the sections about Besov- and Triebel-Lizorkin spaces we saw some duality results, in the sense that for a given function space we were able to give the description of its dual spaces - under some conditions. If we ask ourselves the same question in the framework of Morrey-Besov spaces, the situation is much more complicated. Nevertheless, we shall make use of such a result. More precisely we will use a certain description of the predual spaces of $\mathcal{M}^1$. Before we can state this assertion we have to introduce some function spaces involving the so-called Choquet integral. A general reference for this section is [1] and the references given therein.
We start with the notion of Hausdorff capacity:

Definition 99 (Hausdorff capacity)**.** *Let E be a subset of $\mathbb{R}^n$ and let $\{B_j\}$, $j = 1, 2, \ldots$ be a cover of E, i.e. $\{B_j\}$ is a countable collection of open balls B_j*

with radius r_j such that $E \subset \cup_j B_j$. Then we define the ***Hausdorff capacity of** E **of dimension** d, $0 < d \leq n$ to be the following quantity*

$$H^d_\infty(E) = \inf \sum_j r_j^d$$

where the infimum is taken over all possible covers of E.

Remark 100. The name capacity may lead to confusion. Here we use this expression in the sense of N. Meyers. See [33], page 257.

Once we have this capacity, we can pass to the Choquet integral of $\phi \in C_0(\mathbb{R}^n)^+$:

Definition 101 (Choquet integral and $\mathbf{L^1(H^d_\infty)}$)**.** *Let $\phi \in C_0(\mathbb{R}^n)^+$. Then the* ***Choquet integral*** *of ϕ with respect to the Hausdorff capacity H^d_∞ is defined to be the following Riemann integral:*

$$\int \phi \, dH^d_\infty \equiv \int_0^\infty H^d_\infty[\phi > \lambda] \, d\lambda.$$

The space $\mathbf{L^1(H^d_\infty)}$ is now the completion of $C_0(\mathbb{R}^n)$ under the functional $\int |\phi| \, dH^d_\infty$.

Two important facts about $L^1(H^d_\infty)$ are summarised below, again for instance see [1] and also the references given there.

Remark 102.

- $L^1(H^d_\infty)$ can also be characterised to be the space of all H^d_∞-quasi continuous functions ϕ which satisfy $\int |\phi| \; dH^d_\infty < \infty$, i.e. for all $\varepsilon > 0$ there exists an open set G such that $H^d_\infty[G] < \varepsilon$ and that ϕ restricted to the complement of G is continuous there.

- One can show that $L^1(H^d_\infty)$ is a quasi-Banach space with respect to the quasi-norm $\int |\phi| \; dH^d_\infty$.

Now, we can state the duality result we mentioned earlier. A proof of this assertion is given in [1], but take care of the notation which differs from our notation!

Proposition 103. *([1]) We have $(L^1(H^d_\infty))^* = \mathcal{M}^{\frac{n}{n-d}}$ and in particular the estimate*

$$\Big| \int u \, d\mu \Big| \leq ||u||_{L^1(H^d_\infty)} ||\mu||_{\mathcal{M}^{\frac{n}{n-d}}}$$

holds.

Remark 104. The above proposition is just a spacial case of a more general result which involves also spaces $L^p(H^d_\infty)$, see for instance [2].

Before ending this section we will state some useful remarks for later applications.

Remark 105.

- Observe that $\mathcal{M}^p \subset \mathcal{S}'$ (in particular for $p = \frac{n}{n-d}$). In order to verify this, note that $\mathcal{M}^p \subset N^0_{p,1,\infty} \subset \mathcal{S}'$: Let $\mu \in \mathcal{M}^p$ and let as usual $\varphi \in \Phi(\mathbb{R}^n)$ then we have

$$\begin{aligned}
||\mu|N^0_{p,1,\infty}|| &= \sup_{k\in\mathbb{N}} ||\check{\varphi}_k * \mu|M^p_1|| \\
&= \sup_{k\in\mathbb{N}} ||\check{\varphi}_k * \mu|M^p|| \\
& \text{ note that } \check{\varphi}_k * \mu \in C^\infty \subset L^1_{loc} \text{ since } \mu \in \mathcal{D}' \\
& \text{ and } \check{\varphi}_k * \mu \text{ can be seen as a measure} \\
&\leq \sup_{k\in\mathcal{N}} ||\check{\varphi}_k||_1 ||\mu|M^p|| \\
& \text{ because of lemma 98} \\
&\leq C||\mu|\mathcal{M}^p|| \\
&< \infty \\
& \text{ according to our hypothesis.}
\end{aligned}$$

  Once we have this, we apply the continuous embedding of $N^0_{p,1,\infty}$ into $\mathcal{S}'$ (see e.g. [32]) and conclude that actually $\mathcal{M}^p \subset \mathcal{S}'$.
  Note also that $\mathcal{S} \subset L^1(H^d_\infty)$

- Using the duality asserted above, we can show that $L^1(H^d_\infty) \subset \mathcal{S}'$: We start with $f \in C^\infty_0(\mathbb{R}^n)$. Since $f \in L^\infty$ it is easy to check that $f \in M^p_q$, $1 \leq q \leq p < \infty$, with $||f|M^p_q|| = ||f||_\infty$. Moreover, f even belongs to $\mathcal{M}^p_q$. In order to establish this, it remains to show that there is a constant C, independent on f, such that $\forall x \in \mathbb{R}^n$ and for $1 \leq r$

$$||f||_{L^1(B_r(x))} \leq Cr^{\frac{n}{q}-\frac{n}{p}}.$$

  In fact, it holds $\forall x \in \mathbb{R}^n$ and $\forall r \geq 1$

$$\begin{aligned}
||f||_{L^1(B_r(x))} &\leq ||f||_1 \\
&\leq ||f||_1 r^{\frac{n}{q}-\frac{n}{p}} \\
& \text{ since due to the choice of } p \text{ and } q \text{ we have} \\
& \frac{n}{q} - \frac{n}{p} \geq 0.
\end{aligned}$$

If we put together all this information we find

$$||f|\mathcal{M}^p_q|| \leq ||f||_\infty + ||f||_1.$$

Now, recall that the duality between $L^1(H^d_\infty)$ and $\mathcal{M}^{\frac{n}{n-d}}$ is given by

$$< \mu, u >_{(L^1(H^d_\infty))^* = \mathcal{M}^{\frac{n}{n-d}}, L^1(H^d_\infty)} = \int u \, d\mu$$

where $u \in L^1(H^d_\infty)$ and $\mu \in \mathcal{M}^{\frac{n}{n-d}}$.
In a next step we define the action of $u \in L^1(H_\infty)$ on $f \in C_0^\infty$ as follows

$$< u, f >_{\mathcal{D}', C_0^\infty} := < f, u >_{\mathcal{M}^{\frac{n}{n-d}}, L^1(H^d_\infty}.$$

Last, but not least, we observe that for $\varphi \in \mathcal{S}$ we have

$$||\varphi||_\infty + ||\varphi||_1 \leq C(n)||\varphi||_{\mathcal{S}}.$$

This finally leads to the conclusion that in fact, $L^1(H^d_\infty) \subset \mathcal{S}'$.

This last remark enables us to use the above introduced $L^1(H^d_\infty)$-quasi norm to construct - in analogy to the case of Besov- or Besov-Morrey-spaces - a new space of functions.

Definition 106 (Besov-Choquet spaces). *Let* $\varphi \in \Phi(\mathbb{R}^n)$.
We say that f *belongs to* $\mathbf{B^0_{L^1(H^d_\infty),\infty}}$ *if* $\exists \{f_k(x)\}_{k=0}^\infty \subset L^1(H^d_\infty)$ *such that the following holds*

$$f = \sum_{k=0}^\infty \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k$$

and

$$\sup_k ||f_k|L^1(H^d_\infty)|| < \infty.$$

Moreover we set

$$||f|B^0_{L^1(H^d_\infty),\infty}|| = \inf \sup_k ||f_k|L^1(H^d_\infty)||$$

where the infimum is taken over all admissible representations of f.
Moreover, we denote by $\mathbf{b^0_{L^1(H^d_\infty),\infty}}$ *the closure of* $\mathcal{S}$ *under the construction explained above.*

Remark 107. In complete analogy to the construction of the Besov spaces (respectively the Besov-Morrey-spaces) one could also construct new spaces if we replace the Lebesgue L^p-norms (respectively the Morrey-norms) by $L^p(H^d_\infty)$-quasi-norms.

A.5 Lorentz spaces

In this section we introduce still another class of function spaces.
As in the case of Besov spaces we have to start with some preliminary concepts.

Definition 108 (Non-increasing rearrangement)**.** *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a measurable function. The non-increasing rearrangement of $|f|$ on $[0, |\Omega|)$ is the unique function, denoted by $\mathbf{f}^*$, from $[0, |\Omega|)$ to $\mathbb{R}$ which is non-increasing and such that*

$$|\,\{x \in \Omega |\; |f(x)| \geq s\}\,| = |\,\{t \in (0, |\Omega|) |\; f^*(t) \geq s\}\,|.$$

Moreover we set

$$\mathbf{f}^{**}(\mathbf{t}) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

Once we have this we can proceed to the definition of Lorentz spaces.

Definition 109 (The spaces $L^{p,q}(\Omega; \mathbb{R})$)**.** *Let Ω be an open subset of $\mathbb{R}^m$, $p \in [1, \infty]$, $q \in [1, \infty]$. The Lorentz space $\mathbf{L^{p,q}}(\Omega; \mathbb{R})$ is the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that*

$$|f|_{p,q} = \Big[\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\Big]^{\frac{1}{q}} < \infty, \text{ if } 1 \leq q < \infty \text{ and } 1 \leq p < \infty$$

or

$$|f|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty, \text{ if } 1 \leq p \leq \infty \text{ and } q = \infty.$$

Moreover we define

$$||f||_{p,q} = \Big[\int_0^\infty (t^{\frac{1}{p}} f^{**}(t))^q \frac{dt}{t}\Big]^{\frac{1}{q}}, \text{ if } 1 \leq q < \infty \text{ and } 1 \leq p < \infty$$

or

$$||f||_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), \text{ if } 1 \leq p \leq \infty \text{ and } q = \infty.$$

Both quantities $|f|_{p,q}$ and $||f||_{p,q}$ are important and useful. This fact is underlined by the following theorem.

