Tori in S^3 minimizing locally the conformal volume.

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Abstract : *We prove that the conformal immersions of complex two tori into S^3 which locally minimize their conformal volume in their conformal class all satisfy some elliptic PDE. We prove that they are either minimal tori, CMC flat tori, elliptic conformally constrained minimal tori or critical point of the area under some fixed conformally congruent area. On the way to establish this result we prove that tori which are critical points of the area for perturbations within a given conformal class and which are degenerate points of the conformal class mapping - i.e. isothermic - are either minimal surfaces or flat CMC tori. These results are all proved in the general framework of weak immersions.*

Math. Class. 49Q10, 53A05, 53A30, 35J20

I   Introduction

The notion of conformal volume introduced by P.Li and S.T.Yau in [LiYa] has stimulated a broad interest in mathematics beyond the differential geometry of submanifolds.

Let $G(S^3)$ be the Möbius group of conformal transformations of S^3 we are considering immersions $\vec{\Phi}$ of the two torus T^2 which are critical for the conformal volume

$$V_c(\vec{\Phi}) := \sup_{\Psi \in G(S^3)} A(\Psi \circ \vec{\Phi}) \tag{I.1}$$

within their conformal class. Precisely for any smooth path $\vec{\Phi}_t$ such that $\vec{\Phi}_0 = \vec{\Phi}$ and such that the conformal class defined by the metric $\vec{\Phi}_t^* g_{S^3}$, where g_{S^3} denotes the standard metric on the $3-$sphere S^3, is equal to the one defined by $\vec{\Phi}^* g$ we have

$$\left. \frac{d}{dt} V_c(\vec{\Phi}_t) \right|_{t=0} = 0 \tag{I.2}$$

Absolute minimizers of V_c in a given family of conformally equivalent metrics, when they exist, do satisfy (I.2) for instance. Beside the case of rectangular tori, which is partly solved in [MR], it is not known which conformal class possess a minimizer for the conformal volume. It is expected that not every class posses a minimizer of V_c. Indeed, while taking

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a minimizing sequence $\vec{\Phi}_k$ and the optimal Ψ_k, if it exist, satisfying $V_c(\vec{\Phi}_k) := A(\Psi_k \circ \vec{\Phi}_k)$ one cannot a-priori exclude that the sequence of immersions $\Psi_k \circ \vec{\Phi}_k$ weakly converges to a geodesic sphere. This degeneracy in the minimization process is believed by the author to be the only possible one and, when the minimal conformal volume within a conformal class is strictly larger than 4π, it should be achieved by an immersion of T^2 satisfying (I.2).

The purpose of the present work is to give a characterization of 2-dimensional tori in S^3 which are local minimizers of the conformal volume in their conformal class. In other words we aim to give a characterization of complex 2-tori (tori equipped with a given complex structure) minimizing locally the conformal volume for immersions defining this given complex structure.

In [Ri4] the author derived the infinitesimal characterization of the variations of riemann surfaces : a PDE satisfied by critical points of functional such as the area for variations within a given conformal class was derived in this work. This was a delicate issue at least for isothermic surfaces which are the degenerate points of the conformal class mapping and for which a direct application of Lagrange multiplier theory was excluded a-priori. It is proved that, for any $\vec{\Phi}$ satisfying (I.2) for any variation $\vec{\Phi}_t$ within the conformal class of $\vec{\Phi}$, there exists an holomorphic quadratic form Q of the underlying riemann surface such that

$$H = \Re < Q, h^0 >_{g_{\vec{\Phi}}} \quad , \tag{I.3}$$

where $g_{\vec{\Phi}} := \vec{\Phi}^* g_{S^3}$ is the induced metric on the surface, H is the mean curvature of the immersion and h^0 is the Weingarten form given in complex coordinates by

$$h^0 := 2\,\pi_{\vec{n}}(\partial^2_{z^2}\vec{\Phi})\,\,dz^2 \quad , \tag{I.4}$$

where $\pi_{\vec{n}}$ is the projection onto the normal direction to the surface in TS^3. Solutions to (I.3) are called *conformally constrained minimal surfaces*.

Hopf tori - i.e. the preimages by the Hopf fibration from S^3 into S^2 of closed curves on S^2 - are examples of *conformally constrained minimal surfaces* (see [BPP]). These surfaces are not necessarily smooth and it suffices to have a weak notion of second fundamental form as well as a notion of conformal coordinates in order to give a meaning to equation (I.3). In [Ri3] (see also [Ri1]) the author introduced a notion of weak immersions compatible with variational purposes, for which conformal coordinates exist and for which the mapping assigning the conformal class is smooth for some topology. The motivation originally was to produce a suitable framework for studying the variations of Willmore Lagrangian. It appears moreover that this is a "minimal" requirement for ensuring the smoothness of the mapping assigning the conformal class. This is the framework we shall consider here.

A *weak immersion* of T^2 into S^3 is a map $\vec{\Phi}$ from T^2 into S^3 such that

i)

$$\vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^4) \quad ,$$

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ii) there exists a constant $C_{\vec{\Phi}} > 1$ such that

$$\forall x \in \Sigma \quad \forall X \in T_x\Sigma \qquad C_{\vec{\Phi}}^{-1} \ g_0(X) \leq |d\vec{\Phi}(X)|^2 \leq C_{\vec{\Phi}} \ g_0(X)$$

where g_0 is some given smooth reference metric on T^2 i.e. in other words the metric on T^2 equal to the pull back by $\vec{\Phi}$ of the canonical metric of $\mathbb{R}^4$ is equivalent to any reference metric on T^2,

iii)
$$\vec{n}_{\vec{\Phi}} \in W^{1,2}(T^2, S^3)$$

where $\vec{n}_{\vec{\Phi}}$ is the Gauss map associated to $\vec{\Phi}$ i.e. the unit vector perpendicular to the surface in TS^3 and positively oriented.

This space is usually denoted $\mathcal{E}_{T^2}$. This is a Banach manifold modeled on $W^{1,\infty} \cap W^{2,2}(T^2, \mathbb{R}^3)$. Each element in $\mathcal{E}_{T^2}$ defines uniquely a conformal class and, for a fixed choice of generators of the $\pi_1(T^2)$ the mapping which to $\vec{\Phi}$ assigns the corresponding Teichmüller class is C^1 (see [Ri3]).

The analysis of the equation (I.3) is made particularly complex due to the fact that it includes both *hyperbolic* and *elliptic* regimes. *Strictly elliptic* regime is observed in the domain given by
$$\mathfrak{E}_s(\vec{\Phi}) := \left\{ x \in T^2 \ ; \ 2|Q|_{g_{\vec{\Phi}}}(x) < 1 \right\}$$
The splitting between these two regimes can be seen formally in the following way. On a two torus an holomorphic quadratic form is either zero or never vanish. Then, locally there exist complex coordinates in which $Q = 4^{-1} dz^2$. In these coordinates the equation (I.3) reads
$$\Delta\vec{\Phi} + 2\,\vec{\Phi}\,e^{2\lambda} = \partial_{x_1}(e^{-2\lambda}\,\partial_{x_1}\vec{\Phi}) - \partial_{x_2}(e^{-2\lambda}\,\partial_{x_2}\vec{\Phi})$$
where $g_{\vec{\Phi}} = e^{2\lambda}\,[dx_1^2 + dx_2^2]$. Observe that $|Q|_{g_{\vec{\Phi}}} = 2^{-1}\,e^{-2\lambda}$ hence the domain of strict ellipticity $\mathfrak{E}_s(\vec{\Phi})$ corresponds to the set of points x where $\lambda(x) > 0$ in these special coordinates and hence the principal symbol of the *conformally constrained minimal surfaces* equation which reads

$$\partial_{x_1}\left((1 - e^{-2\lambda})\,\partial_{x_1}\vec{\Phi}\right) + \partial_{x_2}\left((1 + e^{-2\lambda})\,\partial_{x_2}\vec{\Phi}\right) + 2\,\vec{\Phi}\,e^{2\lambda} = 0 \qquad (\text{I.5})$$

is exactly invertible on this domain. Following [MS] and [He], we proved in [Ri1] that for any weak immersion $\vec{\Phi}$ in any conformal coordinates, the induced metric is continuous and hence $\mathfrak{E}_s(\vec{\Phi})$ is an open subset of T^2. The elliptic nature of the *conformally constrained minimal surfaces* equation on the domain of strict ellipticity is reinforced by the following result

Theorem I.1. *Let* $\vec{\Phi}$ *be a weak immersion satisfying the conformally constrained minimal surface equation*

$$H = \Re < Q, h^0 >_{g_{\vec{\Phi}}}$$

for some holomorphic quadratic form on T^2 *equipped with the conformal class defined by* $\vec{\Phi}$. *Then* $\vec{\Phi}$ *is analytic in conformal coordinates within the domain of strict ellipticity* $\mathfrak{E}_s(\vec{\Phi})$ *given by*

$$\mathfrak{E}_s(\vec{\Phi}) := \left\{ x \in T^2 \ ; \ 2 |Q|_{g_{\vec{\Phi}}}(x) < 1 \right\} \quad .$$

$\square$

The theorem is clearly optimal since Hopf tori, which are solutions to the *conformally constrained minimal surface* equation, satisfy $2 |Q|_{g_{\vec{\Phi}}}(x) \equiv 1$ on T^2 and can be nowhere C^2 by taking the lifting of a nowhere C^2 curve Γ on S^2 but still satisfying $\int_\Gamma |\kappa|^2 \, dl < +\infty$ where κ is the geodesic curvature of the curve in S^2 which ensures that this is a weak immersion.

Definition I.1. *A weak immersion is said to satisfy the elliptic conformally constrained minimal surface equation (resp. strictly elliptic) if it satisfies*

$$H = \Re < Q, h^0 >_{g_{\vec{\Phi}}}$$

for some holomorphic quadratic form on T^2 *equipped with the conformal class defined by* $\vec{\Phi}$ *and*

$$\mathfrak{E}(\vec{\Phi}) := \left\{ x \in T^2 \ ; \ 2 |Q|_{g_{\vec{\Phi}}}(x) \leq 1 \right\}$$

concide with the whole torus (resp. $\mathfrak{E}_s(\vec{\Phi}) = T^2$). $\square$

Strictly elliptic conformally constrained minimal surface could be seen as generalization of minimal surfaces sharing many features with minimal surfaces. In particular one deduces from the analysis for proving theorem I.1 that the space of strictly elliptic conformally constrained minimal surface satisfying $2 |Q|_{g_{\vec{\Phi}}}(x) < 1 - \varepsilon_0$ for any $0 < \varepsilon_0 \leq 1$ with uniformly bounded area and controlled conformal class is compact in C^l topology for any $l \in \mathbb{N}$.

Regarding now the similar problem in the euclidian space $\mathbb{R}^3$, in [Ri4] it is proved that critical points of the area among weak immersions of a compact surface Σ into $\mathbb{R}^3$ with prescribed conformal class satisfies also the *conformally constrained minimal surface* equation

$$\vec{H} = \Re < Q, \vec{h^0} >_{g_{\vec{\Phi}}}$$

By multiplying the equation by $\vec{\Phi}$ and integrating by parts one gets the following proposition which is reminiscent to the corresponding result for minimal surfaces.

Proposition I.1. *There exists no weak immersion into* $\mathbb{R}^3$ *of a closed two dimensional manifold satisfying the conformally constrained minimal surface equation.* $\square$

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Observe that an arbitrary cylinder over a plane curve in $\mathbb{R}^3$ is an example of an elliptic (non strictly elliptic) non compact conformally constrained minimal surface satisfying $H = \Re(H^0)$ in cylindrical coordinates in which the Weingarten Operator takes the form

$$\mathbb{I} = \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix} \quad .$$

It is also *isothermic* since $\Im(H^0) \equiv 0$. In [BPP] the authors show that surfaces with rotational symmetry are *conformally constrained minimal* everywhere away from the axis points. Such surfaces are not necessarily smooth and therefore should not be in general *elliptic conformally constrained minimal*.

Finally, it would be interesting to study non compact strictly elliptic conformally constrained minimal surface with ends in $\mathbb{R}^3$ and in particular bounds on their index...etc.

Going back to immersions in S^3, in section IV of the present paper, we establish the following result which identifies the space of *isothermic conformally constrained minimal* immersions of tori.

Theorem I.2. *A weak immersion of the torus T^2 in S^3 is both isothermic and a critical point of the area under constrained conformal class if and only if it is either minimal or realizes a flat CMC torus. Such tori are all solving the strictly elliptic conformally constrained minimal equation.* $\square$

For instance, the *Weingarten Operator* of *Hopf tori* in coordinates given by the Hopf fibers and parallel lifts of the curve Γ in S^2 - whose lift by the *Hopf fibration* is equal to the torus - is given by - see [Pi] identity (21) -

$$\mathbb{I} = \begin{pmatrix} 2\kappa & 1 \\ 1 & 0 \end{pmatrix} \quad .$$

It is clearly solving the conformally constrained minimal surface equation $H = \Re(H^0)$. It is moreover isothermic if and only if $\kappa = \kappa_0 \geq 0$ is constant - i.e. Γ is a circle in S^2 and the torus is a CMC Clifford torus -. It solves the *isothermic* equation

$$\Re((1 + i\,\kappa_0)H^0) = 0 \quad ,$$

together with the *strictly elliptic conformally constrained minimal* equation

$$H = \Re\left(\frac{\kappa_0}{\kappa_0 + i} H^0\right) \quad ,$$

where $\lambda \equiv 0$ and $Q = 4^{-1}\frac{\kappa_0}{\kappa_0 - i}dz^2$ satisfies $2|Q|_g = \sqrt{\frac{\kappa_0^2}{1 + \kappa_0^2}} < 1$. Finally theorem I.2 can be put in perspective with a result by J. Richter, see [Ric] and [BuPP], asserting that

conformally constrained Willmore tori in S^3 which are in addition *isothermic* are CMC surfaces for some constant sectional curvature.

The main result of the present paper is the following theorem which identifies tori minimizing locally the conformal volume within their conformal class.

Theorem I.3. *Let $\vec{\Phi}$ be weak immersion locally minimizing the conformal volume V_c in it's conformal class. Assume $V_c(\vec{\Phi}) \notin 4\pi\mathbb{N}$ and that V_c is differentiable at $\vec{\Phi}$, then the following alternative holds : if there is no other Möbis transformation of $\vec{\Phi}$ realizing $V_c(\vec{\Phi})$ then either*

- *i) $\vec{\Phi}$ is a minimal immersion,*

- *ii) $\vec{\Phi}(T^2)$ is a flat CMC clifford torus congruent to a $S^1 \times \sqrt{1-a^2}\,S^1$ for some $a \in (0,1)$,*

- *iii) $\vec{\Phi}$ is an elliptic conformally constrained minimal surface.*

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In the "degenerate case" of the existence of at least <u>two non isometric</u> Möbius transformations realizing $V_c(\vec{\Phi})$ we have the following proposition.

Proposition I.2. *Let $\vec{\Phi}$ be a weak immersion of T^2 into S^3, solution to the conformally constrained equation (I.3) and realizing it's conformal volume : $A(\vec{\Phi}) = V_c(\vec{\Phi})$. Assume $V_c(\vec{\Phi}) \notin 4\pi\mathbb{N}$ and that V_c is differentiable at $\vec{\Phi}$. If $V_c(\vec{\Phi})$ is also achieved by $\Psi \circ \vec{\Phi}$, a non isometric Möbius transformation of $\vec{\Phi}$ and that any such Möbius transformation satisfy also the conformally constraint minimal equation, then $\vec{\Phi}$ is a critical point of the area A under the constraint that $A(\Psi \circ \vec{\Phi})$ is constant.* $\qquad$ $\square$

The paper is organized as follows. In section II we compute the first and second variations of the area functional within a fixed conformal class. In section III we prove theorem I.1. In section IV we identify the isothermic conformally constrained minimal surfaces. In section V we construct families of deformations which reduces the conformal volume for non-elliptic conformally constrained minimal surfaces. In the last section we collect the informations obtained from the previous ones to prove theorem I.3.

II The first and second variations of a weak immersion locally minimizing the conformal volume in S^3.

II.1 Preliminaries : notations and some computations.

Consider $\vec{\Phi}_t := \vec{\Phi} + t\,\vec{w}$.

We have in local coordinates (that we can choose to be conformal for $\vec{\Phi}$)

$$\vec{n}_t = \star_{\mathbb{R}^4}\left(\vec{\Phi}_t \wedge \frac{\partial_{x_1}\vec{\Phi}_t \wedge \partial_{x_2}\vec{\Phi}_t}{|\partial_{x_1}\vec{\Phi}_t \wedge \partial_{x_2}\vec{\Phi}_t|}\right) \tag{II.6}$$

We have

$$\vec{n}_t = \vec{n} + t\,(a_1\,\vec{e}_1 + a_2\,\vec{e}_2 + b\,\vec{\Phi}) + o(t) \tag{II.7}$$

They can be identified as follows:

$$e^\lambda\,a_1 = \left\langle e^\lambda\,\vec{e}_1, \left.\frac{d}{dt}\vec{n}_t\right|_{t=0}\right\rangle = -\left\langle\left.\frac{d}{dt}\left(\partial_{x_1}\vec{\Phi}_t\right)\right|_{t=0}, \vec{n}\right\rangle = -\langle\partial_{x_1}\vec{w}, \vec{n}\rangle,$$

and similarly one obtains $a_2 = -e^{-\lambda}\,\langle\partial_{x_2}\vec{w}, \vec{n}\rangle$. Let $v := \frac{d\vec{\Phi}}{dt}\cdot\vec{n} = \vec{w}\cdot\vec{n}$. We have

$$b := \vec{\Phi}\cdot\frac{d\vec{n}}{dt} = -v$$

Hence we have

$$\frac{d\vec{n}}{dt} = -e^{-\lambda}\left[\langle\partial_{x_1}\vec{w}, \vec{n}\rangle\,\vec{e}_1 + \langle\partial_{x_2}\vec{w}, \vec{n}\rangle\,\vec{e}_2\right] - v\,\vec{\Phi} \tag{II.8}$$

We now compute the time derivative of the second fundamental form. We have

$$\frac{d\mathbb{I}}{dt} = \sum_{i,j=1}^2\left[\frac{d\vec{n}}{dt}\cdot\partial^2_{x_i x_j}\vec{\Phi} + \vec{n}\cdot\partial^2_{x_i x_j}\vec{w}\right]\,dx_i \otimes dx_j$$

$$= -\sum_{i,j,k=1}^2\left[e^{-2\lambda}\,\partial_{x_k}\vec{w}\cdot\vec{n}\,\partial_{x_k}\vec{\Phi}\cdot\partial^2_{x_i x_j}\vec{\Phi}\right]\,dx_i \otimes dx_j \tag{II.9}$$

$$+v\,g + \sum_{i,j=1}^2\vec{n}\cdot\partial^2_{x_i x_j}\vec{w}\,dx_i \otimes dx_j$$

Recall that

$$e^{-2\lambda}\,\partial_{x_k}\vec{\Phi}\cdot\partial^2_{x_i x_j}\vec{\Phi} = \delta_{jk}\,\partial_{x_i}\lambda + \delta_{ik}\,\partial_{x_j}\lambda - \delta_{ij}\,\partial_{x_k}\lambda \tag{II.10}$$

Combining (II.9) and (II.10) we obtain

$$\frac{d\mathbb{I}}{dt} = - \left(\partial_{x_1} \lambda \, \partial_{x_1} \vec{w} \cdot \vec{n} - \partial_{x_2} \lambda \, \partial_{x_2} \vec{w} \cdot \vec{n} \right) (dx_1^2 - dx_2^2)$$

$$- \left(\partial_{x_1} \lambda \, \partial_{x_2} \vec{w} \cdot \vec{n} + \partial_{x_2} \lambda \, \partial_{x_1} \vec{w} \cdot \vec{n} \right) (dx_1 \otimes dx_2 + dx_2 \otimes dx_1) \tag{II.11}$$

$$+ v \, g + \sum_{i,j=1}^{2} \vec{n} \cdot \partial_{x_i x_j}^2 \vec{w} \, dx_i \otimes dx_j$$

Recall that $H_t = \frac{1}{2} \sum_{i,j} (g_t)^{ij} (\mathbb{I}_t)_{ij}$ Hence

$$\frac{dH}{dt} = \frac{1}{2} \sum_{i,j} (g_t)^{ij} \frac{d(\mathbb{I}_t)_{ij}}{dt} + \frac{1}{2} \sum_{i,j} \frac{d(g_t)^{ij}}{dt} (\mathbb{I}_t)_{ij}$$

Using (II.11) we have

$$\frac{1}{2} \sum_{i,j} (g_t)^{ij} \frac{d(\mathbb{I}_t)_{ij}}{dt} = v + \frac{\vec{n}}{2} \cdot \Delta_g \vec{w} \tag{II.12}$$

We have $(g_t)_{ij} = \langle \partial_{x_i} \vec{\Phi}_t, \partial_{x_j} \vec{\Phi}_t \rangle$, thus

$$\frac{d}{dt} (g_t)_{ij} \Big|_{t=0} = \langle \partial_{x_i} \vec{w}, \partial_{x_j} \vec{\Phi} \rangle + \langle \partial_{x_i} \vec{\Phi}, \partial_{x_j} \vec{w} \rangle. \tag{II.13}$$

Since $\sum_i (g_t)^{ki} (g_t)_{ij} = \delta_{kj}$ and $g_{ij} = e^{2\lambda} I_2$, where I_2 is the (2×2)-identity matrix, we have

$$\frac{d}{dt} (g_t)^{kj} \Big|_{t=0} e^{2\lambda} + e^{-2\lambda} \frac{d}{dt} (g_t)_{kj} \Big|_{t=0} = 0, \tag{II.14}$$

from which we deduce

$$\frac{d}{dt} (g_t)^{kj} \Big|_{t=0} = -e^{-4\lambda} \frac{d}{dt} (g_t)_{kj} \Big|_{t=0} = -e^{-4\lambda} \left(\langle \partial_{x_k} \vec{w}, \partial_{x_j} \vec{\Phi} \rangle + \langle \partial_{x_k} \vec{\Phi}, \partial_{x_j} \vec{w} \rangle \right). \tag{II.15}$$

The previous computation gives then

$$\frac{dH}{dt} = \frac{\vec{n}}{2} \cdot [\Delta_g \vec{w} + 2 \, \vec{w}]$$

$$+ \frac{1}{2} \sum_{i,j=1}^{2} e^{-4\lambda} \left[\partial_{x_i} \vec{w} \cdot \partial_{x_j} \vec{\Phi} + \partial_{x_j} \vec{w} \cdot \partial_{x_i} \vec{\Phi} \right] \, \partial_{x_j} \vec{n} \cdot \partial_{x_i} \vec{\Phi} \tag{II.16}$$

We write

$$\vec{w} = \sigma_1 \, \partial_{x_1} \vec{\Phi} + \sigma_2 \, \partial_{x_2} \vec{\Phi} + v \, \vec{n} \quad . \tag{II.17}$$

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and with these notations we compute in one hand

$$\frac{\vec{n}}{2} \cdot \Delta_g \vec{w} = \frac{\Delta_g v}{2} - \frac{|\mathbb{I}|_g^2}{2}\, v + e^{-2\lambda} \sum_{ij=1}^{2} \partial_{x_i} \sigma_j\ \mathbb{I}_{ij}$$

$$+ e^{-2\lambda} \sum_{i=1}^{2} \sigma_i\, \partial_{x_i}(e^{2\lambda} H) \tag{II.18}$$

and using in particular (II.10) we have

$$\frac{1}{2} \sum_{i,j=1}^{2} e^{-4\lambda} \left[\partial_{x_i} \vec{w} \cdot \partial_{x_j} \vec{\Phi} + \partial_{x_j} \vec{w} \cdot \partial_{x_i} \vec{\Phi} \right]\ \partial_{x_j} \vec{n} \cdot \partial_{x_i} \vec{\Phi}$$

$$= |\mathbb{I}|_g^2\, v - e^{-2\lambda} \sum_{i,j=1}^{2} \mathbb{I}_{ij}\, \partial_{x_i} \sigma_j - 2\,H \sum_{k=1}^{2} \sigma_k\, \partial_{x_k} \lambda \tag{II.19}$$

Combining (II.16), (II.18) and (II.19) we obtain

$$\frac{dH}{dt} = \frac{1}{2} \left[\Delta_g v + (|\mathbb{I}|_g^2 + 2)\, v \right] + \sum_{i=1}^{2} \sigma_i\, \partial_{x_i} H \tag{II.20}$$

Let $\bar{q} := \bar{q}_{ij}\, dx_i \otimes dx_j = a\,[dx_1^2 - dx_2^2] + b\,[dx_1 \otimes dx_2 + dx_2 \otimes dx_1]$ for some functions a and b independent of t. We have

$$\left\langle \frac{d\mathbb{I}^0}{dt}, \bar{q} \right\rangle_g = \left\langle \frac{d\mathbb{I}}{dt}, \bar{q} \right\rangle_g - H \left\langle \frac{dg}{dt}, \bar{q} \right\rangle_g \tag{II.21}$$

We have in one hand

$$-H \left\langle \frac{dg}{dt}, \bar{q} \right\rangle_g = -2\,H\ e^{-4\lambda} \left[a\,(\partial_{x_1} \vec{w} \cdot \partial_{x_1} \vec{\Phi} - \partial_{x_2} \vec{w} \cdot \partial_{x_2} \vec{\Phi}) \right.$$

$$\left. + b\,(\partial_{x_1} \vec{w} \cdot \partial_{x_2} \vec{\Phi} + \partial_{x_2} \vec{w} \cdot \partial_{x_1} \vec{\Phi}) \right] \tag{II.22}$$

In the other hand we have

$$\left\langle \frac{d\mathbb{I}}{dt}, \bar{q} \right\rangle = a\, e^{-4\lambda} \left(\vec{n} \cdot \partial_{x_1^2}^2 \vec{w} - \vec{n} \cdot \partial_{x_2^2}^2 \vec{w} \right) + 2\,b\, e^{-4\lambda}\, \vec{n} \cdot \partial_{x_1 x_2}^2 \vec{w}$$

$$- 2\,a\, e^{-4\lambda} \left(\partial_{x_1} \lambda\, \partial_{x_1} \vec{w} \cdot \vec{n} - \partial_{x_2} \lambda\, \partial_{x_2} \vec{w} \cdot \vec{n} \right) \tag{II.23}$$

$$- 2\,b\, e^{-4\lambda} \left(\partial_{x_1} \lambda\, \partial_{x_2} \vec{w} \cdot \vec{n} + \partial_{x_2} \lambda\, \partial_{x_1} \vec{w} \cdot \vec{n} \right)$$

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Thus, combining (II.21), (II.22) and (II.23) gives

$$\left\langle \frac{d\mathbb{I}^0}{dt}, \vec{q} \right\rangle_g = a\, e^{-4\lambda} \left(\vec{n} \cdot \partial^2_{x_1^2} \vec{w} - \vec{n} \cdot \partial^2_{x_2^2} \vec{w} \right) + 2\, b\, e^{-4\lambda}\, \vec{n} \cdot \partial^2_{x_1 x_2} \vec{w}$$

$$- 2\, a\, e^{-4\lambda} \left(\partial_{x_1} \lambda\, \partial_{x_1} \vec{w} \cdot \vec{n} - \partial_{x_2} \lambda\, \partial_{x_2} \vec{w} \cdot \vec{n} \right)$$

$$\tag{II.24}$$

$$- 2\, b\, e^{-4\lambda} \left(\partial_{x_1} \lambda\, \partial_{x_2} \vec{w} \cdot \vec{n} + \partial_{x_2} \lambda\, \partial_{x_1} \vec{w} \cdot \vec{n} \right)$$

$$- 2\, H\, e^{-4\lambda} \left[a \left(\partial_{x_1} \vec{w} \cdot \partial_{x_1} \vec{\Phi} - \partial_{x_2} \vec{w} \cdot \partial_{x_2} \vec{\Phi} \right) + b \left(\partial_{x_1} \vec{w} \cdot \partial_{x_2} \vec{\Phi} + \partial_{x_2} \vec{w} \cdot \partial_{x_1} \vec{\Phi} \right) \right]$$