Theorem 110. *([68]) If $f \in L^{p,q}$, $1 < p \leq \infty$, then $\exists C > 0$ such that*

$$\frac{1}{C}|f|_{p,q} \leq ||f||_{p,q} \leq C|f|_{p,q}.$$

For instance see [53] and [68].

At this stage it is very natural to ask in how far the well known L^p spaces are related to the new $L^{p,q}$ spaces. The answer is the following

Theorem 111. *([53]) We have*

$$L^p \quad = \quad L^{p,p} \text{ for } 1 \leq p \leq \infty$$

and
$(L^{p,q}, ||\cdot||_{p,q})$ *is a Banach space for* $1 < p \leq \infty$, $1 \leq q \leq \infty$.

Again, proofs can be found in [53].

Apart from that, the following theorem summarises the most important properties of $L^{p,q}$ spaces - among them some similarities between the L^p spaces and the $L^{p,q}$ spaces:

Theorem 112. *([68] et al.)*

i) Assume that $f \in L^{p_1,q_1}$ *and* $g \in L^{p_2,q_2}$ *then* $f \cdot g \in L^{p,q}$ *where*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \text{ and } \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

ii) $(L^{p,q})^* = L^{p',q'}$ *for* $1 < p < \infty$ *and* $1 \leq q \leq \infty$ *where as usual* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$.

iii) $L^{p,q_1} \subset L^{p,q_2}$ *if* $q_1 \leq q_2$.

iv) $\frac{1}{|x|^\mu} \in L^{\frac{n}{\mu},\infty}(\mathbb{R}^n)$.

See for instance [68], [53] and [30], [37].

Last but not least we would like to point out the connection with the distribution function λ:

Definition 113 (Distribution function)**.** *Let* $g(x)$ *be defined on* $\mathbb{R}^n$*. The distribution function* $\lambda(\alpha)$ *of* $|g|$ *is defined to be the measure of the set where* $|g| > \alpha$*, i. e.*

$$\lambda(\alpha) = |\,\{x \mid |g(x)| > \alpha\}\,|.$$

Then we have

Lemma 114. *In the case $L^{2,1}$ we have*

$$||f||_{2,1} = 2|f|_{2,1} = 4\int_0^\infty |\{x|\,|f(x)| > \lambda\}|^{\frac{1}{2}}\,d\lambda$$

Proof of lemma 114:
This lemma is a conclusion of these two calculations:

$$\begin{aligned} ||f||_{2,1} &= \int_0^\infty t^{\frac{1}{2}} f^{**}(t)\frac{dt}{t} \\ &= \int_0^\infty t^{\frac{1}{2}}\frac{1}{t}\int_0^t f^*(s)\,ds\,\frac{dt}{t} \\ &= \int_0^\infty t^{-\frac{3}{2}}\int_0^t f^*(s)\,ds\,dt \\ &= 2\int_0^\infty t^{-\frac{1}{2}} f^*(t)\,dt \\ &= 2|f|_{2,1} \end{aligned}$$

where in the second last step we integrated by parts.
And

$$\begin{aligned} \int_0^\infty t^{-\frac{1}{2}} f^*(t)\,dt &= \int_0^\infty f^*(t)\frac{dt}{\sqrt{t}} \\ &= 2\int_0^\infty f^*(u^2)\,du \\ &= 2\int_0^\infty |\left\{u|\ f^*(u^2) > \lambda\right\}|\,d\lambda \\ &= 2\int_0^\infty |\{t|\ f^*(t) > \lambda\}|^{\frac{1}{2}}\,d\lambda \\ &= 2\int_0^\infty |\{x|\ f(x) > \lambda\}|^{\frac{1}{2}}\,d\lambda \end{aligned}$$

where the second last step holds because of the fact that f^* is non-increasing and the last step holds because of the fact that f and f^* are equimeasurable.

□

Moreover we have that - except at points where either function is discontinuous (and the other is constant on an interval) - the distribution function λ and f^* are inverse to each other. This fact is really well explained in [4], p. 222, and is exploited in the next proposition.

Proposition 115. *Let $\Omega \subset \mathbb{R}^n$ be bounded. Then*

$$L^{2,\infty}(\Omega) \subset L^p(\Omega) = L^{p,p}(\Omega)\ \forall p < 2.$$

Proof of proposition 115:
Since we are on a bounded domain, the distribution function λ is bounded from above by $|\Omega|$. But this implies that

$$\Big(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^p \frac{dt}{t}\Big)^{\frac{1}{p}} = \Big(\int_0^{|\Omega|} (t^{\frac{1}{p}} f^*(t))^p \frac{dt}{t}\Big)^{\frac{1}{p}}.$$

(In one dimension the inverse function is the reflection at the line $x = y$, so $f^*(t)$ does not transgress $t = |\Omega|$.)
Next remember that for $1 < p \leq \infty$ $|f|_{p,p}$ and $||f||_{p,p}$ are equivalent. So it suffices to show that for every $f \in L^{2,\infty}(\Omega)$ $|f|_{p,p} < \infty$:

$$\begin{aligned} |f|_{p,p} &= \Big(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^p \frac{dt}{t}\Big)^{\frac{1}{p}} \\ &= \Big(\int_0^{|\Omega|} (t^{\frac{1}{p}} f^*(t))^p \frac{dt}{t}\Big)^{\frac{1}{p}} \\ &\leq |f|_{2,\infty}\Big(\int_0^{|\Omega|} (t^{\frac{1}{p}} t^{-\frac{1}{2}})^p \frac{dt}{t}\Big)^{\frac{1}{p}} \\ &= |f|_{2,\infty}\Big(\int_0^{|\Omega|} t^{-\frac{p}{2}}\, dt\Big)^{\frac{1}{p}} \\ &< \infty \end{aligned}$$

where the third last step holds because

$$|f|_{2,\infty} = \sup_{t>0} t^{\frac{1}{2}} f^*(t) < \infty \Rightarrow f^*(t) \leq |f|_{2,\infty} t^{-\frac{1}{2}} \ \forall t > 0.$$

The last step holds because of the fact that due to $p < 2$ we have $-\frac{p}{2} > -1$ which gives us the last inequality since we integrate only between zero and $|\Omega|$.

$\square$

For further details about the $L^{p,q}$ spaces see [53], [68], [4] or for those who are especially interested in interpolation theory: [30], [37] and [53].

Appendix B

Two alternative approaches towards Wente's lemma

B.1 Application of the notion of paraproducts towards Wente's result

In this section we will give an alternative proof of the fact that whenever we start with two functions $a \in W^{1,p}(\mathbb{R}^n)$ and $b \in W1, p'(\mathbb{R}^n)$ where $p > 1$ and $\frac{1}{}p + \frac{1}{p'} = 1$ then and two-dimensional determinant belongs not only to $L^1(\mathbb{R}^n)$ but even to the smaller Hardy space $F^0_{1,2}(\mathbb{R}^n)$. This alternative approach can be essentially found in [58], but we prefer to present it here in a slightly different manner, since this was the starting point towards the n-dimensional generalisation we present in the first part.
For the sake of simplicity, in what follows we will use the abbreviation a_z for $\frac{\partial}{\partial z}a$.
In general, for a given function $u \in W^{1,p}$ by Hölder's inequality we can only obtain the a priori information that the product of two first order derivatives $u_{x_i}u_{x_j}$ belongs to $L^{\frac{p}{2}}$. If $p/2 > 1$ then $L^{\frac{p}{2}}(\Omega)$ embeds continuously into $F^0_{1,2}(\Omega)$ for any bounded open C^∞-domain Ω (see also theorem 78). So we conclude that actually $u_{x_i}u_{x_j}$ belongs to $F^0_{1,2;loc}$, i.e. locally our product coincides with a distribution in $F^0_{1,2} = \mathfrak{h}^1$.
Our goal here is to prove a global result, namely the following theorem.

Theorem 116. *Let $a \in \dot{W}^{1,p}(\mathbb{R}^n)$ and $b \in \dot{W}^{1,p'}(\mathbb{R}^n)$ where*

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } p > 1.$$

Then

$$\pi_1(a_x, b_y), \pi_3(a_x, b_y), \pi_1(a_y, b_x) \text{ and } \pi_3(a_y, b_x) \in \dot{F}^0_{1,2}(\mathbb{R}^n) = \mathfrak{H}^1(\mathbb{R}^n)$$

and

$$\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in \dot{B}^0_{1,1}(\mathbb{R}^n) = \dot{F}^0_{1,1}(\mathbb{R}^n) \subset \dot{F}^0_{1,2}(\mathbb{R}^n) = \mathfrak{H}^1(\mathbb{R}^n)$$

where $x = z_i$ *with* $i \in \{1, \dots, n\}$ *and* $y = z_j$ *with* $j \in \{1, \dots, n\}$.

As a warm up we shall first of all prove the following result:

Proposition 117. *Let* $a \in W^{1,p}(\mathbb{R}^n)$ *and* $b \in W^{1,p'}(\mathbb{R}^n)$ *where*

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } p > 1.$$

Then

$$\pi_1(a_x, b_y), \pi_3(a_x, b_y), \pi_1(a_y, b_x) \text{ and } \pi_3(a_y, b_x) \in F^0_{1,2}(\mathbb{R}^n) = \mathfrak{h}^1(\mathbb{R}^n)$$

and

$$\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s \in B^0_{1,1}(\mathbb{R}^n) = F^0_{1,1}(\mathbb{R}^n) \subset F^0_{1,2}(\mathbb{R}^n) = \mathfrak{h}^1(\mathbb{R}^n)$$

where $x = z_i$ *with* $i \in \{1, \dots, n\}$ *and* $y = z_j$ *with* $j \in \{1, \dots, n\}$.

Recall that $\dot{W}^{k,p}(\mathbb{R}^n)$ is the closure of C_0^∞ in the norm $||\cdot||_{\dot{W}^{k,p}} = \sum_{|s|=k} ||\nabla^s \cdot||_{L^p}$ and that the notation "$\subset$" stand for a continuous embedding.
Note that using a result of Coifman, Lions, Meyer and Semmes (see [16]) one can immediately find a similar result namely that

$$a_x b_y - a_y b_x \in \dot{F}^0_{1,2} = \mathfrak{H}^1(\mathbb{R}^n)$$

whenever $a \in \dot{W}^{k,p}$ and $b \in \dot{W}^{m,p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ with $p > 1$ and $k, m \geq 1$.