We have then after some computations

$$\vec{n} \cdot \partial^2_{x_1^2} \vec{w} - \vec{n} \cdot \partial^2_{x_2^2} \vec{w} = \partial_{x_1} (\vec{n} \cdot \partial_{x_1} \vec{w}) - \partial_{x_2} (\vec{n} \cdot \partial_{x_2} \vec{w}) - \partial_{x_1} \vec{n} \cdot \partial_{x_1} \vec{w} + \partial_{x_2} \vec{n} \cdot \partial_{x_2} \vec{w}$$

$$= \partial^2_{x_1^2} v - \partial^2_{x_2^2} v - v \left(|\partial_{x_1} \vec{n}|^2 - |\partial_{x_2} \vec{n}|^2 \right) + H\, e^{2\lambda} (\partial_{x_1} \sigma_1 - \partial_{x_2} \sigma_2)$$

$$\tag{II.25}$$

$$+ e^{-2\lambda}\, \mathbb{I}^0_{11} \left(\partial_{x_1}(e^{2\lambda} \sigma_1) + \partial_{x_2}(e^{2\lambda} \sigma_2) \right) + e^{-2\lambda}\, \mathbb{I}^0_{12} \left(\partial_{x_1}(e^{2\lambda} \sigma_2) - \partial_{x_2}(e^{2\lambda} \sigma_1) \right)$$

$$+ \partial_{x_1} (\mathbb{I}_{11} \sigma_1 + \mathbb{I}_{12} \sigma_2) - \partial_{x_2} (\mathbb{I}_{12} \sigma_1 + \mathbb{I}_{22} \sigma_2)$$

Similarly we have

$$2\, \vec{n} \cdot \partial^2_{x_1 x_2} \vec{w} = \partial_{x_1} (\vec{n} \cdot \partial_{x_2} \vec{w}) + \partial_{x_2} (\vec{n} \cdot \partial_{x_1} \vec{w}) - \partial_{x_1} \vec{n} \cdot \partial_{x_2} \vec{w} - \partial_{x_2} \vec{n} \cdot \partial_{x_1} \vec{w}$$

$$= 2\, \partial^2_{x_1 x_2} v - 2\, \partial_{x_1} \vec{n} \cdot \partial_{x_2} \vec{n}\, v + H\, e^{2\lambda} (\partial_{x_2} \sigma_1 + \partial_{x_1} \sigma_2)$$

$$\tag{II.26}$$

$$+ e^{-2\lambda}\, \mathbb{I}^0_{11} \left(\partial_{x_2}(e^{2\lambda} \sigma_1) - \partial_{x_1}(e^{2\lambda} \sigma_2) \right) + e^{-2\lambda}\, \mathbb{I}^0_{12} \left(\partial_{x_1}(e^{2\lambda} \sigma_1) + \partial_{x_2}(e^{2\lambda} \sigma_2) \right)$$

$$+ \partial_{x_1} (\mathbb{I}_{12} \sigma_1 + \mathbb{I}_{22} \sigma_2) + \partial_{x_2} (\mathbb{I}_{11} \sigma_1 + \mathbb{I}_{12} \sigma_2)$$

We have also

$$\partial_{x_1} \lambda\, \partial_{x_1} \vec{w} \cdot \vec{n} - \partial_{x_2} \lambda\, \partial_{x_2} \vec{w} \cdot \vec{n} = \partial_{x_1} \lambda\, \partial_{x_1} v - \partial_{x_2} \lambda\, \partial_{x_2} v$$

$$+ e^{2\lambda} H\, (\sigma_1 \partial_{x_1} \lambda - \sigma_2 \partial_{x_2} \lambda) + \mathbb{I}^0_{11} (\sigma_1 \partial_{x_1} \lambda + \sigma_2 \partial_{x_2} \lambda) + \mathbb{I}^0_{12} (\sigma_2 \partial_{x_1} \lambda - \sigma_1 \partial_{x_2} \lambda)$$

$$\tag{II.27}$$

and

$$\partial_{x_1} \lambda\, \partial_{x_2} \vec{w} \cdot \vec{n} + \partial_{x_2} \lambda\, \partial_{x_1} \vec{w} \cdot \vec{n} = \partial_{x_1} \lambda\, \partial_{x_2} v + \partial_{x_2} \lambda\, \partial_{x_1} v$$

$$+ e^{2\lambda} H\, (\sigma_1 \partial_{x_2} \lambda + \sigma_2 \partial_{x_1} \lambda) + \mathbb{I}^0_{11} (\sigma_1 \partial_{x_2} \lambda - \sigma_2 \partial_{x_1} \lambda) + \mathbb{I}^0_{12} (\sigma_1 \partial_{x_1} \lambda + \sigma_2 \partial_{x_2} \lambda)$$

$$\tag{II.28}$$

Finally we have in one hand

$$\partial_{x_1} \vec{w} \cdot \partial_{x_1} \vec{\Phi} - \partial_{x_2} \vec{w} \cdot \partial_{x_2} \vec{\Phi} = -2\, v\, \mathbb{I}^0_{11} + e^{2\lambda} (\partial_{x_1} \sigma_1 - \partial_{x_2} \sigma_2) \tag{II.29}$$

and in the other hand

$$\partial_{x_1}\vec{w}\cdot\partial_{x_2}\vec{\Phi} + \partial_{x_2}\vec{w}\cdot\partial_{x_1}\vec{\Phi} = -\,2\,v\,\mathbb{I}^0_{12} + e^{2\lambda}\,(\partial_{x_1}\sigma_2 + \partial_{x_2}\sigma_1) \qquad\qquad (\text{II.30})$$

Combining (II.24)...(II.30) we obtain

$$\left\langle \frac{d\mathbb{I}^0}{dt},\bar{q}\right\rangle_g = a\,e^{-2\lambda}\left(\partial_{x_1}(e^{-2\lambda}\partial_{x_1}v) - \partial_{x_2}(e^{-2\lambda}\partial_{x_2}v)\right) + b\,e^{-2\lambda}\left(\partial_{x_1}(e^{-2\lambda}\partial_{x_2}v) + \partial_{x_2}(e^{-2\lambda}\partial_{x_1}v)\right)$$

$$- e^{-4\lambda}\,H\,a\,(\partial_{x_1}(e^{2\lambda}\sigma_1) - \partial_{x_2}(e^{2\lambda}\sigma_2)) - e^{-4\lambda}\,H\,b\,(\partial_{x_1}(e^{2\lambda}\sigma_2) + \partial_{x_2}(e^{2\lambda}\sigma_1))$$

$$+ e^{-4\lambda}\,(a\,\mathbb{I}^0_{11} + b\,\mathbb{I}^0_{12})\,(\partial_{x_1}\sigma_1 + \partial_{x_2}\sigma_2) + e^{-4\lambda}\,(a\,\mathbb{I}^0_{12} - b\,\mathbb{I}^0_{11})\,(\partial_{x_1}\sigma_2 - \partial_{x_2}\sigma_1)$$

$$+ e^{-4\lambda}\,a\,[\partial_{x_1}(\mathbb{I}_{11}\sigma_1 + \mathbb{I}_{12}\sigma_2) - \partial_{x_2}(\mathbb{I}_{12}\sigma_1 + \mathbb{I}_{22}\sigma_2)]$$

$$+ e^{-4\lambda}\,b\,[\partial_{x_1}(\mathbb{I}_{12}\sigma_1 + \mathbb{I}_{22}\sigma_2) + \partial_{x_2}(\mathbb{I}_{11}\sigma_1 + \mathbb{I}_{12}\sigma_2)]$$

$$(\text{II.31})$$

Assume now that $a - ib$ is holomorphic - i.e. $\partial_{x_1}a + \partial_{x_2}b = 0$ and $\partial_{x_1}b - \partial_{x_2}a = 0$ then we have for instance

$$a\left(\partial_{x_1}(e^{-2\lambda}\partial_{x_1}v) - \partial_{x_2}(e^{-2\lambda}\partial_{x_2}v)\right) + b\left(\partial_{x_1}(e^{-2\lambda}\partial_{x_2}v) + \partial_{x_2}(e^{-2\lambda}\partial_{x_1}v)\right)$$

$$= \partial_{x_1}(a\,e^{-2\lambda}\partial_{x_1}v) - \partial_{x_2}(a\,e^{-2\lambda}\partial_{x_2}v) + \partial_{x_1}(b\,e^{-2\lambda}\partial_{x_2}v) + \partial_{x_2}(b\,e^{-2\lambda}\partial_{x_1}v)$$

or we have also

$$a\,[\partial_{x_1}(\mathbb{I}_{11}\sigma_1 + \mathbb{I}_{12}\sigma_2) - \partial_{x_2}(\mathbb{I}_{12}\sigma_1 + \mathbb{I}_{22}\sigma_2)] + b\,[\partial_{x_1}(\mathbb{I}_{12}\sigma_1 + \mathbb{I}_{22}\sigma_2) + \partial_{x_2}(\mathbb{I}_{11}\sigma_1 + \mathbb{I}_{12}\sigma_2)]$$

$$= \partial_{x_1}(a\,[\mathbb{I}_{11}\sigma_1 + \mathbb{I}_{12}\sigma_2] + b\,[(\mathbb{I}_{12}\sigma_1 + \mathbb{I}_{22}\sigma_2)]) - \partial_{x_2}(a\,[\mathbb{I}_{12}\sigma_1 + \mathbb{I}_{22}\sigma_2] - b\,[\mathbb{I}_{11}\sigma_1 + \mathbb{I}_{12}\sigma_2])$$

and hence

$$\left\langle \frac{d\mathbb{I}^0}{dt},\bar{q}\right\rangle_g = e^{-2\lambda}\left[\partial_{x_1}(a\,e^{-2\lambda}\partial_{x_1}v) - \partial_{x_2}(a\,e^{-2\lambda}\partial_{x_2}v) + \partial_{x_1}(b\,e^{-2\lambda}\partial_{x_2}v) + \partial_{x_2}(b\,e^{-2\lambda}\partial_{x_1}v)\right]$$

$$- e^{-4\lambda}\,H\,a\,(\partial_{x_1}(e^{2\lambda}\sigma_1) - \partial_{x_2}(e^{2\lambda}\sigma_2)) - e^{-4\lambda}\,H\,b\,(\partial_{x_1}(e^{2\lambda}\sigma_2) + \partial_{x_2}(e^{2\lambda}\sigma_1))$$

$$+ e^{-4\lambda}\,(a\,\mathbb{I}^0_{11} + b\,\mathbb{I}^0_{12})\,(\partial_{x_1}\sigma_1 + \partial_{x_2}\sigma_2) + e^{-4\lambda}\,(a\,\mathbb{I}^0_{12} - b\,\mathbb{I}^0_{11})\,(\partial_{x_1}\sigma_2 - \partial_{x_2}\sigma_1)$$

$$+ e^{-4\lambda}\,\partial_{x_1}\left(a\,[\mathbb{I}^0_{11}\sigma_1 + \mathbb{I}^0_{12}\sigma_2] + b\,[(\mathbb{I}^0_{12}\sigma_1 - \mathbb{I}^0_{11}\sigma_2])\right)$$

$$- e^{-4\lambda}\,\partial_{x_2}\left(a\,[\mathbb{I}^0_{12}\sigma_1 - \mathbb{I}^0_{11}\sigma_2] - b\,[\mathbb{I}^0_{11}\sigma_1 + \mathbb{I}^0_{12}\sigma_2]\right)$$

$$+ e^{-4\lambda}\left[\partial_{x_1}\left(e^{2\lambda}\,H\,(a\,\sigma_1 + b\,\sigma_2)\right) + \partial_{x_2}\left(e^{2\lambda}\,H\,(b\,\sigma_1 - a\,\sigma_2)\right)\right]$$

$$(\text{II.32})$$

We now compute

$$\frac{d}{dt} < \overline{q}, \mathbb{I}^0 >_g = \frac{d}{dt} \left(\sum_{ijkl=1}^{2} g^{ki} g^{jl} \overline{q}_{kl} \mathbb{I}_{ij}^0 \right) = 2 \, e^{-2\lambda} \sum_{ijk=1}^{2} \frac{dg^{ij}}{dt} \mathbb{I}_{ik}^0 \overline{q}_{jk} + \left\langle \frac{d\mathbb{I}^0}{dt}, \overline{q} \right\rangle_g \quad \text{(II.33)}$$

Using (II.15) we obtain, using also (II.10),

$$\frac{dg^{ij}}{dt} = 2 \, v \, \mathbb{I}_{ij} \, e^{-4\lambda} - e^{-4\lambda} \sum_{l=1}^{2} \left[\partial_{x_i}(\sigma_l \, \partial_{x_l}\vec{\Phi}) \cdot \partial_{x_j}\vec{\Phi} + \partial_{x_j}(\sigma_l \, \partial_{x_l}\vec{\Phi}) \cdot \partial_{x_i}\vec{\Phi} \right]$$

$$= 2 \, v \, \mathbb{I}_{ij} \, e^{-4\lambda} - e^{-2\lambda} \left(\partial_{x_i}\sigma_j + \partial_{x_j}\sigma_i \right) - 2 \, e^{-2\lambda} \delta_{ij} \sum_{l=1}^{2} \sigma_l \, \partial_{x_l}\lambda \quad \text{(II.34)}$$

Combining (II.33) and (II.34) we obtain

$$\frac{d}{dt} < \overline{q}, \mathbb{I}^0 >_g = 4 \, e^{-6\lambda} v \sum_{ijk=1}^{2} \mathbb{I}_{ij} \mathbb{I}_{ik}^0 \overline{q}_{jk} + \left\langle \frac{d\mathbb{I}^0}{dt}, \overline{q} \right\rangle_g$$

$$-2 \, e^{-4\lambda} \sum_{i,j,k} (\partial_{x_i}\sigma_j + \partial_{x_j}\sigma_i) \, \mathbb{I}_{ik}^0 \, \overline{q}_{jk} - 4 \, < \overline{q}, \mathbb{I}^0 >_g \sum_{l=1}^{2} \sigma_l \, \partial_{x_l}\lambda \quad \text{(II.35)}$$

Observe that

$$\sum_{i,j,k=1}^{2} \mathbb{I}_{ki}^0 \, \overline{q}_{kj} \, \mathbb{I}_{ij}^0 = \sum_{i,j=1}^{2} \mathbb{I}_{ji}^0 \, \overline{q}_{jj} \, \mathbb{I}_{ij}^0 + \sum_{i,j=1}^{2} \mathbb{I}_{j+1i}^0 \, \overline{q}_{j+1j} \, \mathbb{I}_{ij}^0$$

$$= a \sum_{i=1}^{2} \mathbb{I}_{1i}^0 \, \mathbb{I}_{i1}^0 - a \sum_{i=1}^{2} \mathbb{I}_{2i}^0 \, \mathbb{I}_{i2}^0 - b \sum_{i,j=1}^{2} \mathbb{I}_{j+1i}^0 \, \mathbb{I}_{ij}^0$$

Since $\mathbb{I}_{11}^0 = -\mathbb{I}_{22}^0$ and $\mathbb{I}_{12}^0 = \mathbb{I}_{21}^0$ we deduce

$$\sum_{i,j,k=1}^{2} \mathbb{I}_{ki}^0 \, \overline{q}_{kj} \, \mathbb{I}_{ij}^0 = -b \sum_{i,j=1}^{2} \mathbb{I}_{j+1i}^0 \, \mathbb{I}_{ij}^0$$

$$= -b \sum_{i=1}^{2} \mathbb{I}_{1i}^0 \, \mathbb{I}_{i2}^0 - b \sum_{i=1}^{2} \mathbb{I}_{2i}^0 \, \mathbb{I}_{i1}^0 = -b \, \mathbb{I}_{11}^0 \, \mathbb{I}_{12}^0 - b \, \mathbb{I}_{12}^0 \, \mathbb{I}_{22}^0 - b \, \mathbb{I}_{21}^0 \, \mathbb{I}_{11}^0 - b \, \mathbb{I}_{22}^0 \, \mathbb{I}_{21}^0 = 0$$

Combining this identity and (II.35) we obtain

$$\frac{d}{dt} < \overline{q}, \mathbb{I}^0 >_g = 4 \, H \, v \, < \overline{q}, \mathbb{I}^0 >_g + \left\langle \frac{d\mathbb{I}^0}{dt}, \overline{q} \right\rangle_g$$

$$-2 \, e^{-4\lambda} \sum_{i,j,k} (\partial_{x_i}\sigma_j + \partial_{x_j}\sigma_i) \, \mathbb{I}_{ik}^0 \, \overline{q}_{jk} - 4 \, < \overline{q}, \mathbb{I}^0 >_g \sum_{l=1}^{2} \sigma_l \, \partial_{x_l}\lambda \quad \text{(II.36)}$$

A short computation gives

$$-2\,e^{-4\lambda}\sum_{i,j,k}(\partial_{x_i}\sigma_j+\partial_{x_j}\sigma_i)\,\mathbb{I}^0_{ik}\,\overline{q}_{jk}=-\,2\,(\partial_{x_1}\sigma_1+\partial_{x_2}\sigma_2)\ <\overline{q},\mathbb{I}^0>_g \tag{II.37}$$

Combining (II.36) and (II.37) one obtains

$$\frac{d}{dt}<\overline{q},\mathbb{I}^0>_g=4\,H\,v\ <\overline{q},\mathbb{I}^0>_g+\left\langle\frac{d\mathbb{I}^0}{dt},\overline{q}\right\rangle_g$$
$$-2\,e^{-2\lambda}\,(\partial_{x_1}(e^{2\lambda}\,\sigma_1)+\partial_{x_2}(e^{2\lambda}\,\sigma_2))\ <\overline{q},\mathbb{I}^0>_g \tag{II.38}$$

Together with (II.32) this gives

$$\frac{d}{dt}<\overline{q},\mathbb{I}^0>_g=4\,H\,v\ <\overline{q},\mathbb{I}^0>_g-2\,e^{-2\lambda}\,(\partial_{x_1}(e^{2\lambda}\,\sigma_1)+\partial_{x_2}(e^{2\lambda}\,\sigma_2))\ <\overline{q},\mathbb{I}^0>_g$$

$$+e^{-2\lambda}\left[\partial_{x_1}(a\,e^{-2\lambda}\partial_{x_1}v)-\partial_{x_2}(a\,e^{-2\lambda}\partial_{x_2}v)+\partial_{x_1}(b\,e^{-2\lambda}\partial_{x_2}v)+\partial_{x_2}(b\,e^{-2\lambda}\partial_{x_1}v)\right]$$

$$-e^{-4\lambda}\,H\,a\,(\partial_{x_1}(e^{2\lambda}\sigma_1)-\partial_{x_2}(e^{2\lambda}\sigma_2))-e^{-4\lambda}\,H\,b\,(\partial_{x_1}(e^{2\lambda}\sigma_2)+\partial_{x_2}(e^{2\lambda}\sigma_1))$$

$$+e^{-4\lambda}\,(a\,\mathbb{I}^0_{11}+b\,\mathbb{I}^0_{12})\,(\partial_{x_1}\sigma_1+\partial_{x_2}\sigma_2)+e^{-4\lambda}\,(a\,\mathbb{I}^0_{12}-b\,\mathbb{I}^0_{11})\,(\partial_{x_1}\sigma_2-\partial_{x_2}\sigma_1)$$

$$+e^{-4\lambda}\,\partial_{x_1}\left(a\left[\mathbb{I}^0_{11}\sigma_1+\mathbb{I}^0_{12}\sigma_2\right]+b\left[(\mathbb{I}^0_{12}\sigma_1-\mathbb{I}^0_{11}\sigma_2\right]\right)$$

$$-e^{-4\lambda}\,\partial_{x_2}\left(a\left[\mathbb{I}^0_{12}\sigma_1-\mathbb{I}^0_{11}\sigma_2\right]-b\left[\mathbb{I}^0_{11}\sigma_1+\mathbb{I}^0_{12}\sigma_2\right]\right)$$

$$+e^{-4\lambda}\left[\partial_{x_1}\left(e^{2\lambda}\,H\,(a\,\sigma_1+b\,\sigma_2)\right)+\partial_{x_2}\left(e^{2\lambda}\,H\,(b\,\sigma_1-a\,\sigma_2)\right)\right] \tag{II.39}$$

Observe

$$\frac{d}{dt}\left(dvol_{g_t}\right)\Big|_{t=0}=\frac{d}{dt}\left(det(g_t)_{ij}\right)^{1/2}\Big|_{t=0}dx_1\wedge dx_2$$
$$=\frac{1}{2}e^{-2\lambda}\,\frac{d}{dt}\left((g_t)_{11}\,(g_t)_{22}-(g_t)^2_{12}\right)\Big|_{t=0}dx_1\wedge dx_2$$
$$=\frac{1}{2}\left(\frac{d}{dt}(g_t)_{11}\Big|_{t=0}+\frac{d}{dt}(g_t)_{22}\Big|_{t=0}\right)dx_1\wedge dx_2$$
$$=\left(\partial_{x_1}\vec{\Phi}\cdot\partial_{x_1}\vec{w}+\partial_{x_2}\vec{\Phi}\cdot\partial_{x_2}\vec{w}\right)dx_1\wedge dx_2$$
$$=-2\vec{H}\cdot\vec{w}\,dvol_g+\left(\partial_{x_1}(e^{2\lambda}\sigma_1)+\partial_{x_2}(e^{2\lambda}\sigma_2)\right)dx_1\wedge dx_2. \tag{II.40}$$

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II.1.1 The first and second derivatives of the area under constrained conformal class

We consider $\vec{\Phi}$ a weak immersion in $\mathcal{E}_\Sigma$. We fix generators of the π_1 and we denote by $\mathcal{C}(\vec{\Phi})$ the corresponding Teichmüller class of immersions which are conformal isotopic to $\vec{\Phi}$ - immersions $\vec{\Xi}$ such that there exists a Lipschitz diffeomorphism Ψ isotopic to the identity and conformal from $(\Sigma, \vec{\Phi}^* g_{S^3})$ into $(\Sigma, \vec{\Xi}^* g_{S^3})$ -. Let $D_0((-1,1), \mathcal{E}_\Sigma)$ be the space of mapping from $(-1,1)$ into $\mathcal{E}_\Sigma$ which are continuous and differentiable at 0. We say that $\vec{\Phi}$ is a critical point of the area under constrained conformal class if

$$\begin{cases} \forall \, \vec{\Phi}_t \in D_0((-1,1), \mathcal{E}_\Sigma) \quad \text{s.t.} \quad \forall \, t \in (-1,1) \quad \mathcal{C}(\vec{\Phi}_t) = \mathcal{C}(\vec{\Phi}) \\ \\ \text{then} \quad \dfrac{d}{dt} A(\vec{\Phi}_t)(0) = 0 \end{cases} \tag{II.41}$$

As shown in [Ri4], a weak immersion of Σ is a critical point of the area under constrained conformal class if and only if there exists an holomorphic quadratic form Q such that

$$H = \Re < Q, h^0 >_{wp} \quad . \tag{II.42}$$

writing locally $Q = (Q_1 + i\, Q_2)\, dz^2$, since

$$h^0 = e^{2\lambda} [H_\Re^0 + i\, H_\Im^0]\, dz^2 = (\mathbb{I}_{11}^0 - i\, \mathbb{I}_{12}^0)\, dz^2 = -(\mathbb{I}_{22}^0 + i\, \mathbb{I}_{12}^0)\, dz^2$$

, we have that

$$\Re < Q, h^0 >_{wp} = 4\, e^{-4\lambda} \Re \left((Q_1 - i\, Q_2)\, (\mathbb{I}_{11}^0 - i\, \mathbb{I}_{12}^0) \right) = 4\, e^{-4\lambda} [Q_1 \mathbb{I}_{11}^0 - Q_2 \mathbb{I}_{12}^0]$$

where we are using that $|dz^2|_g^2 = 4\, e^{-4\lambda}$. Let

$$\overline{q} := 2\, \Re(Q) = 2\, Q_1\, [dx_1^2 - dx_2^2] - 2\, Q_2\, [dx_1\, dx_2 + dx_2\, dx_1]$$

then we have

$$< \overline{q}, \mathbb{I}^0 >_g = 4\, e^{-4\lambda} [Q_1\, \mathbb{I}_{11}^0 - Q_2 \mathbb{I}_{12}^0] = \Re < Q, h^0 >_{wp}$$

Hence $\vec{\Phi}$ is a critical point of the area under constrained conformal class if and only if there exists $\overline{q} := 2\, Q_1\, [dx_1^2 - dx_2^2] - 2\, Q_2\, [dx_1\, dx_2 + dx_2\, dx_1]$ such that $Q_1 + i\, Q_2$ is holomorphic and

$$H = < \overline{q}, \mathbb{I}^0 >_g \quad . \tag{II.43}$$

Let $\vec{\Phi}_t$ be a mapping in $C^2((-1,1), \mathcal{E}_\Sigma)$ such that $\mathcal{C}(\vec{\Phi}_t) \equiv \mathcal{C}(\vec{\Phi})$ for $t \in (-1,1)$ and such that $\vec{\Phi}_0 = \vec{\Phi}$ is a critical point of the area under constrained conformal class. We are now computing the second derivative of $A(\vec{\Phi}_t)$ at 0. Using (II.40) we obtain

$$\frac{dA(\vec{\Phi}_t)}{dt} = \frac{d}{dt} \left[\int_{\Sigma^2} dvol_{g_t} \right] = -2 \int_\Sigma \vec{H}_t \cdot \frac{d\vec{\Phi}_t}{dt}\, dvol_{g_t} \quad , \tag{II.44}$$

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where g_t denotes the metric on Σ given by $g_t := \vec{\Phi}_t^* g_{S^3}$ and $\vec{H}_t$ is the mean curvature vector of the immersion $\vec{\Phi}_t$.

$$
\frac{d^2 A(\vec{\Phi}_t)}{dt^2}(0) = -2 \int_\Sigma \frac{dH}{dt}(0) \, v \, dvol_g - 2 \int_\Sigma H \frac{d\vec{n}}{dt} \cdot \vec{w} \, dvol_g
$$
$$
-2 \int_\Sigma H \, \vec{n} \cdot \frac{d}{dt} \left(\frac{d\vec{\Phi}_t}{dt} \, dvol_{g_t} \right)
$$
(II.45)

Combining (II.45) and (II.20) gives

$$
\frac{d^2 A(\vec{\Phi}_t)}{dt^2}(0) = \int_\Sigma |dv|_g^2 \, dvol_g - \int_\Sigma [|\mathbb{I}|_g^2 + 2] \, v^2 \, dvol_g
$$
$$
-2 \int_\Sigma v \sum_{k=1}^2 \sigma_k \, \partial_{x_k} H \, dvol_g - 2 \int_\Sigma H \frac{d\vec{n}}{dt} \cdot \vec{w} \, dvol_g
$$
$$
-2 \int_\Sigma H \, \vec{n} \cdot \frac{d}{dt} \left(\frac{d\vec{\Phi}_t}{dt} \, dvol_{g_t} \right)
$$
(II.46)

Since $\mathcal{C}(\vec{\Phi}_t) \equiv \mathcal{C}(\vec{\Phi})$ we have that

$$
d\mathcal{C}_{\vec{\Phi}_t} \frac{d\vec{\Phi}}{dt} \equiv 0
$$

we have that for any Q holomorphic quadratic form for the conformal class defined by $\vec{\Phi}^* g_{S^3}$

$$
0 \equiv \int_\Sigma \Re < Q, \vec{h}_t^0 >_{wp} \cdot \frac{d\vec{\Phi}_t}{dt} \, dvol_{g_t} = \int_\Sigma < \overline{q}, \vec{\mathbb{I}}_t^0 >_{g_t} \cdot \frac{d\vec{\Phi}_t}{dt} \, dvol_{g_t} \quad .
$$
(II.47)

Taking the derivative of this identity gives

$$
\int_\Sigma \frac{d}{dt} \left(< \overline{q}, \vec{\mathbb{I}}_t^0 >_{g_t} \right) \cdot \frac{d\vec{\Phi}_t}{dt} \, dvol_{g_t} = - \int_\Sigma < \overline{q}, \vec{\mathbb{I}}_t^0 >_{g_t} \cdot \frac{d}{dt} \left(\frac{d\vec{\Phi}_t}{dt} \, dvol_{g_t} \right)
$$
(II.48)

We then have

$$
-2 \int_\Sigma H \, \vec{n} \cdot \frac{d}{dt} \left(\frac{d\vec{\Phi}_t}{dt} \, dvol_{g_t} \right) = 2 \int_\Sigma \frac{d}{dt} \left(< \overline{q}, \mathbb{I}_t^0 >_{g_t} \right) \, v \, dvol_g
$$
$$
+2 \int_\Sigma \left(< \overline{q}, \mathbb{I}_t^0 >_{g_t} \right) \frac{d\vec{n}}{dt} \cdot \vec{w} \, dvol_g
$$
(II.49)

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Combining (II.46) and (II.49) we obtain

$$\frac{d^2 A(\vec{\Phi}_t)}{dt^2}(0) = \int_\Sigma |dv|_g^2 \ dvol_g - \int_\Sigma [|\mathbb{I}|_g^2 + 2] \ v^2 \ dvol_g$$

$$-2 \int_\Sigma v \sum_{k=1}^2 \sigma_k \ \partial_{x_k} H \ dvol_g + 2 \int_\Sigma \frac{d}{dt} \left(< \bar{q}, \mathbb{I}_t^0 >_{g_t} \right) \ v \ dvol_g \tag{II.50}$$

The identity (II.39) gives

$$-2 \int_\Sigma v \sum_{k=1}^2 \sigma_k \ \partial_{x_k} H \ dvol_g + 2 \int_\Sigma \frac{d}{dt} \left(< \bar{q}, \mathbb{I}_t^0 >_{g_t} \right) \ v \ dvol_g$$

$$= \int_\Sigma 8 \, H^2 \, v^2 \ dvol_g - 2 \int_\Sigma < \bar{q}, dv \otimes dv >_g \ dvol_g$$

$$-2 \int_\Sigma v \sum_{k=1}^2 \sigma_k \ \partial_{x_k} H \ dvol_g - 4 \int_\Sigma H \, v \, (\partial_{x_1}(e^{2\lambda} \, \sigma_1) + \partial_{x_2}(e^{2\lambda} \, \sigma_2)) \ dx_1 \wedge dx_2$$

$$-2 \int_\Sigma H \, v \, e^{-2\lambda} \ \left[a \ (\partial_{x_1}(e^{2\lambda}\sigma_1) - \partial_{x_2}(e^{2\lambda}\sigma_2)) + b \ (\partial_{x_1}(e^{2\lambda}\sigma_2) + \partial_{x_2}(e^{2\lambda}\sigma_1)) \right] \ dx_1 \wedge dx_2$$

$$+2 \int_\Sigma v \, e^{-2\lambda} \ \left[(a \, \mathbb{I}_{11}^0 + b \, \mathbb{I}_{12}^0) \ (\partial_{x_1}\sigma_1 + \partial_{x_2}\sigma_2) + (a \, \mathbb{I}_{12}^0 - b \, \mathbb{I}_{11}^0) \ (\partial_{x_1}\sigma_2 - \partial_{x_2}\sigma_1) \right] \ dx_1 \wedge dx_2$$

$$+2 \int_\Sigma v \, e^{-2\lambda} \ \partial_{x_1} \left(a \left[\mathbb{I}_{11}^0 \sigma_1 + \mathbb{I}_{12}^0 \sigma_2 \right] + b \left[(\mathbb{I}_{12}^0 \sigma_1 - \mathbb{I}_{11}^0 \sigma_2) \right] \right) \ dx_1 \wedge dx_2$$

$$-2 \int_\Sigma v \, e^{-2\lambda} \ \partial_{x_2} \left(a \left[\mathbb{I}_{12}^0 \sigma_1 - \mathbb{I}_{11}^0 \sigma_2 \right] - b \left[\mathbb{I}_{11}^0 \sigma_1 + \mathbb{I}_{12}^0 \sigma_2 \right] \right) \ dx_1 \wedge dx_2$$

$$+2 \int_\Sigma v \, e^{-2\lambda} \ \left[\partial_{x_1} \left(e^{2\lambda} \, H \, (a \, \sigma_1 + b \, \sigma_2) \right) + \partial_{x_2} \left(e^{2\lambda} \, H \, (b \, \sigma_1 - a \, \sigma_2) \right) \right] \ dx_1 \wedge dx_2 \tag{II.51}$$

We have in one hand

$$+2 \int_\Sigma v \, e^{-2\lambda} \ \partial_{x_1} \left(a \left[\mathbb{I}_{11}^0 \sigma_1 + \mathbb{I}_{12}^0 \sigma_2 \right] + b \left[(\mathbb{I}_{12}^0 \sigma_1 - \mathbb{I}_{11}^0 \sigma_2) \right] \right) \ dx_1 \wedge dx_2$$

$$-2 \int_\Sigma v \, e^{-2\lambda} \ \partial_{x_2} \left(a \left[\mathbb{I}_{12}^0 \sigma_1 - \mathbb{I}_{11}^0 \sigma_2 \right] - b \left[\mathbb{I}_{11}^0 \sigma_1 + \mathbb{I}_{12}^0 \sigma_2 \right] \right) \ dx_1 \wedge dx_2$$

$$= -2 \int_\Sigma (a \, \mathbb{I}_{11}^0 + b \, \mathbb{I}_{12}^0) \ \left[\partial_{x_1}(v \, e^{-2\lambda}) \ \sigma_1 + \partial_{x_2}(v \, e^{-2\lambda}) \ \sigma_2 \right] \ dx_1 \wedge dx_2$$

$$-2 \int_\Sigma (a \, \mathbb{I}_{12}^0 - b \, \mathbb{I}_{11}^0) \ \left[\partial_{x_1}(v \, e^{-2\lambda}) \ \sigma_2 - \partial_{x_2}(v \, e^{-2\lambda}) \ \sigma_1 \right] \ dx_1 \wedge dx_2 \tag{II.52}$$

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Using the fact that $H = 2\, e^{-4\lambda}\, (a\, \mathbb{I}_{11}^0 + b\, \mathbb{I}_{12}^0)$, a short computation gives

$$
-2 \int_\Sigma H\, v\, (\partial_{x_1}(e^{2\lambda}\, \sigma_1) + \partial_{x_2}(e^{2\lambda}\, \sigma_2))\, dx_1 \wedge dx_2
$$

$$
+2 \int_\Sigma v\, e^{-2\lambda}\, \left[(a\, \mathbb{I}_{11}^0 + b\, \mathbb{I}_{12}^0)\right] (\partial_{x_1}\sigma_1 + \partial_{x_2}\sigma_2)\, dx_1 \wedge dx_2
$$

$$
-2 \int_\Sigma (a\, \mathbb{I}_{11}^0 + b\, \mathbb{I}_{12}^0)\, \left[\partial_{x_1}(v\, e^{-2\lambda})\, \sigma_1 + \partial_{x_2}(v\, e^{-2\lambda})\, \sigma_2\right]\, dx_1 \wedge dx_2
$$

$$
= \int_\Sigma v\, e^{2\lambda}\, [\sigma_1\, \partial_{x_1} H + \sigma_2\, \partial_{x_2} H]\, dx_1 \wedge dx_2
$$

$$(\text{II.53})$$

Another short computation gives also

$$
-2 \int_\Sigma H\, v\, e^{-2\lambda}\, \left[a\, (\partial_{x_1}(e^{2\lambda}\sigma_1) - \partial_{x_2}(e^{2\lambda}\sigma_2)) + b\, (\partial_{x_1}(e^{2\lambda}\sigma_2) + \partial_{x_2}(e^{2\lambda}\sigma_1))\right]\, dx_1 \wedge dx_2
$$

$$
+2 \int_\Sigma v\, e^{-2\lambda}\, \left[\partial_{x_1}\left(e^{2\lambda}\, H\, (a\, \sigma_1 + b\, \sigma_2)\right) + \partial_{x_2}\left(e^{2\lambda}\, H\, (b\, \sigma_1 - a\, \sigma_2)\right)\right]\, dx_1 \wedge dx_2
$$

$$
= 2 \int_\Sigma v\, \left[(a\, \sigma_1 + b\, \sigma_2)\, \partial_{x_1} H + (b\sigma_1 - a\, \sigma_2)\, \partial_{x_2} H\right]\, dx_1 \wedge dx_2
$$

$$(\text{II.54})$$

We have also

$$
-2 \int_\Sigma v \sum_{k=1}^2 \sigma_k\, \partial_{x_k} H\, dvol_g - 2 \int_\Sigma H\, v\, (\partial_{x_1}(e^{2\lambda}\, \sigma_1) + \partial_{x_2}(e^{2\lambda}\, \sigma_2))\, dx_1 \wedge dx_2
$$

$$
= 2 \int_\Sigma H \sum_{k=1}^2 \sigma_k\, \partial_{x_k} v\, dvol_g
$$

$$(\text{II.55})$$

Combining (II.50)....(II.55) gives

$$
\frac{d^2 A(\vec{\Phi}_t)}{dt^2}(0) = \int_\Sigma \left[|dv|_g^2 - 2 < \overline{q}, dv \otimes dv >_g\ -[\|\mathbb{I}\|_g^2 + 2 - 8\, H^2]\, v^2\right]\, dvol_g
$$

$$
+2 \int_\Sigma H \sum_{k=1}^2 \sigma_k\, \partial_{x_k} v\, dvol_g + \int_\Sigma v \sum_{k=1}^2 \sigma_k\, \partial_{x_k} H\, dvol_g
$$

$$
+2 \int_\Sigma v\, \left[(a\, \sigma_1 + b\, \sigma_2)\, \partial_{x_1} H + (b\sigma_1 - a\, \sigma_2)\, \partial_{x_2} H\right]\, dx_1 \wedge dx_2
$$

$$
- \int_\Sigma v\ \Im \left\langle Q, h^0 \right\rangle_{wp}\, (\partial_{x_1}\sigma_2 - \partial_{x_2}\sigma_1)\, dvol_g
$$

$$
+ \int_\Sigma e^{2\lambda}\, \Im \left\langle Q, h^0 \right\rangle_{wp}\, \left[\partial_{x_1}(v\, e^{-2\lambda})\, \sigma_2 - \partial_{x_2}(v\, e^{-2\lambda})\, \sigma_1\right]\, dvol_g
$$

$$(\text{II.56})$$

We denote $\vec{w}_T$ to be the following vector-field tangent to the surface $\vec{w}_T := \vec{w} - \vec{w} \cdot \vec{n}\,\vec{n}$. We have for instance

$$\sum_{k=1}^{2} \sigma_k \, \partial_{x_k} v = (dv, \vec{w}_T)_{T^*\Sigma, T\Sigma} \quad \text{and} \quad \sum_{k=1}^{2} \sigma_k \, \partial_{x_k} H = (dH, \vec{w}_T)_{T^*\Sigma, T\Sigma}$$

Observe that we have

$$- v \, (\partial_{x_1}\sigma_2 - \partial_{x_2}\sigma_1) + e^{2\lambda} \left[\partial_{x_1}(v\, e^{-2\lambda}) \, \sigma_2 - \partial_{x_2}(v\, e^{-2\lambda}) \, \sigma_1 \right]$$

$$= - v \, e^{-2\lambda} \left[\partial_{x_1}(e^{2\lambda} \, \sigma_2) - \partial_{x_2}(e^{2\lambda} \, \sigma_1) \right] + (\partial_{x_1} v \, \sigma_2 - \partial_{x_2} v \, \sigma_1)$$

Denote $\vec{w}_T^*$ the 1-form dual to the vector-field $\vec{w}_T$ for the induced metric on the surface. We have in particular

$$* d\vec{w}_T^* = e^{-2\lambda} \left[\partial_{x_1}(e^{2\lambda} \, \sigma_2) - \partial_{x_2}(e^{2\lambda} \, \sigma_1) \right]$$

Finally we denote by $\bar{q} \llcorner dH = 2\,\Re(Q) \llcorner dH$ the following contraction between the real part of the holomorphic quadratic form $2\,Q$ and the one form dH

$$\bar{q} \llcorner dH = e^{-2\lambda} \, a \, (\partial_{x_1} H \, dx_1 - \partial_{x_2} H \, dx_2) + e^{-2\lambda} \, b \, (\partial_{x_1} H \, dx_2 + \partial_{x_2} H \, dx_1)$$

We can summarize the results obtained so far in the following lemma.

Lemma II.1. *Let $\vec{\Phi}$ be a weak immersion in $\mathcal{E}_\Sigma(S^3)$. Assume that $\vec{\Phi}$ is a critical point of the area within the conformal class defined by $\vec{\Phi}$, then there exists an holomorphic quadratic form Q such that*

$$H = \Re < Q, h^0 >_{wp} \quad . \tag{II.57}$$

Taking now a path such that $\vec{\Phi}_t$ is a mapping in $C^2((-1,1), \mathcal{E}_\Sigma)(S^3)$ satisfying that $\mathcal{C}(\vec{\Phi}_t) \equiv \mathcal{C}(\vec{\Phi})$ for $t \in (-1,1)$ and such that $\vec{\Phi}_0 = \vec{\Phi}$ and $d\vec{\Phi}_t/dt(0) = \vec{w}$, we have

$$\frac{d^2 A(\vec{\Phi}_t)}{dt^2}(0) = \int_\Sigma \left[|dv|_g^2 - 4 < \Re(Q), dv \otimes dv >_g \; - [\|\mathbb{I}\|_g^2 + 2 - 8\,H^2] \, v^2 \right] \; dvol_g$$

$$+ \int_\Sigma \left[2H \, dv + v \, dH + 4\,v\,\Re(Q) \llcorner dH + \Im \left\langle Q, h^0 \right\rangle * dv \right] \cdot \vec{w}_T \; dvol_g \tag{II.58}$$

$$- \int_\Sigma \Im \left\langle Q, h^0 \right\rangle \, v * d\vec{w}_T^* \; dvol_g \quad .$$

where $v := \vec{w} \cdot \vec{n}$, where $\vec{w}_T$ is the vector-field tangent to the surface obtained by projecting $\vec{w}$ orthogonally (i.e. $\vec{w}_T := \vec{w} - \vec{w} \cdot \vec{n}\,\vec{n}$) onto $\vec{\Phi}_* T\Sigma$. We denote $\vec{w}_T^*$ the 1-form dual to $\vec{w}$ for the induced metric g. We have denoted $\Re(Q) \llcorner dH$ the contraction with respect to the induced metric g between the real part of the holomorphic quadratic form Q and dH. The brakets $<,>$ denotes the scalar product between quadratic forms induced by the metric g. Finally $\cdot$ is the canonical contraction between 1-forms and vector-fields on Σ. $\square$

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The computation of the second derivative (II.58) was a bit long due to the fact that we were considering general variations of the form $\vec{\Phi}_t$ where $d\vec{\Phi}_t/dt(0) = \vec{w}$ is not necessarily parallel to $\vec{n}$. This was needed due to regularity issues. Since we are considering variations around a *weak immersion* which is not necessarily smooth $\vec{n}$ is only in $W^{1,2}$ and a deformation such as $\vec{\Phi} + tv\,\vec{n}$ would not be within the class of weak immersions anymore.

III   Proof of theorem I.1.

Let $\vec{\Phi}$ be a weak immersion satisfying the *strictly elliptic conformally constrained minimal surface equation* that is to say, there exists an holomorphic quadratic form Q such that

$$H = \Re < Q, h^0 >_{g_{\vec{\Phi}}} \quad \text{and} \quad 2\,|Q|_{g_{\vec{\Phi}}}(x) < 1 \quad \text{on} \quad T^2$$

We can choose locally complex coordinates such that $Q = 4^{-1}\,dz^2$. In these coordinates that we can assume to be defined on the disc D^2, the strict ellipticity condition $2\,|Q|_{g_{\vec{\Phi}}}(x) < 1$ becomes $\lambda > 0$ where $g_{\vec{\Phi}} := e^{2\lambda}\,[dx_1^2 + dx_2^2]$ and the conformally constrained minimal equation becomes

$$H = e^{-2\lambda}H_\Re^0 \quad . \tag{III.1}$$

From the Codazzi equation (A.4) we then have

$$\Delta(e^{4\lambda}H) = \partial_{x_1}(e^{2\lambda}\partial_{x_1}H) - \partial_{x_2}(e^{2\lambda}\partial_{x_2}H) \tag{III.2}$$

Let $u := e^{4\lambda}H$, assuming $\vec{\Phi}$ is a weak immersion gives that in complex coordinates $\nabla\lambda \in W^{1,1}$ and hence $\lambda \in C^0$ from which we deduce that $u \in L^2$. The function u satisfies moreover the elliptic PDE

$$\partial_{x_1}((1 - e^{-2\lambda})\,\partial_{x_1}u) + \partial_{x_2}((1 + e^{-2\lambda})\partial_{x_2}u) = -2\,\left[\partial_{x_1}(\partial_{x_1}e^{-2\lambda}\,u) - \partial_{x_2}(\partial_{x_2}e^{2\lambda}\,u)\right] \tag{III.3}$$

This PDE [1] is indeed elliptic since $\lambda > 0$ and $\lambda \in C^0$ thus we can assume that there exists $c_0 > 0$ such that $\lambda \geq c$ on D^2 and the symbol in the l.h.s. of (III.3) is invertible. If we can show that $u \in L^p_{loc}(D^2)$ for some $p > 2$, this will make the PDE subcritical which implies imply by standard bootstrap arguments that $\nabla u \in L^{2,1}$ and then $\nabla H \in L^{2,1}$. Bootstraping this information respectively in

$$\Delta\vec{\Phi} + \vec{\Phi}\,|\nabla\vec{\Phi}|^2 = 2\,\vec{H}$$

---

[1] Observe that this equation is critical for $u \in L^2$ or even $u \in L^{2,\infty}$. Indeed, $\nabla e^{2\lambda}$ is in the Lorentz space $L^{2,1}$ (see [Ri1]) and thus $\nabla e^{2\lambda}\,u \in L^1$. The Laplacian of a function being a divergence of an L^1 function implies that this function is in $L^{2,\infty}$ and we are back to the space we are starting from which is the definition of being critical.

using Wente integrability by compensation theory (see [Ri1]) we obtain that $\nabla^2 \vec{\Phi} \in \cap_{p<+\infty} L^p$....which finally implies, using equation (I.5), that $\vec{\Phi}$ is analytic.

We sketch now the proof of the fact that $u \in L^p$ for some $p > 2$ which is the only claim that remains to be proven in order to complete the proof of theorem I.1. Before establishing the fact that $u \in L^p_{loc}(D^2)$ for some $p > 2$ we first prove the existence of $\alpha > 0$ such that

$$\sup_{x^0 \in D^2_{1/2} \,,\, 0<r<1/4} r^{-\alpha} \, \|u\|_{L^{2,\infty}(B_r(x^0))} < +\infty \tag{III.4}$$

Let $x^0 \in D^2_{1/2}$ and $r < r_0$ where r_0 is going to be fixed later in the proof. Let χ be a cut-off function on $\mathbb{R}^2$ such that $\chi \equiv 1$ on $B^2_1(0)$ and $\mathrm{supp}(\chi) \subset B^2_2(0)$. We denote by $\chi_r^{x^0}(x)$ the function given by $\chi_r^{x^0}(x) := \chi(r(x-x^0))$. A short computation gives

$$\partial_{x_1}\left((1 - e^{-2\lambda}) \, \partial_{x_1}(\chi_r^{x^0} \, u)\right) + \partial_{x_2}\left((1 + e^{-2\lambda}) \, \partial_{x_2}(\chi_r^{x^0} \, u)\right)$$

$$= 2 \, \chi_r^{x^0} \left[\partial_{x_1}(\partial_{x_1} e^{-2\lambda} \, u) - \partial_{x_2}(\partial_{x_2} e^{-2\lambda} \, u)\right]$$

$$+ 2 \, \partial_{x_1}\left((1 - e^{-2\lambda}) \, \partial_{x_1}(\chi_r^{x^0}) \, u\right) + 2 \, \partial_{x_2}\left((1 + e^{-2\lambda}) \, \partial_{x_2}(\chi_r^{x^0}) \, u\right) \tag{III.5}$$

$$+ \partial_{x_1}(\chi_r^{x^0}) \, \partial_{x_1} e^{-2\lambda} \, u - \partial_{x_2}(\chi_r^{x^0}) \, \partial_{x_2} e^{-2\lambda} \, u$$

$$- (1 - e^{-2\lambda}) \, \partial^2_{x_1^2}(\chi_r^{x^0}) \, u - (1 + e^{-2\lambda}) \, \partial^2_{x_2^2}(\chi_r^{x^0}) \, u$$

We shall now use the following 3 lemmas.

Lemma III.1. *Let λ measurable such that $\lambda \geq c_0 > 0$. For any $p \in (1,2)$ there exists $C_p(c_0)$ such that for any $X \in L^p(D^2, \mathbb{R}^2)$ and $f \in L^1$ there exists a unique solution $v \in W_0^{1,p}(D^2, \mathbb{R})$ of*

$$\begin{cases} \partial_{x_1}((1 - e^{-2\lambda}) \, \partial_{x_1} v) + \partial_{x_2}((1 + e^{-2\lambda})\partial_{x_2} v) = \operatorname{div} X + f & \text{in } D^2 \\ \\ v = 0 & \text{on } \partial D^2 \end{cases}$$

moreover we have

$$\|\nabla v\|_{L^p(D^2)} \leq C(c_0) \, [\|X\|_{L^p(D^2)} + \|f\|_1] \quad . \tag{III.6}$$

$\square$

The proof of this lemma is standard and can be done using the Stampacchia duality method. We then have the following result

Lemma III.2. *Let λ measurable such that $\lambda \geq c_0 > 0$. Let $v \in W^{1,1}(D^2, \mathbb{R})$ such that*

$$\partial_{x_1}((1 - e^{-2\lambda}) \, \partial_{x_1} v) + \partial_{x_2}((1 + e^{-2\lambda})\partial_{x_2} v) = 0 \quad \text{in } D^2 \quad .$$

Then for any $\delta \in (0, 1)$ there exists $\theta \in (0, 1)$ independent of v such that

$$\|v\|_{L^{2,\infty}(B_\theta^2)} \leq \ \delta \ \|v\|_{L^{2,\infty}(B_1^2)} \quad . \tag{III.7}$$

$\square$

This lemma is a straightforward consequence of the fact solutions to such an elliptic equation are smoother than $L^{2,\infty}$, from De Giorgi-Moser classical result we know that they are $W_{loc}^{1,p}$ for some $p > 2$ with ad-hoc estimates which imply (III.7) Finally we are going to make use of the following third lemma.

Lemma III.3. *Let $\lambda \in L^{2,1}(D^2, \mathbb{R})$ such that $\lambda \geq c_0 > 0$. For any $f \in L^{2,1}(D^2)$ there exists a unique $w \in W^{1,\infty}(D^2, \mathbb{R})$ such that*

$$\begin{cases} \partial_{x_1}((1 - e^{-2\lambda}) \, \partial_{x_1} w) + \partial_{x_2}((1 + e^{-2\lambda}) \partial_{x_2} w) = g & in \ D^2 \\ \\ w = 0 & on \ \partial D^2 \end{cases}$$

moreover we have

$$\|\nabla w\|_{L^\infty(D^2)} \leq C(\|\nabla \lambda\|_{2,1}, c_0) \ \|g\|_{L^{2,1}(D^2)} \quad . \tag{III.8}$$

$\square$

The proof of the previous lemma is classical if one replaces the Lorentz space $L^{2,1}$ by any L^q space for $q > 2$. Since the improved Sobolev embedding gives that identity map going from the space of functions with derivatives in the Lorentz space $L^{2,1}$ into C^0 is continuous and since Calderon Zygmund theory extends from the L^q spaces to the Lorentz $L^{2,1}$ space (see [He]), the proof of lemma III.3 follows easily.

We go back to the proof of (III.4). We introduce $v \in W_0^{1,3/2}(B_{2r}^2(x^0))$ given by lemma III.1 solving

$$\partial_{x_1} \left((1 - e^{-2\lambda}) \, \partial_{x_1} v\right) + \partial_{x_2} \left((1 + e^{-2\lambda}) \, \partial_{x_2} v\right)$$

$$= 2 \partial_{x_1} \left((1 - e^{-2\lambda}) \, \partial_{x_1}(\chi_r^{x^0}) \, u\right) + 2 \partial_{x_2} \left((1 + e^{-2\lambda}) \, \partial_{x_2}(\chi_r^{x^0}) \, u\right) \tag{III.9}$$

$$- (1 - e^{-2\lambda}) \, \partial_{x_1^2}^2(\chi_r^{x^0}) \, u - (1 + e^{-2\lambda}) \, \partial_{x_2^2}^2(\chi_r^{x^0}) \, u$$

Then v satisfies

$$\|v\|_{L^{2,\infty}(B_{2r}^2(x^0))} \leq C(c_0) \ \|u\|_{L^{2\infty}(B_{2r}^2(x^0))} \quad . \tag{III.10}$$

Observe that the r.h.s of (III.9) is supported on the annulus $B_{2r}^2(x^0)) \setminus B_r^2(x^0))$ hence, making use of lemma III.1 we deduce from the previous estimate that

$$\forall \delta \in (0, 1) \quad \exists \theta \in (0, 1) \quad \text{s.t.} \quad \|v\|_{L^{2,\infty}(B_{\theta r}^2(x^0))} \leq \ \delta \ \|u\|_{L^{2\infty}(B_{2r}^2(x^0))} \tag{III.11}$$

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Now for any function $g \in L^{2,1}(B^2_{2r}(x^0))$ we consider $w^{x^0}_r$ given by lemma III.3 for λ on $B^2_{2r}(x^0)$ and satisfying then

$$
\begin{cases}
\partial_{x_1}((1 - e^{-2\lambda})\, \partial_{x_1} w^{x^0}_r) + \partial_{x_2}((1 + e^{-2\lambda})\partial_{x_2} w^{x^0}_r) = g & \text{in } B^2_{2r}(x^0) \\[2mm]
w = 0 & \text{on } \partial B^2_{2r}(x^0) \quad .
\end{cases}
\tag{III.12}
$$

Due to scale dependancies of the norms we have

$$
\|\nabla w^{x^0}_r\|_{L^\infty(B^2_{2r}(x^0))} \leq C(\|\nabla\lambda\|_{L^{2,1}(B^2_{2r}(x^0))}, c_0)\, \|f\|_{L^{2,1}(B^2_{2r}(x^0))} \quad .
\tag{III.13}
$$

Multiplying by $w^{x^0}_r - \overline{w^{x^0}_r}$ equation (III.5) to which we have subtracted equation (III.9) and integrating by parts, where $\overline{w^{x^0}_r}$ is the average of $w^{x^0}_r$ on $B^2_{2r}(x^0)$, gives

$$
\int_{B^2_{2r}(x^0)} f\,(\chi^{x^0}_r\, u - v) \leq C\, \|\nabla\lambda\|_{L^{2,1}(B^2_{2r}(x^0))}\, \|u\|_{L^{2,\infty}(B^2_{2r}(x^0))}\, \|\nabla w^{x^0}_r\|_{L^\infty(B^2_{2r}(x^0))}
\tag{III.14}
$$

Taking the sup over any f supported in $B^2_r(x^0)$ whose $L^{2,1}$ norm is less than one gives then

$$
\|u - v\|_{L^{2,\infty}(B^2_r(x^0))} \leq C\, \|\nabla\lambda\|_{L^{2,1}(B^2_{2r}(x^0))}\, \|u\|_{L^{2,\infty}(B^2_{2r}(x^0))} \quad .
\tag{III.15}
$$

We choose now r_0 small enough in such a way that $\sup_{x \in D^2_{1/2}} C\, \|\nabla\lambda\|_{L^{2,1}(B^2_{2r_0}(x^0))} < 1/4$ and choosing θ in (III.10) for $\delta = 1/4$, we have found $\theta \in (0,1)$ independent of r such that

$$
\|u\|_{L^{2,\infty}(B^2_{\theta r}(x^0))} \leq\ 2^{-1}\, \|u\|_{L^{2\infty}(B^2_{2r}(x^0))}
$$

A classical iteration procedure of this inequality implies (III.4) and theorem I.1 is proved.