Remark 118.

- Note that since on any bounded domain Ω the norms $||\cdot||_{W^{k,p}} = \sum_{|s|\leq k} ||\nabla^s \cdot||_{L^p}$ and $||\cdot||_{\dot{W}^{k,p}} = \sum_{|s|=k} ||\nabla^s \cdot||_{L^p}$ are equivalent our a priori estimate reduces to

$$||a_x b_y - a_y b_x | F^0_{1,2}|| \leq C ||\nabla a||_p ||\nabla b||_{p'}$$

  if both a and b have bounded support.

- Theorem 78 below asserts that a local improvement is possible:

$$J_{ij}(a,b) \in B^0_{1,1;loc}(\mathbb{R}^n) = F^0_{1,1;loc}.$$

Proof of proposition 117:
First of all, let us point out the two important facts we have, namely that the quantity we want to study is a determinant, which gives us a certain (algebraic) structure, and moreover that the functions involved in this determinant are (weak) derivatives.
We start with a system $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$ constructed as in the example of remark 69 and define $\mathbf{f^j(x)} = \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(x)$.
The fact that $\sum_{j=0}^{\infty} f^j = f$ in $\mathcal{S}'(\mathbb{R}^n)$ leads us to the notion of the product of tempered distributions.

Definition 119 (Product of two tempered distributions)**.** *Let f, g be two tempered distributions. The* ***product*** *of f and g is given by*

$$f \cdot g = \lim_{j\to\infty} \Big(\sum_{i=0}^{j} f^i\Big)\Big(\sum_{l=0}^{j} g^l\Big)$$

whenever the right hand side of the above equation exists in $\mathcal{S}'(\mathbb{R}^n)$.

The following decomposition into so-called paraproducts will turn out to be helpful.

Definition 120 (**Paraproducts**)**.** *Let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^n)$ be two functions supported around the origin.*
We consider the following bowline map

$$\pi(f,g)(x) := \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\psi_1(2^{-j}\cdot)\mathcal{F}f)(x)\mathcal{F}^{-1}(\psi_2(2^{-j}\cdot)\mathcal{F}g)(x)$$

where $f, g \in \mathcal{S}'(\mathbb{R}^n)$.
Operators of this type are called ***paraproducts*** *or* ***paramultiplication operators****.*

We use the following paraproducts:

$$\begin{aligned}
\pi_1(f,g) &:= \sum_{k=2}^{\infty}\sum_{l=0}^{k-2} f^l g^k \\
&= \sum_{k=2}^{\infty} \mathcal{F}^{-1}(\varphi_0(2^{-k+2}\cdot)\mathcal{F}f)\mathcal{F}^{-1}(\varphi_k \mathcal{F}g) \\
\pi_2(f,g) &:= \sum_{k=0}^{\infty}\sum_{l=k-1}^{k+1} f^l g^k \\
&= \sum_{k=2}^{\infty} \mathcal{F}^{-1}((\varphi_0(2^{-k-1}\cdot) - \varphi_0(2^{-k+1}\cdot))\mathcal{F}f)\mathcal{F}^{-1}(\varphi_k \mathcal{F}g) \\
\pi_3(f,g) &:= \sum_{l=2}^{\infty}\sum_{k=0}^{l-2} f^l g^k \\
&= \sum_{l=2}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F}f)\mathcal{F}^{-1}(\varphi_0(2^{-l+2}\cdot)\mathcal{F}g)
\end{aligned}$$

where $f^i = 0$ for $i \leq -1$ and similarly for g.

The reason why we work with these paraproduct is the following:
Assume that $\pi_1(f,g)$, $\pi_2(f,g)$ and $\pi_3(f,g)$ exist, moreover we assume that $f \in A^{s_1}_{p_1,q_1}$ and $g \in A^{s_2}_{p_2,q_2}$ where $A^s_{p,q}$ denotes a Besov or a Triebel-Lizorkin space. Then the following computations show that in fact we can control also the product $f \cdot g$.

$$\begin{aligned}
& \pi_1(f,g) + \pi_2(f,g) + \pi_3(f,g) \\
= & \lim_{j\to\infty} \sum_{k=2}^{j}\sum_{l=0}^{k-2} f^l g^k + \lim_{j\to\infty} \sum_{k=0}^{j}\sum_{l=k-1}^{k+1} f^l g^k + \lim_{j\to\infty} \sum_{l=2}^{j}\sum_{k=0}^{l-2} f^l g^k \\
= & \lim_{j\to\infty} \Big(\sum_{k=2}^{j}\sum_{l=0}^{k-2} f^l g^k + \sum_{k=0}^{j}\sum_{l=k-1}^{k+1} f^l g^k + \sum_{l=2}^{j}\sum_{k=0}^{l-2} f^l g^k - f^{j+1} g^j \Big) \\
= & \lim_{j\to\infty} \Big(\sum_{l=0}^{j}\sum_{k=0}^{j} f^l g^k \Big) \\
= & \lim_{j\to\infty} \Big(\sum_{l=0}^{j} f^l \Big) \Big(\sum_{l=0}^{j} g^l \Big) \\
= & f \cdot g
\end{aligned}$$

where in the second step we used the fact that $f^{j+1}g^j \to 0$ in $\mathcal{S}'$:
First of all, we note that

$$\begin{aligned} ||2^{(j+1)s_1} f^{j+1}||_{p_1} &\leq C||f^{j+1}|A^{s_1}_{p_1,q_1}|| \\ ||2^{js_2} g^j||_{p_2} &\leq C||g^j|A^{s_2}_{p_2,q_2}||. \end{aligned}$$

Then we have for any test function $\phi \in \mathcal{S}$

$$\begin{aligned} \Big| < f^{j+1}g^j, \phi > \Big| &= \Big| \int f^{j+1} g^j \phi \Big| \\ &\leq \int |f^{j+1} g^j \phi| \\ &= ||f^{j+1} g^j \phi||_1 \\ &\leq ||2^{(j+1)s_1} f^{j+1}||_{p_1} \cdot ||2^{js_2} g^j||_{p_2} \cdot ||2^{-j(s_1+s_2)} \phi||_{\frac{p_1p_2}{p_1p_2-p_2-p_1}} \\ &\quad \text{because of Hölder's inequality} \\ &\leq ||2^{(j+1)s_1} f^{j+1}||_{p_1} \cdot ||2^{js_2} g^j||_{p_2} \cdot ||\phi||_{\frac{p_1p_2}{p_1p_2-p_2-p_1}} \\ &\quad \text{due to the necessary condition } s_1 + s_2 \geq 0 \\ &\quad \text{(see proposition below)} \\ &\leq ||f^{j+1}|A^{s_1}_{p_1,q_1}|| \cdot ||g^j|A^{s_2}_{p_2,q_2}|| \cdot ||\phi||_{\frac{p_1p_2}{p_1p_2-p_2-p_1}} \\ &\longrightarrow 0. \end{aligned}$$

The last step in the above calculation holds because from the definition of the Besov spaces and the Triebel-Lizorkin spaces we immediately deduce that

$$||f^j|A^{s_1}_{p_1,q_1}|| \to 0 \text{ as } j \to \infty.$$

Proposition 121. *([43]) Assume that we have*

$$A^{s_1}_{p_1,q_1} \cdot A^{s_2}_{p_2,q_2} \hookrightarrow A^s_{p,q}$$

where $A^a_{b,c} = B^s_{b,c}$ *with* $a \in \mathbb{R}$, $0 < b, c \leq \infty$ *or*
$A^a_{b,c} = F^s_{b,c}$ *with* $a \in \mathbb{R}$, $0 < b < \infty$ *and* $0 < c \leq \infty$.
This implies

i) $s_1 + s_2 \geq 0$

ii) $s_1 - \frac{n}{p_1} \geq s - \frac{n}{p}$

iii) $s_1 + s \geq \frac{n}{p_1} + \frac{n}{p} - n$

iv) $A^{s_1}_{p_1,q_1} \hookrightarrow L^\infty$, *so* $s_1 - \frac{n}{p_1} \geq 0$.

For instance see [43], p. 160/161.
According to the above introduced decomposition into paraproducts, in what follows we will analyse

$$\pi_1(a_x, b_y)\ ,\ \pi_1(a_y, b_x)\ ,\pi_3(a_x, b_y)\ ,\pi_3(a_y, b_x) \text{ and } \sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s. \quad \text{(B.1)}$$

Note that in stead of $\pi_2(a_x, b_y)$ respectively $\pi_2(a_y, b_x)$ we will study $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$ because we want to take into account cancellation phenomena!

We start with the following observation concerning the supports (and think of $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$ as given in the example in remark 69):
We have

$$\mathcal{F}\Big(\sum_{i=0}^{l-2} a_x^i b_y^l\Big) = \mathcal{F}\Big(\mathcal{F}^{-1}\big(\sum_{i=0}^{l-2}\varphi_i \mathcal{F} a_x\big)\mathcal{F}^{-1}(\varphi_l \mathcal{F} b_y)\Big)$$

which implies that

$$supp\ \mathcal{F}\Big(\sum_{i=0}^{l-2} a_x^i b_y^l\Big) \subset supp\ \sum_{i=0}^{l-2}\varphi_i + supp\ \varphi_l$$

since we have convolutive sets (see for instance [8], p. 132), and after a short and straightforward computation we have

$$supp\ \mathcal{F}\Big(\sum_{i=0}^{l-2} a_x^i b_y^l\Big) \subset \big\{\xi \mid 2^{l-3} \leq |\xi| \leq 2^{l+3}\big\} \text{ for } l \geq 2. \quad \text{(B.2)}$$

Mutatis mutandis we have

$$supp\ \mathcal{F}\Big(\sum_{i=l-1}^{l+1} a_x^i b_y^l\Big) \subset \big\{\xi \mid |\xi| \leq 5\cdot 2^l\big\} \text{ for } l \geq 0. \quad \text{(B.3)}$$

So for each term in (B.1) we have one of the two estimates concerning the supports.
The following analysis of the terms appearing in (B.1) is splitted into two cases: In the first one we discuss terms of the form $\sum_{s=2}^{\infty}\sum_{t=0}^{s-2} a_x^t b_y^s$ (note the symmetry in π_1 and π_3 !) in the second one we treat $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$.

Analysis of terms of the form $\sum_{s=2}^{\infty}\sum_{t=0}^{s-2} a_x^t b_y^s$

The idea is to use the following results

Definition 122 ($\mathcal{B}^p(\mathbb{R}^n)$)**.** *Let $0 < p \leq \infty$. $\mathcal{B}^{\mathbf{p}}(\mathbb{R}^n)$ is the set of all sequences c with the following properties. $c = \{c_k\}_{k=0}^{\infty}$ is a sequence of elements $c_k \in \mathcal{S}'(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that*

$$supp\ \mathcal{F}c_0 \subset \{\xi \mid |\xi| \leq 2\}$$

and

$$supp\ \mathcal{F}c_k \subset \left\{\xi \mid 2^{k-1} \leq |\xi| \leq 2^{k+1}\right\}\ for\ k \geq 1$$

Proposition 123. *([43]) Let $s \in \mathbb{R}$ and suppose $c \in \mathcal{B}^p(\mathbb{R}^n)$.*

i) Let $0 < p < \infty$ and $0 < q \leq \infty$.
If $||2^{js}c_j|L^p(\mathbb{R}^n, l^q)|| = A < \infty$, then the series $\sum_{j=0}^{\infty} c_j$ converge in $\mathcal{S}'(\mathbb{R}^n)$ to a limit $f \in F^s_{p,q}(\mathbb{R}^n)$, and the estimate $||f|F^s_{p,q}(\mathbb{R}^n)|| \leq CA$ takes place with some constant C independent of c.

ii) Let $0 < p \leq \infty$ and $0 < q \leq \infty$.
If $||2^{js}c_j|l^q(L^p(\mathbb{R}^n))|| = A < \infty$, then the series $\sum_{j=0}^{\infty} c_j$ converge in $\mathcal{S}'(\mathbb{R}^n)$ to a limit $f \in B^s_{p,q}(\mathbb{R}^n)$, and the estimate $||f|B^s_{p,q}(\mathbb{R}^n)|| \leq CA$ takes place with some constant C independent of b.

Remark 124. If one goes carefully over the proof of proposition 123 (see e.g. [43], p. 59) one sees that the assertion still holds if we replace the assumption on the supports by the following one

$$supp\ \mathcal{F}b_0 \subset \{\xi \mid |\xi| \leq A2\}$$

and

$$supp\ \mathcal{F}b_k \subset \left\{\xi \mid B2^{k-1} \leq |\xi| \leq C2^{k+1}\right\} \text{ for } k \geq 1$$

where A, B and C are positive constants.

This remark enables us to work directly with the sequence $c_k = \sum_{t=0}^{k-2} a^t_x b^k_y$. Otherwise we had to modify it (see Appendix).
Note that once we have these classes $\mathcal{B}^p(\mathbb{R}^n)$ they are useful in view of the following alternative characterisation of Besov and Triebel-Lizorkin spaces (see e.g. [61]).

Theorem 125. *([61]) Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $0 < p \leq \infty$, then $B^s_{p,q}(\mathbb{R}^n)$ consist of all $f \in \mathcal{S}'$ such that $\exists \{f_k(x)\}_{k=0}^{\infty} \subset \mathcal{B}^p(\mathbb{R}^n)$ such that the following holds*

$$f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k\ in\ \mathcal{S}'(\mathbb{R}^n)$$

and

$$||2^{sk} f_k | l^q(L^p(\mathbb{R}^n)|| < \infty.$$

Moreover

$$\inf ||2^{sk} f_k | l^q(L^p(\mathbb{R}^n)||$$

is an equivalent (quasi-) norm in $B^s_{p,q}(\mathbb{B}^n)$*, where the infimum is taken over all admissible representations of* f*. For the Triebel-Lizorkin the same assertion holds as long as* $0 < p < \infty$*.*

Lemma 126. *([43])*

i) We have

$$|| \sup_{s \geq 0} | \sum_{i=0}^{s} f^i | \ |L^p(\mathbb{R}^n)|| \leq C||f|F^0_{p,2}(\mathbb{R}^n)||$$

for all $f \in F^0_{p,2}(\mathbb{R}^n)$ *if* $p < \infty$*. In the case* $p = \infty$ *we have*

$$|| \sup_{s \geq 0} | \sum_{i=0}^{s} f^i | \ |L^\infty(\mathbb{R}^n)|| \leq C||f|L^\infty(\mathbb{R}^n)||$$

for all $f \in L^\infty(\mathbb{R}^n)$*.*

ii) Let $s < 0$*. We have*

$$||2^{sl} | \sum_{i=0}^{l} f^i | \ |L^p(\mathbb{R}^n, l^q)|| \leq C||f|F^s_{p,q}(\mathbb{R}^n)|| = ||2^{js} f^j |L^p(\mathbb{R}^n, l^q)||$$

for all $f \in F^s_{p,q}(\mathbb{R}^n)$ *if* $p < \infty$*.*

iii) Let $s < 0$*. We have*

$$||2^{sl} | \sum_{i=0}^{l} f^i | \ |l^q(L^p(\mathbb{R}^n))|| \leq C||f|B^s_{p,q}(\mathbb{R}^n)|| = ||2^{js} f^j | l^q(L^p(\mathbb{R}^n))||$$

for all $f \in B^s_{p,q}(\mathbb{R}^n)$*.*

All these results can be found in [43].

In view of proposition 123 it remains to show that $||c_k \mid L^1(l^2)|| < \infty$:

First note that $\sup_{k \geq 0} | \sum_{t=0}^{k} a^t_x | \in L^p$. This holds since $a_x \in L^p = F^0_{p,2}$ which together with lemma 126 gives that $|| \sup_{k \geq 0} | \sum_{t=0}^{k} a^t_x | \ ||_p \leq C||a_x||_p < \infty$.

Apart from that we have that $||(\sum_{k=0}^{\infty}(b_y^k)^2)^{\frac{1}{2}}||_{p'} = ||b_y \mid F^0_{p',2}|| \leq C||b_y||_{p'}$. So we can estimate

$$\begin{aligned} ||c_k \mid L^1(l^2)|| &= ||(\sum_{k=0}^{\infty}(c_k)^2)^{\frac{1}{2}}||_1 \\ &= ||(\sum_{k=0}^{\infty}(\sum_{t=0}^{k-2} a_x^t b_y^k)^2)^{\frac{1}{2}}||_1 \\ &\leq ||\sup_{s\geq 0}|\sum_{t=0}^{s} a_x^t|(\sum_{k=0}^{\infty}(b_y^k)^2)^{\frac{1}{2}}||_1 \\ &\leq C||a_x||_p\, ||b_y||_{p'} \end{aligned}$$

where in the last step we used Hölder's inequality.

So summarised we have the following estimate for $\sum_{s=2}^{\infty}\sum_{t=0}^{s-2} a_x^t b_y^s$

$$||\sum_{s=2}^{\infty}\sum_{t=0}^{s-2} a_x^t b_y^s||_{\mathfrak{h}^1} = C||\sum_{s=2}^{\infty}\sum_{t=0}^{s-2} a_x^t b_y^s | F^0_{1,2}|| \leq C||c_k \mid L^1(l^2)|| \leq C||a_x||_p\, ||b_y||_{p'}.$$

$\pi_3(a_x, b_y)$ $\pi_1(a_y, b_x)$ and $\pi_3(a_y, b_x)$ can be estimated in exactly the same way.

Remark 127. In the derivation of the estimates for π_1 and π_3 we only used the fact that $a_x \in L^p = F^0_{p,2}$ and $b_y \in L^{p'} = F^0_{p',2}$. At this stage one may ask whether it is possible to start with the assumption that $a_x \in F^0_{1,2} = \mathfrak{h}^1$ and $b_y \in F^0_{\infty,2} = bmo$. But unfortunately the space of (pointwise) multipliers of $F^0_{1,2}$, denoted by $M(F^0_{1,2})$ is smaller than *bmo*. In fact we have

$$M(bmo) = M(h_1) = BMO_{log^{-1}t} \cap L^{\infty},$$

see for instance [43].

Analysis of $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$

Before we come to the actual estimate of the term $\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$ let us state and recall some results we will use.

First of all remember that we have $(B^0_{1,1})^* = B^0_{\infty,\infty}$ (for instance see [61] or [43]).

The other results are summarised in the following lemmas. We start with a lemma which relates differentiation with multiplying with appropriate weights:

Lemma 128. *Let* $f \in W^{1,p}(\mathbb{R}^n)$ *then we have*

$$||(\sum_{i=0}^{\infty} 2^{2ki}(f^i)^2)^{\frac{1}{2}}||_p \simeq ||\nabla^k f||_p + ||f||_p \text{ for } 1 < p < \infty.$$

Proof of lemma 128:
We have in fact

$$\begin{aligned} ||(\sum_{i=0}^{\infty} 2^{2ki}(f^i)^2)^{\frac{1}{2}}||_p &= ||f|F^k_{p,2}|| \\ &\simeq ||f|\dot{F}^k_{p,2}|| + ||f|| \\ &\quad \text{by proposition 85} \\ &\simeq ||f|\dot{W}^{k,p}|| + ||f||_p \\ &\quad \text{by theorem 84} \\ &= ||\nabla^k f||_p + ||f||_p. \end{aligned}$$

□

Next, we present the crucial estimate:

Lemma 129. *Let* $h \in B^0_{\infty,\infty}(\mathbb{R}^n)$ *and let* $\{f_k(x)\}_{k=0}^{\infty} \subset L^\infty(\mathbb{R}^n)$ *be a representation of* h, *i.e.* $h = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k$ *in* $\mathcal{S}'$, *such that*

$$||f_k|L^\infty(\mathbb{R}^n, l^\infty)|| \leq 2||h|B^0_{\infty,\infty}||.$$

Moreover, as usual let $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$. *Then*

$$\Big|\Big|\sum_{k=0}^{s+3} \frac{\partial}{\partial x}\check{\varphi}_k * f_k\Big|\Big|_\infty = \Big|\Big|\frac{\partial}{\partial x}\Big(\sum_{k=0}^{s+3} \check{\varphi}_k * f_k\Big)\Big|\Big|_\infty \leq C2^s||h|B^0_{\infty,\infty}||.$$

Proof of 129:
First of all, remember that

$$\check{\varphi}_k * f_k = \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k)$$

Now, the assertion is an immediate consequence of the following computations.