IV   Isothermic conformally constrained surfaces.

This section is devoted to the proof of theorem I.2.

Let $\vec{\Phi}$ be a weak immersion of the torus T^2 equipped with a conformal structure c into S^3. We assume that $\vec{\Phi}$ satisfies simultaneously the conformally constrained équation

$$
H = \Re < Q, h^0 >_{wp}
\tag{IV.1}
$$

for some holomorphic quadratic differential Q of (T^2, c) and the isothermic condition

$$
\Re < Q', h^0 >_{wp} = 0
\tag{IV.2}
$$

for another non zero holomorphic quadratic differential Q'.

Assume first $H \equiv 0$, that is $\vec{\Phi}$ is minimal, it is clearly a critical point of the area under constrained conformal class. The Codazzi equation implies that the Weingarten

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quadratic form $h^0 = \vec{n} \cdot \partial_{z^2}^2 \vec{\Phi} \, dz^2$ is a holomorphic. h^0 cannot be identically zero, otherwise one would contradict Liouville equation, since $H \equiv 0$. Taking $Q' := i \, h^0 \neq 0$, we have (IV.2) which implies that $\vec{\Phi}$ is isothermic.

We shall exclude the case $H \equiv 0$ in this part from now on. Assuming (IV.1), since $Q \neq 0$ and since the vector space of holomorphic quadratic differential on T^2 is a complex one dimensional space, we can then assume that there exist $\theta \in \mathbb{R}$ such that

$$Q' = e^{i\theta} \, Q$$

Since we are excluding the special case where $H \equiv 0$ we have $\theta \notin \pi \, \mathbb{Z}$. Modulo composition with a diffeomorphism we can assume that $\vec{\Phi}$ is conformal from $\mathbb{R}^2/\Lambda$ equipped with the flat metric where $\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$ with $\omega_j \neq 0$ and $\omega_2/\omega_1 \notin \mathbb{R}$ into S^3. In the canonical complex chart of $\mathbb{R}^2 \simeq \mathbb{C} : z = x_1 + i x_2$ we have that $Q = f(z) \, dz^2$ where f is holomorphic and periodic on $\mathbb{C}$ and therefore constant. Modulo rotations we can assume that $f(z) = 1/4$ and then

$$\Re < Q, h^0 >_{wp} = e^{-2\lambda} \, H_\Re^0$$

where $g = e^{2\lambda} \, (dx_1^2 + dx_2^2)$ and the following system is satisfied

$$\begin{cases} H = e^{-2\lambda} \, H_\Re^0 \\[2ex] \cos\theta \, H_\Re^0 - \sin\theta \, H_\Im^0 = 0 \end{cases} \tag{IV.3}$$

and $\sin\theta > 0$. Let $t := \cos\theta / \sin\theta$, the Codazzi identity (A.4) gives then

$$(1 + i \, t) \, \partial_{\bar{z}} \left(e^{2\lambda} \, H_\Re^0 \right) = e^{2\lambda} \, \partial_z \left(e^{-2\lambda} \, H_\Re^0 \right) \quad . \tag{IV.4}$$

We have $2 \, (1 + i \, t) \, \partial_{\bar{z}} = (\partial_{x_1} - t \, \partial_{x_2}) + i \, (t \, \partial_{x_1} + \partial_{x_2})$. Hence (IV.4) becomes

$$\begin{cases} (\partial_{x_1} - t \, \partial_{x_2}) \mathbb{I}_{11}^0 = e^{-2\lambda} \, \partial_{x_1} \mathbb{I}_{11}^0 + e^{2\lambda} \, \partial_{x_1}(e^{-4\lambda}) \, \mathbb{I}_{11}^0 \\[2ex] (t \, \partial_{x_1} + \partial_{x_2}) \mathbb{I}_{11}^0 = -e^{-2\lambda} \, \partial_{x_2} \mathbb{I}_{11}^0 - e^{2\lambda} \, \partial_{x_2}(e^{-4\lambda}) \, \mathbb{I}_{11}^0 \quad . \end{cases} \tag{IV.5}$$

This gives

$$\begin{cases} (1 - e^{-2\lambda}) \, \partial_{x_1} \mathbb{I}_{11}^0 - t \, \partial_{x_2} \mathbb{I}_{11}^0 = 2 \, \partial_{x_1}(e^{-2\lambda}) \, \mathbb{I}_{11}^0 \\[2ex] t \, \partial_{x_1} \mathbb{I}_{11}^0 + (1 + e^{-2\lambda}) \, \partial_{x_2} \mathbb{I}_{11}^0 = -2 \, \partial_{x_2}(e^{-2\lambda}) \, \mathbb{I}_{11}^0 \quad . \end{cases} \tag{IV.6}$$

From this identity we deduce

$$\begin{cases} (1 + t^2 - e^{-4\lambda}) \, \partial_{x_1} \mathbb{I}_{11}^0 = 2 \, (1 + e^{-2\lambda}) \, \partial_{x_1}(e^{-2\lambda}) \, \mathbb{I}_{11}^0 - 2 \, t \, \partial_{x_2}(e^{-2\lambda}) \, \mathbb{I}_{11}^0 \\[2ex] (1 + t^2 - e^{-4\lambda}) \, \partial_{x_2} \mathbb{I}_{11}^0 = -2 \, t \, \partial_{x_1}(e^{-2\lambda}) \, \mathbb{I}_{11}^0 - 2 \, (1 - e^{-2\lambda}) \, \partial_{x_2}(e^{-2\lambda}) \, \mathbb{I}_{11}^0 \quad . \end{cases} \tag{IV.7}$$

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This implies the following conservation laws

$$\begin{cases} \partial_{x_1}\left[(1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11}\right] = 2\,\partial_{x_1}(e^{-2\lambda})\,\mathbb{I}^0_{11} - 2\,t\,\partial_{x_2}(e^{-2\lambda})\,\mathbb{I}^0_{11} \\[2mm] \partial_{x_2}\left[(1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11}\right] = -2\,t\,\partial_{x_1}(e^{-2\lambda})\,\mathbb{I}^0_{11} - 2\,\partial_{x_2}(e^{-2\lambda})\,\mathbb{I}^0_{11} \end{cases} \tag{IV.8}$$

The fact that we are dealing with conservation laws can be seen as follows. Since, for a weak immersion, the conformal factor is continuous, we have on the open set $\Omega := T^2\setminus(e^{-4\lambda})^{-1}\{1+t^2\} = T^2\setminus(e^{2\lambda})^{-1}\{\sin\theta\}$

$$\begin{cases} \partial_{x_1}\left[(1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11}\right] = [\partial_{x_1}F - t\,\partial_{x_2}F]\,(1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11} \\[2mm] \partial_{x_2}\left[(1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11}\right] = [-t\,\partial_{x_1}F - \partial_{x_2}F]\,(1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11} \end{cases} \tag{IV.9}$$

where

$$F(x) := \sin\theta\,\log\left[\frac{e^{2\lambda}+\sin\theta}{e^{2\lambda}-\sin\theta}\right] \tag{IV.10}$$

Since $\nabla\lambda\in L^{2,1}(T^2)$ where $L^{2,1}$ is the Lorentz space whose dual is the weak Marcinkiewicz space $L^{2,\infty}$ (see the previous section) we have that $G := (1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11}$ satisfies on Ω the following PDE

$$\partial_{\bar z}G = A(z)\,G(z) \tag{IV.11}$$

where $A(z)\in L^{2,1}_{loc}(\Omega)$. This equation implies $\Delta G = 4\,\partial_z(A\,G)$ which is critical for $G\in L^{2,\infty}$. The same approach as the one we employed for proving theorem I.1 implies that $\nabla G\in L^{2,1}_{loc}$ and $G\in L^\infty_{loc}$. Hence $(1+t^2-e^{-4\lambda})\,\mathbb{I}^0_{11}\in L^\infty_{loc}(\Omega)$ and using (IV.3) together with the fact that $\lambda\in L^\infty$, the Liouville equation gives that $\Delta\lambda\in L^\infty_{loc}(\Omega)$. Hence a standard bootstrap equation gives that $\vec\Phi$ is smooth on Ω. Thus both G and A are smooth on Ω.

Carleman unique continuation argument applied to Beltrami equation (IV.11) gives that G, and hence H, has only isolated zeros (we are excluding the fact $H\equiv 0$ since the beginning of this section. Denote $\hat\Omega$ the open set obtained by removing to Ω the possibly existing isolated zeros of H, we then have

$$\begin{cases} \partial_{x_1}\left[\log G\right] = [\partial_{x_1}F - t\,\partial_{x_2}F] \\[2mm] \partial_{x_2}\left[\log G\right] = [-t\,\partial_{x_1}F - \partial_{x_2}F] \end{cases} \tag{IV.12}$$

Or in other words we have

$$\partial_z\log G = (1+i\,t)\,\partial_{\bar z}F \quad. \tag{IV.13}$$

This implies in particular

$$t\left[\partial^2_{x_1^2}F - \partial^2_{x_2^2}F\right] + 2\,\partial^2_{x_1 x_2}F = 0 \qquad \text{on } \hat\Omega \tag{IV.14}$$

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Since $\nabla F \in L^2_{loc}(\Omega)$, we have $t \left[\partial^2_{x_1^2} F - \partial^2_{x_2^2} F \right] + 2 \partial^2_{x_1 x_2} F \in H^{-1}_{loc}(\Omega)$ and is supported at isolated points. A standard argument gives then

$$t \left[\partial^2_{x_1^2} F - \partial^2_{x_2^2} F \right] + 2 \partial^2_{x_1 x_2} F = 0 \qquad \text{on } \Omega \qquad \text{(IV.15)}$$

Let σ such that $\sinh 2\sigma = -t$, which is satisfied for

$$\cosh^2 \sigma = 2^{-1}(1 + \sin^{-1} \theta) \quad \text{and} \quad \sinh^2 \sigma = 2^{-1}(-1 + \sin^{-1} \theta)$$

and let $\tau \in \mathbb{R}$ such that

$$\cos^2 \tau = \frac{\cosh^2 \sigma}{\cosh^2 \sigma + \sinh^2 \sigma} = \frac{1}{2}(1 + \sin \theta)$$

and

$$\sin^2 \tau = \frac{\sinh^2 \sigma}{\cosh^2 \sigma + \sinh^2 \sigma} = \frac{1}{2}(1 - \sin \theta)$$

In particular we choose $\cos \tau > 0$ and $\mathrm{sgn}(\sin \tau) = \mathrm{sgn}(-t)$. Hence, since we are assuming $\theta \in (0, \pi)$, we take

$$\tau = \frac{\theta}{2} - \frac{\pi}{4}$$

With these notations equation (IV.15) becomes

$$(\cos \tau \, \partial_{x_1} + \sin \tau \, \partial_{x_2})(- \sin \tau \, \partial_{x_1} + \cos \tau \, \partial_{x_2}) \, F = 0 \qquad \text{on } \Omega \qquad \text{(IV.16)}$$

We proceed to a rotation of $\mathbb{R}^2$ by τ and denote (y_1, y_2) the canonical coordinates after this rotation :

$$\begin{cases} \dfrac{\partial}{\partial y_1} = \cos \tau \, \dfrac{\partial}{\partial x_1} + \sin \tau \, \dfrac{\partial}{\partial x_2} \\[2mm] \dfrac{\partial}{\partial y_2} = - \sin \tau \, \dfrac{\partial}{\partial x_1} + \cos \tau \, \dfrac{\partial}{\partial x_2} \end{cases} \qquad \text{(IV.17)}$$

Then we have in these rotated coordinates

$$\partial^2_{y_1 y_2} F = 0 \qquad \text{on } \Omega$$

Hence we deduce the existence of two functions R and S which are smooth on Ω such that

$$F(y_1, y_2) = \sin \theta \, \log \left[\frac{e^{2\lambda} + \sin \theta}{e^{2\lambda} - \sin \theta} \right] = \sin \theta \, [R(y_1) + S(y_2)] \qquad . \qquad \text{(IV.18)}$$

The system (IV.17) can be rewritten as follows $\partial_{x_1} + i \, \partial_{x_2} = e^{i\tau} \left(\partial_{y_1} + i \, \partial_{y_2} \right)$ or $\partial_{\bar{z}} = e^{i\tau} \partial_{\bar{w}}$ where we denote respectively $z = x_1 + i \, x_2$ and $w = y_1 + i \, y_2$. We have $\partial_z = e^{-i\tau} \partial_w$ hence $dz = e^{i\tau} \, dw$ which implies

$$- i \, e^{i\theta} \, dw^2 = dz^2$$

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Let $\hat{H}^0 = \hat{H}^0_{\Re} + i\,\hat{H}^0_{\Im}$ be the expression of the Weingarten quadratic form in the w coordinates. The intrinsic nature of the Weingarten quadratic form gives

$$\hat{H}^0\,dw^2 = H^0\,dz^2 \quad\Longrightarrow\quad \hat{H}^0 = -i\,e^{i\theta}\,H^0$$

This gives

$$\begin{cases} \hat{H}^0_{\Re} = \sin\theta\,H^0_{\Re} + \cos\theta\,H^0_{\Im} = (\sin\theta)^{-1}\,H^0_{\Re} \\[2mm] \hat{H}^0_{\Im} = -\cos\theta\,H^0_{\Re} + \sin\theta\,H^0_{\Im} = 0 \quad. \end{cases} \tag{IV.19}$$

Hence, y_1 and y_2 are principal directions for $\mathbb{I}$ and the principal curvatures are given by

$$\begin{cases} \kappa_1 = \hat{H}^0_{\Re} + H = \dfrac{e^{-2\lambda}}{\sin\theta}\,(e^{2\lambda} + \sin\theta)\,H^0_{\Re} \\[3mm] \kappa_2 = -\hat{H}^0_{\Re} + H = \dfrac{e^{-2\lambda}}{\sin\theta}\,(-e^{2\lambda} + \sin\theta)\,H^0_{\Re} \end{cases} \tag{IV.20}$$

In these new coordinates the equation (IV.13) becomes

$$\partial_w \log G = (1 + i\,t)\,e^{2\,i\,\tau}\,\partial_{\overline{w}} F \quad. \tag{IV.21}$$

We have $(1 + i\,t) = (\sin\theta)^{-1}\,i\,e^{-i\theta}$ and since $e^{2\,i\,\tau} = -i\,e^{i\theta}$, we have

$$\partial_w \log G = (\sin\theta)^{-1}\partial_{\overline{w}} F = 2^{-1}\left[\dot{R}(y_1) + i\dot{S}(y_2)\right] \quad. \tag{IV.22}$$

Hence, for each connected component of Ω, we deduce the existence of a constant $C \in \mathbb{R}$ such that

$$\log G = R(y_1) - S(y_2) + C \tag{IV.23}$$

The identity (IV.18) gives

$$e^{2\lambda} = \sin\theta\,\frac{e^{R+S} + 1}{e^{R+S} - 1} \tag{IV.24}$$

and then

$$1 + t^2 - e^{-4\lambda} = \frac{4}{\sin^2\theta}\,\frac{e^{R+S}}{(e^{R+S} + 1)^2} \tag{IV.25}$$

Liouville equation reads

$$-\Delta\lambda = e^{2\lambda}\left[1 + H^2 - e^{-4\lambda}\,[(\mathbb{I}^0_{11})^2 + (\mathbb{I}^0_{12})^2]\right] \tag{IV.26}$$

Using the fact that $H = e^{-4\lambda}\,\mathbb{I}^0_{11}$ and that $\mathbb{I}^0_{12} = t\,\mathbb{I}^0_{11}$ we have

$$-\Delta\lambda = e^{2\lambda}\left[1 - e^{-4\lambda}\,(\mathbb{I}^0_{11})^2\,[1 + t^2 - e^{-4\lambda}]\right] \tag{IV.27}$$

Recall that $G = (1 + t^2 - e^{-4\lambda}) \, \mathbb{I}_{11}^0$, thus combining (IV.23), (IV.24) and ((IV.25) we obtain

$$e^{-4\lambda} \, (\mathbb{I}_{11}^0)^2 \, [1 + t^2 - e^{-4\lambda}] = e^{-4\lambda} \, \frac{G^2}{1 + t^2 - e^{-4\lambda}} = \frac{e^{2C}}{4} \, e^{R-S} \, (e^R - e^{-S})^2 \qquad \text{(IV.28)}$$

Using (IV.24) a short computation gives

$$\nabla \lambda = \frac{2 \, e^{R+S}}{e^{2(R+S)} - 1} \begin{pmatrix} \dot{R} \\ \dot{S} \end{pmatrix} \qquad \text{(IV.29)}$$

and hence

$$\Delta \lambda = \frac{2 \, e^{R+S}}{e^{2(R+S)} - 1} \left[\ddot{R} + \ddot{S} - [(\dot{R})^2 + (\dot{S})^2] \frac{e^{2(R+S)} + 1}{e^{2(R+S)} - 1} \right] \qquad \text{(IV.30)}$$

So the following "double ODE" is satisfied

$$\ddot{R} + \ddot{S} - [(\dot{R})^2 + (\dot{S})^2] \frac{e^{2(R+S)} + 1}{e^{2(R+S)} - 1} = \frac{\sin \theta}{2} \frac{(e^{R+S} + 1)^2}{e^{R+S}} \left[1 - \frac{e^{R+2C}}{4 \, e^S} \, (e^R - e^{-S})^2 \right] \quad , \tag{IV.31}$$

where R depends on y_1 and S depends on y_2. Hence both R and S are analytic functions. Assume both R and S are non constant on each connected component of Ω then each of them would satisfy infinitely many independent second order ODE which is a contradiction. So, on each connected component of Ω, λ and $\mathbb{I}$ depend either on y_1 or on y_2 exclusively.

Assume that R is constant and that both λ and $\mathbb{I}$ only depend on y_2 in a connected component of Ω. For any y_2^0 the curves $\Gamma_{y_2^0}$ given by

$$\Gamma_{y_2^0} := \left\{ \vec{\Phi}(y_1, y_2^0) \quad ; \quad (y_1, y_2^0) \in \Omega \right\}$$

are made of portions of planar circles. Indeed the unit tangent direction to $\Gamma_{y_2^0}$ are given by $\tau(y_1) = e^{-\lambda(y_2)} \, \partial_{y_1} \vec{\Phi}(y_1, y_2^0)$. We have

$$\partial_{y_1^2}^2 \vec{\Phi} = -\dot{\lambda}(y_2^0) \, \partial_{y_2} \vec{\Phi}(y_1, y_2^0) + \kappa_1(y_2^0) \, e^{2\lambda(y_2^0)} \, \vec{n}(y_1, y_2^0) - e^{2\lambda(y_2^0)} \, \vec{\Phi}(y_1, y_2^0) \quad ,$$

and

$$\partial_{y_1^3}^3 \vec{\Phi} = -\dot{\lambda}(y_2^0) \, \partial_{y_1 y_2}^2 \vec{\Phi}(y_1, y_2^0) + \kappa_1(y_2^0) \, e^{2\lambda(y_2^0)} \, \partial_{y_1} \vec{n}(y_1, y_2^0) - e^{2\lambda(y_2^0)} \, \partial_{y_1} \vec{\Phi}(y_1, y_2^0) \quad ,$$

We have

$$\partial_{y_1 y_2}^2 \vec{\Phi}(y_1, y_2^0) = e^{-2\lambda(y_2^0)} \, \partial_{y_1 y_2}^2 \vec{\Phi}(y_1, y_2^0) \cdot \partial_{y_1} \vec{\Phi} \, \partial_{y_1} \vec{\Phi} = \dot{\lambda}(y_2^0) \, \partial_{y_1} \vec{\Phi}$$

and

$$\partial_{y_1} \vec{n}(y_1, y_2^0) = e^{-2\lambda(y_2^0)} \, \partial_{y_1} \vec{\Phi} \cdot \partial_{y_1} \vec{n}(y_1, y_2^0) \, \partial_{y_1} \vec{\Phi} = -\kappa_1(y_2^0) \, \partial_{y_1} \vec{\Phi} \quad .$$

Thus

$$\partial^3_{y_1^3} \vec{\Phi} = - \left[\dot{\lambda}^2(y_2^0) + \kappa_1^2 + e^{2\lambda} \right] \partial_{y_1} \vec{\Phi}(y_1, y_2^0)$$

So the $k-$th osculating plane for $k \geq 2$ is constant equal to the 2 dimensional 2nd osculating plane. The fundamental theorem of curves in euclidian spaces asserts that each connected component is included in a 2-dimensional plane. Moreover it has a constant curvature, so it is a portion of circle. The boundary of Ω is made of points where $e^{2\lambda} = \sin\theta$ and λ is continuous on $\mathbb{R}^2$, so if $\lambda(y_2^0) \neq 2^{-1} \log \sin\theta$, which is always the case for y_2^0 for which there exists $y_1 \in \mathbb{R}$ with $(y_1, y_2^0) \in \Omega$, the closure of $\Gamma_{y_2^0}$ does not intersect the boundary of Ω and $\Gamma_{y_2^0}$ is a closed planar circle in S^3. Hence we have proved that each connected component of Ω is a union of lines $y_2 = cte$ or $y_1 = cte$. Both situations cannot coexist and since again λ is C^0 on $\mathbb{R}^2$ and since λ is constant on each component of the complement of Ω and equal to $2^{-1} \log \sin\theta$, we have proved that λ is either a function of y_2 globally or a function of y_1 globally. In any case, the immersed torus posses a foliation by planar circles giving principal directions.

We claim that these planar circles belong to parallel 2-planes. We have first

$$\partial^2_{y_2 y_1} \vec{\Phi} = \dot{\lambda}(y_2) \, e^{-2\lambda} \, \partial_{y_1} \vec{\Phi} \quad . \tag{IV.32}$$

Then we have

$$\partial^3_{y_2 y_1^2} \vec{\Phi} = - \left[\ddot{\lambda} + \dot{\lambda}^2 + \kappa_1 \kappa_2 \, e^{2\lambda} + e^{2\lambda} \right] \partial_{y_2} \vec{\Phi} + e^{2\lambda} \left[\dot{\kappa}_1 + 2 \, \kappa_1 \, \dot{\lambda} - \kappa_2 \, \dot{\lambda} \right] \vec{n} - \dot{\lambda} \, e^{2\lambda} \, \vec{\Phi} \quad .$$

The Liouville equation reads

$$-\ddot{\lambda} = \kappa_1 \, \kappa_2 \, e^{2\lambda} + e^{2\lambda} \quad ,$$

and the Codazzi equation reads

$$\dot{\kappa}_1 = -\dot{\lambda} \, (\kappa_1 - \kappa_2)$$

Thus we have proved

$$\partial^3_{y_2 y_1^2} \vec{\Phi} = \dot{\lambda} \, \partial^2_{y_1^2} \vec{\Phi} \quad . \tag{IV.33}$$

The identities (IV.32) and (IV.33) imply that there exist generators of the osculating 2-planes, $\partial_{y_1} \vec{\Phi}$ and $\partial^2_{y_1^2} \vec{\Phi}$, whose y_2 derivatives still belong to the osculating 2-plane. We deduce that these 2-planes are independent of y_2 and the immersed torus is then axially symmetric.

Since in the y coordinates we have $H = e^{-2\lambda} H^0_\Re = \sin\theta \, e^{-2\lambda} \, \hat{H}^0_\Re$ we have the following system

$$\begin{cases} -\ddot{\lambda} = \kappa_1 \, \kappa_2 \, e^{2\lambda} + e^{2\lambda} \\[2mm] \dot{\kappa}_1 = -\dot{\lambda} \, (\kappa_1 - \kappa_2) \\[2mm] \kappa_1 + \kappa_2 = \sin\theta \, e^{-2\lambda} \, (\kappa_1 - \kappa_2) \end{cases} \tag{IV.34}$$

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the last equation implies

$$\kappa_2 = \frac{\sin\theta - e^{2\lambda}}{\sin\theta + e^{2\lambda}}\,\kappa_1 \qquad\qquad\qquad (IV.35)$$

Thus in particular

$$\kappa_2 - \kappa_1 = -\frac{2\,e^{2\lambda}}{\sin\theta + e^{2\lambda}} \qquad\qquad\qquad (IV.36)$$

Substituting (IV.36) in the second equation of (IV.34) gives

$$\frac{\dot\kappa_1}{\kappa_1} = -\frac{2\,e^{2\lambda}\,\dot\lambda}{\sin\theta + e^{2\lambda}}$$

from which we deduce the existence of $C \in \mathbb{R}$ such that

$$(e^{2\lambda} + \sin\theta)\,\kappa_1 = e^C \qquad\qquad\qquad (IV.37)$$

Combining (IV.35) and (IV.37) gives

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{\sin\theta}{\sin\theta + e^{2\lambda}}\,\kappa_1 = e^{-C}\,\sin\theta\,\kappa_1^2 \quad . \qquad\qquad (IV.38)$$

Combining (IV.36) and (IV.37) gives

$$\kappa_2 - \kappa_1 = -2 + 2\,e^{-C}\,\sin\theta\,\kappa_1 \qquad\qquad\qquad (IV.39)$$

Combining (IV.38) and (IV.39) gives

$$\kappa_1 = 1 + e^{-C}\sin\theta\,(\kappa_1^2 - \kappa_1) \quad . \qquad\qquad\qquad (IV.40)$$

Hence we deduce that κ_1 is constant on each connected component of Ω and λ is constant too. So $\vec\Phi$ is flat on that component and λ is constant with $e^{2\lambda} \neq \sin\theta$. The boundary of Ω is made of points where $e^{2\lambda} = \sin\theta$, since λ is continuous on $\mathbb{R}^2$, Ω has no boundary and λ and $\mathbb{I}$ are constant on $\mathbb{R}$. So $\vec\Phi$ is a flat CMC torus. So the immersion $\vec\Phi$ is a conformal covering of the rectangular embeddings isometric to a torus of the form $a\,S^1 \times \sqrt{1 - a^2}\,S^1$ (see [?] and [?]).

Assuming now $e^{2\lambda} = \sin\theta$ on the whole torus. Equations (IV.3) and (IV.19) give

$$\hat{H}^0_{\Re} = H \qquad \text{and} \qquad \hat{H}^0_{\Re} = 0 \quad .$$

Hence we deduce that the Gaussian curvature is identically equal to 1 which contradicts Liouville equation (IV.26). Hence we have proved theorem I.2. $\qquad\qquad\square$

V Pairs of conformally constrained minimal surfaces which are conformally congruent.

Let $\vec{\Phi}$ be a weak immersion of the torus T^2 which is a conformally constrained minimal surface in S^3 and assume there exists a conformal transformation Ψ of S^3 which is not an isometry and such that $\Psi \circ \vec{\Phi}$ is again a conformally constrained minimal surface. Hence there exists $\vec{a} \in B^4 \setminus \{0\}$ such that $\Psi_{\vec{a}} \circ \vec{\Phi}$ is a conformally constrained minimal surface where

$$\Psi_{\vec{a}}(\vec{y}) = (1 - |\vec{a}|^2) \frac{\vec{y} - \vec{a}}{|\vec{y} - \vec{a}|^2} - \vec{a} \quad . \tag{V.1}$$

Its conformal factor is given by

$$\forall \, \vec{Y} \in T_{\vec{y}} S^3 \qquad |d\Psi_{\vec{a}} \cdot \vec{Y}| = \frac{1 - |\vec{a}|^2}{1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{y}} \, |\vec{Y}| \quad . \tag{V.2}$$

Denote $\mu_{\vec{a}}(x)$ the function on T^2 given by

$$e^{\mu_{\vec{a}}(x)} = \frac{1 - |\vec{a}|^2}{1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}(x)}$$

From now on we shall omit the subscript $\vec{a}$ and simply write Ψ and μ. We introduce the following parallelisation of S^3

$$\forall \, j \in \mathbb{Z}_3 \qquad \vec{\eta}_j(y) = y_{j+1}\, \partial_{y_{j-1}} - y_{j-1}\, \partial_{y_{j+1}} + y_j\, \partial_{y_4} - y_4\, \partial_{y_j}$$

One has

$$D_{\vec{\eta}_{j+1}} \vec{\eta}_j = \vec{\eta}_{j-1} \quad , \qquad D_{\vec{\eta}_{j-1}} \vec{\eta}_j = -\vec{\eta}_{j+1} \quad \text{and} \quad D_{\vec{\eta}_j} \vec{\eta}_j = 0 \tag{V.3}$$

Where D is the covariant derivative on S^3 associated to the Levi-Civita connection for the standard metric. One also verifies that $(\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3)$ realizes an orthonormal frame of S^3. Hence for each $y \in S^3$ the three unit vectors $e^{-\mu} \partial_{\vec{\eta}_j} \Psi(y)$ realizes an orthonormal basis of $T_{\Psi(y)} S^3$. Let a_{kl}^j be the real numbers such that

$$D_{\vec{\eta}_l} \partial_{\vec{\eta}_k} \Psi = \sum_{j=1}^{3} a_{lk}^j\, \partial_{\vec{\eta}_j} \Psi \quad .$$

Or in other words

$$a_{lk}^j = e^{-2\mu}\, D_{\vec{\eta}_l} \partial_{\vec{\eta}_k} \Psi \cdot \partial_{\vec{\eta}_j} \Psi = -\,e^{-2\mu}\, D_{\vec{\eta}_l} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_k} \Psi + 2\, \delta_{jk}\, \partial_{\vec{\eta}_l} \mu$$

Let π^{S^3} be the projection onto the tangent plane at $y \in S^3$ to S^3, we have

$$D_{\vec{\eta}_l} \partial_{\vec{\eta}_k} \Psi = \sum_{i,j=1}^{4} \pi^{S^3} \left(\partial_{y_j} (\partial_{y_i} \Psi \, \eta_k^i) \, \eta_l^j \right)$$

$$= \pi^{S^3} \left[\sum_{i,j=1}^{4} \left(\partial_{y_i} (\partial_{y_j} \Psi \, \eta_l^j) \, \eta_k^i - \partial_{y_j} \Psi \, \eta_k^i \, \partial_{y_i} \eta_l^j + \partial_{y_i} \Psi \, \eta_l^j \, \partial_{y_j} \eta_k^i \right) \right] \quad \text{(V.4)}$$

$$= D_{\vec{\eta}_k} \partial_{\vec{\eta}_l} \Psi - \partial_{D_{\vec{\eta}_k} \vec{\eta}_l} \Psi + \partial_{D_{\vec{\eta}_l} \vec{\eta}_k} \Psi$$

$$= D_{\vec{\eta}_k} \partial_{\vec{\eta}_l} \Psi + 2 \, \partial_{D_{\vec{\eta}_l} \vec{\eta}_k} \Psi$$

We are first computing $D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_j} \Psi$. We have

$$a_{j+1j}^{j} = e^{-2\mu} \, D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_j} \Psi = \partial_{\vec{\eta}_{j+1}} \mu \quad \text{(V.5)}$$

and, using (V.3) and (V.4)

$$a_{j+1j}^{j+1} = e^{-2\mu} \, D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j+1}} \Psi = e^{-2\mu} \, D_{\vec{\eta}_j} \partial_{\vec{\eta}_{j+1}} \Psi \cdot \partial_{\vec{\eta}_{j+1}} \Psi = \partial_{\vec{\eta}_j} \mu \quad \text{(V.6)}$$

Regarding the third coordinate a_{j+1j}^{j-1} we have in one hand using (V.3) and (V.4) again

$$D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j-1}} \Psi = -D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_{j-1}} \Psi \cdot \partial_{\vec{\eta}_j} \Psi$$

$$= -D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_{j+1}} \Psi \cdot \partial_{\vec{\eta}_j} \Psi - 2 \, \partial_{D_{\vec{\eta}_{j+1}} \vec{\eta}_{j-1}} \Psi \cdot \partial_{\vec{\eta}_j} \Psi \quad \text{(V.7)}$$

$$= -D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_{j+1}} \Psi \cdot \partial_{\vec{\eta}_j} \Psi + 2 \, e^{2\mu} \quad .$$

In the other hand we have

$$D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j-1}} \Psi = D_{\vec{\eta}_j} \partial_{\vec{\eta}_{j+1}} \Psi \cdot \partial_{\vec{\eta}_{j-1}} \Psi + 2 \, \partial_{D_{\vec{\eta}_{j+1}} \vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j-1}} \Psi$$

$$= -D_{\vec{\eta}_j} \partial_{\vec{\eta}_{j-1}} \Psi \cdot \partial_{\vec{\eta}_{j+1}} \Psi + 2 \, e^{2\mu}$$

$$= -D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j+1}} \Psi - \partial_{D_{\vec{\eta}_j} \vec{\eta}_{j-1}} \Psi \cdot \partial_{\vec{\eta}_{j+1}} \Psi + 2 \, e^{2\mu} \quad \text{(V.8)}$$

$$= -D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j+1}} \Psi$$

Summing (V.7) and (V.8) gives then

$$a_{j+1j}^{j-1} = e^{-2\mu} \, D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j-1}} \Psi = 1 \quad . \quad \text{(V.9)}$$

Combining now (V.5), (V.6) and (V.9) gives then

$$D_{\vec{\eta}_{j+1}} \partial_{\vec{\eta}_j} \Psi = \partial_{\vec{\eta}_{j+1}} \mu \, \partial_{\vec{\eta}_j} \Psi + \partial_{\vec{\eta}_j} \mu \, \partial_{\vec{\eta}_{j+1}} \Psi + \partial_{\vec{\eta}_{j-1}} \Psi \quad \text{(V.10)}$$

We are first computing $D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_j} \Psi$. We have

$$a^j_{j-1j} = e^{-2\mu} D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_j} \Psi = \partial_{\vec{\eta}_{j-1}} \mu \qquad \text{(V.11)}$$

and, using (V.3) and (V.4)

$$a^{j-1}_{j-1j} = e^{-2\mu} D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j-1}} \Psi = e^{-2\mu} D_{\vec{\eta}_j} \partial_{\vec{\eta}_{j-1}} \Psi \cdot \partial_{\vec{\eta}_{j-1}} \Psi = \partial_{\vec{\eta}_j} \mu \qquad \text{(V.12)}$$

Regarding the third coordinate a^{j+1}_{j-1j} we have seen

$$a^{j+1}_{j-1j} = e^{-2\mu} D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_j} \Psi \cdot \partial_{\vec{\eta}_{j+1}} \Psi = -1 \qquad \text{(V.13)}$$

Combining now (V.11), (V.12) and (V.13) gives then

$$D_{\vec{\eta}_{j-1}} \partial_{\vec{\eta}_j} \Psi = \partial_{\vec{\eta}_{j-1}} \mu \; \partial_{\vec{\eta}_j} \Psi + \partial_{\vec{\eta}_j} \mu \; \partial_{\vec{\eta}_{j-1}} \Psi - \partial_{\vec{\eta}_{j+1}} \Psi \qquad \text{(V.14)}$$

A short computation gives also

$$D_{\vec{\eta}_j} \partial_{\vec{\eta}_j} \Psi = \partial_{\vec{\eta}_j} \mu \; \partial_{\vec{\eta}_j} \Psi - \partial_{\vec{\eta}_{j-1}} \mu \; \partial_{\vec{\eta}_{j-1}} \Psi - \partial_{\vec{\eta}_{j+1}} \mu \; \partial_{\vec{\eta}_{j+1}} \Psi \quad . \qquad \text{(V.15)}$$

We shall now establish the following lemma which is well known to experts in conformal geometry.

Lemma V.1. *Let $\vec{\Phi}$ be a weak immersion of the torus T^2 into S^3 and let Ψ be a conformal transformation of the sphere S^3. Denote by $\vec{h}^0_{\vec{\Phi}}$ and by $\vec{h}^0_{\Psi \circ \vec{\Phi}}$ the Weingarten operators respectively for the immersions $\vec{\Phi}$ and $\Psi \circ \vec{\Phi}$ then the following identity holds*

$$\vec{h}^0_{\Psi \circ \vec{\Phi}} = \Psi_* \vec{h}^0_{\vec{\Phi}} \quad . \qquad \text{(V.16)}$$

$\square$

Proof of lemma V.1. We can assume that $\vec{\Phi}$ is conformal from a riemann surface into S^3. In some local complex coordinates we denote by $e^{2\lambda} = |\partial_{x_1}\vec{\Phi}|^2 = |\partial_{x_2}\vec{\Phi}|^2$. We have $\vec{h}^0_{\vec{\Phi}} = e^{2\lambda} D_z(e^{-2\lambda} \partial_z \vec{\Phi}) \, dz^2$ where D is the covariant derivative in S^3 issued from the Levi-Civita connection of the standard metric. Let $e^{2\mu}$ be the conformal factor of the conformal transformation Ψ. We have then $\vec{h}^0_{\Psi \circ \vec{\Phi}} = e^{2\mu+2\lambda} D_z(e^{-2\mu-2\lambda}\partial_z(\Psi \circ \vec{\Phi})) \, dz^2$. We compute

$$D_z(e^{-2\mu-2\lambda}\partial_z(\Psi \circ \vec{\Phi})) = -2\, \partial_z\mu \, e^{-2\mu-2\lambda} \, \partial_z(\Psi \circ \vec{\Phi})$$

$$+ e^{-2\mu} \sum_{j=1}^{3} D_z \left(e^{-2\lambda} \, \partial_{\vec{\eta}_j} \Psi \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} \right) \qquad \text{(V.17)}$$

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We have

$$\sum_{j=1}^{3} D_z \left(e^{-2\lambda} \, \partial_{\vec{\eta}_j} \Psi \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} \right) = \Psi_* \left(D_z \left(e^{-2\lambda} \, \partial_z \vec{\Phi} \right) \right)$$

$$+ e^{-2\lambda} \sum_{j=1}^{3} D_z \left(\partial_{\vec{\eta}_j} \Psi \right) \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} + e^{-2\lambda} \sum_{j=1}^{3} \partial_{\vec{\eta}_j} \Psi \; D_z \vec{\eta}_j \cdot \partial_z \vec{\Phi}$$

(V.18)

Observe first that, using (V.3), we have for any $j \in \mathbb{Z}_3$

$$D_z \vec{\eta}_j \cdot \partial_z \vec{\Phi} = D_{\vec{\eta}_{j-1}} \vec{\eta}_j \cdot \partial_z \vec{\Phi} \; \vec{\eta}_{j-1} \cdot \partial_z \vec{\Phi} + D_{\vec{\eta}_{j+1}} \vec{\eta}_j \cdot \partial_z \vec{\Phi} \; \vec{\eta}_{j+1} \cdot \partial_z \vec{\Phi}$$

$$= -\vec{\eta}_{j+1} \cdot \partial_z \vec{\Phi} \; \vec{\eta}_{j-1} \cdot \partial_z \vec{\Phi} + \vec{\eta}_{j-1} \cdot \partial_z \vec{\Phi} \; \vec{\eta}_{j+1} \cdot \partial_z \vec{\Phi} = 0$$

(V.19)

Hence

$$\sum_{j=1}^{3} D_z \left(e^{-2\lambda} \, \partial_{\vec{\eta}_j} \Psi \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} \right) = \Psi_* \left(D_z \left(e^{-2\lambda} \, \partial_z \vec{\Phi} \right) \right)$$

$$+ e^{-2\lambda} \sum_{j=1}^{3} D_z \left(\partial_{\vec{\eta}_j} \Psi \right) \; \vec{\eta}_j \cdot \partial_z \vec{\Phi}$$

(V.20)

We have

$$D_z \left(\partial_{\vec{\eta}_j} \Psi \right) = D_{\vec{\eta}_{j-1}} \left(\partial_{\vec{\eta}_j} \Psi \right) \; \vec{\eta}_{j-1} \cdot \partial_z \vec{\Phi} + D_{\vec{\eta}_j} \left(\partial_{\vec{\eta}_j} \Psi \right) \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} + D_{\vec{\eta}_{j+1}} \left(\partial_{\vec{\eta}_j} \Psi \right) \; \vec{\eta}_{j+1} \cdot \partial_z \vec{\Phi} \quad \text{(V.21)}$$

We make now use of (V.10), (V.14) and (V.15) and we deduce

$$D_z \left(\partial_{\vec{\eta}_j} \Psi \right) = \partial_z \mu \; \partial_{\vec{\eta}_j} \Psi$$

$$+ \partial_{\vec{\eta}_{j-1}} \Psi \; \left[\partial_{\vec{\eta}_j} \mu \; \vec{\eta}_{j-1} \cdot \partial_z \vec{\Phi} - \partial_{\vec{\eta}_{j-1}} \mu \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} + \vec{\eta}_{j+1} \cdot \partial_z \vec{\Phi} \right]$$

(V.22)

$$+ \partial_{\vec{\eta}_{j+1}} \Psi \; \left[\partial_{\vec{\eta}_j} \mu \; \vec{\eta}_{j+1} \cdot \partial_z \vec{\Phi} - \partial_{\vec{\eta}_{j+1}} \mu \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} - \vec{\eta}_{j-1} \cdot \partial_z \vec{\Phi} \right]$$

A relatively short computation gives

$$\sum_{j=1}^{3} D_z \left(\partial_{\vec{\eta}_j} \Psi \right) \; \vec{\eta}_j \cdot \partial_z \vec{\Phi} = 2 \, \partial_z \mu \; \partial_z (\Psi \circ \vec{\Phi}) - \sum_{j,k=1}^{3} \partial_{\vec{\eta}_j} \Psi \; \partial_{\vec{\eta}_k} \Psi \; (\vec{\eta}_k \cdot \partial_z \vec{\Phi})^2 \quad \text{(V.23)}$$

Observe that

$$\sum_{k=1}^{3} (\vec{\eta}_k \cdot \partial_z \vec{\Phi})^2 = 4^{-1} \sum_{k=1}^{3} \left[(\partial_{x_1} \vec{\Phi} \cdot \vec{\eta}_k)^2 - (\partial_{x_2} \vec{\Phi} \cdot \vec{\eta}_k)^2 - 2 \, i \, \partial_{x_1} \vec{\Phi} \cdot \vec{\eta}_k \, \partial_{x_2} \vec{\Phi} \cdot \vec{\eta}_k \right]$$

$$= 4^{-1} \left[|\partial_{x_1} \vec{\Phi}|^2 - |\partial_{x_2} \vec{\Phi}|^2 - 2 \, i \, \partial_{x_1} \vec{\Phi} \cdot \partial_{x_2} \vec{\Phi} \right] = 0$$

(V.24)

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Combining (V.20), (V.23) and (V.24) we obtain finally

$$\sum_{j=1}^{3} D_z \left(e^{-2\lambda} \, \partial_{\vec{\eta}_j} \Psi \, \vec{\eta}_j \cdot \partial_z \vec{\Phi} \right) = \Psi_* \left(D_z \left(e^{-2\lambda} \, \partial_z \vec{\Phi} \right) \right) + 2 \, \partial_z \mu \, \partial_z (\Psi \circ \vec{\Phi}) \qquad \text{(V.25)}$$

Combining (V.17) and (V.25) gives (V.16) and lemma V.1 is proved. $\qquad\square$

We establish now the following lemma.

Lemma V.2. *Let $\vec{\Phi}$ be a weak immersion of the torus T^2 into S^3 and let Ψ be a conformal transformation of the sphere S^3. Denote by $\vec{H}_{\vec{\Phi}}$ and by $\vec{H}_{\Psi \circ \vec{\Phi}}$ the mean curvature vectors respectively for the immersions $\vec{\Phi}$ and $\Psi \circ \vec{\Phi}$ then the following identity holds*

$$\vec{H}_{\Psi \circ \vec{\Phi}} = e^{-2\mu} \, \Psi_* \left[\vec{H}_{\vec{\Phi}} - \nabla_{\vec{n}_{\Phi}} \mu \, \vec{n}_{\Phi} \right] \qquad \text{(V.26)}$$

where $e^{2\mu} g_{S^3} = \Psi^* g_{S^3}$ is the conformal factor of the conformal transformation Ψ, $\vec{n}_{\vec{\Phi}}$ is the Gauss map of the immersion $\vec{\Phi}$ and $\nabla \mu$ is the gradient of the function μ over S^3 for the standard metric on S^3. $\qquad\square$

Proof of lemma V.2. We can assume that $\vec{\Phi}$ is conformal from a riemann surface into S^3. In some local complex coordinates we denote by $e^{2\lambda} = |\partial_{x_1} \vec{\Phi}|^2 = |\partial_{x_2} \vec{\Phi}|^2$. We have

$$\vec{H}_{\Psi \circ \vec{\Phi}} = 2 \, e^{-2\mu - 2\lambda} \, D_{\bar{z}} \partial_z (\Psi \circ \vec{\Phi}) \quad . \qquad \text{(V.27)}$$

We compute

$$D_{\bar{z}} \partial_z (\Psi \circ \vec{\Phi}) = \psi_* D_{\bar{z}} \partial_z \vec{\Phi} + \sum_{j=1}^{3} \partial_{\vec{\eta}_j} \Psi \, \partial_z \vec{\Phi} \cdot \partial_{\bar{z}} \vec{\eta}_j$$

$$+ \sum_{j,k=1}^{3} D_{\vec{\eta}_k} \partial_{\vec{\eta}_j} \Psi \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_k \, \partial_z \vec{\Phi} \cdot \vec{\eta}_j \qquad \text{(V.28)}$$

Using (V.3) we obtain in one hand

$$\sum_{j=1}^{3} \partial_{\vec{\eta}_j} \Psi \, \partial_z \vec{\Phi} \cdot \partial_{\bar{z}} \vec{\eta}_j = \sum_{j=1}^{3} \partial_{\vec{\eta}_j} \Psi \, \left[\partial_z \vec{\Phi} \cdot \vec{\eta}_{j-1} \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_{j+1} - \partial_z \vec{\Phi} \cdot \vec{\eta}_{j+1} \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_{j-1} \right] \qquad \text{(V.29)}$$

Using (V.10), (V.14) and (V.15) we obtain in the other hand

$$\sum_{j,k=1}^{3} D_{\vec{\eta}_k} \partial_{\vec{\eta}_j} \Psi \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_k \, \partial_z \vec{\Phi} \cdot \vec{\eta}_j = \sum_{j,k=1}^{3} \partial_{\vec{\eta}_j} \Psi \, \partial_z \vec{\Phi} \cdot \vec{\eta}_j \, \partial_{\vec{\eta}_k} \mu \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_k$$

$$+ \sum_{j,k=1}^{3} \partial_{\vec{\eta}_j} \Psi \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_j \, \partial_{\vec{\eta}_k} \mu \, \partial_z \vec{\Phi} \cdot \vec{\eta}_k - \sum_{j,k=1}^{3} \partial_{\vec{\eta}_j} \Psi \, \partial_{\vec{\eta}_j} \mu \, \partial_z \vec{\Phi} \cdot \vec{\eta}_k \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_k \qquad \text{(V.30)}$$

$$- \sum_{j=1}^{3} \partial_{\vec{\eta}_j} \Psi \, \left[\partial_z \vec{\Phi} \cdot \vec{\eta}_{j-1} \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_{j+1} - \partial_z \vec{\Phi} \cdot \vec{\eta}_{j+1} \, \partial_{\bar{z}} \vec{\Phi} \cdot \vec{\eta}_{j-1} \right]$$

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Combining (V.28), (V.29) and (V.30) gives then

$$D_{\bar{z}}\partial_z(\Psi \circ \vec{\Phi}) = \psi_* D_{\bar{z}}\partial_z \vec{\Phi} + \partial_z(\Psi \circ \vec{\Phi}) \, \partial_{\bar{z}}\mu + \partial_{\bar{z}}(\Psi \circ \vec{\Phi}) \, \partial_z \mu$$

$$- |\partial_z \vec{\Phi}|^2 \sum_{j=1}^{3} \partial_{\vec{\eta}_j}\mu \, \partial_{\vec{\eta}_j} \Psi \quad . \tag{V.31}$$

Observe that we have

$$\partial_z \vec{\Phi} \, \partial_{\bar{z}}\mu + \partial_{\bar{z}}\vec{\Phi} \, \partial_z \mu = 2^{-1} \left[\partial_{x_1}\mu \, \partial_{x_1}\vec{\Phi} + \partial_{x_2}\mu \, \partial_{x_2}\vec{\Phi} \right] \tag{V.32}$$

By definition $\nabla \mu := \sum_{j=1}^{3} \partial_{\vec{\eta}_j}\mu \, \vec{\eta}_j$. Denote π_T the orthogonal projection onto the 2-plane tangent to $\vec{\Phi}(T^2)$. We have at each point of the immersed surface

$$e^{2\lambda} \pi_T(\nabla \mu) = \sum_{i=1}^{2} \sum_{j=1}^{3} \partial_{\vec{\eta}_j}\mu \, \vec{\eta}_j \cdot \partial_{x_i}\vec{\Phi} \, \partial_{x_i}\vec{\Phi} = \sum_{i=1}^{2} \partial_{x_i}\mu \, \partial_{x_i}\vec{\Phi} \quad . \tag{V.33}$$

Combining (V.32) and (V.33) gives

$$\partial_z \vec{\Phi} \, \partial_{\bar{z}}\mu + \partial_{\bar{z}}\vec{\Phi} \, \partial_z \mu = 2^{-1}e^{2\lambda} \pi_T(\nabla \mu) \quad . \tag{V.34}$$

Thus

$$\partial_z(\Psi \circ \vec{\Phi}) \, \partial_{\bar{z}}\mu + \partial_{\bar{z}}(\Psi \circ \vec{\Phi}) \, \partial_z \mu - |\partial_z \vec{\Phi}|^2 \sum_{j=1}^{3} \partial_{\vec{\eta}_j}\mu \, \partial_{\vec{\eta}_j} \Psi$$

$$= 2^{-1}e^{2\lambda} \Psi_* \pi_{\vec{n}_{\vec{\Phi}}}(\nabla \mu) = 2^{-1}e^{2\lambda} \Psi_* \left[\nabla_{\vec{n}_{\vec{\Phi}}}\mu \, \vec{n}_{\vec{\Phi}} \right] \tag{V.35}$$

Combining (V.27), (V.31) and (V.35) gives (V.26) and the lemma V.2 is proved. $\qquad \square$

We shall now prove the following lemma.

Lemma V.3. *Let $\vec{\Phi}$ be a weak immersion of the torus T^2. Assume $\vec{\Phi}$ is a conformally constrained minimal surface which is not isothermic, assume moreover it realizes it's conformal volume $A(\vec{\Phi}) = V_c(\vec{\Phi})$ which is also assumed not to be a multiple of 4π . Assume moreover $\vec{\Phi}$ is a differentiable point of the conformal volume in $W^{1,\infty} \cap W^{2,2}(T^2)$ and that there exists a conformal transformation Ψ of S^3 which is not an isometry such that $A(\Psi \circ \vec{\Phi}) = A(\vec{\Phi})$ and such that any such $\Psi \circ \vec{\Phi}$ also satisfy the conformally constraint equation . Then $\vec{\Phi}$ is a critical point of the area under the constraint $A(\Psi \circ \vec{\Phi})$ being constant.* $\qquad \square$

Proof of lemma V.3. Let Q and Q_Ψ be the two holomorphic quadratic differential of the torus T^2 equipped with the conformal class defined by $\vec{\Phi}^* g_{S^3}$, or equivalently $\vec{\Phi}^* \Psi^* g_{S^3}$,such that

$$\begin{cases} H_{\vec{\Phi}} = \Re < Q, h^0_{\vec{\Phi}} >_{g_\Phi} \\[2mm] H_{\Psi \circ \vec{\Phi}} = \Re < Q_\Psi, h^0_{\Psi \circ \vec{\Phi}} >_{g_{\Psi \circ \vec{\Phi}}} \end{cases} \tag{V.36}$$

We now exclude the case $H_{\vec{\Phi}} = 0$ or $H_{\Psi \circ \vec{\Phi}} = 0$. Indeed if $\vec{\Phi}$ is minimal it is then isothermic and we are not considering this case and if $\Psi \circ \vec{\Phi}$ is minimal, it is well known that it is a strict maximum of the it's conformal volume and we could not have $A(\vec{\Phi}) = A(\Psi \circ \vec{\Phi})$. Since we are excluding from now on the cases $H_{\vec{\Phi}} = 0$ or $H_{\Psi \circ \vec{\Phi}} = 0$, then there exists $c \in \mathbb{C}^*$ such that

$$Q = c\, Q_\Psi$$

Assume first $c \in \mathbb{R}$. From lemma V.1 we have

$$\Re < Q_\Psi, \vec{h}^0_{\Psi \circ \vec{\Phi}} >_{g_{\Psi \circ \vec{\Phi}}} = c\; e^{-4\mu}\, \Psi_*(\Re < Q, h^0_{\vec{\Phi}} >_{g_\Phi})$$

Therefore for any $\vec{w} \in W^{1,\infty} \cap W^{2,2}(T^2)$ such that $\vec{w} \cdot \vec{\Phi} = 0$ on T^2

$$dA_{\Psi \circ \vec{\Phi}} \cdot \Psi_* \vec{w} = -2\, c \int_{T^2} \vec{w} \cdot \Re < Q, \vec{h}^0_{\vec{\Phi}} >_{g_\Phi}\; dvol_g = c\; dA_{\vec{\Phi}} \cdot \vec{w} \qquad \text{(V.37)}$$

So the differential of $\vec{\Phi} \to A(\Psi \circ \vec{\Phi})$ and $\vec{\Phi} \to A(\vec{\Phi})$ are proportional to each other. This implies the lemma in the case when $c \in \mathbb{R}^*$.