First of all, note that due to the fact that the sum is finite we have

$$\begin{aligned}
\Big|\Big|\frac{\partial}{\partial x}\Big(\sum_{k=0}^{s+3}\check{\varphi}_k * f_k\Big)\Big|\Big|_\infty &= \Big|\Big|\sum_{k=0}^{s+3}\frac{\partial}{\partial x}\check{\varphi}_k * f_k\Big|\Big|_\infty \\
&\leq \sum_{k=0}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * f_k\Big|\Big|_\infty \\
&\leq \sum_{k=0}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k\Big|\Big|_1 ||f_k||_\infty \\
&\leq 2||h|B^0_{\infty,\infty}||\sum_{k=0}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k\Big|\Big|_1 \\
&\quad \text{because } ||f_k||_\infty \leq ||f_k|\ L^\infty(\mathbb{R}^n, l^\infty)|| \leq 2||h|B^0_{\infty,\infty}||.
\end{aligned}$$

For this last quantity we have

$$\begin{aligned}
2||h|B^0_{\infty,\infty}||\sum_{k=0}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k\Big|\Big|_1 &= 2||h|B^0_{\infty,\infty}||\sum_{k=0}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\Big(\check{\varphi}_1(2^k\cdot)\Big)2^{nk}\Big|\Big|_1 \\
&\quad \text{remember that } \varphi_k(\cdot) = \varphi_1(2^{-k}\cdot) \\
&= 2||h|B^0_{\infty,\infty}||\sum_{k=0}^{s+3}\Big|\Big|2^{nk}2^k\Big(\frac{\partial}{\partial x}\check{\varphi}_1\Big)(2^k\cdot)\Big|\Big|_1 \\
&= 2||h|B^0_{\infty,\infty}||\sum_{k=0}^{s+3}2^k\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_1\Big|\Big|_1 \\
&= 2||h|B^0_{\infty,\infty}||\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_1\Big|\Big|_1\sum_{k=0}^{s+3}2^k \\
&= 2||h|B^0_{\infty,\infty}||\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_1\Big|\Big|_1(2\cdot 2^{s+3}-1) \\
&\leq C2^s||h|B^0_{\infty,\infty}||
\end{aligned}$$

□

Remark 130. Obviously the assertion of the above lemma remains true if we look at derivatives in other directions.

Last, but not least we will need the next two technical lemmas.
The first one enables us to interchange the order of summation and integration.

Lemma 131. *Let a and b belong to $C_0^\infty(\mathbb{R}^n)$, $t = s + j$ where $j \in \{-1, 0, 1\}$ and $h \in B^0_{\infty,\infty}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \Big(\sum_{s=0}^{\infty} a_x^t b_y^s - a_y^t b_x^s \Big) h = \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} (a_x^t b_y^s - a_y^t b_x^s) h.$$

Proof of lemma 131:
First of all, define for $z \in \mathbb{R}^n$

$$f_n(z) := \sum_{s=0}^{n} (a_x^t(z) b_y^s(z) - a_y^t(z) b_x^s(z)) h(z).$$

Our goal is to apply dominated convergence in order to prove the claim. Obviously we have

$$|f_n(z)| \le \sum_{s=0}^{\infty} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)| \big) |h(z)| =: g(z).$$

Next, we will show that $g \in L^1(\mathbb{R}^n)$.
Remember that we may choose a representation of h, i.e. a sequence in L^∞, such that $h = \sum_{k=0}^{\infty} \mathcal{F}^{-1} \varphi_k \mathcal{F} f_k$ in $\mathcal{S}'$ and

$$||f_k|L^\infty(\mathbb{R}^n, l^\infty)|| \le 2||h|B^0_{\infty,\infty}||.$$

Moreover, keep in mind what ideas we used in order to prove lemma 129. Some ideas will be used here with some small changes! Then we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |g(z)| &= \int_{\mathbb{R}^n} \sum_{s=0}^{\infty} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)| \big) |h(z)| \\
&= \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)| \big) |h(z)| \\
&\quad \text{by monotone convergence} \\
&\quad \text{more generally we have } L^p(\mathbb{R}^n, l^p) = l^p(L^p(\mathbb{R}^n)) \ \forall 1 \le p \le \infty \\
&= \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)| \big) \Big| \sum_{k=0}^{s+3} \check{\varphi}_k * f_k(z) \Big| \\
&\quad \text{by lemma 132 below} \\
&\le \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)| \big) \Big|\Big| \sum_{k=0}^{s+3} \check{\varphi}_k * f_k(z) \Big|\Big|_\infty \\
&\le \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)| \big) \sum_{k=0}^{s+3} ||\check{\varphi}_k||_1 ||f_k||_\infty,
\end{aligned}$$

and finally

$$\int_{\mathbb{R}^n} |g(z)| \leq C \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)|\big) ||h|B^0_{\infty,\infty}|| \sum_{k=0}^{s+3} ||\check{\varphi}_k||_1.$$

Now, we continue our estimates as follows

$$\begin{aligned}
\int_{\mathbb{R}^n} |g(z)| &\leq C \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)|\big) ||h|B^0_{\infty,\infty}|| 2^{s+3} \\
&\leq C 2^s ||h|B^0_{\infty,\infty}|| \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \big(|a_x^t(z)||b_y^s(z)| + |a_y^t(z)||b_x^s(z)|\big) \\
&\leq C ||h|B^0_{\infty,\infty}|| \int_{\mathbb{R}^n} \sum_{s=0}^{\infty} |a_x^t(z)| 2^s |b_y^s(z)| + \sum_{s=0}^{\infty} |a_y^t(z)| 2^s |b_x^s(z)| \\
&\quad \text{by monotone convergence} \\
&\leq C ||h|B^0_{\infty,\infty}|| \int_{\mathbb{R}^n} \Big(\sum_{s=0}^{\infty} |a_x^t(z)|^2\Big)^{1/2} \Big(\sum_{s=0}^{\infty} 2^{2s} |b_y^s(z)|^2\Big)^{1/2} \\
&\quad + C ||h|B^0_{\infty,\infty}|| \int_{\mathbb{R}^n} \Big(\sum_{s=0}^{\infty} |a_y^t(z)|^2\Big)^{1/2} \Big(\sum_{s=0}^{\infty} 2^{2s} |b_x^s(z)|^2\Big)^{1/2} \\
&\quad \text{by Hölder's inequality applied for the series} \\
&\leq C ||h|B^0_{\infty,\infty}|| \big(||a_x||_2 (||b_y||_2 + ||\nabla b_y||_2) + ||a_y||_2 (||b_x||_2 + ||\nabla b_x||_2 \big) \\
&\quad \text{by lemma 128} \\
&\leq C ||h|B^0_{\infty,\infty}|| \; ||\nabla a||_2 ||b|W^{2,2}|| \\
&< \infty \\
&\quad \text{since we assumed that } a, b \in C_0^\infty(\mathbb{R}^n).
\end{aligned}$$

In a next step we want to show that

$$\sum_{s=0}^{n} (a_x^t b_y^s - a_y^t b_x^s) \longrightarrow \sum_{s=0}^{\infty} (a_x^t b_y^s - a_y^t b_x^s) \text{ in } \mathcal{S}.$$

It is enough to show that $\forall i \in \mathbb{N}$

$$\mathcal{N}_i \Big(\sum_{s=n}^{m} a_x^t b_y^s - a_y^t b_x^s \Big) \longrightarrow 0 \text{ as } n, m \to \infty.$$

Note that we have

$$\begin{aligned}\mathcal{N}_i\Big(\sum_{s=n}^{m} a_x^t b_y^s - a_y^t b_x^s\Big) &= \sum_{|\alpha|\ ,\ |\beta|\le i} \Big\|x^\alpha \partial^\beta \Big(\sum_{s=n}^{m} a_x^t b_y^s - a_y^t b_x^s\Big)\Big\|_\infty \\ &\le (i+1)^2\Big(\sup_{n-1\le t\le m+1} \mathcal{N}_i(a_x^t)\mathcal{N}_i(\sum_n^m b_y^s) \\ &\quad + \sup_{n-1\le t\le m+1} \mathcal{N}_i(a_y^t)\mathcal{N}_i(\sum_n^m b_x^s)\Big) \\ &\to 0 \text{ as } n, m \to \infty.\end{aligned}$$

In the second last step $i+1$ indicates the number of possible choices of $|\alpha|$ and $|\beta|$.
In the last step we used the following reasoning:
We know that (recall that $\sum_{j=0}^{\infty} f^j = f$ in $\mathcal{S}'(\mathbb{R}^n)$)

$$\sum_{s=0}^{n} a_x^s \longrightarrow a_x \text{ in } \mathcal{S} \text{ as } n, m \to \infty.$$

But this implies that

$$\mathcal{N}_i\Big(\sum_n^m a_x^s\Big) \longrightarrow 0 \text{ in } \mathcal{S} \text{ as } n, m \to \infty\ \forall\ i$$

and also

$$\mathcal{N}_i(a_x^t) \longrightarrow 0 \text{ in } \mathcal{S} \text{ as } t \to \infty\ \forall\ i.$$

Similar conclusions hold also for b_y, a_y and b_x.
This finally enables us to apply dominated convergence which completes the proof.

□

The second and last lemma tells us that in order to analyse the whole sum in $f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k$ we just have to study an appropriate part of it.