Now we are going to rule out the case $c \in \mathbb{C}^* \setminus \mathbb{R}^*$. Let $\vec{w} \in W^{1,\infty} \cap W^{2,2}(T^2)$ such that $\vec{w} \cdot \vec{\Phi} = 0$ on T^2 and $\vec{a} \in B^4_1$ and denote for t small enough $\vec{\Phi}_t := (\vec{\Phi} + t\vec{w})/|\vec{\Phi} + t\vec{w}|$. We denote $\vec{\Psi}_{\vec{a}}$ the conformal transformation given by (V.1). Using (V.2) we have for t small

$$A(\vec{\Psi}_{\vec{a}} \circ \vec{\Phi}_t) - A(\vec{\Psi}_{\vec{a}} \circ \vec{\Phi})$$

$$= (1 - |\vec{a}|^2)^2 \left[\int_{T^2} \frac{dvol_{g_{\vec{\Phi}_t}}}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_t)^2} - \int_{T^2} \frac{dvol_{g_{\vec{\Phi}}}}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi})^2} \right] \qquad \text{(V.38)}$$

Hence we have

$$\left| A(\Psi_{\vec{a}} \circ \vec{\Phi}_t) - A(\Psi_{\vec{a}} \circ \vec{\Phi}) - \left| \frac{1 - |\vec{a}|^2}{1 + |\vec{a}|^2} \right|^2 (A(\vec{\Phi}_t) - A(\vec{\Phi})) \right|$$

$$\leq\; C_{\vec{\Phi}}\, |t|\, |\vec{a}|\, [\|\vec{w}\|_\infty + \|\nabla \vec{w}\|_\infty] \qquad \text{(V.39)}$$

We have

$$\left| A(\Psi_{\vec{a}} \circ \vec{\Phi}_t) - A(\Psi_{\vec{a}} \circ \vec{\Phi}) + 2\, t \int_{T^2} \vec{w} \cdot \Re < Q, \vec{h}^0_{\vec{\Phi}} >_{g_{\vec{\Phi}}}\; dvol_{g_{\vec{\Phi}}} \right|$$

$$\leq\; C_{\vec{\Phi}}\, |t|\, |\vec{a}|\, [\|\vec{w}\|_\infty + \|\nabla \vec{w}\|_\infty] + o(t) \qquad \text{(V.40)}$$

Taking now the variation $\vec{\Phi}_t$ since $V_c(\vec{\Phi}) \notin 4\pi \mathbb{Z}$, for t small enough there exists $\vec{a}_t$ such that $A(\Psi_{\vec{a}_t} \circ \vec{\Phi}) = V_c(\vec{\Phi}_t)$ and $\vec{a}_t$ converges to the set $U \subset B^4$ of $\vec{a} \in B^4$ such that

$A(\Psi_{\vec{a}} \circ \vec{\Phi}) = A(\vec{\Phi}) = V_c(\vec{\Phi})$. From our hypothesis for all $\vec{a} \in U$ $\Psi_{\vec{a}} \circ \vec{\Phi}$ is a conformally constrained minimal surface. Then from [Ri4] there exist for each $\vec{a}$ an holomorphic quadratic form $Q_{\vec{a}} = c_{\vec{a}} Q$ such that

$$H_{\Psi_{\vec{a}} \circ \vec{\Phi}} = \Re < Q_{\vec{a}}, h^0_{\Psi_{\vec{a}} \circ \vec{\Phi}} >_{g_{\Psi_{\vec{a}} \circ \vec{\Phi}}} \tag{V.41}$$

Recall that we are assuming $U \neq \{0\}$ and there exists $\vec{a}_0 \in U \setminus \{0\}$ such that $\Psi = \Psi_{\vec{a}_0}$ and with our notations we have $Q_{\vec{a}_0} = Q_{\Psi}$. It is also clear that U is closed.

We are assuming that V_c is differentiable at $\vec{\Phi}$ and hence there exists a linear form L on $W^{1,\infty} \cap W^{2,2}$ such that

$$|V_c(\vec{\Phi}_t) - V_c(\vec{\Phi}) - t\, L(\vec{w})| = o(t) \quad .$$

We claim now that

$$L(\vec{w}) = \max_{\vec{a} \in U} \left\{ -2 \int_{T^2} \vec{w} \cdot \Re < Q_{\vec{a}}, \vec{h}^0_{\vec{\Phi}} >_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}}} \right\} \tag{V.42}$$

Proof of the claim. Let $\vec{a}_t$ such that

$$V_c(\vec{\Phi}_t) = A(\Psi_{\vec{a}_t} \circ \vec{\Phi}_t)$$

such a sequence exists for t small enough since we are assuming $V_c(\vec{\Phi}) \notin 4\pi\mathbb{Z}$. In order to simplify the presentation we can assume that there exists $t_k \to 0$ such that $\vec{a}_{t_k}$ converges to 0 since all the $\Psi_{\vec{a}}$ for $\vec{a} \in U$ play the same role. We also omit to mention the subscript k. Using (V.40) we have

$$\left| V_c(\vec{\Phi}_t) - A(\Psi_{\vec{a}_t} \circ \vec{\Phi}) + 2\,t \int_{T^2} \vec{w} \cdot \Re < Q, \vec{h}^0_{\vec{\Phi}} >_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}}} \right|$$
$$\leq C_{\vec{\Phi}} \, |t| \, |\vec{a}_t| \, [\|\vec{w}\|_\infty + \|\nabla \vec{w}\|_\infty] + o(t) \tag{V.43}$$

We deduce from this estimate

$$V_c(\vec{\Phi}_t) \leq V_c(\vec{\Phi}) - 2\,t \int_{T^2} \vec{w} \cdot \Re < Q, \vec{h}^0_{\vec{\Phi}} >_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}}} + o(t) \tag{V.44}$$

This implies then

$$L(\vec{w}) \leq \max_{\vec{a} \in U} \left\{ -2 \int_{T^2} \vec{w} \cdot \Re < Q_{\vec{a}}, \vec{h}^0_{\vec{\Phi}} >_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}}} \right\} \tag{V.45}$$

Let now $\vec{a}$ such that

$$\max_{\vec{a} \in U} \left\{ -2 \int_{T^2} \vec{w} \cdot \Re < Q_{\vec{a}}, \vec{h}^0_{\vec{\Phi}} >_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}}} \right\} = -2 \int_{T^2} \vec{w} \cdot \Re < Q_{\vec{a}}, \vec{h}^0_{\vec{\Phi}} >_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}}} \tag{V.46}$$

We can assume without loss of generality that $\vec{a} = 0$. We have

$$V_c(\vec{\Phi}_t) \geq A(\vec{\Phi}_t) = A(\vec{\Phi}) - 2 \int_{T^2} \vec{w} \cdot \Re < Q, \vec{h}_{\vec{\Phi}}^0 >_{g_{\vec{\Phi}}} \ dvol_{g_{\vec{\Phi}}} + o(t)$$

This implies

$$L(\vec{w}) \geq \max_{\vec{a} \in U} \left\{ -2 \int_{T^2} \vec{w} \cdot \Re < Q_{\vec{a}}, \vec{h}_{\vec{\Phi}}^0 >_{g_{\vec{\Phi}}} \ dvol_{g_{\vec{\Phi}}} \right\} \tag{V.47}$$

and the claim is proved.

Since we are assuming that Q_Ψ and Q are not $\mathbb{R}-$parallel and since $\vec{\Phi}$ is assumed not to be isothermic, the space generated by the linear forms $-2 \int_{T^2} \vec{w} \cdot \Re < Q_{\vec{a}}, \vec{h}_{\vec{\Phi}}^0 >_{g_{\vec{\Phi}}} \ dvol_{g_{\vec{\Phi}}}$ for $\vec{a}$ in U is two dimensional. Then there exist two independent linear forms $L_1(\vec{w})$ and $L_2(\vec{w})$ and two functions α and β on U such that such that

$$\forall \vec{a} \in U \quad \forall \vec{w} \in W^{1,\infty} \cap W^{2,2}(T^2, \mathbb{R}^4)$$

$$-2 \int_{T^2} \vec{w} \cdot \Re < Q_{\vec{a}}, \vec{h}_{\vec{\Phi}}^0 >_{g_{\vec{\Phi}}} \ dvol_{g_{\vec{\Phi}}} = \alpha(\vec{a}) \, L_1(\vec{w}) + \beta(\vec{a}) \, L_2(\vec{w})$$

Since the space is bi-dimensional there exist $\vec{w} \neq \vec{w}'$ and $\vec{a} \neq \vec{a}'$ such that

$$\alpha(\vec{a}') \, L_1(\vec{w}) + \beta(\vec{a}') \, L_2(\vec{w}) < \alpha(\vec{a}) \, L_1(\vec{w}) + \beta(\vec{a}) \, L_2(\vec{w}) = L(\vec{w}) \tag{V.48}$$

and

$$\alpha(\vec{a}) \, L_1(\vec{w}') + \beta(\vec{a}) \, L_2(\vec{w}') < \alpha(\vec{a}') \, L_1(\vec{w}') + \beta(\vec{a}') \, L_2(\vec{w}') = L(\vec{w}') \tag{V.49}$$

For any $s, t \in \mathbb{R}$ one has, using the maximality property of L we have for instance

$$L(s \, \vec{w} + t \, \vec{w}') = s \, L(\vec{w}) + t \, L(\vec{w}') \geq \alpha(\vec{a}) \, L_1(s\vec{w} + t\vec{w}') + \beta(\vec{a}) \, L_2(s\vec{w} + t\vec{w}')$$

The three previous identities imply

$$t \, (\alpha(\vec{a}') \, L_1(\vec{w}') + \beta(\vec{a}') \, L_2(\vec{w}')) \geq t \, (\alpha(\vec{a}') \, L_1(\vec{w}') + \beta(\vec{a}') \, L_2(\vec{w}'))$$

This contradicts (V.49) for $t < 0$. Hence the assumption that Q_Ψ and Q are not $\mathbb{R}-$parallel contradicts the fact that V_c is differentiable at $\vec{\Phi}$. Lemma V.3 is then proved. $\qquad \square$

VI Bounding the norm of the Lagrange Multiplier for Tori minimizing locally their conformal volume in their conformal class.

Let $\vec{\Phi}$ be a weak immersion of the Torus i.e. an element in $\mathcal{E}_{T^2}$. We denote by

$$\mathcal{T}_{T^2}(\vec{\Phi})$$

the subspace of $\mathcal{E}_{T^2}$ made of weak immersion which define the same Teichmüller class for some fixed generator of the $\pi_1(T^2)$.

In this subsection $\vec{\Phi}$ shall denote a weak immersion of T^2 minimizing locally the conformal volume in it's Teichmüller class.

$$A(\vec{\Phi}) = \inf_{\vec{\xi} \in \mathcal{U}} \sup_{\Psi \in G(S^3)} A(\Psi \circ \vec{\xi}) \qquad (\text{VI.1})$$

where $G(S^3)$ denotes the Möbius group of conformal transformation of the sphere S^3 and $\mathcal{U}$ is an open neighborhood of $\vec{\Phi}$ in $\mathcal{T}_{T^2}(\vec{\Phi})$.

VI.1 Families decreasing the area for small variations in the Möbius group in a neighborhood of a non isothermic constrained minimal surface.

The goal of the present subsection is to prove the following lemma

Lemma VI.1. *Let $\vec{\Phi}$ be a weak immersion of the torus. Assume $\vec{\Phi}$ is a critical point of the area under constrained conformal class which is not isothermic. Let Q be an holomorphic quadratic differential such that*

$$\vec{H} = \Re(< Q, \vec{h^0} >_{wp})$$

Then for $\mathcal{H}^2-$almost every point x^0 such that $2\,|Q|_{g_{\vec{\Phi}}}(x^0) > 1$ there exists $\varepsilon_{x^0,\vec{\Phi}} > 0$, $\nu_{x^0,\vec{\Phi}} > 0$ and $\delta_{x^0,\vec{\Phi}} > 0$ such that

$$\forall \varepsilon < \varepsilon_{x^0,\vec{\Phi}} \quad \forall \vec{a} \in B^4_{\nu_{x^0,\vec{\Phi}}}(0) \quad A(\Psi_{\vec{a}} \circ \vec{\Phi}^\varepsilon_{x^0}(\delta_{x^0,\vec{\Phi}}\,\varepsilon)) < A(\Psi_{\vec{a}} \circ \vec{\Phi}) - \varepsilon^4\,C(x_0,\vec{\Phi}) \quad (\text{VI.2})$$

for some positive constant $C(x_0,\vec{\Phi}) > 0$ depending only on x^0 and $\vec{\Phi}$ and not on ε and where $\vec{\Phi}^\varepsilon_{x^0}(t)$ is the family defined in lemma A.2. $\qquad\square$

Proof of lemma VI.1. Let x_0 be a point in T^2 which is a Lebesgue point for $\nabla^2\vec{\Phi}$ and for $\nabla\vec{\Phi}$. Let $\mathfrak{U}_{x^0,\vec{\Phi}}$ be the open set given by lemma A.2 and denote by $\vec{\Phi}^\varepsilon_{x^0}(t)$ the family of weak immersions of T^2 given by this lemma for (t,ε) in $\mathfrak{U}_{x^0,\vec{\Phi}}$. We have

$$\frac{dA}{dt}(0) = 0 \qquad (\text{VI.3})$$

and, due to lemma II.1, $\vec{\Phi}$ satisfies the constraint minimal surfaces equation

$$\vec{H} = \Re < Q, \vec{h^0} >_{wp} \quad . \qquad (\text{VI.4})$$

for some holomorphic quadratic form Q. Hence in particular

$$A(\vec{\Phi}^{\varepsilon}_{x^0}(t)) = A(\vec{\Phi}) + \int_0^t (t-\tau)\, \frac{d^2 A(\vec{\Phi}^{\varepsilon}_{x^0}(\tau))}{d\tau^2}\, d\tau$$

$$= A(\vec{\Phi}) - 2 \int_0^t (t-\tau)\, d\tau\, \frac{d}{d\tau}\left[\int_\Sigma \vec{H}^{\varepsilon}_{x^0}(\tau) \cdot \frac{d\vec{\Phi}^{\varepsilon}_{x^0}}{d\tau}\, dvol_{g^{\varepsilon}_{x^0}(\tau)}\right]$$

$$= A(\vec{\Phi}) - 2 \int_0^t (t-\tau)\, d\tau \int_\Sigma \frac{d\vec{H}^{\varepsilon}_{x^0}}{d\tau} \cdot \frac{d\vec{\Phi}^{\varepsilon}_{x^0}}{d\tau}\, dvol_{g^{\varepsilon}_{x^0}(\tau)}$$

$$- 2 \int_0^t (t-\tau)\, d\tau \int_\Sigma \vec{H}^{\varepsilon}_{x^0}(\tau) \cdot \frac{d}{d\tau}\left[\frac{d\vec{\Phi}^{\varepsilon}_{x^0}}{d\tau}\, dvol_{g^{\varepsilon}_{x^0}(\tau)}\right] \tag{VI.5}$$

Thus

$$A(\vec{\Phi}^{\varepsilon}_{x^0}(t)) = A(\vec{\Phi}) - t^2 \int_\Sigma \frac{d\vec{H}^{\varepsilon}_{x^0}}{d\tau}(0) \cdot \frac{d\vec{\Phi}}{d\tau}(0)\, dvol_g$$

$$- 2 \int_0^t (t-\tau)\, d\tau \int_0^\tau ds \int_\Sigma \frac{d^2 \vec{H}^{\varepsilon}_{x^0}}{ds^2} \cdot \frac{d\vec{\Phi}}{ds}\, dvol_{g(s)}$$

$$- 2 \int_0^t (t-\tau)\, d\tau \int_0^\tau ds \int_\Sigma \frac{d\vec{H}^{\varepsilon}_{x^0}}{ds} \cdot \frac{d}{ds}\left[\frac{d\vec{\Phi}}{ds}\, dvol_{g(s)}\right] \tag{VI.6}$$

$$- 2 \int_0^t (t-\tau)\, d\tau \int_\Sigma [\vec{H}^{\varepsilon}_{x^0}(\tau) - \vec{H}^{\varepsilon}_{x^0}(0)] \cdot \frac{d}{d\tau}\left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)}\right]$$

$$- 2 \int_0^t (t-\tau)\, d\tau \int_\Sigma \vec{H}^{\varepsilon}_{x^0}(0) \cdot \frac{d}{d\tau}\left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)}\right]$$

Since $\vec{H}^{\varepsilon}_{x^0}(0) = \vec{H}$ satisfies (VI.4) we have, omitting to write the subscript x^0 and the superscript ε from now on ,

$$- 2 \int_0^t (t-\tau)\, d\tau \int_\Sigma \vec{H}(0) \cdot \frac{d}{d\tau}\left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)}\right]$$

$$= -2 \int_0^t (t-\tau)\, d\tau \int_\Sigma \Re < Q, \vec{h}^0(0) >_{g(0)} \cdot \frac{d}{d\tau}\left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)}\right]$$

$$= -2 \int_0^t (t-\tau)\, d\tau \int_\Sigma \Re < Q, \vec{h}^0(\tau) >_{g(\tau)} \cdot \frac{d}{d\tau}\left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)}\right]$$

$$- 2 \int_0^t (t-\tau)\, d\tau \int_\Sigma \Re\left[< Q, \vec{h}^0(0) >_{g(0)} - < Q, \vec{h}^0(\tau) >_{g(\tau)}\right] \cdot \frac{d}{d\tau}\left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)}\right] \tag{VI.7}$$

Since $\mathcal{C}(\vec{\Phi}(t)) \equiv \mathcal{C}(\vec{\Phi})$, using (II.47), we have then

$$
\begin{aligned}
-2 &\int_0^t (t-\tau)\, d\tau \int_\Sigma \vec{H}(0) \cdot \frac{d}{d\tau} \left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)} \right] \\
= 2 &\int_0^t (t-\tau)\, d\tau \int_\Sigma \frac{d}{d\tau} \left[\Re < Q, \vec{h}^0(\tau) >_{g(\tau)} \right] \cdot \frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)} \\
-2 &\int_0^t (t-\tau)\, d\tau \int_\Sigma \Re \left[< Q, \vec{h}^0 >_{g(0)} - < Q, \vec{h}^0(\tau) >_{g(\tau)} \right] \cdot \frac{d}{d\tau} \left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)} \right]
\end{aligned}
\tag{VI.8}
$$

and then

$$
\begin{aligned}
-2 &\int_0^t (t-\tau)\, d\tau \int_\Sigma \vec{H}(0) \cdot \frac{d}{d\tau} \left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)} \right] \\
= t^2 &\int_\Sigma \frac{d}{d\tau} \left[\Re < Q, \vec{h}^0(\tau) >_{g(\tau)} \right] (0) \cdot \frac{d\vec{\Phi}}{d\tau}(0)\, dvol_g \\
+2 &\int_0^t (t-\tau)\, d\tau \int_0^\tau ds\, \frac{d}{ds} \left[\int_\Sigma \frac{d}{ds} \left[\Re < Q, \vec{h}^0(s) >_{g(s)} \right] \cdot \frac{d\vec{\Phi}}{ds}\, dvol_{g(s)} \right] \\
-2 &\int_0^t (t-\tau)\, d\tau \int_\Sigma \Re \left[< Q, \vec{h}^0 >_{g(0)} - < Q, \vec{h}^0(\tau) >_{g(\tau)} \right] \cdot \frac{d}{d\tau} \left[\frac{d\vec{\Phi}}{d\tau}\, dvol_{g(\tau)} \right]
\end{aligned}
\tag{VI.9}
$$

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Combining (VI.6) and (VI.9) gives then

$$
A(\vec{\Phi}^{\varepsilon}_{x^0}(t)) = A(\vec{\Phi}) - t^2 \int_{\Sigma} \frac{d\vec{H}}{d\tau}(0) \cdot \frac{d\vec{\Phi}}{d\tau}(0) \ dvol_g
$$

$$
+ t^2 \int_{\Sigma} \frac{d}{d\tau} \left[\Re < Q, \vec{h}^0(\tau) >_{g(\tau)} \right](0) \cdot \frac{d\vec{\Phi}}{d\tau}(0) \ dvol_g
$$

$$
- 2 \int_0^t (t-\tau) \, d\tau \int_0^\tau ds \int_\Sigma \frac{d^2 \vec{H}}{ds^2} \cdot \frac{d\vec{\Phi}}{ds} \ dvol_{g(s)}
$$

$$
- 2 \int_0^t (t-\tau) \, d\tau \int_0^\tau ds \int_\Sigma \frac{d\vec{H}}{ds} \cdot \frac{d}{ds} \left[\frac{d\vec{\Phi}^{\varepsilon}_{x^0}}{ds} \ dvol_{g(s)} \right]
$$

$$
- 2 \int_0^t (t-\tau) \, d\tau \int_\Sigma [\vec{H}(\tau) - \vec{H}(0)] \cdot \frac{d}{d\tau} \left[\frac{d\vec{\Phi}}{d\tau} \ dvol_{g^{\varepsilon}_{x^0}(\tau)} \right]
$$

$$
+ 2 \int_0^t (t-\tau) \, d\tau \int_0^\tau ds \int_\Sigma \frac{d^2}{ds^2} \left[\Re < Q, \vec{h}^0(s) >_{g(s)} \right] \cdot \frac{d\vec{\Phi}}{ds} \ dvol_{g(s)}
$$

$$
+ 2 \int_0^t (t-\tau) \, d\tau \int_0^\tau ds \int_\Sigma \frac{d}{ds} \left[\Re < Q, \vec{h}^0(s) >_{g(s)} \right] \cdot \frac{d}{ds} \left[\frac{d\vec{\Phi}}{ds} \ dvol_{g(s)} \right]
$$

$$
- 2 \int_0^t (t-\tau) \, d\tau \int_\Sigma \Re \left[< Q, \vec{h}^0 >_{g(0)} - < Q, \vec{h}^0(\tau) >_{g(\tau)} \right] \cdot \frac{d}{d\tau} \left[\frac{d\vec{\Phi}}{d\tau} \ dvol_{g(\tau)} \right]
$$

(VI.10)

Using (II.46), (II.49) and (II.58) we obtain

$$
- 2 \int_\Sigma \left[\frac{d\vec{H}}{d\tau}(0) - \frac{d}{d\tau} \left[\Re < Q, \vec{h}^0(\tau) >_{g(\tau)} \right](0) \right] \cdot \frac{d\vec{\Phi}}{d\tau}(0) \ dvol_g
$$

$$
= \int_\Sigma \left[|dv|_g^2 - 2 < \overline{q}, dv \otimes dv >_g + 8\, H^2 \, v^2 - (|\vec{\mathbb{I}}|_g^2 + 2)\, v^2 \right] \ dvol_g \quad .
$$

$$
+ \int_\Sigma \left[2H \, dv + v \, dH + 4 \, v \, \Re(Q) \llcorner dH + \Im \left\langle Q, h^0 \right\rangle * dv \right] \cdot \vec{w}_T \ dvol_g
$$

$$
- \int_\Sigma \Im \left\langle Q, h^0 \right\rangle \, v * d\vec{w}_T^* \ dvol_g \quad .
$$

(VI.11)

where $v = \frac{d\vec{\Phi}^{\varepsilon}_{x^0}}{d\tau}(0) \cdot \vec{n}$. Using the explicit formula of $\vec{\Phi}^{\varepsilon}_{x^0}(t)$ we obtain

$$
v = \beta^{\varepsilon}_{x^0}(x) \, \chi^{\varepsilon}_{x^0}(x) \, \vec{n}(x^0) \cdot \vec{n} + \beta^{\varepsilon}_{x^0}(x) \sum_{j=1}^2 \alpha_j(0, \varepsilon) \, \vec{a}_j \cdot \vec{n}
$$

(VI.12)

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and
$$|\vec{w}_T| = |\vec{w} \wedge \vec{n}| \le C \, \varepsilon \, |\vec{n} - \vec{n}(x^0)| \, \mathbf{1}^{\varepsilon}_{x^0} + O_{\vec{\Phi}, x^0}(\varepsilon^3)$$

Moreover
$$|\nabla^g \vec{w}_T| \le C \left(|\vec{n} - \vec{n}(x^0)| \ + \varepsilon \, |\vec{\mathbb{I}}|_g \right) \, \mathbf{1}^{\varepsilon}_{x^0} + O_{\vec{\Phi}, x^0}(\varepsilon^3) \, |\vec{\mathbb{I}}|_g$$

Inserting the previous expressions in (VI.11) gives then

$$-2 \int_{\Sigma} \left[\frac{d\vec{H}}{d\tau}(0) - \frac{d}{d\tau} \left[\Re < Q, \vec{h}^0(\tau) >_{g(\tau)} \right] (0) >_{g(\tau)} \right] \cdot \frac{d\vec{\Phi}^{\varepsilon}_{x^0}}{d\tau}(0) \; dvol_g$$

$$= \varepsilon^2 \int_{\mathbb{C}} |\nabla \chi|^2 \; dx^2 - 4 \varepsilon^2 \, e^{-2\lambda(x^0)} \, Q_1(x^0) \int_{\mathbb{C}} |\partial_{x_1} \chi|^2 - |\partial_{x_2} \chi|^2 \; dx^2 \tag{VI.13}$$

$$- 4 \varepsilon^2 \, e^{-2\lambda(x^0)} \, Q_2(x^0) \int_{\mathbb{C}} 2 \, \partial_{x_1} \chi \, \partial_{x_2} \chi \; dx^2 + o_{\vec{\Phi}, x^0}(\varepsilon^2)$$

We proceed to a rotation of the domain in such a way that $Q_2(x^0) = 0$. Combining (VI.10), the estimates in lemma A.2 and (VI.13), one has finally

$$\left| A(\vec{\Phi}^{\varepsilon}_{x^0}(t)) - A(\vec{\Phi}) - t^2 \, \varepsilon^2 \; F_{\chi}(x_0) \right| \le C_{\vec{\Phi}, x^0} \, \varepsilon \, t^3 + t^2 \, o_{\vec{\Phi}, x^0}(\varepsilon^2) \quad , \tag{VI.14}$$

where $B(x^0)$ is the number given by

$$2 \, F_{\chi}(x^0) := \int_{\mathbb{C}} |\nabla \chi|^2 \; dx^2 - 4 \, e^{-2\lambda(x^0)} \, Q_1(x^0) \int_{\mathbb{C}} |\partial_{x_1} \chi|^2 - |\partial_{x_2} \chi|^2 \; dx^2 \quad . \tag{VI.15}$$

For any $\vec{a} \in B^4$ we denote by $\Psi_{\vec{a}}$ the following element of the Möbius group

$$\Psi_{\vec{a}}(\vec{y}) = (1 - |\vec{a}|) \frac{\vec{y} - \vec{a}}{|\vec{y} - \vec{a}|^2} - \vec{a} \quad .$$

Its conformal factor is given by

$$\forall \, \vec{Y} \in T_{\vec{y}} S^3 \qquad |d\Psi_{\vec{a}} \cdot \vec{Y}| = \frac{1 - |\vec{a}|}{1 + |\vec{a}|^2 - 2\vec{a} \cdot \vec{y}} \, |\vec{Y}| \quad .$$

Hence we have

$$A(\Psi_{\vec{a}} \circ \vec{\Phi}) = \int_{\Sigma} \frac{(1 - |\vec{a}|)^2}{(1 + |\vec{a}|^2 - 2 \, \vec{a} \cdot \vec{\Phi})^2} \; dvol_g \quad .$$

For (t, ε) in the neighborhood of the origin $\mathfrak{U}_{x^0, \vec{\Phi}}$ given by lemma A.2 we estimate for $|\vec{a}| < \nu < 1$ the following difference

$$A(\Psi_{\vec{a}} \circ \vec{\Phi}_{x^0}^{\varepsilon}(t)) - A(\Psi_{\vec{a}} \circ \vec{\Phi})$$

$$= (1 - |\vec{a}|)^2 \left[\int_{\Sigma} \frac{dvol_{g_{x^0}^{\varepsilon}(t)}}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t))^2} - \frac{dvol_g}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi})^2} \right]$$

$$= (1 - |\vec{a}|)^2 \left[\int_{\Sigma} \frac{dvol_{g_{x^0}^{\varepsilon}(t)} - dvol_g}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t))^2} \right] \tag{VI.16}$$

$$+ (1 - |\vec{a}|)^2 \left[\int_{\Sigma} \left(\frac{1}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t))^2} - \frac{1}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi})^2} \right) dvol_g \right]$$

We have in one hand

$$\int_{\Sigma} \frac{dvol_{g_{x^0}^{\varepsilon}(t)} - dvol_g}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t)(x))^2}$$

$$= \frac{1}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t)(x^0))^2} \int_{\Sigma} dvol_{g_{x^0}^{\varepsilon}(t)} - dvol_g \tag{VI.17}$$

$$- \int_{\Sigma} \left[\frac{dvol_{g_{x^0}^{\varepsilon}(t)} - dvol_g}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t)(x^0))^2} - \frac{dvol_{g_{x^0}^{\varepsilon}(t)} - dvol_g}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t)(x))^2} \right]$$