Lemma 132. *Let a and b belong to $C_0^\infty(\mathbb{R}^n)$, $t = s + j$ where $j \in \{-1, 0, 1\}$ and $h \in B^0_{\infty,\infty}(\mathbb{R}^n)$ with representation $f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k$ as above. Then*

$$\begin{aligned}&\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s)h - \frac{\partial}{\partial y}(a^t b_x^s)h \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}\Big(a^t b_y^s\Big)\Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k)\Big) - \frac{\partial}{\partial y}\Big(a^t b_x^s\Big)\Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k)\Big).\end{aligned}$$

Proof of lemma 132:
First of all, note that $h \in \mathcal{S}'$ and $a^t b_y^s$ and $a^t b_x^s$ belong to $\mathcal{S}$ independently of the choices of s and t.
We now calculate

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s)h - \frac{\partial}{\partial y}(a^t b_x^s)h &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s)h - \int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b_x^s)h \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \\
&\quad - \int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b_x^s) \sum_{k=0}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \Big[\sum_{k=0}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) + \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big] \\
&\quad - \int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b_x^s) \Big[\sum_{k=0}^{s+4} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) + \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big].
\end{aligned}$$

These calculations show that we have to prove that

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) = 0$$

and

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial y}(a^t b_x^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) = 0.$$

In what follows, we will only discuss the first integral because the second one can be analysed in exactly the same way.
So from now on we look at

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k).$$

Here we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \sum_{k=s+4}^{\infty} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \mathcal{F}^{-1}\Big(\sum_{k=s+4}^{\infty} \varphi_k \mathcal{F} f_k \Big) \\
&\quad \text{sine the sum is locally finite} \\
&= \int_{\mathbb{R}^n} \frac{\partial}{\partial x}(a^t b_y^s) \mathcal{F}\mathcal{F}^{-1}\mathcal{F}^{-1}\Big(\sum_{k=s+4}^{\infty} \varphi_k \mathcal{F} f_k \Big) \\
&= (2\pi)^n \int_{\mathbb{R}^n} \mathcal{F}\Big(\frac{\partial}{\partial x}(a^t b_y^s)\Big) \sum_{k=s+4}^{\infty} \varphi_k(-\,\cdot)\mathcal{F} f_k(-\,\cdot) \\
&\quad \text{because } \frac{\partial}{\partial x}(a^t b_y^s) \in \mathcal{S} \text{ and } \sum_{k=s+4}^{\infty} \varphi_k \mathcal{F} f_k) \in \mathcal{S}' \\
&= 0.
\end{aligned}$$

In the last step of the above calculations we used the fact that

$$supp\ \mathcal{F}(\frac{\partial}{\partial x}(a^t b_y^s)) \subset \{\xi |\ |\xi| \leq 5 \cdot 2^s\}$$

and

$$supp \sum_{k=s+4}^{\infty} \varphi_k(-\,\cdot) \subset \big\{\xi |\ 2^{s+3} \leq |\xi|\big\}$$

imply that

$$supp\ \mathcal{F}\Big(\frac{\partial}{\partial x}(a^t b_y^s)\Big) \cap supp \sum_{k=s+4}^{\infty} \varphi_k = \emptyset.$$

This completes the proof.

□

Now we can start with the estimate of $\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$. Our goal is to show that $\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$ belongs to $B_{1,1}^0$. Making use of the duality between $B_{1,1}^0$ and $B_{\infty,\infty}^0$ it suffices to show that for all $h \in B_{\infty,\infty}^0$ the following inequality holds

$$\int_{\mathbb{R}^n} \Big(\sum_{s=0}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s\Big) h < \infty.$$

First of all let us fix $t = s + j$ where $j \in \{-1, 0, 1\}$. Moreover we will assume that a and b belong to $C_0^{\infty}(\mathbb{R}^n)$ by density.
In a first step we will estimate

$$\int_{\mathbb{R}^n} (\sum_{s=0}^{\infty} a_x^t b_y^s - a_y^t b_x^s) h.$$

In this case we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \Big(\sum_{s=0}^{\infty} a_x^t b_y^s - a_y^t b_x^s \Big) h &= \int_{\mathbb{R}^n} \sum_{s=0}^{\infty} a_x^t b_y^s h - a_y^t b_x^s h \\
&= \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} a_x^t b_y^s h - a_y^t b_x^s h \\
& \quad \text{because of lemma 131} \\
&= \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \frac{\partial}{\partial x} \big(a^t b_y^s \big) h - \frac{\partial}{\partial y} \big(a^t b_x^s \big) h \\
&= \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial x} \big(a^t b_y^s \big) \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1} (\varphi_k \mathcal{F} f_k) \Big) \\
& \quad - \frac{\partial}{\partial y} \big(a^t b_x^s \big) \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1} (\varphi_k \mathcal{F} f_k) \Big) \Big] \\
& \quad \text{because of lemma 132} \\
&= \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \Big[- a^t b_y^s \frac{\partial}{\partial x} \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1} (\varphi_k \mathcal{F} f_k) \Big) \\
& \quad + a^t b_x^s \frac{\partial}{\partial y} \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1} (\varphi_k \mathcal{F} f_k) \Big) \Big] \\
& \quad \text{by a simple integration by parts} \\
&\leq \sum_{s=0}^{\infty} \int_{\mathbb{R}^n} \Big[|a^t||b_y^s| \, \Big|\Big| \frac{\partial}{\partial x} \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1} (\varphi_k \mathcal{F} f_k) \Big) \Big|\Big|_\infty \\
& \quad + |a^t||b_x^s| \, \Big|\Big| \frac{\partial}{\partial y} \Big(\sum_{k=0}^{s+3} \mathcal{F}^{-1} (\varphi_k \mathcal{F} f_k) \Big) \Big|\Big|_\infty \Big] \\
&\leq \sum_{s=0}^{\infty} \int_{\mathbb{R}} |a^t||b_y^s| 2^s C ||h| B^0_{\infty,\infty}|| + |a^t||b_x^s| 2C ||h| B^0_{\infty,\infty}|| \\
& \quad \text{due to lemma 129} \\
&\leq C||h|B^0_{\infty,\infty}|| \Big[\int_{\mathbb{R}^n} \sum_{s=0}^{\infty} 2^s |a^{s+j}||b_y^s| + \sum_{s=0}^{\infty} 2^s |a^{s+j}||b_x^s| \Big] \\
& \quad \text{where we used monotone convergence} \\
&\leq C||h|B^0_{\infty,\infty}|| \Big[\int_{\mathbb{R}^n} \Big(\sum_{s=0}^{\infty} 2^{2s} |a^{s+j}|^2 \Big)^{1/2} \Big(\sum_{s=0}^{\infty} |b_y^s|^2 \Big)^{1/2} \\
& \quad + \Big(\sum_{s=0}^{\infty} 2^{2s} |a^{s+j}|^2 \Big)^{1/2} \Big(\sum_{s=0}^{\infty} |b_x^s|^2 \Big)^{1/2} \Big] \\
& \quad \text{by Cauchy Schwarz inequality}
\end{aligned}
$$

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\sum_{s=0}^{\infty} a_x^t b_y^s - a_y^t b_x^s\Big)h &\leq C||h|B^0_{\infty,\infty}||\big[(||a||_p+||\nabla a||_p)||b_y||_{p'} \\ &\quad +(||a||_p+||\nabla a||_p)||b_x||_{p'}\big] \\ &\quad \text{by lemma 128} \\ &\leq C||h|B^0_{\infty,\infty}||\ ||a|W^{1,p}||\ ||b|W^{1,p'}|| \\ &< \infty.\end{aligned}$$

The last step holds since we assumed that $a \in W^{1,p}$ and $b \in W^{1,p'}$.

Now, since the above estimate is independent of the choice of j we immediately conclude that

$$\int_{\mathbb{R}^n}\Big(\sum_{s=0}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s\Big)h \leq C||h|B^0_{\infty,\infty}||\ ||a|W^{1,p}||\ ||b|W^{1,p'}||.$$

This completes the proof of proposition 117.

□

Proof of theorem 116:
The proof of theorem 116 is quite similar to the one we gave for proposition 117, but nevertheless let us explain the differences. We will not restate what trivially is the same in both settings!
First of all, we have to adapt our notion of a product of two tempered distributions to the homogeneous setting, this is done by the following modification:
What concerns notation, we will use the following:
Assume that $\varphi = \{\varphi_j(x)\}_{j=-\infty}^{\infty} \in \dot{\Phi}(\mathbb{R}^n)$ then set $\mathbf{f^j}(\mathbf{x}) = \mathcal{F}(\varphi_j \mathcal{F} f)(x)$. Now we can proceed to the definition mentioned before:

Definition 133 (Product in homogeneous function spaces)**.** *Let f, g be two tempered distributions lying in some homogeneous function spaces. The* ***(homogeneous) product*** *of f and g is given by*

$$f \cdot g = \lim_{j\to\infty}\Big(\sum_{i=-\infty}^{j} f^i\Big)\Big(\sum_{l=-\infty}^{j} g^l\Big)$$

whenever the right hand side of the above equation exists in $\mathcal{S}'(\mathbb{R}^n)$.

A first natural question which arises from this definition is: Assume that $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$. Does the (homogeneous) product defined above coincide with the usual point-wise product of f and g? The answer is yes, which we summarise in the following lemma. Recall that for $1 < p < \infty$ we have $L^p = \dot{F}^0_{p,2}$.

Lemma 134. *Let $1 < p < \infty$ and assume that $f \in L^p$ and $g \in L^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\lim_{j\to\infty} \left(\sum_{i=-\infty}^{j} f^i\right)\left(\sum_{l=-\infty}^{j} g^l\right)$ exists in $\mathcal{D}'(\Omega)$ for all open bounded sets Ω and the restriction of $\lim_{j\to\infty} \left(\sum_{i=-\infty}^{j} f^i\right)\left(\sum_{l=-\infty}^{j} g^l\right)$ to Ω is a regular distribution which coincides with the usual point-wise multiplication there.*

The proof of this lemma is the same - of course with the necessary changes - as the one for the corresponding non-homogeneous assertion which can be found e.g. in [43].
Next, we modify our paraproducts such that they fit into the homogeneous framework. We will use the following ones:

$$\begin{aligned}
\tilde{\pi}_1(f,g) &:= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{k-2} f^l g^k \\
\tilde{\pi}_2(f,g) &:= \sum_{k=-\infty}^{\infty} \sum_{l=k-1}^{k+1} f^l g^k \\
\tilde{\pi}_3(f,g) &:= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{l-2} f^l g^k
\end{aligned}$$

As in the case of proposition 117 we will study

$$\tilde{\pi}_1(a_x, b_y)\ ,\ \tilde{\pi}_1(a_y, b_x)\ ,\tilde{\pi}_3(a_x, b_y)\ ,\tilde{\pi}_3(a_y, b_x) \text{ and } \sum_{s=-\infty}^{\infty} \sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s.$$

Analysis of terms of the form $\sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{s-2} a_x^t b_y^s$

A careful look at the proof of proposition 123 shows that the corresponding assertion holds also in the homogeneous framework.
What concerns lemma 126 it translates into the homogeneous setting as follows:

Lemma 135. *We have*

$$||\sup_{s\in\mathbb{Z}}|\sum_{i=-\infty}^{s} f^i| \ |L^p(\mathbb{R}^n)|| \leq C||f|\dot{F}^0_{p,2}(\mathbb{R}^n)||$$

for all $f \in \dot{F}^0_{p,2}(\mathbb{R}^n)$ *if* $1 < p < \infty$.

Proof of lemma 135:
The claim follows immediately from the facts that

- $\mathfrak{H}^p = L^p = \dot{F}^0_{p,2}$ for $1 < p < \infty$, see theorem 84.
- $\sum_{j=-\infty}^{s} \varphi_j(x) = \psi(2^{-s}x)$ for all $s \in \mathbb{Z}$, recall also the example from 82.

Apart from that, the proof is the same as in the non-homogeneous case (cf. e.g. [43]).

□

As before, it is enough to show that $||\sum_{t=-\infty}^{k-2} a_x^t b_y^k \mid L^1(l^2)|| < \infty$:

Now, we put together all the information we have:
First note that $\sup_{k\in\mathbb{Z}}|\sum_{t=-\infty}^{k} a_x^t| \in L^p$. This holds since $a_x \in L^p = \dot{F}^0_{p,2}$ which together with lemma 135 gives that $||\sup_{k\in\mathbb{Z}}|\sum_{t=-\infty}^{k} a_x^t| \ ||_p \leq C||a_x||_p < \infty$.
Apart from that we have that $||(\sum_{k=-\infty}^{\infty}(b_y^k)^2)^{\frac{1}{2}}||_{p'} = ||b_y \mid \dot{F}^0_{p',2}|| \leq C||b_y||_{p'}$.
So we estimate

$$\begin{aligned} ||\sum_{t=-\infty}^{k-2} a_x^t b_y^k \mid L^1(l^2)|| &= ||(\sum_{k=-\infty}^{\infty}(\sum_{t=-\infty}^{k-2} a_x^t b_y^k)^2)^{\frac{1}{2}}||_1 \\ &\leq ||\sup_{s\in\mathbb{Z}}|\sum_{t=-\infty}^{s} a_x^t|(\sum_{k=-\infty}^{\infty}(b_y^k)^2)^{\frac{1}{2}}||_1 \\ &\leq C||a_x||_p \, ||b_y||_{p'} \end{aligned}$$

where in the last step we used Hölder's inequality.

So summarised we have the following estimate for $\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{s-2} a_x^t b_y^s$

$$\begin{aligned} ||\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{s-2} a_x^t b_y^s||_{\mathfrak{H}^1} &= C||\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{s-2} a_x^t b_y^s|\dot{F}^0_{1,2}|| \\ &\leq C||\sum_{t=-\infty}^{k-2} a_x^t b_y^k \mid L^1(l^2)|| \leq C||a_x||_p \, ||b_y||_{p'}. \end{aligned}$$

$\tilde{\pi}_3(a_x, b_y)$ $\tilde{\pi}_1(a_y, b_x)$ and $\tilde{\pi}_3(a_y, b_x)$ can be estimated in exactly the same way.

Analysis of $\sum_{s=-\infty}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$

Before we come to the actual estimate of the term $\sum_{s=-\infty}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$ let us state and recall some results we will use.

First of all remember that we have $(\dot{B}^0_{1,1})^* = \dot{B}^0_{\infty,\infty}$ (for instance see [22]).

The crucial estimate in the homogeneous case is stated as follows.

Lemma 136. *Let* $h \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$ *and let* $\{f_k(x)\}_{k=-\infty}^{\infty} \subset L^\infty(\mathbb{R}^n)$ *be a representation of* h, *i.e.* $h = \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k$ *in* $\mathcal{Z}'$, *such that*

$$||f_k|L^\infty(\mathbb{R}^n, l^\infty)|| \le 2||h|\dot{B}^0_{\infty,\infty}||.$$

Moreover, as usual let $\varphi = \{\varphi_j(x)\}_{j=-\infty}^{\infty} \in \dot{\Phi}(\mathbb{R}^n)$. *Then*

$$\Big|\Big|\sum_{k=-\infty}^{s+3}\frac{\partial}{\partial x}\check{\varphi}_k * f_k\Big|\Big|_\infty = \Big|\Big|\frac{\partial}{\partial x}\Big(\sum_{k=-\infty}^{s+3}\check{\varphi}_k * f_k\Big)\Big|\Big|_\infty \le C2^s||h|\dot{B}^0_{\infty,\infty}||.$$

Proof of lemma 136:
First of all, remember that

$$\check{\varphi}_k * f_k = \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k)$$

Now, the assertion is an immediate consequence of the following computations.
First of all, we have the trivial estimates

$$\begin{aligned}\Big|\Big|\frac{\partial}{\partial x}\Big(\sum_{k=-\infty}^{s+3}\check{\varphi}_k * f_k\Big)\Big|\Big|_\infty &= \Big|\Big|\sum_{k=-\infty}^{s+3}\frac{\partial}{\partial x}\check{\varphi}_k * f_k\Big|\Big|_\infty \\ &\le \sum_{k=-\infty}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k * f_k\Big|\Big|_\infty \\ &\le \sum_{k=-\infty}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k\Big|\Big|_1 ||f_k||_\infty \\ &\le 2||h|\dot{B}^0_{\infty,\infty}||\sum_{k=-\infty}^{s+3}\Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k\Big|\Big|_1\end{aligned}$$

because $||f_k||_\infty \le ||f_k|\ L^\infty(\mathbb{R}^n, l^\infty)|| \le 2||h|\dot{B}^0_{\infty,\infty}||$.

For this last quantity we have

$$
\begin{aligned}
2||h|\dot{B}^0_{\infty,\infty}|| \sum_{k=-\infty}^{s+3} \Big|\Big|\frac{\partial}{\partial x}\check{\varphi}_k\Big|\Big|_1 &= 2||h|\dot{B}^0_{\infty,\infty}|| \sum_{k=-\infty}^{s+3} \Big|\Big|\frac{\partial}{\partial x}\Big(\check{\phi}(2^k\cdot)\Big)2^{nk}\Big|\Big|_1 \\
&\quad \text{remember that } \varphi_k(\cdot) = \phi(2^{-k}\cdot) \\
&= 2||h|\dot{B}^0_{\infty,\infty}|| \sum_{k=-\infty}^{s+3} \Big|\Big|2^{nk}2^k\Big(\frac{\partial}{\partial x}\check{\phi}\Big)(2^k\cdot)\Big|\Big|_1 \\
&= 2||h|\dot{B}^0_{\infty,\infty}|| \sum_{k=-\infty}^{s+3} 2^k\Big|\Big|\frac{\partial}{\partial x}\check{\phi}\Big|\Big|_1 \\
&= 2||h|\dot{B}^0_{\infty,\infty}||\Big|\Big|\frac{\partial}{\partial x}\check{\phi}\Big|\Big|_1 \sum_{k=-\infty}^{s+3} 2^k \\
&\leq C2^s||h|\dot{B}^0_{\infty,\infty}||.
\end{aligned}
$$

The last step holds since if $s+3 \leq 0$ then

$$\sum_{k=-\infty}^{s+3} 2^k \leq \sum_{k=-\infty}^{0} 2^k = 2$$

and if $s+3 \geq 1$ then

$$
\begin{aligned}
\sum_{k=-\infty}^{s+3} 2^k &= \sum_{k=-\infty}^{0} 2^k + \sum_{k=1}^{s+3} 2^k \\
&\leq 2 + C2^s \\
&\quad \text{as in the non-homogeneous case} \\
&\leq C2^s.
\end{aligned}
$$

□

Lemmas 131 and 132 can easily restated and reproved in the homogeneous setting with the same ideas, so we do not rewrite what we did earlier.
Now we can start with the estimate of $\sum_{s=-\infty}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$. Our goal is to show that $\sum_{s=-\infty}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s$ belongs to $\dot{B}^0_{1,1}$. Making use of the duality between $\dot{B}^0_{1,1}$ and $\dot{B}^0_{\infty,\infty}$ it suffices to show that for all $h \in \dot{B}^0_{\infty,\infty}$ the following inequality holds

$$\int_{\mathbb{R}^n}\Big(\sum_{s=-\infty}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s\Big)h < \infty.$$

First of all let us fix $t = s + j$ where $j \in \{-1, 0, 1\}$. Moreover we will assume that a and b belong to $C_0^\infty(\mathbb{R}^n)$ by density.
In a first step we will estimate

$$\int_{\mathbb{R}^n} \big(\sum_{s=-\infty}^{\infty} a_x^t b_y^s - a_y^t b_x^s \big) h.$$