Hence, using the estimate (A.12), we obtain for $|\vec{a}| < 1/4$

$$\left| \int_{\Sigma} \frac{dvol_{g_{x^0}^{\varepsilon}(t)} - dvol_g}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t)(x))^2} - \frac{dvol_{g_{x^0}^{\varepsilon}(t)} - dvol_g}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t)(x^0))^2} \right|$$

$$\leq C_{\vec{\Phi}, x^0} \, t \int_{\Sigma} \left[\mathbf{1}_{x^0}^{\varepsilon}(|\vec{n} - \vec{n}^0| + t) + \varepsilon^3 + t\,\varepsilon^2 \right] |\vec{a}| \, |\vec{\Phi}_{x^0}^{\varepsilon}(t)(x) - \vec{\Phi}_{x^0}^{\varepsilon}(t)(x^0)| \, dvol_g \tag{VI.18}$$

$$\leq 4\, |\vec{a}| \, C_{\vec{\Phi}, x^0} \int_{\Sigma} [t\,\varepsilon^3 + t^2\,\varepsilon^2] \, dvol_g + |\vec{a}| \, \varepsilon \, t \, C_{\vec{\Phi}} \int_{B_{\varepsilon}(x^0)} (|\vec{n} - \vec{n}^0| + t) \, dvol_g$$

$$\leq C_{\vec{\Phi}, x^0} \, |\vec{a}| \, (t\,\varepsilon^3 + t^2\,\varepsilon^2)$$

In the other hand we have, using the explicit expression of $\vec{\Phi}_{x^0}^{\varepsilon}(t)$ and the estimate (A.10),

$$\left| \int_{\Sigma} \left(\frac{1}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi}_{x^0}^{\varepsilon}(t))^2} - \frac{1}{(1 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{\Phi})^2} \right) dvol_g \right|$$

$$\leq |\vec{a}| \left[\int_{B_{\varepsilon}(x^0)} \varepsilon \, t \, \|\chi\|_{\infty} dvol_g + C_{\vec{\Phi}, x^0} \int_{\Sigma} t\,\varepsilon^3 \, dvol_g \right] \leq C_{\vec{\Phi}, x^0} \, |\vec{a}| \, t\,\varepsilon^3 \tag{VI.19}$$

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Combining (VI.14), (VI.16), (VI.17), (VI.18) and (VI.19) we obtain for $|\vec{a}| < 1/4$ and $(t, \varepsilon) \in \mathfrak{U}_{x^0, \vec{\Phi}}$

$$\left| A(\Psi_{\vec{a}} \circ \vec{\Phi}^{\varepsilon}_{x^0}(t)) - A(\Psi_{\vec{a}} \circ \vec{\Phi}) - \frac{t^2 \, \varepsilon^2 \, F_\chi(x^0)}{(1 + |\vec{a}|^2 - 2 \, \vec{a} \cdot \vec{\Phi}^{\varepsilon}_{x^0}(t)(x^0))^2} \right| \tag{VI.20}$$

$$\leq \; C_{\vec{\Phi}, x^0} \, \varepsilon \, t^3 + t^2 \, o_{\vec{\Phi}, x^0}(\varepsilon^2) + C_{\vec{\Phi}, x^0} \, |\vec{a}| \, (t \, \varepsilon^3 + t^2 \, \varepsilon^2)$$

Let now $\varphi(s) \in C_0^\infty(\mathbb{R})$ be an arbitrary non zero function compactly supported on $\mathbb{R}$. For any $\tau \in \mathbb{R}_+^*$ we denote

$$\chi_\tau(x_1, x_2) := \varphi(\tau \, x_1) \, \varphi(x_2)$$

We have

$$F_{\chi_\tau}(x^0) = \int_{\mathbb{R}} \dot{\varphi}^2(s) \, ds \int_{\mathbb{R}} \varphi^2(s) \, ds \left[\frac{\tau}{2} \, (1 - 4 \, e^{-2\lambda(x^0)} \, Q_1(x^0)) + \frac{1}{2 \, \tau} \, (1 + 4 \, e^{-2\lambda(x^0)} \, Q_1(x^0)) \right]$$

Assuming

$$|4 \, e^{-2\lambda(x^0)} \, Q_1(x^0)| > 1$$

which corresponds to assume $4 \, |Q|_g^2 > 1$ since $Q_2 = 0$ and $|dz^2|_g^2 = 4 \, e^{-4\lambda}$, we can choose $\tau \in \mathbb{R}_+^*$ such that $F_{\chi_\tau}(x^0) < 0$. Hence by choosing $t = \delta_{\vec{\Phi}, x^0} \, \varepsilon$ for $\varepsilon < \varepsilon^0_{\vec{\Phi}, x^0}$ and $\varepsilon^0_{\vec{\Phi}, x^0}$ is chosen small enough and $|\vec{a}| < \nu_{\vec{\Phi}, x^0}$ for some $\nu_{\vec{\Phi}, x^0} > 0$, also chosen small enough but independent of ε, we obtain from (VI.20) the lemma VI.1. $\qquad \square$

VII   Proof of theorem I.3.

Let $\vec{\Phi}$ be a weak immersion of the torus T^2 in S^3 and assume that $\vec{\Phi}$ is a local minimizer of the conformal volume in it's conformal class and that there is no non isometric Möbius transformation Ψ such that $V_c(\vec{\Phi}) = A(\Psi \circ \vec{\Phi})$. Since $\vec{\Phi}$ is assumed to achieve uniquely it's conformal volume, classical variations arguments combined with the computations from section II give that $\vec{\Phi}$ is a conformally constrained minimal surface. If $\vec{\Phi}$ is additionally isothermic then we apply section IV and the theorem is proved in that case. We assume now that $\vec{\Phi}$ is not isothermic. Since it achieves uniquely it's conformal volume $\vec{\Phi}$ satisfies

$$\forall \, \vec{a} \in B^4(0) \setminus \{0\} \qquad A(\vec{\Phi}) > A(\Psi_{\vec{a}} \circ \vec{\Phi}) \quad,$$

then, since $A(\vec{\Phi}) \notin 4\pi\mathbb{N}$, for all $\nu > 0$ there exists $\delta > 0$ such that

$$\forall \, \vec{a} \in B^4(0) \setminus B^4_\nu(0) \qquad A(\vec{\Phi}) > A(\Psi_{\vec{a}} \circ \vec{\Phi}) + \delta \quad. \tag{VII.1}$$

Assume there would exists a non zero measure set of points $x^0 \in T^2$ such that $2 \, |Q|_{g_{\vec{\Phi}}}(x^0) > 1$ then, using lemma VI.1, there exists $\varepsilon_0 > 0$, $\nu_0 > 0$ and a family of deformations $\vec{\Phi}^\varepsilon$ in

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the same conformal class defined by $\vec{\Phi}$ such that such that $\vec{\Phi}^\varepsilon$ converges to $\vec{\Phi}$ as $\varepsilon \to 0$ in the space of weak immersions and

$$\forall \varepsilon < \varepsilon_0 \quad \forall \vec{a} \in B^4_{\nu_0}(0) \quad A(\Psi_{\vec{a}} \circ \vec{\Phi}^\varepsilon) < A(\Psi_{\vec{a}} \circ \vec{\Phi}) - \varepsilon^4 \, C_0 \tag{VII.2}$$

for some positive constant $C_0 > 0$ which is independent of ε. Let $\nu := \nu_0$ and let $\delta_0 > 0$ such that (VII.1) holds. Then there exists $\varepsilon_1 > 0$ such that

$$\forall \, \vec{a} \in B^4(0) \setminus B^4_{\nu_0}(0) \quad \forall \varepsilon < \varepsilon_1 \qquad A(\vec{\Phi}) > A(\Psi_{\vec{a}} \circ \vec{\Phi}^\varepsilon) + \delta/2 \quad . \tag{VII.3}$$

Hence for any $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ we would have

$$V_c(\vec{\Phi}^\varepsilon) < V_c(\vec{\Phi}) \quad .$$

This contradicts the assumption that $\vec{\Phi}$ locally minimizes the conformal volume in it's conformal class and theorem I.3 is proved in all cases.

A   Appendix

A.1   Codazzi identity.

Recall the definition of the Weingarten form of the immersion $\vec{\Phi}$

$$\vec{h}^0 := 2 \, \pi_{\vec{n}}(\partial^2_{z^2}\vec{\Phi}) \, dz \otimes dz = e^{2\lambda} \, \vec{H}^0 \, dz \otimes dz$$

where

$$\vec{H}^0 = 2 \, \partial_z \left(e^{-2\lambda} \partial_z \vec{\Phi} \right) = 2^{-1} e^{-2\lambda} \, \pi_{\vec{n}}(\partial^2_{x_1^2}\vec{\Phi} - \partial^2_{x_2^2}\vec{\Phi}) - i \, e^{-2\lambda} \, \pi_{\vec{n}}(\partial^2_{x_1 x_2}\vec{\Phi}) = [H^0_\Re + i \, H^0_\Im] \, \vec{n} \quad ,$$

where $\partial_z = 2^{-1}(\partial_{x_1} - i \, \partial_{x_2})$. This gives

$$\begin{cases} H^0_\Re = -\dfrac{e^{-2\lambda}}{2} \left[(\partial_{x_1}\vec{n}, \partial_{x_1}\vec{\Phi}) - (\partial_{x_2}\vec{n}, \partial_{x_2}\vec{\Phi}) \right] = e^{-2\lambda}\mathbb{I}^0_{11} = -e^{-2\lambda}\mathbb{I}^0_{22} \\[4mm] H^0_\Im = e^{-2\lambda} \, (\partial_{x_1}\vec{n}, \partial_{x_2}\vec{\Phi}) = e^{-2\lambda} \, (\partial_{x_2}\vec{n}, \partial_{x_1}\vec{\Phi}) = -e^{-2\lambda}\mathbb{I}^0_{12} \quad . \end{cases} \tag{A.1}$$

We have

$$\partial_{\bar{z}}\vec{H}^0 = 2 \, \partial_{\bar{z}}\partial_z \left(e^{-2\lambda} \, \partial_z\vec{\Phi} \right) = 2 \, \partial_z\partial_{\bar{z}} \left(e^{-2\lambda} \, \partial_z\vec{\Phi} \right)$$

$$= -4 \, \partial_z \left(e^{-2\lambda} \, \partial_{\bar{z}}\lambda \, \partial_z\vec{\Phi} \right) + 2 \, \partial_z \left(e^{-2\lambda} \partial^2_{z\bar{z}}\vec{\Phi} \right) = -4 \, \partial_z \left(e^{-2\lambda} \, \partial_{\bar{z}}\lambda \, \partial_z\vec{\Phi} \right) + 2^{-1} \, \partial_z \left(e^{-2\lambda} \, \Delta\vec{\Phi} \right)$$

Hence we have obtained the following identity so far

$$\partial_{\bar{z}}\vec{H}^0 = -4 \, \partial_z \left(e^{-2\lambda} \, \partial_{\bar{z}}\lambda \, \partial_z\vec{\Phi} \right) + \partial_z\vec{H} - \partial_z\vec{\Phi} \quad . \tag{A.2}$$

Taking the scalar product with $\vec{n}$ gives then

$$\partial_{\bar{z}} H^0 = -2\, \partial_{\bar{z}}\lambda\, H^0 + \partial_z H \qquad\qquad (A.3)$$

from which we deduce

$$\partial_{\bar{z}}\left(e^{2\lambda}\, H^0\right) = e^{2\lambda}\, \partial_z H \qquad . \qquad\qquad (A.4)$$

This can also be rewritten locally as follows

$$\begin{cases} \partial_{x_1}\mathbb{I}^0_{11} + \partial_{x_2}\mathbb{I}^0_{12} = e^{2\lambda}\, \partial_{x_1} H \\[2mm] \partial_{x_2}\mathbb{I}^0_{11} - \partial_{x_1}\mathbb{I}^0_{12} = -e^{2\lambda}\, \partial_{x_2} H \end{cases} \qquad\qquad (A.5)$$

Hence we have proved the following

Lemma A.1. *Let $h^0 := 2\,\vec{n}\cdot\partial^2_{z^2}\vec{\Phi}\, dz^2$ and denote $g_{\mathbb{C}} := e^{2\lambda}\, d\bar{z}\otimes dz$ we have*

$$\overline{\partial} h^0 = g_{\mathbb{C}} \otimes \partial H \qquad . \qquad\qquad (A.6)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A.2 Construction of infinitesimal perturbations within a conformal class.

Recall that $\mathcal{T}_{T^2}$ denotes the Teichmüller Space of T^2 and, having fixed generators of the $\pi_1(T^2)$, for any metric g on T^2 we denote by $[g]$ the Teichmüller class associated to g. Let $\vec{\Phi}$ be a weak immersion of T^2 and assume it is <u>not isothermic</u>. Using [Ri3], we decuce the existence of two maps $\vec{a}_j$ in the space $W^{2,2}\cap W^{1,\infty}(T^2,\mathbb{R}^4)$ such that

$$(d\mathcal{C}_{\vec{\Phi}}\cdot\vec{a}_j)_{j=1,2} \qquad \text{forms a basis to the tangent space at } [\vec{\Phi}^* g_{\mathbb{R}^3}] \text{ of } \mathcal{T}_{T^2}$$

and $\vec{a}_j(x)\cdot\vec{\Phi}(x) \equiv 0$ on T^2.

Lemma A.2. *Let x_0 be a point in T^2 which is a Lebesgue point for $\nabla^2\vec{\Phi}$ and for $\nabla\vec{\Phi}$. Let χ be a smooth function on $\mathbb{C}$ supported in $B_1(0)$. In some fixed conformal chart in a neighborhood of x^0 we denote $\chi^\varepsilon_{x^0}(x) := \varepsilon\,\chi(\varepsilon^{-1}(x-x^0))$. There exists a neighborhood of 0 in $\mathbb{R}^2$, $\mathfrak{U}_{x^0,\vec{\Phi}}$, and there exists two C^1 maps $\sigma_j(t,\varepsilon)$ such that*

$$\forall\,(t,\varepsilon)\in\mathfrak{U}_{x^0,\vec{\Phi}} \qquad \mathcal{C}\left(\vec{\Phi}^\varepsilon_{x^0}(t)\right)\equiv\mathcal{C}(\vec{\Phi})$$

$$\text{where} \quad \vec{\Phi}^\varepsilon_{x^0}(t) := \beta^\varepsilon(x,t)\left[\vec{\Phi}(x) + t\,\chi^\varepsilon_{x^0}(x)\,\vec{n}(x_0) + t\,\sum_{j=1}^2\alpha_j(t,\varepsilon)\,\vec{a}_j(x)\right] \quad , \qquad (A.7)$$

$$|\alpha_j(0,\varepsilon)| = O_{\vec{\Phi},x^0}(\varepsilon^3) \quad , \qquad\qquad (A.8)$$

$$\beta^\varepsilon(x,t) \text{ is chosen such that } |\vec{\Phi}_{x^0}^\varepsilon(t)| \equiv 1 \tag{A.9}$$

Moreover $\alpha_j(t,\varepsilon)$ is C^2 with respect to t for $\varepsilon > 0$. Finally the following estimates hold for all $(t,\varepsilon) \in \mathfrak{U}_{x^0,\vec{\Phi}}$ and $k = 0,1,2$

$$\sum_{j=1}^2 \left| \frac{d^k}{dt^k}\left(t\,\alpha_j(t,\varepsilon)\right) \right| \leq C_{\vec{\Phi},x^0}\,\varepsilon^2\,\left[\inf\{\varepsilon\,|t|^{1-k},1\} + |t|^{2-k}\right] \tag{A.10}$$

and

$$\left| \frac{d}{dt}\nabla\vec{\Phi}_{x^0}^\varepsilon(t) \right| \leq C_{\vec{\Phi}}\,\left[\mathbf{1}_{x^0}^\varepsilon + C_{\vec{\Phi},x^0}\,(\varepsilon^3 + |t|\,\varepsilon^2)\right] \quad. \tag{A.11}$$

and

$$\left| \frac{dg_{\vec{\Phi}_{x^0}^\varepsilon(t)}}{dt} \right| \leq C_{\vec{\Phi}}\,\left[\mathbf{1}_{x^0}^\varepsilon(|\vec{n} - \vec{n}^0| + t) + O_{x^0,\vec{\Phi}}(\varepsilon^3 + |t|\,\varepsilon^2)\right] \tag{A.12}$$

and

$$\left| \frac{d}{dt}\nabla^2\vec{\Phi}_{x^0}^\varepsilon(t) \right| + \left| \frac{d}{dt}\mathbb{I}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \right| \leq C_{\vec{\Phi}}\,\mathbf{1}_{x^0}^\varepsilon\,\left[\varepsilon^{-1} + \varepsilon\,|\nabla^2\vec{\Phi}|\right]$$

$$+ C_{\vec{\Phi},x^0}\,(\varepsilon^3 + |t|\,\varepsilon^2)\,\left[1 + \sum_{j=1}^2 |\nabla^2\vec{a}_j|\right] \quad. \tag{A.13}$$

and finally

$$\left| \frac{d^2}{dt^2}\nabla^2\vec{\Phi}_{x^0}^\varepsilon(t) \right| \leq C_{\vec{\Phi},x^0}\,\mathbf{1}_{x^0}^\varepsilon + C_{\vec{\Phi},x^0}\,[1 + |\nabla^2\vec{\Phi}|]\,(\varepsilon^2 + |t|\,\varepsilon) \quad, \tag{A.14}$$

where $C_{\vec{\Phi}} > 0$ is independent of $(t,\varepsilon) \in \mathfrak{U}_{x^0,\vec{\Phi}}$, where $\mathbf{1}_{x^0}^\varepsilon$ is the characteristic function of the geodesic ball of center x^0 and radius $\varepsilon > 0$ for g_0, the constant scalar curvature metric of volume 1, and $C_{\vec{\Phi},x^0}$ is a constant depending only on χ, on

$$\sup_{\varepsilon > 0} \varepsilon^{-2} \int_{B_\varepsilon(x^0)} [|\mathbb{I}_{\vec{\Phi}}|_g^2 + 1]\,dvol_g$$

and on $\|\nabla a_j\|_{L^\infty(\Sigma)} + \|\log|\nabla\vec{\Phi}|\|_{L^\infty(\Sigma)}$. $\qquad\qquad\square$

Proof of Lemma A.2. We Assume

$$\varepsilon < \varepsilon_0 \qquad \text{and} \qquad |t| \leq 4^{-1}\,\|e^\lambda\|_{L^\infty(B_{\varepsilon_0}(x^0))} \tag{A.15}$$

for some $\varepsilon_0 > 0$, where the ball $B_\varepsilon(x^0)$ is the ball of radius ϵ for the flat metric of volume one g_0 conformally equivalent to $g_{\vec{\Phi}}$. We use a chart in a neighborhood of x^0 in which $g_0 = dx_1^2 + dx_2^2$ and $g_{\vec{\Phi}} = e^{2\lambda}\,g_0$. ε_0 has been then chosen sufficiently small in order for this chart $x = (x_1, x_2)$ to contain the support of $\chi_{x^0}^\varepsilon(x)$.

Denote

$$\vec{n}^0 := \vec{n}(x^0) \quad .$$

Under these assumptions, for ε_0 small enough, the perturbation of $\vec{\Phi}$

$$\vec{\Phi} + t\,\chi^\varepsilon_{x^0}\,\vec{n}(x_0)$$

is still an immersion. Consider now $\alpha_j \in \mathbb{R}$ such that

$$\sum_{j=1}^2 |\alpha_j|\,\|\vec{a}_j\|_{L^\infty(\Sigma)} < 4^{-1}\|e^{-\lambda}\|_\infty \quad . \tag{A.16}$$

Under the assumptions (A.15) and (A.16), the map

$$\vec{\Phi}^\varepsilon_{x^0}(t,\alpha_1,\alpha_2) := \beta(t,\varepsilon,\alpha_1,\alpha_2)(x)\left[\vec{\Phi} + t\,\chi^\varepsilon_{x^0}\,\vec{n}^0 + t\sum_{j=1}^2 \alpha_j\,\vec{a}_j\right]$$

is again an immersion where β is chosen in such a way that $|\vec{\Phi}^\varepsilon_{x^0}| \equiv 1$ on T^2 :

$$\begin{aligned}
\beta(t,\varepsilon,\alpha_1,\alpha_2)(x) = &\left[1 + 2\,t\,\vec{\Phi}\cdot\vec{n}^0\,\chi^\varepsilon_{x^0} + t^2(\chi^\varepsilon_{x^0})^2 + \left[\sum_{j=1}^2 t\,\alpha_j\,\vec{a}_j\right]^2\right. \\
&\left.+ 2\,t\,\chi^\varepsilon_{x^0}\left[\sum_{j=1}^2 t\,\alpha_j\,\vec{a}_j\cdot\vec{n}^0\right]\right]^{-1/2}
\end{aligned} \tag{A.17}$$

For $(t,\varepsilon,\alpha_1,\alpha_2)$ in the open neighborhood $\mathcal{U}_{\vec{\Phi}}$ of $(0,0,0,0)$ given by (A.15) and (A.16) we define $\Gamma(t,\varepsilon,\alpha_1,\alpha_2)$ by

$$\forall\,(t,\varepsilon,\alpha_1,\alpha_2)\in\mathcal{U}_{\vec{\Phi}} \quad , \quad t\neq 0 \quad \Gamma(t,\varepsilon,\alpha_1,\alpha_2) := \mathcal{C}(\vec{\Phi}^\varepsilon_{x^0}(t,\alpha_1,\alpha_2))$$

For any $\varepsilon < \varepsilon_0$ we denote by $(\alpha_1^0(\varepsilon),\alpha_2^0(\varepsilon))$ the unique element of $\mathbb{R}^2$ such that

$$d\mathcal{C}_{\vec{\Phi}}\cdot\left[\chi^\varepsilon_{x^0}\,\vec{n}^0 + \sum_{j=1}^2 \alpha_j^0(\varepsilon)\,\vec{a}_j\right] = 0$$

We are using in this assertion the fact that $d\mathcal{C}_{\vec{\Phi}}\cdot\vec{a}_j$ realizes a basis of the tangent space to the Teichmüller space $\mathcal{T}_{T^2}$ at $\mathcal{C}(\vec{\Phi})$. Let (Q^1,Q^2) be a pair of 2 independent holomorphic quadratic forms of $(T^2,[\vec{\Phi}^*g_{\mathbb{R}^3}])$, chosen to be orthonormal with respect to the Weil-Peterson metric. One has (see [Ri3]), denoting by g_0 the flat metric of volume one on $(T^2,[\vec{\Phi}^*g_{\mathbb{R}^3}])$

$$\begin{aligned}
d\mathcal{C}_{\vec{\Phi}}\cdot[\chi^\varepsilon_{x^0}\,\vec{n}] &= \sum_{j=1}^2 Q^j\int_\Sigma \chi^\varepsilon_{x^0}\,\Re<Q^j,h^0>_{wp,g_0}\,\vec{n}\cdot\vec{n}(x_0)dvol_{g_0} \\
&= \sum_{j=1}^2 Q^j\int_{B_\varepsilon(x^0)} \chi^\varepsilon_{x^0}\,<\bar{q}^j,\mathbb{I}^0_{\vec{\Phi}}>_{g_{\vec{\Phi}}}\,\vec{n}\cdot\vec{n}(x_0)\,dvol_{g_{\vec{\Phi}}}
\end{aligned}$$

where $\bar{q}^j$ is the following trace free symmetric real 2-differential given in any conformal coordinates by

$$\bar{q}^j := 2\, Q_1^j\, [dx_1^2 - dx_2^2] - 2\, Q_2^j\, [dx_1\, dx_2 + dx_2\, dx_1] \quad \text{where} \quad Q^j = (Q_1^j + i\, Q_2^j)\, dz^2 \quad .$$

and then in any conformal chart one has (see previous sections)

$$< \bar{q}^j, \mathbb{I}_{\vec{\Phi}}^0 >_{g_{\vec{\Phi}}} = 4\, e^{-4\lambda}\, [Q_1^j\, \mathbb{I}_{11}^0 - Q_2^j\, \mathbb{I}_{12}^0]$$

where $g_{\vec{\Phi}} = e^{2\lambda}\, [dx_1^2 + dx_2^2]$ in these coordinates. Therefore we have

$$|d\mathcal{C}_{\vec{\Phi}} \cdot [\chi_{x^0}^\varepsilon\, \vec{n}(x_0)]| \leq C\, \|e^{-2\lambda}\|_\infty\, \varepsilon^2\, \left[\int_{B_\varepsilon(x^0)} |\mathbb{I}_{\vec{\Phi}}^0|^2\right]^{1/2} \leq O_{\vec{\Phi},x^0}(\varepsilon^3) \quad , \tag{A.18}$$

Since x_0 is chosen to be a Lebesgue point for the second fundamental form we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{B_\varepsilon^{g_0}(x^0)} |\mathbb{I}_{\vec{\Phi}}^0|^2 dvol_{g_{\vec{\Phi}}} = \pi\, |\mathbb{I}_{\vec{\Phi}}^0|^2(x_0)\, e^{2\lambda(x_0)} \tag{A.19}$$

So we deduce from (A.18) and (A.19)

$$|\alpha_j^0| = O_{\vec{\Phi},x^0}(\varepsilon^3) \quad . \tag{A.20}$$

We claim that the mapping Γ is C^1 in $\mathcal{U}_{\vec{\Phi}}$. The C^1 property of Γ at $(t, 0, \alpha_1, \alpha_2)$ requires a justification : We have

$$\partial_\varepsilon \Gamma(t, \varepsilon, \alpha_1, \alpha_2) = d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon}\, \left[\partial_\varepsilon \log \beta\, \vec{\Phi}_{x^0}^\varepsilon + \beta\, \varepsilon^{-1} \nabla \chi(\varepsilon^{-1}(x - x^0)) \cdot (x - x^0)\right] \quad .$$

Observe that

$$\|\varepsilon^{-1} \nabla \chi(\varepsilon^{-1}(x - x^0)) \cdot (x - x^0)\|_{L^\infty(B_\varepsilon(x^0))} \leq \|\nabla \chi\|_\infty$$

This implies in particular using the explicit expression of β given by (A.17)

$$|\partial_\varepsilon \log \beta| \leq |t|\, \|\nabla \chi\|_\infty [1 + |t|]\, \left[1 + \sum_{j=1}^2 |\alpha_j|\, \|a_j\|_\infty\right]$$

and using the explicit expression of $d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon}$ we obtain that

$$|\partial_\varepsilon \Gamma(t, \varepsilon, \alpha_1, \alpha_2)| \leq C\, \|e^{2\lambda}\|_\infty\, \varepsilon\, \|\nabla \chi\|_\infty [1 + |t|]^2\, \left[1 + \sum_{j=1}^2 |\alpha_j|\, \|a_j\|_\infty\right] \left[\int_{B_\varepsilon(x^0)} |\mathbb{I}_{\vec{\Phi}}^0|^2\right]^{1/2}$$

thus $\partial_\varepsilon \Gamma(t, \varepsilon, \alpha_1, \alpha_2)$ extends continuously by 0 at $(t, 0, \alpha_1, \alpha_2)$ and the claim is proved.