In this case we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Big(\sum_{s=-\infty}^{\infty} a_x^t b_y^s - a_y^t b_x^s \Big) h &= \int_{\mathbb{R}^n} \sum_{s=-\infty}^{\infty} a_x^t b_y^s h - a_y^t b_x^s h \\
&= \sum_{s=-\infty}^{\infty} \int_{\mathbb{R}^n} a_x^t b_y^s h - a_y^t b_x^s h \\
&= \sum_{s=-\infty}^{\infty} \int_{\mathbb{R}^n} \frac{\partial}{\partial x} \big(a^t b_y^s\big) h - \frac{\partial}{\partial y} \big(a^t b_x^s\big) h \\
&= \sum_{s=-\infty}^{\infty} \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial x} \big(a^t b_y^s\big) \Big(\sum_{k=-\infty}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \\
&\quad - \frac{\partial}{\partial y} \big(a^t b_x^s\big) \Big(\sum_{k=-\infty}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \Big] \\
&= \sum_{s=-\infty}^{\infty} \int_{\mathbb{R}^n} \Big[- a^t b_y^s \frac{\partial}{\partial x} \Big(\sum_{k=-\infty}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \\
&\quad + a^t b_x^s \frac{\partial}{\partial y} \Big(\sum_{k=-\infty}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \Big] \\
&\quad \text{by a simple integration by parts} \\
&\leq \sum_{s=-\infty}^{\infty} \int_{\mathbb{R}^n} \Big[|a^t| |b_y^s| \, \Big|\Big| \frac{\partial}{\partial x} \Big(\sum_{k=-\infty}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \Big|\Big|_\infty \\
&\quad + |a^t| |b_x^s| \, \Big|\Big| \frac{\partial}{\partial y} \Big(\sum_{k=-\infty}^{s+3} \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_k) \Big) \Big|\Big|_\infty \Big] \\
&\leq \sum_{s=-\infty}^{\infty} \int_{\mathbb{R}} |a^t| |b_y^s| 2^s C ||h| \dot{B}^0_{\infty,\infty}|| + |a^t| |b_x^s| 2C ||h| \dot{B}^0_{\infty,\infty}|| \\
&\quad \text{due to lemma 136}
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^n}\Big(\sum_{s=-\infty}^{\infty} a_x^t b_y^s - a_y^t b_x^s\Big) &\leq C||h|\dot{B}^0_{\infty,\infty}||\Big[\int_{\mathbb{R}^n}\sum_{s=-\infty}^{\infty} 2^s|a^{s+j}||b_y^s| + \sum_{s=-\infty}^{\infty} 2^s|a^{s+j}||b_x^s|\Big] \\
&\quad \text{where we used monotone convergence} \\
&\leq C||h|\dot{B}^0_{\infty,\infty}||\Big[\int_{\mathbb{R}^n}\Big(\sum_{s=-\infty}^{\infty} 2^{2s}|a^{s+j}|^2\Big)^{1/2}\Big(\sum_{s=-\infty}^{\infty}|b_y^s|^2\Big)^{1/2} \\
&\quad +\Big(\sum_{s=-\infty}^{\infty} 2^{2s}|a^{s+j}|^2\Big)^{1/2}\Big(\sum_{s=-\infty}^{\infty}|b_x^s|^2\Big)^{1/2}\Big] \\
&\quad \text{by Cauchy Schwarz inequality} \\
&\leq C||h|\dot{B}^0_{\infty,\infty}||\big(||\nabla a||_p||b_y||_{p'} + ||\nabla a||_p||b_x||_{p'}\big) \\
&\quad \text{by theorem 84} \\
&\leq C||h|\dot{B}^0_{\infty,\infty}||\ ||a|\dot{W}^{1,p}||\ ||b|\dot{W}^{1,p'}|| \\
&< \infty.
\end{aligned}$$

The last step holds since we assumed that $a \in \dot{W}^{1,p}$ and $b \in \dot{W}^{1,p'}$.

Now, since the above estimate is independent of the choice of j we immediately conclude that

$$\int_{\mathbb{R}^n}\Big(\sum_{s=-\infty}^{\infty}\sum_{t=s-1}^{s+1} a_x^t b_y^s - a_y^t b_x^s\Big)h \leq C||h|\dot{B}^0_{\infty,\infty}||\ ||a|\dot{W}^{1,p}||\ ||b|\dot{W}^{1,p'}||.$$

This completes the proof of theorem 116.

□

B.2 Application of the commutator estimate due to Coifman, Rochberg and Weiss ([18])

Again we assume that $a, b \in W^{1,2}(\mathbb{R}^n)$. Our goal is to show that

$$a_x b_y - a_y b_x \in \mathfrak{h}^1.$$

The idea here is to use - apart from the famous duality between *bmo* and $\mathfrak{h}^1$ which we are already familiar with - the commutator estimate due to Coifman, Rochberg and Weiss (CRW commutator estimate).
Let us start with the following definition

Definition 137 (Pseudodifferential operator)**.** *A pseudodifferential operator* **P** *is an operator of the following form*

$$Pf(x) = p(x,D)f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} p(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi = \mathcal{F}^{-1}(\widehat{f}p(x,\xi))$$

where

$$p(x,\xi) = \sum_{|\alpha|\leq k} a_\alpha(x)\xi^\alpha.$$

We say that P belongs to **OPS** *if the symbol $p(x,\xi)$ belongs to the class of symbols S. In particular we have the following classes of symbols:*
Assume that $\rho, \delta \in [0,1]$ and $m \in \mathbb{R}$. Then $\mathbf{S^m_{\rho,\delta}}$ consists of all C^∞-functions satisfying

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all α, β where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$.

For a detailed discussion of such operators see [57], vol. II, or [59].

Lemma 138. *([57]) If $\delta < 1$, then*

$$p(x,\xi) : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

A proof of this lemma can be found for example in [57], vol. II, p. 3.

Theorem 139 (CRW commutator estimate,[18])**.** *Given $P \in OPS^0_{1,0}$, $1 < p < \infty$, we have*

$$||f(Pu) - P(fu)||_p \leq C_p||f||_{bmo}||u||_p.$$

A proof of this theorem can be found in [18] or [6].

In a first step we map

$$a \mapsto \widetilde{a} = \Lambda a = \mathcal{F}^{-1}((1+|\xi|^2)^{\frac{1}{2}}\widehat{a})$$

and similarly for b.
In view of theorem 139 we define appropriate pseudodifferential operators in a second step:

$$P = \begin{pmatrix} \partial_x\Lambda^{-1} \\ \partial_y\Lambda^{-1} \end{pmatrix}$$

and

$$Q = \begin{pmatrix} \partial_y\Lambda^{-1} \\ -\partial_x\Lambda^{-1} \end{pmatrix}.$$

Then we have

- $$P\widetilde{a} \cdot Q\widetilde{b} = a_x b_y - a_y b_x$$

  which is exactly the quantity we are interested in.

- $$Q^t P = (\partial_y \Lambda^{-1} \quad -\partial_x \Lambda^{-1}) \begin{pmatrix} \partial_x \Lambda^{-1} \\ \partial_y \Lambda^{-1} \end{pmatrix} = \partial_y \Lambda^{-1} \partial_x \Lambda^{-1} - \partial_x \Lambda^{-1} \partial_y \Lambda^{-1} = 0$$

Now, as before we want to show that for all $h \in bmo$ the following inequality holds

$$\int (a_x b_y - a_y b_x) h < \infty$$

where this time the left hand side can be rewritten as

$$\begin{aligned} (h, P\widetilde{a} \cdot Q\widetilde{b}) &= \int (a_x b_y - a_y b_x) h \\ &= \int h(P\widetilde{a}) \cdot (Q\widetilde{b}) \\ &= \int \widetilde{b} \cdot Q^t (hP\widetilde{a}) \\ &= (\widetilde{b}, [Q^t, m_h] P\widetilde{a}) + (h\widetilde{b}, Q^t P\widetilde{a}) \\ &= (\widetilde{b}, [Q^t, m_h] P\widetilde{a}) \end{aligned}$$

where in the second last step $[.,.]$ denotes the usual commutator and m_h denotes the multiplication with h. The last step holds because of the fact that $Q^t P = 0$. Now we use theorem 139 to estimate

$$\begin{aligned} \int (a_x b_y - a_y b_x) h &= (\widetilde{b}, [Q^t, m_h] P\widetilde{a}) \\ &\leq ||\widetilde{b}||_2 \, || \, [Q^t, m_h] P\widetilde{a}||_2 \\ &\leq ||\widetilde{b}||_2 C_2 ||h||_{bmo} ||P\widetilde{a}||_2 \\ &\leq C(||b||_2 + ||\nabla b||_2) ||h||_{bmo} ||\nabla a||_2 \end{aligned}$$

where in the last step we used the following calculation

$$\begin{aligned} ||(1 + |\xi|^2)^{\frac{1}{2}} \widehat{b}||_2 &\leq ||(1 + |\xi|) \widehat{b}||_2 \\ &\leq ||(1 + C \sum_{i=0}^{n} |\xi_i|) \widehat{b}||_2 \\ &= ||\widehat{b} + C \sum_{i=0}^{n} |\widehat{\partial_{x_i} b}| \, ||_2 \end{aligned}$$

which implies

$$||\widetilde{b}||_2 \leq C(||b||_2 + ||\nabla b||_2).$$

The last thing that remains is to show that our operators P and Q satisfy the hypothesis of theorem 139, i. e. we have to show that they belong to $S^0_{1,0}$. Let us show this for the operator P, since for Q it is the same. First of all note that the symbol corresponding to P is

$$p(x,\xi) = \begin{pmatrix} \xi_1(1+|\xi|^2)^{-\frac{1}{2}} \\ \xi_2(1+|\xi|^2)^{-\frac{1}{2}} \end{pmatrix}$$

since Λ^{-1} is given by

$$\Lambda^{-1}a = \mathcal{F}^{-1}((1+|\xi|^2)^{-\frac{1}{2}}\widehat{a}).$$

But this immediately implies

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| = |D^\alpha_\xi \xi_i(1+|\xi|^2)^{-\frac{1}{2}}| \leq C_\alpha(1+|\xi|^2)^{-\frac{|\alpha|}{2}} = C_\alpha\langle\xi\rangle^{-|\alpha|}$$

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Curriculum Vitae

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Education

2006, Oct - 2011, Apr:	PhD studies at ETH, Zurich, Switzerland Title of the thesis: “Integrability by compensation in Besov-Morrey spaces and applications to Schrödinger systems with antisymmetric potentials” Advisor: Prof. Dr. T. Rivière
2006, Apr:	Diploma in Mathematics (equivalent to Master) at ETH, Zurich, Switzerland Title of the diploma thesis: ”Local analysis of nonlinearities arising in geometry” Advisor: Prof. Dr. Tristan Rivière
2001 - 2006:	Studies in Mathematics at ETH, Zurich, Switzerland main emphasis: Theoretical Physics
2001, Jan:	Matura (with Latin, best of the school)
1994 - 2001:	Secondary education at Kantonsschule Freudenberg in Zurich, Switzerland
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Professional experience

2010, Oct -2011, Aug:	Assistant at the Institute for Mathematical Research (FIM) at ETH, Zurich, Switzerland
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