Observe that we have

$$\partial_{\alpha_j}\Gamma(0,0,0,0) = d\mathcal{C}_{\vec{\Phi}} \cdot \vec{a}_j \tag{A.21}$$

By assumption on $(\vec{a}_j)_{j=1,2}$ the pair $(\partial_{\alpha_j}\Gamma(0,0,0,0))_{j=1,2}$ generates the tangent space to the Teichmüller space $\mathcal{T}_\Sigma$ at $\mathcal{C}(\vec{\Phi})$. We can then apply the local inversion theorem and deduce the existence of a neighborhood $\mathfrak{U}$ of $(0,0)$ and the existence of two C^1 functions $\alpha_1(t,\varepsilon)$ and $\alpha_2(t,\varepsilon)$ defined on $\mathfrak{U}$ such that there exists a neighborhood $\mathcal{U}$ of $(0,0,0,0)$ included in $\mathcal{U}_\Phi$ for which

$$\mathcal{U} \cap \mathcal{C}^{-1}(\mathcal{C}(\vec{\Phi})) = \{(t,\varepsilon,\alpha_1(t,\varepsilon),\alpha_2(t,\varepsilon))\} \quad .$$

It is clear that

$$\alpha_j(0,\varepsilon) = \alpha_j^0(\varepsilon) \quad .$$

Denote $\beta^\varepsilon(x,t) := \beta(t,\varepsilon,\alpha_1(t,\varepsilon),\alpha_2(t,\varepsilon))$ and

$$\vec{\Phi}_{x^0}^\varepsilon(t) := \beta^\varepsilon(x,t) \left[\vec{\Phi} + t\,\chi_{x^0}^\varepsilon\,\vec{n}^0 + t\sum_{j=1}^2 \alpha_j(t,\varepsilon)\,\vec{a}_j \right]$$

We have for any (t,ε) in $\mathfrak{U}$

$$0 \equiv d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \cdot \frac{\partial \vec{\Phi}_{x^0}^\varepsilon(t)}{\partial t} = \beta_{x^0}^\varepsilon(x,t)\,d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \cdot \left[\chi_{x^0}^\varepsilon\,\vec{n}^0 + \sum_{j=1}^2 \frac{d(t\,\alpha_j(t,\varepsilon))}{dt}\,\vec{a}_j \right] \tag{A.22}$$

where we have used the fact that

$$d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon} \vec{\Phi}_{x^0}^\varepsilon = \sum_{j=1}^2 Q^j \int_\Sigma \vec{\Phi}_{x^0}^\varepsilon \cdot < \bar{q}^j, \vec{\mathbb{I}}_{\vec{\Phi}}^0 >_{g_{\vec{\Phi}}} dvol_{g_{\vec{\Phi}_{x^0}^\varepsilon}} = 0$$

Hence

$$\sum_{j=1}^2 \frac{d(t\,\alpha_j(t,\varepsilon))}{dt}\,d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \cdot \vec{a}_j = d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \cdot \chi_{x^0}^\varepsilon\,\vec{n}^0 \tag{A.23}$$

For any $\vec{w} \in L^2$ the map

$$d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \cdot \vec{w} = \sum_{j=1}^2 Q^j \int_\Sigma \vec{w} \cdot < \bar{q}^j, \vec{\mathbb{I}}_{\vec{\Phi}_{x^0}^\varepsilon}^0 >_{g_{\vec{\Phi}_{x^0}^\varepsilon}} dvol_{g_{\vec{\Phi}_{x^0}^\varepsilon}}$$

Hence we have the existence of two maps $\vec{F}^j(x,p,q)$ for $x \in \Sigma$, $p \in \mathbb{R}^2 \otimes \mathbb{R}^4$ and $q \in \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^4$ which is algebraic in p and linear in q and such that for any $0 < \delta < 1$ and for $p = p_1 \otimes p_2$ satisfying $\delta|p|^2 \le p_1 \wedge p_2 \le \delta^{-1}|p|^2$

$$|F^j(x,p,q)| \le C_\delta\,|p|^{-2}\,|q|$$

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where C_δ only depends on δ and such that

$$d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{w} = \sum_{j=1}^{2} Q^j \int_\Sigma \vec{w} \cdot \vec{F}^j(x, \nabla \vec{\Phi}^\varepsilon_{x^0}(t), \nabla^2 \vec{\Phi}^\varepsilon_{x^0}(t)) \ dvol_{g_0}$$

Combining this fact and the explicit expression of $\vec{\Phi}^\varepsilon_{x^0}(t)$ we deduce that, for ε positive,

$$t \longrightarrow d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{w}$$

is a C^1 function. Thus the following functions are C^1 with respect to t as long as $\varepsilon > 0$:

$$d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{a}_j \quad \text{and} \quad d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \chi^\varepsilon_{x^0} \ \vec{n}^0$$

Since $d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(0)} \cdot \vec{a}_j = d\mathcal{C}_{\vec{\Phi}} \cdot \vec{a}_j$ forms a basis of the tangent space to $\mathcal{T}_{T^2}$ at $\mathcal{C}(\vec{\Phi})$, for t small enough $(d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{a}_j)$ is a C^1 map into the space of basis of $\mathcal{T}_{T^2}$ at $\mathcal{C}(\vec{\Phi})$. The functions $\frac{d(t\,\alpha_j(t,\varepsilon))}{dt}$ are then the coordinates of the C^1 map $d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \chi^\varepsilon_{x^0} \ \vec{n}^0$ in the C^1 frame $(d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{a}_j)_{j=1,2}$. Thus $\frac{d(t\,\alpha_j(t,\varepsilon))}{dt}$ are C^1 functions of t in a neighborhood of 0 for $\varepsilon > 0$.

We shall establish now the estimates (A.10), (A.12), (A.13) and (A.14). We take $t > 0$ to simplify the notations. Using the definition (A.17) denoting by $\mathbf{1}^\varepsilon_{x^0}$ the characteristic function of the geodesic ball $B_\varepsilon(x^0)$ for the metric g_0, we have

$$\left| \frac{d\beta^\varepsilon_{x^0}}{dt}(x,t) \right| \le C\,\varepsilon\,\mathbf{1}^\varepsilon_{x^0} \left[|\vec{n}(x) - \vec{n}^0| + t\,\varepsilon + \sum_{j=1}^{2} |t\,\alpha_j| + t \sum_{j=1}^{2} \left| \frac{d[t\,\alpha_j]}{dt} \right| \right]$$
$$+ C \sum_{j=1}^{2} |t\,\alpha_j| \left| \frac{d[t\,\alpha_j]}{dt} \right| \tag{A.24}$$

and

$$\left| \frac{d^2\beta^\varepsilon_{x^0}}{dt^2}(x,t) \right| \le C\,\varepsilon\,\mathbf{1}^\varepsilon_{x^0} \left[\varepsilon + \sum_{j=1}^{2} \left| \frac{d[t\,\alpha_j]}{dt} \right| + t \sum_{j=1}^{2} \left| \frac{d^2[t\,\alpha_j]}{dt^2} \right| \right]$$
$$+ C \sum_{j=1}^{2} |t\,\alpha_j| \left| \frac{d^2[t\,\alpha_j]}{dt^2} \right| + C \sum_{j=1}^{2} \left| \frac{d[t\,\alpha_j]}{dt} \right|^2 + C \left| \frac{d\beta^\varepsilon}{dt}(x,t) \right|^2 \tag{A.25}$$

We have moreover

$$|\nabla \beta^\varepsilon_{x^0}| \le C_{\vec{\Phi}}\,\mathbf{1}^\varepsilon_{x^0} \left[\varepsilon\,t\,e^\lambda |\vec{n}(x) - \vec{n}^0| + t\,|\vec{n}(x) - \vec{n}^0| + \varepsilon\,t^2 + t \sum_{j=1}^{2} |t\,\alpha_j| \right]$$
$$+ C_{\vec{\Phi}} \sum_{j=1}^{2} |t\,\alpha_j|^2 \tag{A.26}$$

and

$$
\begin{aligned}
|\nabla^2 \beta_{x^0}^\varepsilon| \leq\ & C_{\vec{\Phi}}\, \mathbf{1}_{x^0}^\varepsilon \left[t\,\varepsilon^{-1}\,|\vec{n}(x) - \vec{n}^0| + t\,\varepsilon\,|\nabla^2\vec{\Phi}| + t^2 + t\,\varepsilon^{-1} \sum_{j=1}^{2} |t\,\alpha_j| \right] \\
& + C_{\vec{\Phi}} \sum_{j=1}^{2} |t\,\alpha_j|^2 \left[1 + |\nabla^2 a_j| \right] + C\,|\nabla\beta_{x^0}^\varepsilon|^2
\end{aligned}
\tag{A.27}
$$

We have also

$$
\begin{aligned}
\left| \nabla \frac{d\beta_{x^0}^\varepsilon}{dt} \right| \leq\ & C_{\vec{\Phi}}\, \mathbf{1}_{x^0}^\varepsilon \left[\left[\, |\vec{n}(x) - \vec{n}^0| + \varepsilon\,t + \sum_{j=1}^{2} |t\,\alpha_j| + t\,\sum_{j=1}^{2} \left| \frac{d}{dt}[t\,\alpha_j] \right| \right] \right] \\
& + C_{\vec{\Phi}} \sum_{j=1}^{2} |t\,\alpha_j| \left| \frac{d[t\,\alpha_j]}{dt} \right| + |\nabla\beta_{x^0}^\varepsilon| \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right|
\end{aligned}
\tag{A.28}
$$

and

$$
\begin{aligned}
\left| \nabla \frac{d^2\beta_{x^0}^\varepsilon}{dt^2} \right| \leq\ & C_{\vec{\Phi}}\, \mathbf{1}_{x^0}^\varepsilon \left[\varepsilon + \sum_{j=1}^{2} \left| \frac{d[t\,\alpha_j]}{dt} \right| + t\,\sum_{j=1}^{2} \left| \frac{d^2[t\,\alpha_j]}{dt^2} \right| \right] \\
& + C_{\vec{\Phi}} \sum_{j=1}^{2} |t\,\alpha_j| \left| \frac{d^2[t\,\alpha_j]}{dt^2} \right| + C \sum_{j=1}^{2} \left| \frac{d[t\,\alpha_j]}{dt} \right|^2 \\
& + |\nabla\beta_{x^0}^\varepsilon| \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right|^2 + |\nabla\beta_{x^0}^\varepsilon| \left| \frac{d^2\beta_{x^0}^\varepsilon}{dt^2}(x,t) \right| + \left| \nabla \frac{d\beta_{x^0}^\varepsilon}{dt} \right| \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right|
\end{aligned}
\tag{A.29}
$$

Finally, regarding this time the second derivatives of $\beta_{x^0}^\varepsilon$ we have the following pointwise estimates

$$
\begin{aligned}
\left| \nabla^2 \frac{d\beta_{x^0}^\varepsilon}{dt} \right| \leq\ & C_{\vec{\Phi}}\, \mathbf{1}_{x^0}^\varepsilon \left[\varepsilon^{-1}\,|\vec{n}(x) - \vec{n}^0| + \varepsilon\,|\nabla^2\vec{\Phi}| + t + \varepsilon^{-1} \sum_{j=1}^{2} |t\,\alpha_j| + t\,\sum_{j=1}^{2} \left| \frac{d}{dt}[t\,\alpha_j] \right| \right] \\
& + C_{\vec{\Phi}} \sum_{j=1}^{2} |t\,\alpha_j| \left| \frac{d[t\,\alpha_j]}{dt} \right| \left[1 + |\nabla^2 a_j| \right] \\
& + |\nabla\beta_{x^0}^\varepsilon|^2 \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right| + |\nabla^2\beta_{x^0}^\varepsilon| \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right| + |\nabla\beta_{x^0}^\varepsilon| \left| \nabla \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right|
\end{aligned}
\tag{A.30}
$$

and

$$\left| \nabla^2 \frac{d^2 \beta_{x^0}^\varepsilon}{dt^2} \right| \leq C_{\vec{\Phi}} \ \mathbf{1}_{x^0}^\varepsilon \left[1 + \varepsilon^{-1} \sum_{j=1}^2 \left| \frac{d}{dt}[t \, \alpha_j] \right| + \varepsilon^{-1} t \sum_{j=1}^2 \left| \frac{d^2}{dt^2}[t \, \alpha_j] \right| \right]$$

$$+ C_{\vec{\Phi}} \sum_{j=1}^2 [|t \, \alpha_j| \left| \frac{d^2[t \, \alpha_j]}{dt^2} \right| + C_{\vec{\Phi}} \sum_{j=1}^2 [1 + |\nabla^2 a_j|] \left| \frac{d[t \, \alpha_j]}{dt} \right|^2$$

$$+ |\nabla^2 \beta_{x^0}^\varepsilon| \left| \frac{d^2 \beta_{x^0}^\varepsilon}{dt^2}(x,t) \right| + \left| \nabla \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right| \left| \nabla \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right| + |\nabla \beta_{x^0}^\varepsilon| \left| \nabla \frac{d^2 \beta_{x^0}^\varepsilon}{dt^2}(x,t) \right|$$

$$+ \left| \nabla^2 \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right| \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right| + |\nabla \beta_{x^0}^\varepsilon|^2 \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right|^2 + |\nabla \beta_{x^0}^\varepsilon|^2 \left| \frac{d^2 \beta_{x^0}^\varepsilon}{dt^2}(x,t) \right|$$

$$+ |\nabla^2 \beta_{x^0}^\varepsilon| \left| \frac{d\beta_{x^0}^\varepsilon}{dt}(x,t) \right|^2$$

$$(A.31)$$

We shall now estimate successively $(t \, \alpha_j(t, \varepsilon))$, $d/dt(t \, \alpha_j(t, \varepsilon))$ and $d^2/dt^2(t \, \alpha_j(t, \varepsilon))$. Since $d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon(0)} \cdot \vec{a}_j = d\mathcal{C}_{\vec{\Phi}} \cdot \vec{a}_j$ forms a basis of the tangent space to $\mathcal{T}_{T^2}$ at $\mathcal{C}(\vec{\Phi})$, using (A.18) and the pointwize inequality

$$\left| \mathbb{I}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \right| \leq |\mathbb{I}_{\vec{\Phi}}| + C_\Phi \left[t \, \varepsilon^{-1} + t \sum_{k=1}^2 |\alpha_k(t)| \, [|\nabla^2 a_k| + 1] \right]$$

we have

$$\left| \frac{d}{dt}(t \, \alpha_j(t, \varepsilon)) \right| \leq C \left| d\mathcal{C}_{\vec{\Phi}_{x^0}^\varepsilon(t)} \cdot \vec{n}^0 \, \chi_{x^0}^\varepsilon \right| = O_{x^0, \vec{\Phi}}(\varepsilon^3 + t \, \varepsilon^2) \tag{A.32}$$

Inserting this estimate in (A.24) gives

$$\left| \frac{d\beta_{x^0}^\varepsilon}{dt} \right| \leq C_{\vec{\Phi}} \, \varepsilon \, \mathbf{1}_{x^0}^\varepsilon \, \left[|\vec{n}(x) - \vec{n}^0| + t \, \varepsilon \right] + t \, O_{x^0, \vec{\Phi}}((\varepsilon^3 + t \, \varepsilon^2)^2) \quad . \tag{A.33}$$

in (A.26) in also gives

$$|\nabla \beta_{x^0}^\varepsilon| \leq C_{\vec{\Phi}} \, \mathbf{1}_{x^0}^\varepsilon \, \left[t \, |\vec{n}(x) - \vec{n}^0| + t^2 \, \varepsilon \right] + t^2 \, O_{x^0, \vec{\Phi}}((\varepsilon^3 + t \, \varepsilon^2)^2) \quad , \tag{A.34}$$

and

$$\left| \nabla^2 \beta_{x^0}^\varepsilon \right| \leq C_{\vec{\Phi}} \, \mathbf{1}_{x^0}^\varepsilon \, \left[t \, \varepsilon^{-1} \, |\vec{n}(x) - \vec{n}^0| + t \, \varepsilon \, |\nabla^2 \vec{\Phi}| + t^2 \right]$$

$$+ t^2 \, O_{x^0, \vec{\Phi}}((\varepsilon^3 + t \, \varepsilon^2)^2) \, [1 + |\nabla^2 a|] \quad . \tag{A.35}$$

Inserting (A.24) and the previous two estimates in (A.28) give

$$\left| \nabla \frac{d\beta_{x^0}^\varepsilon}{dt} \right| \leq C_{\vec{\Phi}} \, \mathbf{1}_{x^0}^\varepsilon \, \left[|\vec{n}(x) - \vec{n}^0| + t \, \varepsilon \right] + t \, O_{x^0, \vec{\Phi}}((\varepsilon^3 + t \, \varepsilon^2)^2) \quad . \tag{A.36}$$

Finally inserting (A.24)and the previous two estimates in (A.30) give

$$\left| \nabla^2 \frac{d\beta_{x^0}^\varepsilon}{dt} \right| \le C_{\vec{\Phi}} \, \mathbf{1}_{x^0}^\varepsilon \, \left[\varepsilon^{-1} \, |\vec{n}(x) - \vec{n}^0| + \varepsilon \, |\nabla^2 \vec{\Phi}| + t \right]$$
$$+ t \, O_{x^0,\vec{\Phi}}((\varepsilon^3 + t\,\varepsilon^2)^2)[1 + |\nabla^2 a|] \quad . \tag{A.37}$$

We shall now estimate $d(g_{\vec{\Phi}_{x^0}^\varepsilon(t)})/dt$ as well as $d(\mathbb{I}_{\vec{\Phi}_{x^0}^\varepsilon(t)}^0)/dt$. We first have

$$\left| \frac{d}{dt} \nabla \vec{\Phi}_{x^0}^\varepsilon(t) \right| \le \left| \frac{d}{dt} \nabla \beta_{x^0}^\varepsilon(t) \right| + C_{\vec{\Phi}} \left| \frac{d}{dt} \beta_{x^0}^\varepsilon(t) \right| \, [1 + |t\,\alpha_j(t,\varepsilon)|]$$
$$+ C_{\vec{\Phi}} \left[\mathbf{1}_{x^0}^\varepsilon + \left| \frac{d}{dt}(t\,\alpha_j(t,\varepsilon)) \right| + |\nabla \beta_{x^0}^\varepsilon| \, \left[\varepsilon \, \mathbf{1}_{x^0}^\varepsilon + \left| \frac{d}{dt}(t\,\alpha_j(t,\varepsilon)) \right| \right] \right] \quad . \tag{A.38}$$

From which we deduce using the previous estimates

$$\left| \frac{d}{dt} \nabla \vec{\Phi}_{x^0}^\varepsilon(t) \right| \le C_{\vec{\Phi}} \, \left[\mathbf{1}_{x^0}^\varepsilon + O_{x^0,\vec{\Phi}}(\varepsilon^3 + t\,\varepsilon^2) \right] \quad . \tag{A.39}$$

Regarding the metric $g_{\vec{\Phi}_{x^0}^\varepsilon(t)}$ we have

$$g_{\vec{\Phi}_{x^0}^\varepsilon(t)} = \nabla \vec{\Phi}_{x^0}^\varepsilon(t) \dot\otimes \nabla \vec{\Phi}_{x^0}^\varepsilon(t)$$

So we deduce

$$\left| \frac{dg_{ij}}{dt} - \beta^2 \frac{d}{dt} \left(\partial_{x_i}(\beta^{-1} \vec{\Phi}_{x^0}^\varepsilon(t)) \cdot \partial_{x_j}(\beta^{-1} \vec{\Phi}_{x^0}^\varepsilon(t)) \right) \right| \le C_{\vec{\Phi}} \left[\left| \nabla \frac{d\beta}{dt} \right| + \left| \frac{d\beta}{dt} \right| + |\nabla \beta| \, \left| \frac{d}{dt} \nabla \vec{\Phi}_{x^0}^\varepsilon(t) \right| \right]$$

Observe that we have

$$\left(\partial_{x_i}(\beta^{-1} \vec{\Phi}_{x^0}^\varepsilon(t)) \cdot \partial_{x_j}(\beta^{-1} \vec{\Phi}_{x^0}^\varepsilon(t)) \right) = \delta_{ij} \, e^{2\lambda}$$

$$+ t \, [\vec{n}^0 - \vec{n}] \, \left[\partial_{x_i}\chi(\varepsilon^{-1}(x - x^0)) \, \partial_{x_j}\vec{\Phi} + \partial_{x_j}\chi(\varepsilon^{-1}(x - x^0)) \, \partial_{x_i}\vec{\Phi} \right]$$

$$+ t \sum_{k=1}^{2} \alpha_k(t) \, [\partial_{x_j}\vec{a}_k \cdot \partial_{x_i}\vec{\Phi} + \partial_{x_i}\vec{a}_k \cdot \partial_{x_j}\vec{\Phi}]$$

$$+ t^2 \sum_{k=1}^{2} \left[\partial_{x_i}\chi(\varepsilon^{-1}(x - x^0)) \, \vec{n}^0 \cdot \partial_{x_j}\vec{a}_k + \partial_{x_j}\chi(\varepsilon^{-1}(x - x^0)) \, \vec{n}^0 \cdot \partial_{x_i}\vec{a}_k \right]$$

$$+ t^2 \partial_{x_i}\chi(\varepsilon^{-1}(x - x^0))\partial_{x_j}\chi(\varepsilon^{-1}(x - x^0)) + t^2 \sum_{k=1}^{2} \alpha_k(t) \, \partial_{x_i}\vec{a}_k \cdot \sum_{k=1}^{2} \alpha_k(t) \, \partial_{x_j}\vec{a}_k$$

We deduce from the previous identities

$$\left| \frac{dg_{\vec{\Phi}^\varepsilon_{x^0}(t)}}{dt} \right| \leq C_{\vec{\Phi}} \left[\mathbf{1}^\varepsilon_{x^0}(|\vec{n} - \vec{n}^0| + t) + O_{x^0,\vec{\Phi}}(\varepsilon^3 + t\,\varepsilon^2) \right] \tag{A.41}$$

Now we have

$$\left| \frac{d}{dt} \nabla^2 \vec{\Phi}^\varepsilon_{x^0}(t) \right| \leq \left| \frac{d}{dt} \nabla^2 \beta^\varepsilon_{x^0}(t) \right| + C_{\vec{\Phi}} \left| \nabla \beta^\varepsilon_{x^0} \right| \left[\mathbf{1}^\varepsilon_{x^0} + \left| \frac{d}{dt}(t\,\alpha_j(t,\varepsilon)) \right| \right]$$

$$+ C_{\vec{\Phi}} \left| \frac{d}{dt} \nabla \beta^\varepsilon_{x^0}(t) \right| + C_{\vec{\Phi}} \left[\varepsilon^{-1} \mathbf{1}^\varepsilon_{x^0} + \left| \frac{d}{dt}(t\,\alpha_j(t,\varepsilon)) \right| |\nabla^2 a_j| \right] \tag{A.42}$$

$$+ C_{\vec{\Phi}} \left| \nabla^2 \beta^\varepsilon_{x^0} \right| \left[\varepsilon\, \mathbf{1}^\varepsilon_{x^0} + \left| \frac{d}{dt}(t\,\alpha_j(t,\varepsilon)) \right| \right]$$

From which we deduce using the previous estimates

$$\left| \frac{d}{dt} \nabla^2 \vec{\Phi}^\varepsilon_{x^0}(t) \right| \leq C_{\vec{\Phi}} \, \mathbf{1}^\varepsilon_{x^0} \left[\varepsilon^{-1} + \varepsilon\, |\nabla^2 \vec{\Phi}| \right] + O_{x^0,\vec{\Phi}}(\varepsilon^3 + t\,\varepsilon^2) \left[1 + \sum_{j=1}^2 |\nabla^2 \vec{a}_j| \right] \quad . \tag{A.43}$$

This gives then

$$\left| \frac{d}{dt} \mathbb{I}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \right| \leq C_{\vec{\Phi}} \, \mathbf{1}^\varepsilon_{x^0} \left[\varepsilon^{-1} + \varepsilon\, |\nabla^2 \vec{\Phi}| \right] + O_{x^0,\vec{\Phi}}(\varepsilon^3 + t\,\varepsilon^2) \left[1 + \sum_{j=1}^2 |\nabla^2 \vec{a}_j| \right] \quad . \tag{A.44}$$

Taking now the t derivative of the identity (A.23)

$$\sum_{j=1}^2 \frac{d^2(t\,\alpha_j(t,\varepsilon))}{dt^2} \, dC_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{a}_j = - \sum_{j=1}^2 \frac{d(t\,\alpha_j(t,\varepsilon))}{dt} \, \frac{d\left(dC_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{a}_j \right)}{dt}$$

$$+ \frac{d\left(dC_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \chi^\varepsilon_{x^0}\,\vec{n}^0 \right)}{dt} \tag{A.45}$$

Since again $dC_{\vec{\Phi}^\varepsilon_{x^0}(0)} \cdot \vec{a}_j = dC_{\vec{\Phi}} \cdot \vec{a}_j$ forms a basis of the tangent space to $\mathcal{T}_{T^2}$ at $\mathcal{C}(\vec{\Phi})$ we deduce that for $j = 1, 2$

$$\left| \frac{d^2(t\,\alpha_j(t,\varepsilon))}{dt^2} \right| \leq \sum_{j=1}^2 \left| \frac{d(t\,\alpha_j(t,\varepsilon))}{dt} \right| \left| \frac{d\left(dC_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{a}_j \right)}{dt} \right|$$

$$+ \left| \frac{d\left(dC_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \chi^\varepsilon_{x^0}\,\vec{n}^0 \right)}{dt} \right| \tag{A.46}$$

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Using (A.41) and (A.44) we obtain in one hand

$$\left| \frac{d\left(d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \vec{a}_j\right)}{dt} \right| \leq O_{x^0,\vec{\Phi}}(\varepsilon) \tag{A.47}$$

and in the other hand

$$\left| \frac{d\left(d\mathcal{C}_{\vec{\Phi}^\varepsilon_{x^0}(t)} \cdot \chi^\varepsilon_{x^0} \, \vec{n}^0\right)}{dt} \right| \leq O_{x^0,\vec{\Phi}}(\varepsilon^2) \tag{A.48}$$

Hence combining (A.46), (A.47) and (A.48) we obtain

$$\left| \frac{d^2(t\,\alpha_j(t,\varepsilon))}{dt^2} \right| \leq O_{x^0,\vec{\Phi}}(\varepsilon^2) \tag{A.49}$$

We bound now $d^2(\nabla^2\vec{\Phi}^\varepsilon_{x^0})/dt^2$. We have

$$\left| \frac{d^2}{dt^2}\nabla^2\vec{\Phi}^\varepsilon_{x^0}(t) \right| \leq \left| \frac{d^2}{dt^2}\nabla^2\beta^\varepsilon_{x^0}(t) \right| + \left| \frac{d}{dt}\nabla^2\beta^\varepsilon_{x^0}(t) \right| \left[|\nabla\vec{\Phi}| + C\,t\,\mathbf{1}^\varepsilon_{x^0} + C\,|t\,\alpha_j(t,\varepsilon)| \right]$$

$$+ |\nabla^2\beta^\varepsilon_{x^0}| \left[\left| \frac{d^2}{dt^2}(t\,\alpha_j(t,\varepsilon)) \right| \right] + \left| \frac{d^2}{dt^2}\nabla\beta^\varepsilon_{x^0}(t) \right| \left[|\nabla\vec{\Phi}| + C\,t\,\mathbf{1}^\varepsilon_{x^0} + C\,|t\,\alpha_j(t,\varepsilon)| \right]$$

$$+ \left| \frac{d}{dt}\nabla\beta^\varepsilon_{x^0}(t) \right| \left[C\,\mathbf{1}^\varepsilon_{x^0} + C\,\left| \frac{d(t\,\alpha_j(t,\varepsilon))}{dt} \right| \right] + C\,|\nabla\beta^\varepsilon_{x^0}(t)| \left| \frac{d^2(t\,\alpha_j(t,\varepsilon))}{dt^2} \right|$$

$$+ \left| \frac{d^2}{dt^2}\beta^\varepsilon_{x^0}(t) \right| \left[|\nabla\vec{\Phi}| + C\,t\,\varepsilon^{-1}\,\mathbf{1}^\varepsilon_{x^0} + C\,|t\,\alpha_j(t,\varepsilon)| \right]$$

$$+ \left| \frac{d}{dt}\beta^\varepsilon_{x^0}(t) \right| \left[C\,\varepsilon^{-1}\,\mathbf{1}^\varepsilon_{x^0} + C\,\left| \frac{d(t\,\alpha_j(t,\varepsilon))}{dt} \right| \right] + C\,\left| \frac{d^2}{dt^2}(t\,\alpha_j(t,\varepsilon)) \right| \tag{A.50}$$

Inserting the previous estimates in this inequality gives, after a lengthy computation,

$$\left| \frac{d^2}{dt^2}\nabla^2\vec{\Phi}^\varepsilon_{x^0}(t) \right| \leq C_{\vec{\Phi},x^0}\,\mathbf{1}^\varepsilon_{x^0} + [1 + |\nabla^2\vec{\Phi}|]\,O_{x^0,\vec{\Phi}}(\varepsilon^2 + t\,\varepsilon) \quad . \tag{A.51}$$

which concludes the proof of lemma A.2. $\qquad\square$

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